Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness and stability results for a quasilinear periodic Muskat problem

Anca-Voichita Matioc [a], Bogdan-Vasile Matioc [b,*]

[a] *Leibniz Universität Hannover, Institut für Angewandte Mathematik, Welfengarten 1, 30167 Hannover, Germany*
[b] *Universität Regensburg, Fakultät für Mathematik, D-93040 Regensburg, Germany*

Received 3 March 2018; revised 24 August 2018

Abstract

We study the Muskat problem describing the spatially periodic motion of two fluids with equal viscosities under the effect of gravity in a vertical unbounded two-dimensional geometry. We first prove that the classical formulation of the problem is equivalent to a nonlocal and nonlinear evolution equation expressed in terms of singular integrals and having only the interface between the fluids as unknown. Secondly, we show that this evolution equation has a quasilinear structure, which is at a formal level not obvious, and we also disclose the parabolic character of the equation. Exploiting these aspects, we establish the local well-posedness of the problem for arbitrary initial data in $H^s(\mathbb{S})$, with $s \in (3/2, 2)$, determine a new criterion for the global existence of solutions, and uncover a parabolic smoothing property. Besides, we prove that the zero steady-state solution is exponentially stable.
© 2018 Elsevier Inc. All rights reserved.

MSC: 35B35; 35B65; 35K59; 35Q35; 42B20

Keywords: Muskat problem; Singular integral; Well-posedness; Parabolic smoothing; Stability

* Corresponding author.
E-mail addresses: matioca@ifam.uni-hannover.de (A.-V. Matioc), bogdan.matioc@ur.de (B.-V. Matioc).

https://doi.org/10.1016/j.jde.2018.10.038
0022-0396/© 2018 Elsevier Inc. All rights reserved.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

1. Introduction and the main results

We study the contour-integral formulation

$$\begin{cases} \partial_t f(t,x) = -\dfrac{k\Delta_\rho}{4\pi\mu} f'(t,x)\,\mathrm{PV}\displaystyle\int_{-\pi}^{\pi} f'(t,x-s)\frac{(T_{[x,s]}f(t))(1+t_{[s]}^2)}{t_{[s]}^2+(T_{[x,s]}f(t))^2}\,ds \\ \qquad\qquad -\dfrac{k\Delta_\rho}{4\pi\mu}\,\mathrm{PV}\displaystyle\int_{-\pi}^{\pi} f'(t,x-s)\frac{t_{[s]}[1-(T_{[x,s]}f(t))^2]}{t_{[s]}^2+(T_{[x,s]}f(t))^2}\,ds, \qquad t>0,\ x\in\mathbb{R}, \\ f(0)=f_0 \end{cases} \tag{1.1}$$

of the Muskat problem, that describes the evolution of the interface $[y = f(t,x)]$ separating two periodic immiscible fluid layers of unbounded height and with equal viscosity constants in a homogeneous porous medium with permeability k (or a vertical Hele–Shaw cell), the fluid system being close to the rest state far away from the interface, see Section 2.[1] The unknown $f = f(t,x)$ is thus assumed to be 2π-periodic with respect to the variable x. For this reason, we use $\mathbb{S}$ to denote the unit circle $\mathbb{R}/2\pi\mathbb{Z}$, functions depending on $x\in\mathbb{S}$ being 2π-periodic with respect to the real variable x. We further denote by g the Earth's gravity, $\rho_\pm$ is the density of the fluid $\pm$ which is located at $\Omega_\pm(t)$, where

$$\Omega_-(t) := [y < f(t,x)] \qquad \text{and} \qquad \Omega_+(t) := [y > f(t,x)],$$

and $\mu_- = \mu_+ =: \mu$ is the viscosity coefficient of the fluids. For brevity, we write $(\cdot)'$ for the spatial derivative ∂_x and PV stands for the principle value which is taken at $s = 0$. Our analysis is in the regime where the Rayleigh–Taylor condition

$$\Delta_\rho := g(\rho_- - \rho_+) > 0 \tag{1.2}$$

is satisfied. Furthermore, in order to keep the notation short we have set

$$\delta_{[x,s]}f := f(x) - f(x-s), \qquad T_{[x,s]}f = \tanh\Big(\frac{\delta_{[x,s]}f}{2}\Big), \qquad t_{[s]} = \tan\Big(\frac{s}{2}\Big).$$

Our analysis covers only the Muskat problem without surface tension, because we expect that when surface tension is taken into account we obtain the very same results as in the general case when the viscosity constants are not necessarily equal (this issue will be addressed in a subsequent paper). Such a behavior is not expected when neglecting surface tension, cf. [38,39].

Formulated by the American petroleum engineer M. Muskat in 1934 (see [41]) to describe the intrusion of water into an oil sand, the Muskat problem is a classical model in the petroleum engineering and as such it has received much attention also in the field of applied mathematics. Because of the complexity of the mathematical formulation most of the progress in the analysis of this problem is quite recent, see the references [5,6,8,10,12–23,25,27,29–34,39,43–46,48].

[1] In fact, the problem (1.1) is also a model for the evolution of a fluid system that moves vertically with velocity $(0, V)$, for some $V\in\mathbb{R}$, and for which the interface between the fluids is parameterized as the graph $[y = f(t,x) + Vt]$, cf. [38].

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Moreover, a large body of literature considers the special case when the two fluids have equal viscosities and the surface tension effects are neglected, as in this particular context the equations of motion (see (2.1)) have an elegant contour-integral formulation (see (1.1) in the periodic setting).

With respect to the goals of this paper, we mention the following results pertaining to (1.1) and compare them to our main results:

(*i*) **Local existence and uniqueness:** In [21] the authors establish the local existence and uniqueness of solutions to (1.1) for general initial data in $f_0 \in H^3(\mathbb{S})$. This result was improved recently in [15] where, in the more general context of fluids with general (not necessarily equal) viscosities, initial data $f_0 \in H^2(\mathbb{S})$ that are small with respect to some $H^{3/2+\varepsilon}(\mathbb{S})$-norm, where $\varepsilon \in (0,1)$ is arbitrarily small, are considered. Besides, it was shown in [15] that solutions corresponding to small initial data in $H^2(\mathbb{S})$ exist globally. In our first main result, that is Theorem 1.1, we establish the well-posedness of the evolution problem (1.1) – that is, the local existence and uniqueness of classical solutions as well as the continuous dependence of the solutions on the initial data – in the Sobolev space $H^s(\mathbb{S})$, with $s \in (3/2, 2)$. Hence, a more general class of initial data is considered and no smallness assumptions are required. Moreover, Theorem 1.1 provides also a new criterion for the global existence of solutions.

(*ii*) **Exponential stability:** In contrast to the case of confined (that is bounded) geometries considered in [24–26,29,44], stability results for the zero solution to (1.1) are not available. Related to this issue we mention the exponential stability result for the zero solution to the one-phase inhomogeneous Muskat problem in $H^2(\mathbb{S})$, established only recently in [33], and the decaying estimates with respect to $H^r(\mathbb{S})$-norms, $r \in [0,2)$, of solutions to (1.1) corresponding to small initial data in $H^2(\mathbb{S})$, see [15]. Theorem 1.3 provides the exponential stability of the zero solution to the unconfined Muskat problem and shows how the physical properties of the fluids influence the convergence rate of solutions to (1.1), determined by small data in $H^2(\mathbb{S})$, towards the zero solution.

(*iii*) **Instantaneous real-analyticity:** In [39] it was shown, in the nonperiodic case, that the solutions to the Muskat problem that start in $H^s(\mathbb{R})$, $s > 3/2$, become instantly real-analytic, and they stay so until they cease to exist. A similar result is available in the periodic case only for the one-phase problem, see [15]. Theorem 1.1 below establishes also the expected parabolic smoothing property for (1.1).

We start our analysis by proving in Proposition 2.1, in a quite general setting, that the classical formulation (2.1) of the Muskat problem and the contour-integral formulation (1.1) are equivalent. Our analysis is motivated by the previous paper [39] where analogous results to Theorem 1.1 and Proposition 2.1 have been established in the nonperiodic case. The main ingredients exploited in [39] are: (I) the quasilinearity of the nonperiodic version

$$\partial_t f(t,x) = \frac{k\Delta_\rho}{2\pi\mu}\,\mathrm{PV}\int_{\mathbb{R}} \frac{y(f'(t,x)-f'(t,x-y))}{y^2+(f(t,x)-f(t,x-y))^2}\,dy \qquad \text{for } t>0,\ x\in\mathbb{R}, \tag{1.3}$$

of $(1.1)_1$, a property which is obvious at least at formal level since the integrand in the latter equation is linear with respect to the highest order spatial derivative of f, that is f', and; (II) the parabolic character of the problem when assuming that the Rayleigh–Taylor condition (1.2) is

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

satisfied. With respect to (I) we point out that the situation changes in the periodic case considered herein as the right-hand side of $(1.1)_1$ is not linear with respect to f', and it cannot be modified to become linear in f'. Despite this, we are still able to formulate (1.1) in a suitable functional analytic setting as a quasilinear evolution problem (see Section 3). Concerning (II), this aspect has been firstly evinced in a bounded geometry in [27], and subsequently in [38,39] for the unconfined Muskat problem for fluids with equal or different viscosities. It is shown in Section 4 that the parabolicity property is a defining feature also for (1.1). These two properties are also crucial in this paper as they enable us to use theory developed by H. Amann [1–3] (see Theorem A.4 in the Appendix A) and A. Lunardi [36,37] (the nonlinear principle of linearized stability) in the proofs of our main results, see Sections 4–5.

Compared to the nonperiodic case studied in [39], the analysis in this paper is more involved for two clear reasons: firstly because, as mentioned above, the equation $(1.1)_1$ is not linear with respect to f'; and, secondly because a deep result from harmonic analysis (see the main theorem in [40]) which was used in an essential way in [39] has no correspondent in the periodic setting.

The first main result of this paper is the following theorem.

Theorem 1.1. *Let $\Delta_\rho > 0$ and $s \in (3/2, 2)$ be given. Then, the following hold:*

(*i*) ***Well-posedness in*** $H^s(\mathbb{S})$***:*** *The problem (1.1) possesses for each $f_0 \in H^s(\mathbb{S})$ a unique maximal solution*

$$f := f(\cdot; f_0) \in \mathrm{C}([0, T_+(f_0)), H^s(\mathbb{S})) \cap \mathrm{C}((0, T_+(f_0)), H^2(\mathbb{S})) \cap \mathrm{C}^1((0, T_+(f_0)), H^1(\mathbb{S})),$$

with $T_+(f_0) \in (0, \infty]$, and $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $H^s(\mathbb{S})$.

(*ii*) ***Global existence/blow-up criterion:*** *If*

$$\sup_{[0, T_+(f_0)) \cap [0, T]} \|f(t)\|_{H^s(\mathbb{S})} < \infty \qquad \text{for all } T > 0,$$

then $T_+(f_0) = \infty$.

(*iii*) ***Parabolic smoothing:*** *The mapping $[(t, x) \mapsto f(t, x)] : (0, T_+(f_0)) \times \mathbb{R} \to \mathbb{R}$ is real-analytic. In particular, $f(t)$ is a real-analytic function for all $t \in (0, T_+(f_0))$.*

With respect to Theorem 1.1 we add the following.

Remark 1.2.

(*i*) We conjecture (in both periodic and nonperiodic settings) that $s = 3/2$ is (similarly as for the Camassa–Holm equation [11]) the critical Sobolev index for the local well-posedness of the Muskat problem for fluids with equal viscosities. We expect that proving the ill-posedness of (1.1) for $s < 3/2$ is more difficult than for the Camassa–Holm equation, as the proof in [11] exploits the availability of some explicit solutions, which have no correspondent for (1.1). Besides, the structure of the Camassa–Holm equation is simpler as the highest order term is a local one.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

(*ii*) We point out that if $T_+(f_0) < \infty$ for some $f_0 \in H^s(\mathbb{S})$, $s \in (3/2, 2)$, then Theorem 1.1 implies

$$\inf_{r\in(3/2,s)} \sup_{[0,T_+(f_0))} \|f(t)\|_{H^r(\mathbb{S})} = \infty.$$

(*iii*) The analysis in the nonperiodic case in [38] suggests that the case when the viscosities are not equal is more involved. The quasilinear character of the problem is lost in this regime and the parabolicity property holds only if the initial data satisfy the Rayleigh–Taylor condition (which is in this setting a nonlocal and nonlinear condition on the initial data and no longer so simple as in (1.2)). However, we expect that a similar stability result as in Theorem 1.3 can be established also when $\mu_- \neq \mu_+$.

The second main result of the paper is the following asymptotic stability result.

Theorem 1.3 *(Exponential stability). Let $\Delta_\rho > 0$. Then, the zero solution to (1.1) is exponentially stable. More precisely, given $\omega \in (0, k\Delta_\rho/2\mu)$, there exist constants $r > 0$ and $M > 0$, with the property that if $f_0 \in H^2(\mathbb{S})$ satisfies*

$$\|f_0\|_{H^2(\mathbb{S})} < r \qquad \text{and} \qquad \int_{-\pi}^{\pi} f_0\, dx = 0,$$

then $T_+(f_0) = \infty$ and

$$\|f(t)\|_{H^2(\mathbb{S})} + \|\partial_t f(t)\|_{H^1(\mathbb{S})} \leq Me^{-\omega t}\|f_0\|_{H^2(\mathbb{S})} \qquad \text{for all } t \geq 0.$$

In particular, Theorem 1.3 shows how the exponential stability of the zero solution is influence by the physical properties of the fluids. Solutions that correspond to small data converge at a faster rate towards the zero solution if the density jump or the permeability are larger or if the viscosity is smaller. We point out that due to the fact that the integral mean of the initial data is preserved by the flow, see Section 5, only solutions with zero integral mean may converge to the zero steady-state.

2. The equations of motion and the equivalent formulation

In this section we present the equations governing the evolution of the fluids system and we rigorously prove that the latter are equivalent to the contour-integral formulation (1.1). Since for flows in porous media the conservation of momentum equation can be replaced by Darcy's law, cf. e.g. [9], in the fluid layers the dynamic is governed by the following equations

$$\begin{cases} \operatorname{div} v_\pm(t) = 0 & \text{in } \Omega_\pm(t), \\ v_\pm(t) = -\dfrac{k}{\mu}\big(\nabla p_\pm(t) + (0, \rho_\pm g)\big) & \text{in } \Omega_\pm(t) \end{cases} \tag{2.1a}$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

for $t > 0$, where $v_\pm(t) := (v_\pm^1(t), v_\pm^2(t))$ denotes the velocity vector and $p_\pm(t)$ the pressure of the fluid $\pm$. These equations are supplemented by the natural boundary conditions at the free interface

$$\begin{cases} p_+(t) - p_-(t) = 0 & \text{on } [y = f(t,x)], \\ \langle v_+(t) | \nu(t) \rangle = \langle v_-(t) | \nu(t) \rangle & \text{on } [y = f(t,x)], \end{cases} \tag{2.1b}$$

where $\nu(t)$ is the unit normal at $[y = f(t,x)]$ pointing into $\Omega_+(t)$ and $\langle \cdot \,|\, \cdot \rangle$ the inner product in $\mathbb{R}^2$. Furthermore, we impose the far-field boundary condition

$$v_\pm(t,x,y) \to 0 \quad \text{for } |y| \to \infty \text{ and uniformly with respect to } x, \tag{2.1c}$$

which states that the fluid motion is localized, the fluids being close to the rest state far way from the free interface $[y = f(t,x)]$. The motion of this interface is coupled to that of the fluids through the kinematic boundary condition

$$\partial_t f(t) = \langle v_\pm(t) | (-f'(t), 1) \rangle \qquad \text{on } [y = f(t,x)]. \tag{2.1d}$$

As we consider periodic flows, it is also assumed that $f(t)$, $v_\pm(t)$, and $p_\pm(t)$ are 2π-periodic with respect to x for all $t > 0$. Finally, the interface at time $t = 0$ is given

$$f(0) = f_0. \tag{2.1e}$$

The equations (2.1) are known as the Muskat problem and they determine entirely the dynamic of the fluid system. We now show in Proposition 2.1 that the Muskat problem (2.1) is equivalent to the system (1.1) presented in the introduction. This is the first rigorous equivalent result for the two formulations.

Proposition 2.1 *(Equivalence result). Let $T \in (0, \infty]$ be given. The following assertions are equivalent:*

(i) The triple $(f, v_\pm, p_\pm)$ with regularity $f \in \mathrm{C}([0,T), L_2(\mathbb{S})) \cap \mathrm{C}^1((0,T), L_2(\mathbb{S}))$ and

$$f(t) \in H^2(\mathbb{S}), \quad v_\pm(t) \in \mathrm{C}(\overline{\Omega}_\pm(t)) \cap \mathrm{C}^1(\Omega_\pm(t)), \quad p_\pm(t) \in \mathrm{C}^1(\overline{\Omega}_\pm(t)) \cap \mathrm{C}^2(\Omega_\pm(t))$$

for all $t \in (0,T)$ is a solution to the Muskat problem (2.1);

(ii) f with regularity $f \in \mathrm{C}([0,T), L_2(\mathbb{S})) \cap \mathrm{C}^1((0,T), L_2(\mathbb{S}))$ and $f(t) \in H^2(\mathbb{S})$ for all $t \in (0,T)$ is a solution to (1.1).

Proof. In the arguments that follow the dependence on time is not written explicitly. Given a test function $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, it is easy to verify that the vorticity

$$\omega := \mathrm{rot}\, v := \partial_x v^2 - \partial_y v^1 \in \mathcal{D}'(\mathbb{R}^2)$$

associated to the global velocity field $v := (v^1, v^2) := v_- \mathbf{1}_{[y \le f(x)]} + v_+ \mathbf{1}_{[y > f(x)]}$ satisfies

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$\langle \omega, \varphi \rangle = \int_{\mathbb{R}} \overline{\omega}(x)\varphi(x, f(x))\, dx,$$

where

$$\overline{\omega} := -\frac{k\Delta_\rho}{\mu} f' \in H^1(\mathbb{S}). \tag{2.2}$$

We next prove that the velocity v is determined by the function f. More precisely, we show that v coincides in $\mathbb{R}^2 \setminus [y = f(x)]$ with the velocity field $\widetilde{v}$ introduced via

$$\begin{aligned} \widetilde{v}^1(x, y) &:= -\frac{1}{4\pi} \int_{\mathbb{S}} \overline{\omega}(s) \frac{\tanh((y - f(s))/2)\big[1 + \tan^2((x - s)/2)\big]}{\tan^2((x - s)/2) + \tanh^2((y - f(s))/2)}\, ds, \\ \widetilde{v}^2(x, y) &:= \frac{1}{4\pi} \int_{\mathbb{S}} \overline{\omega}(s) \frac{\tan((x - s)/2)\big[1 - \tanh^2((y - f(s))/2)\big]}{\tan^2((x - s)/2) + \tanh^2((y - f(s))/2)}\, ds \end{aligned} \tag{2.3}$$

for $(x, y) \in \mathbb{R}^2 \setminus [y = f(x)]$. To this end, we first observe that $\widetilde{v}$ is well-defined away from $[y = f(x)]$ and 2π-periodic with respect to x. Recalling (2.2), it is not difficult to see that the far-field boundary condition (2.1c) is also satisfied by $\widetilde{v}$. Furthermore, $\widetilde{v}^i \in \mathrm{C}^1(\mathbb{R}^2 \setminus [y = f(x)])$, $i = 1, 2$, and, letting $\widetilde{v}_\pm := \widetilde{v}|_{\Omega_\pm}$, it holds that

$$\operatorname{div} \widetilde{v}_\pm = \operatorname{rot} \widetilde{v}_\pm = 0 \qquad \text{in } \Omega_\pm. \tag{2.4}$$

As a further step, we show that $\widetilde{v}_\pm \in \mathrm{C}(\overline{\Omega}_\pm)$ and that the value of $\widetilde{v}_\pm$ on the boundary $[y = f(x)]$ is given by the formulas

$$\begin{aligned} \widetilde{v}^1_\pm(x, f(x)) &= -\frac{1}{4\pi} \operatorname{PV} \int_{\mathbb{S}} \overline{\omega}(s) \frac{\tanh((f(x) - f(s))/2)\big[1 + \tan^2((x - s)/2)\big]}{\tan^2((x - s)/2) + \tanh^2((f(x) - f(s))/2)}\, ds \\ &\quad \mp \frac{1}{2} \frac{\overline{\omega}(x)}{1 + f'^2(x)}, \\ \widetilde{v}^2_\pm(x, f(x)) &= \frac{1}{4\pi} \operatorname{PV} \int_{\mathbb{S}} \overline{\omega}(s) \frac{\tan((x - s)/2)\big[1 - \tanh^2((f(x) - f(s))/2)\big]}{\tan^2((x - s)/2) + \tanh^2((f(x) - f(s))/2)}\, ds \\ &\quad \mp \frac{1}{2} \frac{\overline{\omega}(x) f'(x)}{1 + f'^2(x)}, \end{aligned} \tag{2.5}$$

where

$$\Big(\frac{\overline{\omega}(x)}{1 + f'^2(x)}, \frac{\overline{\omega}(x) f'(x)}{1 + f'^2(x)}\Big) = \frac{\overline{\omega}(x)(1, f'(x))}{1 + f'^2(x)}$$

is a vector tangent to the free surface $[y = f(x)]$. Moreover, in (2.5) the principle value is taken at $s = x$ (if we integrate over $(x - \pi, x + \pi)$). Changing variables, (2.5) can be compactly written as

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$\widetilde{v}_{\pm}(x,f(x))=\frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi}\overline{\omega}(x-s)\frac{\big(-(T_{[x,s]}f)(1+t_{[s]}^2),\,t_{[s]}[1-(T_{[x,s]}f)^2]\big)}{t_{[s]}^2+(T_{[x,s]}f)^2}\,ds \mp\frac{1}{2}\frac{\overline{\omega}(x)(1,f'(x))}{1+f'^2(x)}, \tag{2.6}$$

and now the PV is taken at zero. Using the fact that $\overline{\omega}\in \mathrm{C}^{1/2}(\mathbb{S})$ and $f\in \mathrm{C}^{3/2}(\mathbb{S})$, it is matter of direct computation to see, in view of the inequalities $x\le\tan(x)$ for $x\in[0,\pi/2)$ and $\tanh(x)\le x$ for $x\ge 0$, that the principal value integrals (2.5) exist. In order to prove that the functions defined in (2.3) extend continuously up to the free surface $[y=f(x)]$, it suffices to show that the limits $\widetilde{v}_{-}(x,f(x))$ and $\widetilde{v}_{+}(x,f(x))$ of $\widetilde{v}$ at $(x,f(x))$, when we approach this point from below the interface $[y=f(x)]$ or from above, respectively, exist and that they are as given in (2.5). To this end, we first note that

$$\widetilde{v}_{\pm}(z)=\overline{\frac{1}{4\pi i}\int_{\Gamma}\frac{g(w)}{\tan((w-z)/2)}\,dw}\qquad\text{for } z=(x,y)\notin[y=f(x)],$$

where we integrate over

$$\Gamma:=\{(s,f(s)):x-\pi\le s<x+\pi\}$$

and where $g:\Gamma\to\mathbb{C}$ is given by

$$g(s,f(s)):=g(s):=-\frac{\overline{\omega}(s)(1-if'(s))}{1+f'^2(s)},\qquad s\in\mathbb{R}.$$

It holds that $g\in \mathrm{C}^{1/2}(\mathbb{S},\mathbb{R}^2)$ and additionally

$$\begin{aligned}\overline{\widetilde{v}_{\pm}}(z)&=\frac{1}{4\pi i}\int_{\Gamma}\frac{g(w)}{\tan((w-z)/2)}\,dw\\&=\frac{1}{4\pi i}\int_{\Gamma}g(w)\Big[\frac{1}{\tan((w-z)/2)}-\frac{2}{w-z}\Big]dw+\frac{1}{2\pi i}\int_{\Gamma}\frac{g(w)}{w-z}\,dw.\end{aligned}$$

Observing that $|\tan(z)-z|\le C|z|^3$ for all $|z|\le 1$, the first integral is not singular for $z=(x,f(x))$ and the limit $z\to(x,f(x))$ can be performed by using classical arguments. For the second integral we may use the Plemelj theorem, see e.g. [35], to pass to the limit $z\to(x,f(x))$ and to conclude in this way that indeed $\overline{\widetilde{v}_{\pm}}(z)\to\widetilde{v}_{\pm}(x,f(x))$, together with $\widetilde{v}_{\pm}\in \mathrm{C}(\overline{\Omega}_{\pm})$. Setting $V_{\pm}:=v_{\pm}-\widetilde{v}_{\pm}$, the function $V:=(V^1,V^2):=V_{-}\mathbf{1}_{[y\le f(x)]}+V_{+}\mathbf{1}_{[y>f(x)]}$ belongs to $\mathrm{BC}(\mathbb{R}^2)$. Let

$$\psi_{\pm}(x,y):=\int_{f(x)}^{y}V_{\pm}^1(x,s)\,ds-\int_{0}^{x}\langle V_{\pm}(s,f(s))|(-f'(s),1)\rangle\,ds\qquad\text{for } (x,y)\in\overline{\Omega}_{\pm},$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

be the stream function associated to $V_\pm$. It then follows from (2.4) that $\psi := \psi_- \mathbf{1}_{[y\leq f]} + \psi_+ \mathbf{1}_{[y>f]}$ belongs to $\mathrm{C}^1(\mathbb{R}^2)$ and satisfies $\Delta\psi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence, ψ is the real part of a holomorphic function $u : \mathbb{C} \to \mathbb{C}$. Since u' is also holomorphic and $u' = \partial_x\psi - i\partial_y\psi = -(V^2, V^1)$ is bounded and vanishes for $|y| \to \infty$ it follows that $u' = 0$, hence $V = 0$. This proves that $v_\pm = \widetilde{v}_\pm$. The kinematic boundary condition (2.1e) together with (2.6) imply that f solves the evolution problem (1.1).

Vice versa, given a solution f to (1.1), we define the velocity fields $v_\pm \in \mathrm{C}(\overline{\Omega}_\pm) \cap \mathrm{C}^1(\Omega_\pm)$ by (2.5) and the pressures $p_\pm \in \mathrm{C}^1(\overline{\Omega}_\pm) \cap \mathrm{C}^2(\Omega_\pm)$ according to

$$p_\pm(x,y) := c_\pm - \frac{\mu}{k}\int_0^x v_\pm^1(s,\pm d)\,ds - \frac{\mu}{k}\int_{\pm d}^y v_\pm^2(x,s)\,ds - \rho_\pm g y, \qquad (x,y) \in \overline{\Omega}_\pm$$

for some $d > \|f\|_\infty$ and $c_\pm \in \mathbb{R}$. We note that $p_\pm$ are 2π-periodic with respect to the x-variable if and only if

$$\int_0^{2\pi} v_\pm^1(s,\pm d)\,ds = 0. \tag{2.7}$$

In order to establish (2.7), we infer from (2.4) that

$$\int_0^{2\pi} (\partial_x v_\pm^2 - \partial_y v_\pm^1)(x,\pm y)\,dx = 0 \qquad \text{for all } y > \|f\|_\infty,$$

and, exploiting the periodicity of $v_\pm^2$, we find constants $C_\pm \in \mathbb{R}$ such that

$$\int_0^{2\pi} v_\pm^1(x,\pm y)\,dx = C_\pm \qquad \text{for all } y > \|f\|_\infty.$$

Recalling (2.1c), we conclude that $C_\pm = 0$, and this proves the 2π-periodicity of $p_\pm$. Finally, it is easy to see, in view of (2.6), that the constants $c_\pm$ can be chosen in such a way that the triple $(f, v_\pm, p_\pm)$ solves (2.1). □

3. The Muskat problem as a quasilinear evolution equation

The goal of this section is to show that the Muskat problem (1.1) can indeed be formulated as a quasilinear evolution problem (see (3.1)). This is the first stage in the analysis of (1.1). As mentioned in the introduction, this essential property of (1.1) is not obvious at all, in contrast to the nonperiodic case where it is evident (at least at a formal level). The departure from the nonperiodic case is due to the fact that an argument from [39] that uses integration by parts is not applicable to $(1.1)_1$ in order to further transform this evolution equation. The key point in the analysis presented below is to relate the equation $(1.1)_1$ to the contour-integral formulation (1.3)

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

obtained in the nonperiodic case and to enforce the integration by parts argument from [39]. This procedure will give rise to several integral terms of "lower order".[2]

To begin, we first note that (1.1) can be expressed, after rescaling the time appropriately, in the following compact form

$$\partial_t f = \Phi(f)[f], \quad t > 0, \qquad f(0) = f_0, \tag{3.1}$$

where $\Phi(f)$ is a linear operator which is decomposed as

$$\Phi(f) = \Phi_0(f) - \Phi_1(f) - \Phi_2(f), \tag{3.2}$$

with

$$\Phi_0(f)[h](x) := \mathrm{PV} \int_{-\pi}^{\pi} \frac{\delta_{[x,s]} h'}{t_{[s]}} \frac{1}{1 + \left(T_{[x,s]} f / t_{[s]}\right)^2} \, ds, \tag{3.3}$$

$$\Phi_1(f)[h](x) := h'(x) \, \mathrm{PV} \int_{-\pi}^{\pi} \left[\frac{f'(x-s)}{t_{[s]}} \frac{T_{[x,s]} f}{t_{[s]}} + \frac{1}{t_{[s]}} \right] \frac{1}{1 + \left(T_{[x,s]} f / t_{[s]}\right)^2} \, ds, \tag{3.4}$$

$$\begin{aligned} \Phi_2(f)[h](x) := {} & h'(x) \int_{-\pi}^{\pi} f'(x-s) \frac{T_{[x,s]} f}{1 + \left(T_{[x,s]} f / t_{[s]}\right)^2} \, ds \\ & - \int_{-\pi}^{\pi} h'(x-s) \frac{\left(T_{[x,s]} f / t_{[s]}\right) T_{[x,s]} f}{1 + \left(T_{[x,s]} f / t_{[s]}\right)^2} \, ds. \end{aligned} \tag{3.5}$$

In the following the function f is chosen to be an element of the periodic Sobolev space $H^s(\mathbb{S})$, $s \in (3/2, 2)$, and the appropriate space for h is $H^2(\mathbb{S})$. However, when dealing with "lower order" terms we sometimes choose $h \in H^s(\mathbb{S})$. We first note that Φ_2 encompasses only integral terms that are not singular and therefore the PV is not needed. As we shall see below, cf. Lemma 3.3, the integral defined in (3.4) and which appears to be singular can be decomposed as a sum of three integral terms which are all not singular. Finally, the integral in (3.3) is singular and quite similar to the right-hand side of (1.3), and this term proves to be the most important one in the analysis.

The main result of this section is the regularity property stated in Proposition 3.1, which shows that (3.1) is indeed a quasilinear evolution problem.

Proposition 3.1. *Given $s \in (3/2, 2)$, it holds that*

$$\Phi \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))). \tag{3.6}$$

Proof. The assertion is a direct consequence of (3.2) and of the Lemmas 3.2–3.4 below. □

[2] As it is clear from Theorem 1.1, the Muskat problem (1.1) is a first order evolution problem.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Similarly as in the nonperiodic case [39], it can be shown by using the Lemmas 3.2–3.4 and other classical arguments, that the operator Φ is actually real-analytic, that is

$$\Phi \in C^{\omega}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))) \qquad \text{for each } s \in (3/2, 2). \tag{3.7}$$

The lengthy details are left to the interested reader.

To proceed, we first study the mapping properties of the operator Φ_2.

Lemma 3.2. *Given $s \in (3/2, 2)$, it holds that*

$$\Phi_2 \in C^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))). \tag{3.8}$$

Proof. We decompose $\Phi_2(f)[h] = h'\phi_1(f) - \phi_2(f)[h]$, where, given f, $h \in H^s(\mathbb{S})$, we set

$$\phi_1(f)(x) := \int_{-\pi}^{\pi} f'(x-s) \frac{T_{[x,s]} f}{1 + \big(T_{[x,s]} f / t_{[s]}\big)^2}\, ds, \tag{3.9}$$

$$\phi_2(f)[h](x) := \int_{-\pi}^{\pi} h'(x-s) \frac{\big(T_{[x,s]} f / t_{[s]}\big) T_{[x,s]} f}{1 + \big(T_{[x,s]} f / t_{[s]}\big)^2}\, ds. \tag{3.10}$$

The theorem on the differentiation of parameter integrals shows that $\phi_1(f)$, $\phi_2(f)[h] \in C^1(\mathbb{S})$, $i = 1, 2$, with

$$\begin{aligned}(\phi_1(f))'(x) = {} & \frac{f'(x)}{2} \int_{-\pi}^{\pi} f'(x-s) \frac{\big[1 - (T_{[x,s]} f)^2\big]\big[1 - \big(T_{[x,s]} f / t_{[s]}\big)^2\big]}{\big[1 + \big(T_{[x,s]} f / t_{[s]}\big)^2\big]^2}\, ds, \\ & + \int_{-\pi}^{\pi} f'(x-s) \frac{(1 + t_{[s]}^2)(T_{[x,s]} f)^2}{t_{[s]}^2} \frac{T_{[x,s]} f / t_{[s]}}{\big[1 + \big(T_{[x,s]} f / t_{[s]}\big)^2\big]^2}\, ds,\end{aligned}$$

respectively

$$\begin{aligned}(\phi_2(f)[h])'(x) = {} & f'(x) \int_{-\pi}^{\pi} h'(x-s) \frac{\big[1 - (T_{[x,s]} f)^2\big]\big(T_{[x,s]} f / t_{[s]}\big)}{\big[1 + \big(T_{[x,s]} f / t_{[s]}\big)^2\big]^2}\, ds, \\ & - \frac{1}{2} \int_{-\pi}^{\pi} h'(x-s) \frac{(1 + t_{[s]}^2)(T_{[x,s]} f)^2}{t_{[s]}^2} \frac{1 - \big(T_{[x,s]} f / t_{[s]}\big)^2}{\big[1 + \big(T_{[x,s]} f / t_{[s]}\big)^2\big]^2}\, ds.\end{aligned}$$

The previous formulas imply that $\Phi_2(f) \in \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))$. To be more precise, these formulas actually show that

$$\phi_1 \in C^{1-}(H^s(\mathbb{S}), C^1(\mathbb{S})), \tag{3.11}$$

$$\phi_2 \in C^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^s(\mathbb{S}), C^1(\mathbb{S}))). \tag{3.12}$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

The desired local Lipschitz continuity property (3.8) is a simple consequence of the (3.11)–(3.12). □

Compared to (3.8), we have taken in (3.12) the variable h from the larger space $H^s(\mathbb{S})$, with $s \in (3/2, 2)$. This enables us later on to identify the operator $\phi_2(f)$ as a "lower order" term in the decomposition (3.2) of $\Phi(f)$.

We next consider the operator Φ_1. At first glance, the integral in (3.4) is singular and this could hinder us to write the problem (1.1) as a quasilinear evolution problem. Nevertheless, exploiting the cancellation in a singular integral whose integrand behaves at first order asymptotically for $s \to 0$ in the same way as that in (3.4), cf. (3.18), (we are guided at this point by the analysis in the nonperiodic case [39]), we show below that this singular integral defines a function in $C^1(\mathbb{S})$ when requiring merely that $f \in H^s(\mathbb{R})$. This is an important observation with respect to our goal of expressing $(1.1)_1$ as quasilinear evolution equation.

Lemma 3.3. *Given $s \in (3/2, 2)$ and $f \in H^s(\mathbb{S})$, we let*

$$\phi_3(f)(x) := \mathrm{PV} \int_{-\pi}^{\pi} \Big[\frac{f'(x-s)}{t_{[s]}} \frac{T_{[x,s]} f}{t_{[s]}} + \frac{1}{t_{[s]}} \Big] \frac{1}{1 + \big(T_{[x,s]} f / t_{[s]}\big)^2} \, ds. \tag{3.13}$$

Then, it holds that

$$\phi_3 \in C^{1-}(H^s(\mathbb{S}), C^1(\mathbb{S})) \qquad \text{and} \qquad \Phi_1 \in C^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))). \tag{3.14}$$

Proof. Since $f \in C^{s-1/2}(\mathbb{S})$, standard arguments show that the limit $\phi_3(f)(x)$ exists (in the sense of PV) for all $x \in \mathbb{R}$. We now decompose the function $\phi_3(f)$ as follows. Using the relations

$$|\tanh(x) - x| \le |x|^3, \ x \in \mathbb{R}, \qquad \text{and} \qquad |\tan(x) - x| \le |x| \tan^2(x), \ |x| < \pi/2, \tag{3.15}$$

we may write

$$\phi_3(f) = \phi_{3a}(f) + \phi_{3b}(f) + \frac{1}{2} \phi_{3c}(f),$$

where

$$\phi_{3a}(f)(x) := \int_{-\pi}^{\pi} \frac{f'(x-s)}{(t_{[s]})^2 + (T_{[x,s]} f)^2} \Big[\tanh\Big(\frac{\delta_{[x,s]} f}{2} \Big) - \frac{\delta_{[x,s]} f}{2} \Big] ds, \tag{3.16}$$

$$\phi_{3b}(f)(x) := \int_{-\pi}^{\pi} \frac{1}{(t_{[s]})^2 + (T_{[x,s]} f)^2} \Big[\tan\Big(\frac{s}{2} \Big) - \frac{s}{2} \Big] ds, \tag{3.17}$$

$$\phi_{3c}(f)(x) := \mathrm{PV} \int_{-\pi}^{\pi} \frac{s + f'(x-s) \delta_{[x,s]} f}{(t_{[s]})^2 + (T_{[x,s]} f)^2} \, ds.$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Using the relation

$$\mathrm{PV}\int_{-\pi}^{\pi}\frac{s+f'(x-s)(\delta_{[x,s]}f)}{s^2+(\delta_{[x,s]}f)^2}\,ds=0, \tag{3.18}$$

see [39], we may express $\phi_{3c}(f)$ as

$$\phi_{3c}(f)(x)=\int_{-\pi}^{\pi}[s+f'(x-s)(\delta_{[x,s]}f)]\Big[\frac{1}{(t_{[s]})^2+(T_{[x,s]}f)^2}-\frac{4}{s^2+(\delta_{[x,s]}f)^2}\Big]ds. \tag{3.19}$$

Due to (3.15), the principal value is not needed in (3.16), (3.17), and (3.19). Furthermore, (3.15) leads us to the conclusion that $\phi_{3a}(f)$, $\phi_{3b}(f)$, $\phi_{3c}(f)\in \mathrm{C}^1(\mathbb{S})$, with

$$\begin{aligned}(\phi_{3a}(f))'(x)=&-f'(x)\int_{-\pi}^{\pi}f'(x-s)\frac{(T_{[x,s]}f)\big[1-(T_{[x,s]}f)^2\big]}{\big[t_{[s]}^2+(T_{[x,s]}f)^2\big]^2}\Big[T_{[x,s]}f-\frac{\delta_{[x,s]}f}{2}\Big]ds\\&-\int_{-\pi}^{\pi}f'(x-s)\frac{t_{[s]}\big[1+(t_{[s]})^2\big]}{\big[t_{[s]}^2+(T_{[x,s]}f)^2\big]^2}\Big[T_{[x,s]}f-\frac{\delta_{[x,s]}f}{2}\Big]ds\\&-\frac{f'(x)}{2}\int_{-\pi}^{\pi}f'(x-s)\frac{(T_{[x,s]}f)^2}{(t_{[s]})^2+(T_{[x,s]}f)^2}\,ds,\\(\phi_{3b}(f))'(x)=&-\int_{-\pi}^{\pi}\delta_{[x,s]}f'\frac{t_{[s]}-(s/2)}{\big[t_{[s]}^2+(T_{[x,s]}f)^2\big]^2}(T_{[x,s]}f)\big[1-(T_{[x,s]}f)^2\big]ds,\end{aligned}$$

respectively

$$\begin{aligned}(\phi_{3c}(f))'(x)=&\int_{-\pi}^{\pi}s\,\delta_{[x,s]}f'\Big[\frac{8(\delta_{[x,s]}f)}{\big[s^2+(\delta_{[x,s]}f)^2\big]^2}-\frac{(T_{[x,s]}f)\big[1-(T_{[x,s]}f)^2\big]}{\big[t_{[s]}^2+(T_{[x,s]}f)^2\big]^2}\Big]ds\\&+f'(x)\int_{-\pi}^{\pi}f'(x-s)\Big[\frac{1}{(t_{[s]})^2+(T_{[x,s]}f)^2}-\frac{4}{s^2+(\delta_{[x,s]}f)^2}\Big]ds\\&+\int_{-\pi}^{\pi}f'(x-s)\,\delta_{[x,s]}f\Big[\frac{8s}{\big[s^2+(\delta_{[x,s]}f)^2\big]^2}-\frac{t_{[s]}(1+(t_{[s]})^2)}{\big[(t_{[s]})^2+(T_{[x,s]}f)^2\big]^2}\,ds\\&+\frac{8f'(x)(\delta_{[x,s]}f)}{\big[s^2+(\delta_{[x,s]}f)^2\big]^2}-\frac{f'(x)(T_{[x,s]}f)(1+(T_{[x,s]}f)^2)}{\big[(t_{[s]})^2+(T_{[x,s]}f)^2\big]^2}\Big]ds.\end{aligned}$$

Noticing that $\Phi_1(f)=\phi_3(f)\partial_x$, the local Lipschitz continuity properties stated at (3.14) are direct consequences of the formulas above. □

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

We now turn our attention to the singular integral operator $\Phi_0(f)$ defined (3.3).

Lemma 3.4. *It holds that*

$$\Phi_0 \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))). \tag{3.20}$$

Proof. It is convenient to write

$$\Phi_0 = \Phi_a - \Phi_b + 2\Phi_c,$$

where

$$\Phi_a(f)[h] := h'\phi_4(f), \tag{3.21}$$

$$\Phi_b(f)[h](x) := \int_{-\pi}^{\pi} h'(x-s)\Big[\frac{1}{t_{[s]}}\frac{1}{1+\big(T_{[x,s]}f/t_{[s]}\big)^2} - \frac{1}{s/2}\frac{1}{1+\big(\delta_{[x,s]}f/s\big)^2}\Big]ds, \tag{3.22}$$

$$\Phi_c(f)[h](x) := \mathrm{PV}\int_{-\pi}^{\pi} \frac{\delta_{[x,s]}h'}{s}\frac{1}{1+\big(\delta_{[x,s]}f/s\big)^2}\,ds, \tag{3.23}$$

and

$$\phi_4(f)(x) := \int_{-\pi}^{\pi}\Big[\frac{1}{t_{[s]}}\frac{1}{1+\big(T_{[x,s]}f/t_{[s]}\big)^2} - \frac{1}{s/2}\frac{1}{1+\big(\delta_{[x,s]}f/s\big)^2}\Big]ds. \tag{3.24}$$

The relation (3.15) shows the principal value is not needed in (3.22) and (3.24). Similarly as in the Lemmas 3.2–3.3, it follows that

$$\phi_4 \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathrm{C}^1(\mathbb{S})), \tag{3.25}$$

$$\Phi_b \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^s(\mathbb{S}), \mathrm{C}^1(\mathbb{S}))). \tag{3.26}$$

In order to deal with the operator Φ_c we introduce for each $h \in H^k(\mathbb{S})$, $k \in \mathbb{N}$, the function $\widetilde{h} := h\varphi$, where $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}, [0,1])$ is a fixed function chosen such that $\varphi = 1$ for $|x| \le 2\pi$ and $\varphi = 0$ for $|x| \ge 4\pi$. Then, $\widetilde{h} \in H^k(\mathbb{R})$ and there exists $C = C(k)$ such that

$$\|h\|_{H^k(\mathbb{S})} \le \|\widetilde{h}\|_{H^k(\mathbb{R})} \le C\|h\|_{H^k(\mathbb{S})} \qquad \text{for all } h \in H^k(\mathbb{S}). \tag{3.27}$$

For later use, we notice that (3.27) together with interpolation property (4.21) implies that for each $r \ge 0$ there exists $C = C(r)$ such that

$$\|\widetilde{h}\|_{H^r(\mathbb{R})} \le C\|h\|_{H^r(\mathbb{S})} \qquad \text{for all } h \in H^r(\mathbb{S}). \tag{3.28}$$

It is obvious that for each f, $h \in H^s(\mathbb{S})$ the function $\Phi_c(f)[h]$ is well-defined and it belongs to $\mathrm{C}(\mathbb{S})$ (the PV is actually not needed). Given $x \in (-\pi, \pi)$, we may write

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$\Phi_c(f)[h](x) = \mathrm{PV}\int_{-\pi}^{\pi} \frac{\delta_{[x,s]}\widetilde{h}'}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds = A_1(f)[\widetilde{h}](x) - A_2(f)[h](x), \quad (3.29)$$

where

$$A_1(f)[\widetilde{h}](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,s]}\widetilde{h}'}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds,$$

$$A_2(f)[h](x) := \mathrm{PV}\int_{|s|>\pi} \frac{\delta_{[x,s]}(\varphi h)'}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds.$$

The latter formulas make sense for arbitrary $x\in\mathbb{R}$ and the principal value needs to be taken also at infinity. Our first goal is to prove that $A_1(f)[\widetilde{h}], A_2(f)[h] \in H^1((-\pi,\pi))$. Having shown this property, (3.29) together with $\Phi_c(f)[h]\in \mathrm{C}(\mathbb{S})$ implies that $\Phi_c(f)[h]\in H^1(\mathbb{S})$ and that $(\Phi_c(f)[h])'$ is the periodic extension of $(A_1(f)[\widetilde{h}] - A_2(f)[h])'$ (see Lemma A.2).

Arguing as in the proof of [39, Lemma 3.5], it follows from Lemma A.1, that

$$A_1 \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))), \quad (3.30)$$

with

$$\begin{aligned}(A_1(f)[\widetilde{h}])'(x) = \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,s]}\widetilde{h}''}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds \\ - 2\,\mathrm{PV}\int_{\mathbb{R}} \frac{\left[\delta_{[x,s]}f/s\right]\left[(\delta_{[x,s]}f'/s\right]\left[\delta_{[x,s]}\widetilde{h}'/s\right]}{\left[1+\left(\delta_{[x,s]}f/s\right)^2\right]^2}\,ds.\end{aligned} \quad (3.31)$$

We now turn our attention to the operator A_2, which we decomposed as the difference

$$A_2 = A_{2a} - A_{2b},$$

where, given $x\in(-\pi,\pi)$, we have set

$$A_{2a}(f)[h](x) := h'(x)\,\mathrm{PV}\int_{|s|>\pi} \frac{1}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds,$$

$$A_{2b}(f)[h](x) := \int_{|s|>\pi} \frac{(\varphi h)'(x-s)}{s}\frac{1}{1+\left(\delta_{[x,s]}f/s\right)^2}\,ds)s.$$

It can be easily verified that the function

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$\begin{aligned}\phi_5(f)(x) &:= \mathrm{PV}\int\limits_{|s|>\pi} \frac{1}{s}\frac{1}{1+\big(\delta_{[x,s]}f/s\big)^2}\,ds\\ &= \int\limits_{\pi}^{\infty}\frac{1}{s^3}\frac{(\delta_{[x,-s]}f)^2-(\delta_{[x,s]}f)^2}{[1+\big(\delta_{[x,s]}f/s\big)^2][1+\big(\delta_{[x,-s]}f/s\big)^2]}\,ds\end{aligned} \tag{3.32}$$

satisfies

$$\phi_5 \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathrm{BC}^1(\mathbb{R})), \tag{3.33}$$

and therewith

$$A_{2a} \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^2(\mathbb{S}), H^1((-\pi,\pi)))).$$

Concerning A_{2b}, we recall that $\operatorname{supp}\varphi \subset [-4\pi, 4\pi]$, meaning that

$$A_{2b}(f)[h](x) = \int\limits_{\pi<|s|<5\pi} \frac{(\varphi h)'(x-s)}{s}\frac{1}{1+\big(\delta_{[x,s]}f/s\big)^2}\,ds \qquad \text{for } x\in(-\pi,\pi). \tag{3.34}$$

In view of (3.34), we find that the function $A_{2b}(f)[h]$ is differentiable on $(-\pi,\pi)$ for each $h \in H^s(\mathbb{S})$, with

$$\begin{aligned}(A_{2b}(f)[h])'(x) = -2f'(x)&\int\limits_{\pi<|s|<5\pi}\frac{(\varphi h)'(x-s)}{s^2}\frac{\delta_{[x,s]}f/s}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^2}\,ds\\ &-\int\limits_{\pi<|s|<5\pi}\frac{(\varphi h)'(x-s)}{s^2}\frac{1-\big(\delta_{[x,s]}f/s\big)^2}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^2}\,ds,\end{aligned} \tag{3.35}$$

and therewith we find

$$A_{2b} \in \mathrm{C}^{1-}(H^s(\mathbb{S}), \mathcal{L}(H^s(\mathbb{S}), \mathrm{C}^1([-\pi,\pi]))). \tag{3.36}$$

The desired claim (3.20) is a direct consequence of (3.25), (3.26), (3.29), (3.30), (3.34), and (3.36). □

4. The sectoriality property of the principal part

The first goal of this section is to prove that the operator $\Phi(f)$, regarded as an unbounded operator in $H^1(\mathbb{S})$ with definition domain $H^2(\mathbb{S})$ is, for each $f \in H^s(\mathbb{S})$, $s \in (3/2, 2)$, the generator of a strongly continuous analytic semigroup in $\mathcal{L}(H^1(\mathbb{S}))$, that is (in the notation used in [4])

$$-\Phi(f) \in \mathcal{H}(H^2(\mathbb{S}), H^1(\mathbb{S})).$$

To this end it suffices to show that the complexification of this unbounded operator, which we denote again by $\Phi(f)$, that is the operator

$$\big[h = u + iv \mapsto \Phi(f)[u] + i\Phi(f)[v]\big] : H^2(\mathbb{S}, \mathbb{C}) \subset H^1(\mathbb{S}, \mathbb{C}) \to H^1(\mathbb{S}, \mathbb{C}),$$

generates such a semigroup in $\mathcal{L}(H^1(\mathbb{S}, \mathbb{C}))$, see [37, Corollary 2.1.3].[3]

The desired generator property is established in Theorem 4.3. In fact, we only need to prove that there exist constants $\omega > 0$ and $\kappa \geq 1$ such that

$$\omega - \Phi(f) \in \mathrm{Isom}(H^2(\mathbb{S}), H^1(\mathbb{S})) \tag{4.1}$$

and

$$\kappa \|(\lambda - \Phi(f))[h]\|_{H^1(\mathbb{S})} \geq |\lambda| \cdot \|h\|_{H^1(\mathbb{S})} + \|h\|_{H^2(\mathbb{S})} \tag{4.2}$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \omega$ and $h \in H^2(\mathbb{S})$, cf. [4]. We end the section by presenting the proof of our first main result Theorem 1.1.

To start, we choose for each integer $p \geq 3$ a set $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\} \subset \mathrm{C}^\infty(\mathbb{S}, [0, 1])$ such that

- $\mathrm{supp}\,\pi_j^p = \cup_{n\in\mathbb{Z}}\big(2\pi n + I_j^p\big)$ and $I_j^p := [j - 5/3, j - 1/3]\dfrac{\pi}{2^p}$;
- $\displaystyle\sum_{j=1}^{2^{p+1}} \pi_j^p = 1$ in $\mathrm{C}(\mathbb{S})$.

We call $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$ a *p-partition of unity*. Moreover, let $\{\chi_j^p : 1 \leq j \leq 2^{p+1}\} \subset \mathrm{C}^\infty(\mathbb{S}, [0, 1])$ be an associated set of functions such that

- $\mathrm{supp}\,\chi_j^p = \cup_{n\in\mathbb{Z}}\big(2\pi n + J_j^p\big)$ with $I_j^p \subset J_j^p := [j - 8/3, j + 2/3]\dfrac{\pi}{2^p}$; (4.3)
- $\chi_j^p = 1$ on I_j^p. (4.4)

The integer p should be seen as a parameter. The larger p is, the better can be the operator $\Phi(f)$ locally approximated by certain Fourier multipliers (which are much simpler to analyze), see the proof of Theorem 4.2. The following remark is a simple exercise.

Remark 4.1. Let $k, p \in \mathbb{N}$ with $p \geq 3$ be given and let $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$ be a p-partition of unity. The mapping

$$\Big[h \mapsto \max_{1\leq j\leq 2^{p+1}} \|\pi_j^p h\|_{H^k(\mathbb{S})}\Big] : H^k(\mathbb{S}) \to \mathbb{R}$$

defines a norm on $H^k(\mathbb{S})$ which is equivalent to the standard Sobolev norm.

[3] The variable h is in the following complex valued, while f is arbitrary (but fixed) and real-valued. Having made this convention, we use $H^s(\mathbb{S})$ to denote both Sobolev spaces of real- or complex-valued functions. This applies in the entire section, excepting the proof of Theorem 1.1 where we come back to the setting of real-valued Sobolev functions considered in all of the other sections.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

With respect to our goal (4.1), it is convenient to introduce the continuous path

$$[\tau \mapsto \Phi(\tau f)] : [0,1] \to \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S})),$$

which transforms the operator $\Phi(f)$ into the well-known operator

$$\Phi(0)[h](x) = -\,\mathrm{PV} \int_{-\pi}^{\pi} \frac{h'(x-s)}{t_{[s]}}\, ds = -2\pi H[h'](x),$$

where H denotes the periodic Hilbert transform, see e.g. [47]. Since H is the Fourier multiplier with symbol $(-i\,\mathrm{sign}(m))_{m\in\mathbb{Z}}$, it follows that $\Phi(0) = -2\pi(-\partial_x^2)^{1/2}$ is the Fourier multiplier with symbol $(-2\pi|m|)_{m\in\mathbb{Z}}$. The following theorem is a commutator type result which states that $\Phi(\tau f)$ can be locally approximated by Fourier multipliers that can be explicitly determined.

Theorem 4.2. *Let $f \in H^s(\mathbb{S})$, $s \in (3/2, 2)$, and $\mu > 0$ be given, and set*

$$s' := \max\left\{s, \frac{11-2s}{4}\right\} \in (3/2, 2).$$

Then, there exist $p \geq 3$, a finite p-partition of unity $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$, a constant $K = K(p)$, and for each $j \in \{1, \dots, 2^{p+1}\}$ and $\tau \in [0,1]$ there exist operators

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))$$

such that

$$\|\pi_j^p \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^p h]\|_{H^1(\mathbb{S})} \leq \mu \|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^{s'}(\mathbb{S})} \tag{4.5}$$

for all $j \in \{1, \dots, 2^{p+1}\}$, $\tau \in [0,1]$, and $h \in H^2(\mathbb{S})$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := a_\tau(x_j^p)\partial_x - b_\tau(x_j^p)(-\partial_x^2)^{1/2}, \tag{4.6}$$

where $x_j^p \in I_j^p$ is arbitrary, but fixed. Furthermore,

$$a_\tau := \phi_4(\tau f) + 2\phi_6(\tau f) - \phi_1(\tau f) - \phi_3(\tau f) \qquad \text{and} \qquad b_\tau := \frac{2\pi}{1+\tau^2 f'^2}, \tag{4.7}$$

where ϕ_1, ϕ_3, and ϕ_4 are defined in (3.9), (3.13), and (3.24), respectively, and with

$$\phi_6(f)(x) := \mathrm{PV} \int_{-\pi}^{\pi} \frac{1}{s} \frac{1}{1+(\delta_{[x,s]}f/s)^2}\, ds.$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Proof. Let $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$, with $p \geq 3$ to be fixed later on, be a p-partition of unity and let $\{\chi_j^p : 1 \leq j \leq 2^{p+1}\}$ be a family associated to this p-partition of unity which satisfies (4.3)–(4.4). In the following we let C denote constants which are independent of $p \in \mathbb{N}$, $h \in H^2(\mathbb{S})$, $\tau \in [0,1]$, and $j \in \{1,\dots,2^{p+1}\}$, and the constants denoted by K may depend only upon p.

Step 1: Some "lower order" terms. In view of (3.2), it follows directly from (3.3)–(3.5) and Lemma 3.3, that

$$\|\Phi(\tau f)[h]\|_\infty \leq C\|h\|_{H^s(\mathbb{S})},$$

and therewith

$$\|\pi_j^p \Phi(\tau f)[h]\|_{L_2(\mathbb{S})} \leq C\|h\|_{H^s(\mathbb{S})}.$$

Moreover, combining (3.11), (3.14), (3.25), and (A.1), it holds

$$\sup_{\tau\in[0,1]} \|a_\tau\|_\infty \leq C,$$

and therewith we get

$$\|\mathbb{A}_{j,\tau}[\pi_j^p h]\|_{L_2(\mathbb{S})} \leq K\|h\|_{H^1(\mathbb{S})}.$$

These estimates lead us to

$$\begin{aligned} &\|\pi_j^p \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^p h]\|_{L_2(\mathbb{S})} \leq \|\Phi(\tau f)[h]\|_{L_2(\mathbb{S})} + \|\mathbb{A}_{j,\tau}[\pi_j^p h]\|_2 \leq K\|h\|_{H^s(\mathbb{S})},\\ &\|(\pi_j^p)'\Phi(\tau f)[h]\|_{L_2(\mathbb{S})} \leq K\|h\|_{H^s(\mathbb{S})}, \end{aligned} \tag{4.8}$$

and we are left to estimate the quantity $\|\pi_j^p(\Phi(\tau f)[h])' - (\mathbb{A}_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})}$. To this end we use the decomposition $\Phi(f) = \Phi_0(f) - \Phi_1(f) - \Phi_2(f)$ provided in (3.2) and we establish suitable estimates for each of these three operators separately.

Step 2: The operator $\Phi_1(f)$. Letting

$$\mathbb{A}^1_{j,\tau} := \phi_3(\tau f)(x_j^p)\partial_x,$$

we deduce from Lemma 3.3 and (4.3)–(4.4) that

$$\begin{aligned} &\|\pi_j^p(\Phi_1(\tau f)[h])' - (\mathbb{A}^1_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})}\\ &\quad\leq \|(\phi_3(\tau f) - \phi_3(\tau f)(x_j^p))(\pi_j^p h)''\|_{L_2(\mathbb{S})} + K\|h\|_{H^1(\mathbb{S})}\\ &\quad\leq \|(\phi_3(\tau f) - \phi_3(\tau f)(x_j^p))\chi_j^p\|_\infty \|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^1(\mathbb{S})}\\ &\quad\leq \frac{\mu}{3}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^1(\mathbb{S})} \end{aligned} \tag{4.9}$$

if p is chosen sufficiently large.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Step 3: The operator $\Phi_2(f)$. Letting

$$\mathbb{A}^2_{j,\tau} := \phi_1(\tau f)(x_j^p)\partial_x,$$

we deduce from (3.11) and (3.12), similarly as above, that

$$\begin{aligned}
&\|\pi_j^p(\Phi_2(\tau f)[h])' - (\mathbb{A}^2_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})} \\
&\quad \le \|\pi_j^p(\phi_1(\tau f)h')' - (\mathbb{A}^2_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})} + \|\pi_j^p(\phi_2(\tau f)[h])'\|_{L_2(\mathbb{S})} \\
&\quad \le \|(\phi_1(\tau f) - \phi_1(\tau f)(x_j^p))\chi_j^p\|_\infty \|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})} \\
&\quad \le \frac{\mu}{3}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})}
\end{aligned} \tag{4.10}$$

if p is again sufficiently large.

Step 4: The operator $\Phi_0(f)$. The estimates for this operator are more involved than for the other two. To start, we introduce

$$\mathbb{A}^0_{j,\tau} := [\phi_4(\tau f)(x_j^p) + 2\phi_6(\tau f)(x_j^p)]\partial_x + \frac{2}{1+\tau^2 f'^2(x_j^p)}\Phi_c(0)[h],$$

where Φ_c is defined in (3.23). Recalling Lemma 3.4, it holds that

$$(\Phi_0(\tau f)[h])' = (h'\phi_4(\tau f))' - (\Phi_b(\tau f)[h])' + (2\Phi_c(\tau f)[h])'.$$

Together with (3.25) and (3.26) we obtain, similarly as in the previous 2 steps, for p sufficiently large, that

$$\begin{aligned}
&\|\pi_j^p(\Phi_0(\tau f)[h])' - (\mathbb{A}^0_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})} \\
&\quad \le \|\pi_j^p(h'\phi_4(\tau f))' - \phi_4(\tau f)(x_j^p)(\pi_j^p h)''\|_{L_2(\mathbb{S})} \\
&\qquad + \|(\Phi_b(\tau f)[h])'\|_{L_2(\mathbb{S})} \\
&\qquad + 2\left\|\pi_j^p(\Phi_c(\tau f)[h])' - \phi_6(\tau f)(x_j^p)(\pi_j^p h)'' - \frac{1}{1+\tau^2 f'^2(x_j^p)}(\Phi_c(0)[\pi_j^p h])'\right\|_{L_2(\mathbb{S})} \\
&\quad \le 2\left\|\pi_j^p(\Phi_c(\tau f)[h])' - \phi_6(\tau f)(x_j^p)(\pi_j^p h)'' - \frac{1}{1+\tau^2 f'^2(x_j^p)}(\Phi_c(0)[\pi_j^p h])'\right\|_{L_2((-\pi,\pi))} \\
&\qquad + \frac{\mu}{9}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})}.
\end{aligned} \tag{4.11}$$

In order to estimate the remaining term in (4.11), we recall from Lemma 3.4 that in $(-\pi,\pi)$ the derivative $(\Phi_c(\tau f)[h])'$ can be represented as follows

$$(\Phi_c(\tau f)[h])' = \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[\cdot,s]}(\varphi h)''}{s}\frac{1}{1+\left(\tau\delta_{[\cdot,s]}f/s\right)^2}\,ds$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$- 2\tau^2 \,\mathrm{PV} \int_{\mathbb{R}} \frac{(\delta_{[\cdot,s]} f/s)(\delta_{[\cdot,s]} f'/s)(\delta_{[\cdot,s]}(\varphi h)'/s)}{\left[1+\left(\tau\delta_{[\cdot,s]} f/s\right)^2\right]^2}\, ds$$
$$- (h'\phi_5(\tau f))' + (A_{2b}(\tau f)[h])',$$

where ϕ_5 is defined in (3.32), $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}, [0,1])$ is the function chosen in the proof of Lemma 3.4, and $(A_{2b}(\tau f)[h])'$ is given in (3.35). Using Lemma A.1 (iii) (with $r = s$ and $\tau = 7/4 - s/2$) together with (3.28) and the relation (3.36), we get

$$\left\| \mathrm{PV} \int_{\mathbb{R}} \frac{(\delta_{[\cdot,s]} f/s)(\delta_{[\cdot,s]} f'/s)(\delta_{[\cdot,s]}(\varphi h)'/s)}{\left[1+\left(\tau\delta_{[\cdot,s]} f/s\right)^2\right]^2}\, ds \right\|_{L_2((-\pi,\pi))} + \|(A_{2b}(\tau f)[h])'\|_{L_2((-\pi,\pi))}$$
$$\leq C\|h\|_{H^{s'}(\mathbb{S})}. \tag{4.12}$$

Hence, we infer from (3.33), (4.11), (4.12), and (A.1) that

$$\|\pi_j^p(\Phi_0(\tau f)[h])' - (\mathbb{A}^0_{j,\tau}[\pi_j^p h])'\|_{L_2(\mathbb{S})} \leq \|T_1[h]\|_{L_2(\mathbb{S})} + 2\|T_2[h]\|_{L_2((-\pi,\pi))} + \frac{\mu}{9}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^{s'}(\mathbb{S})}, \tag{4.13}$$

where

$$T_1[h] := (\pi_j^p h)''(\phi_6(\tau f) - \phi_6(\tau f)(x_j^p)),$$
$$T_2[h] := \pi_j^p \,\mathrm{PV} \int_{\mathbb{R}} \frac{(\varphi h)''(\cdot - s)}{s} \frac{1}{1+\left(\tau\delta_{[\cdot,s]} f/s\right)^2}\, ds$$
$$- \frac{1}{1+\tau^2 f'^2(x_j^p)} \,\mathrm{PV} \int_{\mathbb{R}} \frac{(\pi_j^p \varphi h)''(\cdot - s)}{s}\, ds.$$

With regard to $T_1[h]$, it follows from (A.1) that if p is sufficiently large, then

$$\|T_1[h]\|_{L_2(\mathbb{S})} \leq \frac{\mu}{9}\|\pi_j^p h\|_{H^2(\mathbb{S})}. \tag{4.14}$$

We now consider the term $T_2[h]$. Using the notation introduced in Lemma A.1 (i), we have

$$\pi_j^p \,\mathrm{PV} \int_{\mathbb{R}} \frac{(\varphi h)''(\cdot - s)}{s} \frac{1}{1+\left(\tau\delta_{[\cdot,s]} f/s\right)^2}\, ds = B_{0,1}(\tau f)[\pi_j^p(\varphi h)''] + \int_{\mathbb{R}} (\varphi h)''(\cdot - s) \frac{\delta_{[\cdot,s]}\pi_j^p/s}{1+\left(\tau\delta_{[\cdot,s]} f/s\right)^2}\, ds,$$

and integrating the last term by parts we are led to

$$T_2[h] = -B_{1,1}(\tau f)[\pi_j^p, (\varphi h)'] - 2\tau^2 B_{2,2}(\tau f, \tau f)[f, \pi_j^p, f'(\varphi h)'] + 2\tau^2 B_{3,2}(\tau f, \tau f)[f, f, \pi_j^p, (\varphi h)'] - B_{0,1}(\tau f)[(\pi_j^p)''\varphi h + (\pi_j^p)'(\varphi h)'] + B_{0,1}(\tau f)[(\pi_j^p \varphi h)''] - B_{0,1}(\tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[(\pi_j^p \varphi h)''].$$

Lemma A.1 (i) together with (3.28) yields

$$\|T_2[h]\|_{L_2((-\pi,\pi))} \le \|B_{0,1}(\tau f)[(\pi_j^p \varphi h)''] - B_{0,1}(\tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[(\pi_j^p \varphi h)'']\|_{L_2((-\pi,\pi))} + K\|h\|_{H^1(\mathbb{S})}. \tag{4.15}$$

Furthermore, an algebraic computation shows that

$$B_{0,1}(\tau f)[(\pi_j^p \varphi h)''] - B_{0,1}(\tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[(\pi_j^p \varphi h)''] = -\tau^2 B_{2,2}(\tau f, \tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[f - f'(x_j^p)\mathrm{id}_{\mathbb{R}}, f + f'(x_j^p)\mathrm{id}_{\mathbb{R}})(\pi_j^p \varphi h)''],$$

and, recalling (4.4), we may write

$$B_{0,1}(\tau f)[(\pi_j^p \varphi h)''] - B_{0,1}(\tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[(\pi_j^p \varphi h)''] = \tau^2(T_{2a}[h] - T_{2b}[h]),$$

where

$$T_{2a}[h] := \int_{\mathbb{R}} \frac{\big[\delta_{[\cdot,s]}(f - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(f + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big(\delta_{[\cdot,s]}\chi_j^p/s\big)}{\big[1 + \tau^2 f'^2(x_j^p)\big]\big[1 + \tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]}(\pi_j^p \varphi h)''(\cdot - s)\, ds,$$

$$T_{2b}[h] := \chi_j^p B_{2,2}(\tau f, \tau f'(x_j^p)\mathrm{id}_{\mathbb{R}})[f - f'(x_j^p)\mathrm{id}_{\mathbb{R}}, f + f'(x_j^p)\mathrm{id}_{\mathbb{R}}), (\pi_j^p \varphi h)''].$$

Using integration by parts we get, similarly as in the deduction of (4.15), that

$$\|T_{2a}[h]\|_{L_2((-\pi,\pi))} \le K\|h\|_{H^1(\mathbb{S})}, \tag{4.16}$$

and we are left to estimate $\|T_{2b}[h]\|_{L_2((-\pi,\pi))}$. Since $\operatorname{supp}\varphi \subset [-4\pi, 4\pi]$, it holds

$$\|T_{2b}[h]\|_{L_2((-\pi,\pi))} \le \|T_{2c}[h]\|_{L_2((-\pi,\pi))} + \|T_{2d}[h]\|_{L_2((-\pi,\pi))},$$

where

$$T_{2c}[h] := \mathbf{1}_{(-\pi,\pi)}\chi_j^p \times \mathrm{PV}\int_{|s|<\frac{\pi}{2^p}} \frac{\big[\delta_{[\cdot,s]}(f - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(f + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]}{\big[1 + \tau^2 f'^2(x_j^p)\big]\big[1 + \tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]}\frac{(\pi_j^p \varphi h)''(\cdot - s)}{s}\, ds,$$

$$T_{2d}[h] := \chi_j^p \int_{\frac{\pi}{2^p}<|s|<5\pi} \frac{\big[\delta_{[\cdot,s]}(f - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(f + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]}{\big[1 + \tau^2 f'^2(x_j^p)\big]\big[1 + \tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]}\frac{(\pi_j^p \varphi h)''(\cdot - s)}{s}\, ds.$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Integration by parts obviously yields

$$\|T_{2d}[h]\|_{L_2((-\pi,\pi))} \leq K\|h\|_{H^s(\mathbb{S})}. \tag{4.17}$$

To deal with $T_{2c}[h]$, we notice that if $T_{2c}[h](x) \neq 0$, then necessarily

$$x \in (-\pi,\pi) \cap \Big(\cup_{n\in\mathbb{Z}} \big(2\pi n + J_j^p\big)\Big).$$

We distinguish two cases.

(i) For $j \notin \{2^p, 2^p+1, 2^p+2\}$, it holds that $(-\pi,\pi) \cap \big(2\pi n + J_j^p\big) \neq \emptyset$ for a single value n_j^p, which is either -1 or 0, and

$$(-\pi,\pi) \cap \big(2\pi n_j^p + J_j^p\big) = 2\pi n_j^p + J_j^p =: [a_j^p, b_j^p].$$

(ii) For $j \in \{2^p, 2^p+1, 2^p+2\}$, it holds that $(-\pi,\pi) \cap \big(2\pi n + J_j^p\big) \neq \emptyset$ if and only if $n_j^p \in \{-1, 0\}$ and

$$(-\pi,\pi) \cap J_j^p =: [b_j^p, \pi), \qquad (-\pi,\pi) \cap \big(-2\pi + J_j^p\big) =: (-\pi, a_j^p].$$

The values of a_j^p, b_j^p can be computed explicitly, cf. (4.3), in both cases.

We first estimate $T_{2c}[h]$ for $j \notin \{2^p, 2^p+1, 2^p+2\}$. To this end, we let $F_j \in C^{1-}(\mathbb{R})$ be the function defined by

$$F_j = f \quad \text{on } [a_j^p, b_j^p], \qquad F_j' = f'(x_j^p) \quad \text{on } \mathbb{R}\setminus[a_j^p, b_j^p].$$

Then, $\|F_j'\|_\infty \leq \|f'\|_\infty$, and since $(\operatorname{supp}\pi_j^p) \cap [a_j^p - \pi/2^p, b_j^p + \pi/2^p] \subset [a_j^p, b_j^p]$, we conclude that

$$\begin{aligned} T_{2c}[h] &= \mathbf{1}_{(-\pi,\pi)}\chi_j^p \\ &\quad \times \mathrm{PV} \int_{|s|<\frac{\pi}{2^p}} \frac{\big[\delta_{[\cdot,s]}(F_j - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(F_j + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]}{\big[1+\tau^2 f'^2(x_j^p)\big]\big[1+\tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]} \frac{(\pi_j^p\varphi h)''(\cdot - s)}{s}\, ds \\ &= T_{2e}[h] - T_{2f}[h], \end{aligned}$$

where

$$\begin{aligned} T_{2e}[h] &:= \mathbf{1}_{(-\pi,\pi)}\chi_j^p \\ &\quad \times \mathrm{PV} \int_{\mathbb{R}} \frac{\big[\delta_{[\cdot,s]}(F_j - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(F_j + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]}{\big[1+\tau^2 f'^2(x_j^p)\big]\big[1+\tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]} \frac{(\pi_j^p\varphi h)''(\cdot - s)}{s}\, ds, \end{aligned}$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

$$T_{2f}[h] := \mathbf{1}_{(-\pi,\pi)}\chi_j^p \times \int\limits_{\frac{\pi}{2^p}<|s|<5\pi} \frac{\big[\delta_{[\cdot,s]}(F_j - f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]\big[\delta_{[\cdot,s]}(F_j + f'(x_j^p)\mathrm{id}_{\mathbb{R}})/s\big]}{\big[1+\tau^2 f'^2(x_j^p)\big]\big[1+\tau^2\big(\delta_{[\cdot,s]}f/s\big)^2\big]} \frac{(\pi_j^p\varphi h)''(\cdot - s)}{s}\, ds.$$

Integrating the second term by parts, it follows from Lemma A.1 (i) that

$$\begin{aligned}\|T_{2b}[h]\|_{L_2((-\pi,\pi))} &\le \|T_{2e}[h]\|_{L_2((-\pi,\pi))} + \|T_{2f}[h]\|_{L_2((-\pi,\pi))} + K\|h\|_{H^s(\mathbb{S})}\\ &\le C\|F_j' - f'(x_j^p)\|_\infty \|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})}\\ &\le C\|f' - f'(x_j^p)\|_{L_\infty((a_j^p,b_j^p))}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})}\\ &\le \frac{\mu}{18}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^s(\mathbb{S})},\end{aligned} \tag{4.18}$$

if p is sufficiently large.

Noticing that (3.26) implies

$$\|(A_{j,\tau}[\pi_j^p h])' - ((A_{j,\tau}^0 - A_{j,\tau}^1 - A_{j,\tau}^2)[\pi_j^p h])'\|_{L_2(\mathbb{S})} \le \|\Phi_b(0)[\pi_j^p h]\|_{H^1(\mathbb{S})} \le K\|h\|_{H^s(\mathbb{S})},$$

for $j \notin \{2^p, 2^p+1, 2^p+2\}$ the desired conclusion (4.5) follows from (4.8)–(4.10) and (4.13)–(4.18).

In the second case, when $j \in \{2^p, 2^p+1, 2^p+2\}$, we decompose

$$T_{2c}[h] = \mathbf{1}_{(-\pi,a_j^p]}T_{2c}[h] + \mathbf{1}_{[b_j^p,\pi)}T_{2c}[h],$$

and we estimate the new two terms separately, by using similar arguments to those that lead to (4.18). For example, when considering $\mathbf{1}_{[b_j^p,\pi)}T_{2c}[h]$, the function $F_j \in C^{1-}(\mathbb{R})$ is defined by

$$F_j = f \quad \text{on } [b_j^p, \pi + \pi/2^p], \qquad F_j' = f'(x_j^p) \quad \text{on } \mathbb{R}\setminus[b_j^p, \pi+\pi/2^p].$$

The other details are analogous to the ones used above, and therefore we omit them. This completes our argument. □

We are now in the position to prove the desired estimates (4.1) and (4.2), and therewith the desired generator property.

Theorem 4.3. *Given $f \in H^s(\mathbb{S})$, $s \in (3/2, 2)$ it holds that*

$$-\Phi(f) \in \mathcal{H}(H^2(\mathbb{S}), H^1(\mathbb{S})).$$

Proof. Letting a_τ and b_τ denote the functions defined in (4.7), it follows from (3.11), (3.14), (3.25), and (A.1) that there exists a constant $\vartheta \ge 2\pi$ such that

$$\sup_{\tau\in[0,1]}\|a_\tau\|_\infty \le \vartheta \qquad \text{and} \qquad \frac{1}{\vartheta} \le \min_{\tau\in[0,1]} b_\tau \le \max_{\tau\in[0,1]} b_\tau \le \vartheta.$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Given $\tau \in [0,1]$, $3 \leq p \in \mathbb{N}$, a finite p-partition of unity $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$, and $j \in \{1,\dots,2^{p+1}\}$, the operators $A_{j,\tau}$ defined in (4.6) are elements of the set $\{\mathbb{A}_{a,b} : |a| \leq \vartheta, \vartheta^{-1} \leq b \leq \vartheta\}$, where $\mathbb{A}_{a,b}$ denotes the Fourier multiplier

$$\mathbb{A}_{a,b} := a\partial_x - b(-\partial_x^2)^{1/2}.$$

A Fourier series expansion argument shows that there exists a constant $\kappa_0 \geq 1$ such that

$$\lambda - \mathbb{A}_{a,b} \in \mathrm{Isom}(H^2(\mathbb{S}), H^1(\mathbb{S})), \tag{4.19}$$

$$\kappa_0 \|(\lambda - \mathbb{A}_{a,b})[h]\|_{H^1(\mathbb{S})} \geq |\lambda| \cdot \|h\|_{H^1(\mathbb{S})} + \|h\|_{H^2(\mathbb{S})} \tag{4.20}$$

for all $|a| \leq \vartheta$, $\vartheta^{-1} \leq b \leq \vartheta$, $\mathrm{Re}\,\lambda \geq 1$, and $h \in H^2(\mathbb{S})$.

We now chose $\mu = 1/2\kappa_0$ and infer from Theorem 4.2 that there exist $p \geq 3$, a finite p-partition of unity $\{\pi_j^p : 1 \leq j \leq 2^{p+1}\}$, a constant $K = K(p)$, and for each $p \in \{1,\dots,2^{p+1}\}$ and $\tau \in [0,1]$ there exist operators $\mathbb{A}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{S}), H^1(\mathbb{S}))$ such that

$$\|\pi_j^p \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^p h]\|_{H^1(\mathbb{S})} \leq \frac{1}{2\kappa_0}\|\pi_j^p h\|_{H^2(\mathbb{S})} + K\|h\|_{H^{s'}(\mathbb{S})}$$

for all $j \in \{1,\dots,2^{p+1}\}$, $\tau \in [0,1]$, and $h \in H^2(\mathbb{S})$. The latter estimate together with (4.20) yields

$$\begin{aligned}
&\kappa_0 \|\pi_j^p(\lambda - \Phi(\tau f))[h]\|_{H^1(\mathbb{S})} \\
&\quad \geq \kappa_0 \|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^p h]\|_{H^1(\mathbb{S})} - \kappa_0 \|\pi_j^p \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^p h]\|_{H^1(\mathbb{S})} \\
&\quad \geq |\lambda| \cdot \|\pi_j^p h\|_{H^1(\mathbb{S})} + \frac{1}{2}\|\pi_j^p h\|_{H^2(\mathbb{S})} - \kappa_0 K \|h\|_{H^{s'}(\mathbb{S})}.
\end{aligned}$$

The Remark 4.1 together with Young's inequality and (4.21) ensures now the existence of constants $\kappa \geq 1$ and $\omega \geq 1$ such that

$$\kappa \|(\lambda - \Phi(\tau f))[h]\|_{H^1(\mathbb{S})} \geq |\lambda| \cdot \|h\|_{H^1(\mathbb{S})} + \|h\|_{H^2(\mathbb{S})}$$

for all $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \omega$, and $h \in H^2(\mathbb{S})$. Choosing $\tau = 1$, we have established (4.2). Moreover, for the particular choice $\lambda = \omega$ in the latter inequality, the method of continuity, see e.g. [4, Proposition 1.1.1], together with (4.19) implies, in view of $\Phi(0) = \mathbb{A}_{0,2\pi}$, that indeed $\omega - \Phi(f) \in \mathrm{Isom}(H^2(\mathbb{S}), H^1(\mathbb{S}))$. This completes the proof. □

We now come to the proof of our first main result.

Proof of Theorem 1.1. Let $s \in (3/2, 2)$ be fixed and set $\mathbb{E}_1 := H^2(\mathbb{S})$, $\mathbb{E}_0 := H^1(\mathbb{S})$, $\alpha := s - 1$, and $\beta := \overline{s} - 1$, where we fix some $\overline{s} \in (3/2, 2)$. In view of the well-known interpolation property

$$[H^{s_0}(\mathbb{X}), H^{s_1}(\mathbb{X})]_\theta = H^{(1-\theta)s_0 + \theta s_1}(\mathbb{X}), \qquad \theta \in (0,1),\ 0 \leq s_0 \leq s_1 < \infty,\ \mathbb{X} \in \{\mathbb{S}, \mathbb{R}\}, \tag{4.21}$$

it holds

$$\mathbb{E}_\alpha = [\mathbb{E}_0, \mathbb{E}_1]_\alpha = H^s(\mathbb{S}) \qquad \text{and} \qquad \mathbb{E}_\beta = [\mathbb{E}_0, \mathbb{E}_1]_\beta = H^{\overline{s}}(\mathbb{S}).$$

The relation (3.7) together with Theorem 4.3 ensures that the assumptions of Theorem A.4 are fulfilled by the Muskat problem (3.1), and we conclude that (3.1) has, for each $f \in H^s(\mathbb{S})$, a maximal classical solution $f := f(\cdot; f_0)$, with

$$f \in C([0, T_+(f_0)), H^s(\mathbb{S})) \cap C((0, T_+(f_0)), H^2(\mathbb{S})) \cap C^1((0, T_+(f_0)), H^1(\mathbb{S})), \ T_+(f_0) \leq \infty,$$
$$f \in C^{s-\overline{s}}([0, T], H^{\overline{s}}(\mathbb{S})) \quad \text{for all } T \in (0, T_+(f_0)).$$

Concerning the uniqueness claim in Theorem 1.1, it suffices to prove that any function

$$f \in C([0, T], H^s(\mathbb{S})) \cap C((0, T], H^2(\mathbb{S})) \cap C^1((0, T], H^1(\mathbb{S})), \qquad T > 0,$$

that solves (3.1) pointwise, satisfies

$$f \in C^\eta([0, T], H^{\overline{s}}(\mathbb{S})), \tag{4.22}$$

where $\eta := (s - \overline{s})/s \in (0, s - \overline{s})$. To this end, we note that the assumption $f \in C([0, T], H^s(\mathbb{S}))$ ensures that $f([0, T])$ is a compact subset of $H^s(\mathbb{S})$. Using the formulas (3.3)–(3.4) together with Lemma 3.3, it follows that $\partial_t f : (0, T] \to L_2(\mathbb{S})$ is also bounded, hence $\partial_t f \in \mathrm{BC}((0, T], L_2(\mathbb{S}))$, and the property (4.22), and implicitly the claim (i) of the theorem, follows from

$$\|f(t) - f(s)\|_{H^{\overline{s}}(\mathbb{S})} \leq \|f(t) - f(s)\|_{L_2(\mathbb{S})}^{1-\overline{s}/s} \|f(t) - f(s)\|_{H^s(\mathbb{S})}^{\overline{s}/s} \leq C|t - s|^\eta, \qquad t, s \in [0, T].$$

The criterion for global existence stated at (ii) follows directly from Theorem A.4. Finally, the real-analyticity property (iii) is obtained by using (3.7) and a parameter trick which appears, in other forms, also in [7,28,42]. Since the proof of this claim is almost identical to that in the nonperiodic case [39, Theorem 1.3] we omit here the details. □

5. Stability analysis

We now study the stability properties of the stationary solution $f \equiv 0$ to the evolution problem (1.1). To this end we first note that for each constant $c \in \mathbb{R}$, the function $f \equiv c$ defines a stationary solution to (1.1). Therefore, it is suitable to introduce the following Hilbert spaces

$$H_0^r(\mathbb{S}) := \{f \in H^r(\mathbb{S}) : \langle f \rangle = 0\}, \qquad r \geq 0,$$

where

$$\langle f \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

denotes the integral mean of f over one period. Since the value of the quotient $k\Delta_\rho/4\pi\mu$ influences the rate of convergence of the solutions that start in $H_0^2(\mathbb{S})$ and are initially small towards

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

the zero solution, see Theorem 1.3, we do not scale out this constant from the problem, as we did when proving Theorem 1.1.

Lemma 5.1. *Given* $f \in H_0^2(\mathbb{S})$, *let*

$$\Psi(f) := \frac{k\Delta_\rho}{4\pi\mu}\Phi(f)[f]. \tag{5.1}$$

It then holds

$$\Psi \in \mathrm{C}^\infty(H_0^2(\mathbb{S}), H_0^1(\mathbb{S})). \tag{5.2}$$

Proof. Having established that $\langle \Psi(f)\rangle = 0$ for all $f \in H_0^2(\mathbb{S})$, the desired claim (5.2) follows from (3.7). Given $f \in H_0^2(\mathbb{S})$, let

$$v_+ = \widetilde{v}\big|_{[y>f(x)]} \in \mathrm{C}([y \geq f(x)]) \cap \mathrm{C}^1([y > f(x)]),$$

where $\widetilde{v}$ is introduced in (2.3) for $\overline{\omega}$ defined in (2.2). The proof of Proposition 2.1 shows that

$$\Psi(f) = \langle v_+|_{[y=f(x)]}|(-f', 1)\rangle.$$

Using Stokes's theorem, we find in view of $(2.1a)_1$ and of (2.1c) that

$$\int_{-\pi}^{\pi} \langle v_+|_{[y=f(x)]}|(-f', 1)\rangle\, dx = \int\limits_{[y=f(x)]} \langle v_+|\nu\rangle\, d\sigma = -\int\limits_{[y>f(x)]} \operatorname{div} v_+\, d(x, y) = 0,$$

where we integrate over one period of the curve $[y = f(x)]$, respectively over one period of the periodic domain $[y > f(x)]$. This proves the claim. □

Lemma 5.1 shows in particular that the integral mean of the solutions to (1.1) found in Theorem 1.1 is given by the integral mean of the initial data, and this aspect should be taken into account when studying the asymptotic stability of the zero solution. To do so, we observe that for initial data with zero integral mean, (1.1) can be recast as

$$\partial_t f = \Psi(f), \quad t > 0, \qquad f(0) = f_0, \tag{5.3}$$

and our task reduces to studying the asymptotic stability of zero solution to (5.3). In this way we have eliminated the zero eigenvalue from the spectrum of the Fréchet derivative $\partial\Psi(0)$, as the next result shows.

Lemma 5.2. *The Fréchet derivative* $\partial\Psi(0) \in \mathcal{L}(H_0^2(\mathbb{S}), H_0^1(\mathbb{S}))$ *is given by*

$$\partial\Psi(0)[f] = \frac{k\Delta_\rho}{4\pi\mu}\Phi(0)[f] \qquad \text{for } f \in H_0^2(\mathbb{S}),$$

and

$$\sigma(\partial\Psi(0)) = \Big\{ -\frac{k\Delta_\rho m}{2\mu} \,:\, 1 \leq m \in \mathbb{N} \Big\}. \tag{5.4}$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Proof. The first claim is a direct consequence of (3.7) and (5.1). The relation (5.4) follows from the previous observation that $\Phi(0) = -2\pi(-\partial_x^2)^{1/2}$. □

Lemma 5.2 in particular implies

$$-\partial\Psi(0) \in \mathcal{H}(H_0^2(\mathbb{S}), H_0^1(\mathbb{S})). \tag{5.5}$$

The relation (5.5) together with the Lemmas 5.1 and 5.2 are the ingredients needed in order to apply the principle of linearized stability [37, Theorem 9.1.2] in the context of (5.3).

Proof of Theorem 1.3. Noticing that

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\partial\Psi(0))\} = -\frac{k\Delta_\rho}{2\mu} < 0,$$

the assertion of Theorem 1.3 is a straightforward consequence of [37, Theorem 9.1.2] and of Theorem 1.1. □

Acknowledgments

The authors thank the anonymous referee for the valuable suggestions which have improved the quality of the article.

Appendix A. Some technical results

We collect in Lemma A.1 some results on the boundedness of two families of multilinear singular integral operators which are needed in the analysis and which are extensions of some results established in the nonperiodic case in [39].

Lemma A.1. [4]

(*i*) *Given n, $m \in \mathbb{N}$ and functions $a_1, \dots, a_n$, $b_1, \dots, b_m : \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous, the singular integral operator $B_{m,n}(a_1, \dots, a_n)[b_1, \dots, b_m, \cdot\,]$ defined by*

$$B_{m,n}(a_1, \dots, a_n)[b_1, \dots, b_m, h](x) := \mathrm{PV} \int_{\mathbb{R}} \frac{h(x-y)}{y} \frac{\prod_{i=1}^m \left(\delta_{[x,y]} b_i / y\right)}{\prod_{i=1}^n \left[1 + \left(\delta_{[x,y]} a_i / y\right)^2\right]} dy,$$

belongs to $\mathcal{L}(L_2(\mathbb{R}))$ and $\|B_{m,n}(a_1, \dots, a_n)[b_1, \dots, b_m, \cdot\,]\|_{\mathcal{L}(L_2(\mathbb{R}))} \le C \prod_{i=1}^m \|b_i'\|_\infty$, where C is a constant depending only on n, m and $\max_{i=1,\dots,n} \|a_i'\|_\infty$.

(*ii*) *Given $n, m \in \mathbb{N}$, $r \in (3/2, 2)$, $a_1, \dots, a_n$, $b_1, \dots, b_m \in H^r(\mathbb{S})$, and $h \in L_2(\mathbb{R})$ we define*

$$A_{m,n}(a_1, \dots, a_n)[b_1, \dots, b_m, h](x) := \mathrm{PV} \int_{\mathbb{R}} \frac{\prod_{i=1}^m \left(\delta_{[x,y]} b_i / y\right)}{\prod_{i=1}^n \left[1 + \left(\delta_{[x,y]} a_i / y\right)^2\right]} \frac{\delta_{[x,y]} h}{y}\, dy.$$

[4] By convention the empty product is equal to 1.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Then:

(*ii*1) *There exists a constant* C*, depending only on* r*,* n*,* m*, and* $\max_{i=1,\dots,n}\|a_i\|_{H^r(\mathbb{S})}$*, such that*

$$\big\|A_{m,n}(a_1,\dots,a_n)[b_1,\dots,b_m,h]\big\|_{L_2(\mathbb{R})}\leq C\|h\|_{L_2(\mathbb{R})}\prod_{i=1}^{m}\|b_i\|_{H^r(\mathbb{S})}$$

for all $b_1,\dots,b_m\in H^r(\mathbb{S})$ *and* $h\in L_2(\mathbb{R})$.

(*ii*2) $A_{m,n}\in \mathrm{C}^{1-}((H^r(\mathbb{S}))^n,\mathcal{L}_{m+1}((H^r(\mathbb{S}))^m\times L_2(\mathbb{R}),L_2(\mathbb{R}))$.

(*iii*) *Let* $n\in\mathbb{N}$*,* $1\leq m\in\mathbb{N}$*,* $r\in(3/2,2)$*,* $\tau\in(5/2-r,1)$*, and* $a_1,\dots,a_n\in H^r(\mathbb{S})$ *be given. Then:*

(*iii*1) *There exists a constant* C*, depending only on* r *and* τ *such that*

$$\big\|A_{m,n}(a_1,\dots,a_n)[b_1,\dots,b_m,h]\big\|_{L_2(\mathbb{R})}\leq C\|h\|_{H^\tau(\mathbb{R})}\|b_m\|_{H^{r-1}(\mathbb{S})}\prod_{i=1}^{m-1}\|b_i'\|_\infty$$

for all $b_1,\dots,b_m\in H^r(\mathbb{S})$ *and all* $h\in H^1(\mathbb{R})$*. In particular,* $A_{m,n}(a_1,\dots,a_n)$ *extends to a bounded operator*

$$A_{m,n}(a_1,\dots,a_n)\in\mathcal{L}_{m+1}((H^r(\mathbb{S}))^{m-1}\times H^{r-1}(\mathbb{S})\times H^\tau(\mathbb{R}),L_2(\mathbb{R})).$$

(*iii*2) $A_{m,n}\in \mathrm{C}^{1-}((H^r(\mathbb{S}))^n,\mathcal{L}_{m+1}((H^r(\mathbb{S}))^{m-1}\times H^{r-1}(\mathbb{S})\times H^\tau(\mathbb{R}),L_2(\mathbb{R}))$.

Proof. The proof is identical to that of the Lemmas 3.1 and 3.4 (see also Remark 3.3) in [39]. □

The following result is used in the proof of Lemma 3.4.

Lemma A.2. *Let* $f\in \mathrm{C}(\mathbb{S})$ *and assume that* $g:=f|_{(-\pi,\pi)}\in H^1((-\pi,\pi))$*. Then* $f\in H^1(\mathbb{S})$ *and* f' *is the* 2π*-periodic extension of* g'.

Proof. Let $F\in L_2(\mathbb{S})$ be the 2π-periodic extension of g'. Given $\varphi\in \mathrm{C}^\infty(\mathbb{S})$, it holds, in virtue of $g(\pm\pi)=f(\pm\pi)=f(\pi)$, that

$$\int_{\mathbb{S}} f\varphi'\,dx=\int_{-\pi}^{\pi} g\varphi'\,dx=(g\varphi)(\pi)-(g\varphi)(-\pi)-\int_{-\pi}^{\pi} g'\varphi\,dx=-\int_{\mathbb{S}} F\varphi\,dx.$$

This proves the claim. □

The following technical result is employed in the proof of Theorem 4.2.

Lemma A.3. *Given* $f\in H^s(\mathbb{S})$*,* $s\in(3/2,2)$*, let* $\phi_6(f):\mathbb{R}\to\mathbb{R}$ *be defined by*

$$\phi_6(f)(x):=\mathrm{PV}\int_{-\pi}^{\pi}\frac{1}{s}\frac{1}{1+(\delta_{[x,s]}f/s)^2}\,ds.$$

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

Let further $\alpha := s/2 - 3/4 \in (0, 1)$. *It then holds* $\phi_6 \in \mathrm{C}^\alpha(\mathbb{S})$ *and*

$$\sup_{\tau\in[0,1]} \|\phi_6(\tau f)\|_{\mathrm{C}^\alpha} < \infty. \tag{A.1}$$

Proof. The proof is similar to that of [39, Lemma A.1]. □

In the proof of Theorem 1.1 we use results for abstract quasilinear parabolic problem obtained by H. Amann in a more general context, see [3, Section 12] and [1,2], and which we collect in Theorem A.4.

Theorem A.4. *Let* $\mathbb{E}_0, \mathbb{E}_1$ *be Banach spaces with compact dense embedding* $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$, *let* $[\cdot,\cdot]_\theta$ *denote the complex interpolation functor, and let* $\mathbb{E}_\theta := [\mathbb{E}_0, \mathbb{E}_1]_\theta$ *for all* $0 < \theta < 1$. *Let further* $0 < \beta < \alpha < 1$ *and assume that*

$$-\Phi \in C^{1-}(\mathbb{E}_\beta, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)).$$

The following assertions hold for the quasilinear evolution problem

$$\partial_t f = \Phi(f)[f], \quad t > 0, \qquad f(0) = f_0. \tag{QP}$$

Existence: *Given* $f_0 \in \mathbb{E}_\alpha$, *the problem* (QP) *possesses a maximal solution*

$$\begin{aligned} f &:= f(\cdot; f_0) \\ &\in C([0, T_+(f_0)), \mathbb{E}_\alpha) \cap C((0, T_+(f_0)), \mathbb{E}_1) \cap C^1((0, T_+(f_0)), \mathbb{E}_0) \cap C^{\alpha-\beta}([0, T], \mathbb{E}_\beta) \end{aligned}$$

for all $T \in (0, T_+(f_0))$, *with* $T_+(f_0) \in (0, \infty]$.

Uniqueness: *If* $\widetilde{T} \in (0, \infty]$, $\eta \in (0, \alpha - \beta]$, *and* $\widetilde{f} \in C((0, \widetilde{T}), \mathbb{E}_1) \cap C^1((0, \widetilde{T}), \mathbb{E}_0)$ *satisfies*

$$\widetilde{f} \in C^\eta([0, T], \mathbb{E}_\beta) \qquad \text{for all } T \in (0, \widetilde{T})$$

and solves (QP), *then* $\widetilde{T} \le T_+(f_0)$ *and* $\widetilde{f} = f$ *on* $[0, \widetilde{T})$.

Criterion for global existence: *If* $f : [0, T] \cap [0, T_+(f_0)) \to \mathbb{E}_\alpha$ *is bounded for all* $T > 0$, *then*

$$T_+(f_0) = \infty.$$

Continuous dependence of initial data: *The mapping* $[(t, f_0) \mapsto f(t; f_0)]$ *defines a semiflow on* $\mathbb{E}_\alpha$ *and, if* $\Phi \in C^\omega(\mathbb{E}_\beta, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0))$, *then*

$$[(t, f_0) \mapsto f(t; f_0)] : \{(t, f_0) : f_0 \in \mathbb{E}_\alpha, t \in (0, T_+(f_0))\} \to \mathbb{E}_\alpha$$

is a real-analytic map too.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

References

[1] H. Amann, Quasilinear parabolic systems under nonlinear boundary conditions, Arch. Ration. Mech. Anal. 92 (2) (1986) 153–192.
[2] H. Amann, Dynamic theory of quasilinear parabolic equations. I. Abstract evolution equations, Nonlinear Anal. 12 (9) (1988) 895–919.
[3] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: Function Spaces, Differential Operators and Nonlinear Analysis, Friedrichroda, 1992, in: Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993, pp. 9–126.
[4] H. Amann, Linear and Quasilinear Parabolic Problems, vol. I. Abstract Linear Theory, Monogr. Math., vol. 89, Birkhäuser Boston, Inc., Boston, MA, 1995.
[5] D.M. Ambrose, Well-posedness of two-phase Hele–Shaw flow without surface tension, European J. Appl. Math. 15 (5) (2004) 597–607.
[6] D.M. Ambrose, The zero surface tension limit of two-dimensional interfacial Darcy flow, J. Math. Fluid Mech. 16 (1) (2014) 105–143.
[7] S.B. Angenent, Nonlinear analytic semiflows, Proc. Roy. Soc. Edinburgh Sect. A 115 (1–2) (1990) 91–107.
[8] B.V. Bazaliy, N. Vasylyeva, The two-phase Hele–Shaw problem with a nonregular initial interface and without surface tension, Zh. Mat. Fiz. Anal. Geom. 10 (1) (2014) 3–43, 152, 155.
[9] J. Bear, Dynamics of Fluids in Porous Media, Dover Publications, New York, 1988.
[10] L.C. Berselli, D. Córdoba, R. Granero-Belinchón, Local solvability and turning for the inhomogeneous Muskat problem, Interfaces Free Bound. 16 (2) (2014) 175–213.
[11] P. Byers, Existence time for the Camassa–Holm equation and the critical Sobolev index, Indiana Univ. Math. J. 55 (3) (2006) 941–954.
[12] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, Breakdown of smoothness for the Muskat problem, Arch. Ration. Mech. Anal. 208 (3) (2013) 805–909.
[13] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, M. López-Fernández, Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves, Ann. of Math. (2) 175 (2) (2012) 909–948.
[14] A. Castro, D. Córdoba, C.L. Fefferman, F. Gancedo, M. López-Fernández, Turning waves and breakdown for incompressible flows, Proc. Natl. Acad. Sci. USA 108 (12) (2011) 4754–4759.
[15] C.H.A. Cheng, R. Granero-Belinchón, S. Shkoller, Well-posedness of the Muskat problem with H^2 initial data, Adv. Math. 286 (2016) 32–104.
[16] P. Constantin, D. Córdoba, F. Gancedo, L. Rodrígez-Piazza, R.M. Strain, On the Muskat problem: global in time results in 2D and 3D, Amer. J. Math. 138 (2016) 1455–1494.
[17] P. Constantin, D. Córdoba, F. Gancedo, R.M. Strain, On the global existence for the Muskat problem, J. Eur. Math. Soc. (JEMS) 15 (1) (2013) 201–227.
[18] P. Constantin, F. Gancedo, R. Shvydkoy, V. Vicol, Global regularity for 2D Muskat equations with finite slope, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (4) (2017) 1041–1074.
[19] A. Córdoba, D. Córdoba, F. Gancedo, Interface evolution: the Hele–Shaw and Muskat problems, Ann. of Math. (2) 173 (1) (2011) 477–542.
[20] A. Córdoba, D. Córdoba, F. Gancedo, Porous media: the Muskat problem in three dimensions, Anal. PDE 6 (2) (2013) 447–497.
[21] D. Córdoba, F. Gancedo, Contour dynamics of incompressible 3-D fluids in a porous medium with different densities, Comm. Math. Phys. 273 (2) (2007) 445–471.
[22] D. Córdoba, F. Gancedo, Absence of squirt singularities for the multi-phase Muskat problem, Comm. Math. Phys. 299 (2) (2010) 561–575.
[23] D. Córdoba Gazolaz, R. Granero-Belinchón, R. Orive-Illera, The confined Muskat problem: differences with the deep water regime, Commun. Math. Sci. 12 (3) (2014) 423–455.
[24] M. Ehrnström, J. Escher, B.-V. Matioc, Steady-state fingering patterns for a periodic Muskat problem, Methods Appl. Anal. 20 (1) (2013) 33–46.
[25] J. Escher, A.-V. Matioc, B.-V. Matioc, A generalized Rayleigh–Taylor condition for the Muskat problem, Nonlinearity 25 (2012) 73–92.
[26] J. Escher, B.-V. Matioc, On the parabolicity of the Muskat problem: well-posedness, fingering, and stability results, Z. Anal. Anwend. 30 (2) (2011) 193–218.
[27] J. Escher, B.-V. Matioc, C. Walker, The domain of parabolicity for the Muskat problem, Indiana Univ. Math. J. 67 (2) (2018) 679–737.
[28] J. Escher, G. Simonett, Analyticity of the interface in a free boundary problem, Math. Ann. 305 (3) (1996) 439–459.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038

[29] A. Friedman, Y. Tao, Nonlinear stability of the Muskat problem with capillary pressure at the free boundary, Nonlinear Anal. 53 (1) (2003) 45–80.
[30] F. Gancedo, R.M. Strain, Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem, Proc. Natl. Acad. Sci. USA 111 (2) (2014) 635–639.
[31] J. Gómez-Serrano, R. Granero-Belinchón, On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof, Nonlinearity 27 (6) (2014) 1471–1498.
[32] R. Granero-Belinchón, Global existence for the confined Muskat problem, SIAM J. Math. Anal. 46 (2) (2014) 1651–1680.
[33] R. Granero-Belinchón, S. Shkoller, Well-posedness and decay to equilibrium for the Muskat problem with discontinuous permeability, Trans. Amer. Math. Soc. (2016), https://doi.org/10.1090/tran/7335, in press, arXiv:1611.06147.
[34] J. Hong, Y. Tao, F. Yi, Muskat problem with surface tension, J. Partial Differ. Equ. 10 (3) (1997) 213–231.
[35] J.K. Lu, Boundary Value Problems for Analytic Functions, Ser. Pure Math., vol. 16, World Scientific Publishing Co., Inc., River Edge, NJ, 1993.
[36] A. Lunardi, An introduction to geometric theory of fully nonlinear parabolic equations, in: Qualitative Aspects and Applications of Nonlinear Evolution Equations, Trieste, 1990, World Sci. Publ., River Edge, NJ, 1991, pp. 107–131.
[37] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Progr. Nonlinear Differential Equations Appl., vol. 16, Birkhäuser Verlag, Basel, 1995.
[38] B.-V. Matioc, Viscous displacement in porous media: the Muskat problem in 2D, Trans. Amer. Math. Soc. 370 (10) (2018) 7511–7556.
[39] B.-V. Matioc, The Muskat problem in two dimensions: equivalence of formulations, well-posedness, and regularity results, Anal. PDE 12 (2) (2019) 281–332.
[40] T. Murai, Boundedness of singular integral operators of Calderón type. VI, Nagoya Math. J. 102 (1986) 127–133.
[41] M. Muskat, Two fluid systems in porous media. The encroachment of water into an oil sand, Physics 5 (1934) 250–264.
[42] J. Prüss, Y. Shao, G. Simonett, On the regularity of the interface of a thermodynamically consistent two-phase Stefan problem with surface tension, Interfaces Free Bound. 17 (4) (2015) 555–600.
[43] J. Prüss, G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monogr. Math., Birkhäuser Verlag, 2016.
[44] J. Prüss, G. Simonett, On the Muskat flow, Evol. Equ. Control Theory 5 (4) (2016) 631–645.
[45] M. Siegel, R.E. Caflisch, S. Howison, Global existence, singular solutions, and ill-posedness for the Muskat problem, Comm. Pure Appl. Math. 57 (10) (2004) 1374–1411.
[46] S. Tofts, On the existence of solutions to the Muskat problem with surface tension, J. Math. Fluid Mech. 19 (4) (2017) 581–611, https://doi.org/10.1007/s00021-016-0297-y.
[47] A. Torchinsky, Real-Variable Methods in Harmonic Analysis, Dover Publications, Inc., Mineola, NY, 2004, reprint of the 1986 original [Dover, New York; MR0869816].
[48] F. Yi, Local classical solution of Muskat free boundary problem, J. Partial Differ. Equ. 9 (1) (1996) 84–96.

Please cite this article in press as: A.-V. Matioc, B.-V. Matioc, Well-posedness and stability results for a quasilinear periodic Muskat problem, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.038