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A local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs ☆

Seung-Yeal Ha [a,b], Shi Jin [c], Jinwook Jung [d,*], Woojoo Shim [d]

[a] *Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea*

[b] *Korea Institute for Advanced Study, Hoegiro 87, Seoul 02455, Republic of Korea*

[c] *School of Mathematical Sciences, MOE-LSC, and Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai 200240, China*

[d] *Department of Mathematical Sciences, Seoul National University, Seoul, 08826, Republic of Korea*

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Abstract

We present a local sensitivity analysis for the hydrodynamic Cucker-Smale (HCS) model with random inputs. In the absence of random inputs, the HCS model was derived as a macroscopic model for the emergent dynamics of the CS flocking ensemble from the kinetic CS model via the moment method and mono-kinetic ansatz for a closure condition. In this paper, we incorporate the uncertain effects together with the HCS model to result in the random HCS model. For definiteness, we consider the uncertainties in initial data and communication weight function. For this random HCS model, we perform local sensitivity estimates such as the propagation of pathwise well-posedness, pathwise L^2-stability and flocking estimates of solution process.

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* Corresponding author.

E-mail addresses: syha@snu.ac.kr (S.-Y. Ha), shijin-m@sjtu.edu.cn (S. Jin), warp100@snu.ac.kr (J. Jung), cosmo.shim@gmail.com (W. Shim).

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1. Introduction

The purpose of this paper is to continue the systematic study begun in a series of works [1,5,13,16–19] for the synthesis of uncertainty quantification (UQ) and emergent flocking dynamics via flocking and synchronization models with random inputs. For the last decade, there has been lots of intense research in the domain of collective dynamics of many-body systems, which arise from the modeling of self-propelled particle systems [4,11,30,33,36,37,46–49]. Parallel to the advance of collective dynamics, uncertainty quantification has also received lots of attention in diverse disciplines such as the applied mathematics, atmospheric sciences and engineering [24,35,42–44]. Thus, it is natural to synthesize these two emerging disciplines in a common platform. This synthesis idea were already performed in aforementioned works which deal with the local sensitivity analysis for the particle and kinetic CS and Kuramoto models. So far, this marriage was successful only for the particle and kinetic collective models whose global existence of regular solutions has been well established in the absence of random inputs. However, this synthesis has not been made for the hydrodynamic models for collective dynamics yet. Of course, there are some previous works [25,26,32,38–41] on the scalar conservation law and Euler system with random inputs from the point of numerics in the context of UQ.

It is well known that hydrodynamic models arising from the theory of hyperbolic conservation laws and fluid mechanics do not often allow sufficiently smooth solutions enough to implement a local sensitivity analysis. In particular, hyperbolic conservation laws do not allow a global smooth solution for generic initial data. They instead exhibit discontinuous solutions for generic initial data, which makes a UQ program difficult to implement [12]. This is why the local sensitivity theory has not been well studied in the hyperbolic conservation laws. Despite of this, hyperbolic models arising from the modeling of flocking and synchronization admit smooth solutions for well-prepared initial data thanks to the extra nonlocal flux and source terms, which play the role of regularizing mechanism. Thus, it seems plausible to extend our local sensitivity analysis to the hydrodynamic models for collective dynamics.

As a first step toward the successful synthesis of local sensitivity analysis and hyperbolic conservation laws, we consider the pressureless Euler system for the CS ensemble which is a hyperbolic system with a nonlocal source term. In this case, the nonlocal flocking source term acts like a nonlocal damping which suppresses the appearance of the Delta shocks for small solutions. To incorporate random inputs to the HCS model, we consider a random vector z defined on the sample space $\Omega \subset \mathbb{R}^d$ with the probability density function $\pi = \pi(z)$. For the notational simplicity, we will assume that z is an one-dimensional variable. This random variable z registers the uncertain effects in the initial data and communication weights. To fix the idea, we consider an ensemble of collisionless Cucker-Smale flocking particles on the periodic domain $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, $d \geq 1$, and let $\rho := \rho(t,x,z)$ and $u := u(t,x,z)$ be the local mass and bulk velocity of the CS fluid at position $x \in \mathbb{T}^d$, random vector z and time t, respectively. In this setting, the dynamics of macroscopic observables (ρ, u) is governed by the Cauchy problem to the random HCS model:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \ \ x \in \mathbb{T}^d, \ \ z \in \Omega, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \rho \displaystyle\int_{\mathbb{T}^d} \psi(x-y,z)(u(t,y,z) - u(t,x,z))\rho(t,y,z)dy, \\ (\rho, u)(0,x) = (\rho_0, u_0), \end{cases} \tag{1.1}$$

where ∇ is the spatial gradient, and ψ is a non-negative communication weight function measuring the strength of interactions between CS particles. In this paper, we would like to see the dynamic properties of z-variations $(\partial_z^\alpha \rho, \partial_z^\alpha u)$ to the random HCS model (1.1), which is what is called the local sensitivity analysis [45]. Such an analysis is not only of analytical interest. Since it yields regularities in the random space, it is important for numerical methods like stochastic Galerkin or collocation methods [24,26].

Note that for a frozen $z \in \Omega$, system (1.1) becomes the deterministic pressureless Euler system with a flocking dissipation, which has been studied in previous literature, e.g., the global existence of classical solutions and interaction with incompressible fluids [21] and existence of entropic weak solutions in one-dimension [15].

The main results of this paper are three-fold. First, we present the propagation of pathwise well-posedness of the random HCS model (1.1). For $s > \frac{d}{2} + m + 1$, if the initial processes and their z-variations $\{(\partial_z^l \rho_0, \partial_z^l u_0)\}_{l=0}^m$ satisfy the non-vacuum, regularity and smallness conditions, we show that z-variations of solution processes $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$ exist in any finite time interval, and satisfy the desired regularity and smallness conditions (see Theorem 3.1 and Theorem 3.2).

Second, we provide a finite-in-time L^2-stability of the z-variations to system (1.1). More precisely, let (ρ, u) and $(\bar\rho, \bar u)$ be solution processes to (1.1) corresponding to initial processes (ρ_0, u_0) and $(\bar\rho_0, \bar u_0)$, respectively. Then, there exists a positive random function $C = C(T, z)$ such that for each $T \in (0, \infty)$ and $z \in \Omega$,

$$\begin{aligned} &\sup_{0 \le t \le T} \sum_{0 \le l \le m} \left(\|\partial_z^l \rho(t,z) - \partial_z^l \bar\rho(t,z))\|_{H^{m-l}}^2 + \|\partial_z^l u(t,z) - \partial_z^l \bar u(t,z))\|_{H^{m-l+1}}^2 \right) \\ &\qquad \le C(T,z) \sum_{0 \le l \le m} \left(\|\partial_z^l \rho_0(z) - \partial_z^l \bar\rho_0(z)\|_{H^{m-l}}^2 + \|\partial_z^l u_0(z) - \partial_z^l \bar u_0(z)\|_{H^{m-l+1}}^2 \right). \end{aligned}$$

Third, we show that the bulk velocity process and its z-variations $\{\partial_z^l u\}$ exhibit an exponential decay toward the mean-velocity under *a priori* assumptions, which implies the flocking estimate. We assume the uniform-in-time boundedness for solution processes and their z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$, and impose an *a priori* condition for the lower bound of the communication weight function to obtain the exponential decay of $\{\partial_z^l u\}$ toward its mean-velocity.

The rest of this paper is organized as follows. In Section 2, we briefly discuss how the HCS model with random inputs can be derived from the particle CS model, and review global existence, stability and flocking results for the deterministic HCS model. In Section 3, we present the pathwise well-posedness for the random HCS model. In Section 4, we provide L^2-stability estimates for the z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$. In Section 5, we present an exponential decay of the bulk velocity process and its z-variations. Finally, Section 6 is devoted to a brief summary of our results and some remaining issues for future works. In Appendix, we provide tedious and straightforward proofs for Lemma 3.2, Lemma 3.5, Lemma 4.4 and Theorem 5.2.

Notation. For any $k \in \mathbb{N} \cup \{0\}$, we set $H^k := H^k(\mathbb{T}^d)$ and $W^{k,\infty} := W^{k,\infty}(\mathbb{T}^d)$ to be the k-th order L^2- and L^∞-Sobolev spaces on $\mathbb{T}^d$, respectively and $\mathcal{C}^k(I;\mathcal{B})$ is the space of k-times continuously differentiable functions from an interval I into a Banach space $\mathcal{B}$. Moreover, ∇^k denotes any partial derivative ∂^α with respect to x-variable with multi-index α with $|\alpha| = k$.

2. Preliminaries

In this section, we briefly recall a heuristic derivation of the random HCS model from the kinetic CS equation under the mono-kinetic ansatz, and present previous results on the deterministic HCS model.

2.1. The random HCS model

Consider an ensemble of N identical CS flocking particles in a random communication field registered by $\psi := \psi(x,z)$ and let $(x_i(t,z), v_i(t,z)) \in \mathbb{R}^d \times \mathbb{R}^d$ be the spatial and velocity process of the i-th CS particle. Then, the dynamics of the random process $(x_i(t,z), v_i(t,z))$ is governed by the following Cauchy problem to the random ODE system:

$$
\begin{aligned}
&\partial_t x_i(t,z) = v_i(t,z), \quad i = 1,\cdots,N, \quad t > 0,\\
&\partial_t v_i(t,z) = \frac{1}{N}\sum_{j=1}^{N} \psi(x_j(t,z) - x_i(t,z), z)(v_j(t,z) - v_i(t,z)),\\
&(x_i(0,z), v_i(0,z)) = (x_{i0}(z), v_{i0}(z)).
\end{aligned}
\tag{2.1}
$$

Note that for each fixed z, (2.1) becomes a deterministic CS model which has been extensively studied in [3,4,6–10,14,20,22,23] from various points of view. On the other hand, when the number of particles is sufficiently large, the dynamic observables of the CS ensemble can be effectively described by the velocity moments for the one-particle distribution function $f = f(t,x,v,z)$. Via the mean field limit $N \to \infty$ on (2.1), the kinetic density function f satisfies the following Vlasov-type equation:

$$
\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(F_a[f]f) = 0, \quad t > 0, \ \ x, v \in \mathbb{R}^d, \ \ z \in \Omega,\\
&F_a[f](t,x,v,z) = -\int_{\mathbb{R}^{2d}} \psi(x - x_*, z)(v - v_*) f(t, x_*, v_*, z)\, dv_* dx_*,\\
&f(0,x,v,z) = f_0(x,v,z).
\end{aligned}
\tag{2.2}
$$

The well-posedness and flocking estimates for the deterministic analogue of (2.2) and its generalized models have been studied in [27–29,22,23,34], and recently, the local sensitivity analysis for the random CS models (2.1) and (2.2) has been carried out in [16,17]. To derive a hydrodynamic model from (2.2), we introduce the macroscopic observables such as the local mass, momentum and energy densities:

$$
\begin{aligned}
&\rho(t,x,z) := \int_{\mathbb{R}^d} f\,dv, \quad (\rho u)(t,x,z) := \int_{\mathbb{R}^d} v f\,dv, \\
&(\rho E)(x,t) := \frac{1}{2}\rho|u|^2 + \rho e, \quad \rho e := \frac{1}{2}\int_{\mathbb{R}^d} |v-u(x,t)|^2 f\,dv.
\end{aligned}
$$

We multiply 1, v, $|v|^2/2$ to (2.2) and integrate the resulting relations with respect to the velocity variable to derive a system of balance laws for the macroscopic observables (ρ, u, E):

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \quad z \in \Omega, \\
&\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + P) = S^{(1)}, \\
&\partial_t (\rho E) + \nabla \cdot (\rho E u + P u + q) = S^{(2)},
\end{aligned}
\tag{2.3}
$$

where $P = (p_{ik})$ and $q = (q_1, \cdots, q_d)$ are the stress tensor and heat flow, respectively:

$$
p_{ij}(t,x,z) := \int_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j) f\,dv, \quad q_i(t,x,z) := \int_{\mathbb{R}^d} (v_i - u_i)|v-u|^2 f\,dv,
$$

and the source terms are written as follows:

$$
\begin{aligned}
&S^{(1)}(t,x,z) := \rho \int_{\mathbb{R}^d} \psi(x-y,z)(u(t,y,z) - u(t,x,z))\rho(t,y,z)dy, \\
&S^{(2)}(t,x,z) := \rho \int_{\mathbb{R}^d} \psi(x-y,z)(E(t,x,z) + E(t,y,z) - u(t,x,z)\cdot u(t,y,z))\rho(t,y,z)dy.
\end{aligned}
$$

Since system (2.3) is not closed as it is, one may introduce a mono-kinetic ansatz for f as a closure condition:

$$
f(t,x,v,z) = \rho(t,x,z)\delta_{(v-u(x,t))}(v).
$$

With this ansatz, it can be observed that the internal energy, stress tensor and heat flux in (2.3) vanish, and we obtain the following pressureless Euler system with flocking dissipation:

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
&\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = S^{(1)}.
\end{aligned}
$$

Before we move on to next subsection, we present conservation laws associated with system (1.1).

Proposition 2.1. *Let (ρ, u) be a sufficiently smooth periodic solution to* (1.1)*. Then, for a fixed $z \in \Omega$ and $m \geq 0$,*

$$\int_{\mathbb{T}^d} \partial_z^m \rho(t,z)dx = \int_{\mathbb{T}^d} \partial_z^m \rho_0(z)dx, \quad \int_{\mathbb{T}^d} \partial_z^m (\rho u)(t,z)dx = \int_{\mathbb{T}^d} \partial_z^m (\rho_0 u_0)(z)dx, \quad t \geq 0, \quad z \in \Omega.$$

Proof. The proofs follow from the direct integration of (1.1). □

2.2. Previous results

In this subsection, we recall several previous results on the deterministic HCS model:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \rho \displaystyle\int_{\mathbb{T}^d} \psi(x-y)(u(t,y) - u(t,x))\rho(t,y)dy, \\ (\rho, u)(0,x) = (\rho_0, u_0). \end{cases} \tag{2.4}$$

Below, we provide the standing assumptions $(\mathcal{H}1) - (\mathcal{H}2)$ for the well-posedness, stability and flocking estimates for (1.1). For an integer $s > \frac{d}{2} + 1$,

- $(\mathcal{H}1)$: The communication weight function $\psi : \mathbb{T}^d \to \mathbb{R}$ is in $\mathcal{C}^{s+1}$ and satisfies symmetric, positive conditions: for each $x, y \in \mathbb{T}^d$,

$$\psi(x-y) = \psi(y-x) \quad \text{and} \quad \inf_{x \in \mathbb{T}^d} \psi(x) =: \psi_m > 0.$$

- $(\mathcal{H}2)$: The initial data (ρ_0, u_0) satisfy the non-vacuum, regularity and smallness conditions, i.e. for sufficiently small $\varepsilon > 0$,

$$\inf_{x \in \mathbb{T}^d} \rho_0(x) > 0, \quad (\rho_0, u_0) \in H^s \times H^{s+1}, \quad \|\rho_0\|_{H^s} + \|u_0\|_{H^{s+1}} < \varepsilon.$$

Before we state previous results, we introduce a Lyapunov functional $\mathcal{E}_0$ for flocking:

$$\mathcal{E}_0(t) := \int_{\mathbb{T}^d} \rho |u - u_c(t)|^2 dx, \qquad u_c(t) := \frac{\int_{\mathbb{T}^d} \rho u dx}{\int_{\mathbb{T}^d} \rho dx} = u_c(0), \qquad t \geq 0. \tag{2.5}$$

Then, the random HCS model can be summarized in the following theorem:

Theorem 2.1. [21] *For a given positive constant $T > 0$, suppose that conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ hold. Then, there exist positive constants $C = C(T)$ and $0 < \varepsilon \ll 1$ such that the Cauchy problem (2.4) has a unique global-in-time classical solution process (ρ, u) satisfying the following properties:*

(1) (Propagation of the Sobolev regularity): The solution (ρ, u) satisfies the following regularity and uniform-in-time boundedness condition:

$$\inf_{(t,x)\in[0,T]\times\mathbb{T}^d} \rho(t,x) > 0, \quad (\rho(t), u(t)) \in H^s \times H^{s+1}, \quad \textit{for } t \in [0,T],$$
$$\sup_{0\le t\le T} \left(\|\rho(t)\|_{H^s} + \|u(t)\|_{H^{s+1}}\right) < \sqrt{\varepsilon}.$$

(2) *(Finite-in-time stability): For two classical solution processes (ρ, u) and $(\bar\rho, \bar u)$ to (2.4) with initial data (ρ_0, u_0) and $(\bar\rho_0, \bar u_0)$ respectively,*

$$\sup_{0\le t\le T} \left(\|\rho(t)-\bar\rho(t)\|_{L^2}^2 + \|u(t)-\bar u(t)\|_{H^1}^2\right) \le C(T)(\|\rho_0-\bar\rho_0\|_{L^2}^2 + \|u_0-\bar u_0\|_{H^1}^2).$$

(3) *(Exponential pathwise flocking estimate): The functional $\mathcal{E}_0(t)$ decays exponentially pathwise:*

$$\mathcal{E}_0(t) \le e^{-2\psi_m\|\rho_0\|_{L^1}t}\mathcal{E}_0(0), \quad \forall t > 0.$$

Remark 2.1. By Theorem 2.1, the local mass ρ stays positive. Moreover, since the solution is classical, the momentum equations of (2.4) can be rewritten as

$$\partial_t u + u\cdot\nabla u = \int_{\mathbb{T}^d} \psi(x-y)(u(t,y)-u(t,x))\rho(t,y)dy.$$

3. Pathwise well-posedness of z-variations

In this section, we present a global existence of z-variations $(\partial_z^m\rho, \partial_z^m u)$ to system (1.1) using pathwise energy method.

Note that in a non-vacuum regime, system (1.1) can be rewritten as

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho u) = 0, \quad t>0, \ \ x\in\mathbb{T}^d, \ \ z\in\Omega, \\ \partial_t u + u\cdot\nabla u = \displaystyle\int_{\mathbb{T}^d} \psi(x-y,z)(u(t,y,z)-u(t,x,z))\rho(t,y,z)dy, \\ (\rho,u)(0,x,z) = (\rho_0(x,z), u_0(x,z)). \end{cases} \tag{3.1}$$

First, we derive equations for the z-variations by applying z-derivative to (3.1) to obtain

$$\begin{aligned} &\partial_t(\partial_z^m\rho) + \sum_{l=0}^m \binom{m}{l} \nabla\cdot\left(\partial_z^l\rho\,\partial_z^{m-l}u\right) = 0, \\ &\partial_t(\partial_z^m u) + \sum_{l=0}^m \binom{m}{l}\left(\partial_z^l u\cdot\nabla(\partial_z^{m-l}u)\right) \\ &\quad = \sum_{\alpha+\beta+\gamma=m} \frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^d} \partial_z^\alpha\psi(x-y,z)\partial_z^\beta\left(u(t,y,z)-u(t,x,z)\right)\partial_z^\gamma\rho(t,y,z)dy. \end{aligned} \tag{3.2}$$

For a global well-posedness of the z-variations, we provide our standing assumptions $(\mathcal{A}1)-(\mathcal{A}2)$ as follows: For an integer $s > \frac{d}{2}+m+1$,

- $(\mathcal{A}1)$: The communication weight function $\psi : \mathbb{T}^d \times \Omega \to \mathbb{R}$ is in $\mathcal{C}^{s+1}(\mathbb{T}^d \times \Omega)$ and satisfies symmetric, non-negative and boundedness conditions: for each $x, y \in \mathbb{T}^d$ and $z \in \Omega$,

$$\psi(x-y,z) = \psi(y-x,z) \geq 0, \qquad \|\psi\|_s := \max_{|\alpha|+|\beta|\leq s+1} \sup_{(x,z)\in\mathbb{T}^d\times\Omega} |\partial_z^\alpha \partial_x^\beta \psi(x,z)| < \infty.$$

- $(\mathcal{A}2)$: The initial data (ρ_0, u_0) satisfy the non-vacuum, regularity and smallness conditions: for each $z \in \Omega$ and $l = 0, \cdots, m$,

$$\begin{aligned} &\inf_{x\in\mathbb{T}^d} \rho_0(x,z) > 0, \quad (\partial_z^l \rho_0(z), \partial_z^l u_0(z)) \in H^{s-l} \times H^{s-l+1}, \\ &\max_{0\leq l\leq m} \Big(\|\partial_z^l \rho_0(z)\|_{H^{s-l}} + \|\partial_z^l u_0(z)\|_{H^{s-l+1}} \Big) < \varepsilon(z), \end{aligned}$$

where $\varepsilon = \varepsilon(z)$ is a positive random function such that $\sup\limits_{z\in\Omega} \varepsilon(z) \ll 1$.

For the simplicity of notation, we suppress z-dependence in (ρ, u) and ψ, i.e.

$$\rho(t,x) := \rho(t,x,z), \qquad u(t,x) := u(t,x,z), \qquad \psi(x) := \psi(x,z).$$

To derive *a priori* estimates, we employ a mathematical induction on m.

3.1. First-order z-variations

In this subsection, we consider a global well-posedness for the first-order z-variations $(\partial_z\rho, \partial_z u)$ for the initial step of induction process on m. To provide a global well-posedness, we construct a sequence of approximated solutions $(\partial_z\rho^{n+1}, \partial_z u^{n+1})$ to (3.2). For a given solution (ρ, u) and $m = 1$, we may construct the sequence as follows:

$$\begin{aligned} &\partial_t(\partial_z\rho^{n+1}) + \nabla\cdot(\partial_z\rho^{n+1} u) + \nabla(\rho\partial_z u^n) = 0, \quad n = 0,1,2,\cdots \\ &\partial_t(\partial_z u^{n+1}) + \partial_z u^n \cdot \nabla u + u\cdot\nabla(\partial_z u^{n+1}) \\ &\qquad = \int_{\mathbb{T}^d} \partial_z\psi(x-y)(u(t,y)-u(t,x))\rho(t,y)dy \\ &\qquad\quad + \int_{\mathbb{T}^d} \psi(x-y)(\partial_z u^n(t,y) - \partial_z u^n(t,x))\rho(t,y)dy \\ &\qquad\quad + \int_{\mathbb{T}^d} \psi(x-y)(u(t,y)-u(t,x))\partial_z\rho^{n+1}(t,y)dy \\ &(\partial_z\rho^0, \partial_z u^0) = (\partial_z\rho_0, \partial_z u_0), \end{aligned} \tag{3.3}$$

subject to the fixed initial data:

$$(\partial_z\rho^{n+1}(0,x), \partial_z u^{n+1}(0,x)) = (\partial_z\rho_0(x), \partial_z u_0(x)).$$

Since the pathwise well-posedness for (ρ, u) is already given in Theorem 2.1, there is no need for (ρ, u) to be involved in the iteration scheme (3.3). Thus, the iteration procedure in (3.3) will be carried out only for the z-variations $(\partial_z\rho, \partial_z u)$. We proceed by induction on n for the sequence $(\partial_z\rho^n, \partial_z u^n)$. First, we state the results on the uniform-in-n bound estimates.

Lemma 3.1. *Suppose that assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ and induction hypothesis hold: for each $z \in \Omega$,*

$$\sup_{\substack{0\le j\le n\\ 0\le t\le T}} \|\partial_z u^j(t,z)\|_{H^s} < \sqrt{\varepsilon(z)}.$$

Then, there exists a unique $\partial_z\rho^{n+1} = \partial_z\rho^{n+1}(t,z) \in H^{s-1}$ satisfying relation $(3.3)_1$ and a bound:

$$\sup_{0\le t\le T} \|\partial_z\rho^{n+1}(t,z)\|_{H^{s-1}} < \frac{\sqrt{\varepsilon(z)}}{2}.$$

Proof. Since system (3.3) is linear with respect to $\partial_z\rho^{n+1}$, the existence and uniqueness for $\partial_z\rho^{n+1}$ are obvious. Thus, it suffices to show the boundedness of the solution. Here, we split the estimates into the zeroth-order case and higher-order case.

• Step A (The zeroth-order estimates): First, we multiply $(3.3)_1$ by $\partial_z\rho^{n+1}$ and integrate it over $\mathbb{T}^d$ to yield

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z\rho^{n+1}\|_{L^2}^2\\
&= -\frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\partial_z\rho^{n+1}|^2dx - \int_{\mathbb{T}^d}(\nabla\rho\cdot\partial_z u^n)\partial_z\rho^{n+1}dx - \int_{\mathbb{T}^d}\rho\nabla\cdot(\partial_z u^n)\partial_z\rho^{n+1}dx\\
&\le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\partial_z\rho^{n+1}\|_{L^2}^2 + \|\rho\|_{W^{1,\infty}}\|\partial_z u^n\|_{H^1}\|\partial_z\rho^{n+1}\|_{L^2}\\
&\le \left(\frac{\|\nabla\cdot u\|_{L^\infty}}{2} + \frac{\|\rho\|_{W^{1,\infty}}}{2}\right)\|\partial_z\rho^{n+1}\|_{L^2}^2 + \frac{\|\rho\|_{W^{1,\infty}}}{2}\|\partial_z u^n\|_{H^1}^2,\\
&\le \varepsilon^{1/2}\|\partial_z\rho^{n+1}\|_{L^2}^2 + \varepsilon^{3/2},
\end{aligned} \tag{3.4}$$

where we used Young's inequality on the second inequality and Theorem 2.1 on the last inequality. Then, we integrate the above relation (3.4) to derive

$$\|\partial_z\rho^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z\rho^{n+1}(s,z)\|_{L^2}^2 ds + \varepsilon^{3/2}\right). \tag{3.5}$$

• Step B (Higher-order estimates): For higher-order estimates, let $1 \le k \le s-1$. Then, we apply ∇^k to $(3.3)_1$, multiply by $\nabla^k(\partial_z\rho^{n+1})$ and integrate the resulting relation over $\mathbb{T}^d$ to yield

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}^2\\
&=\frac{1}{2}\int_{\mathbb{T}^d}|\nabla^k(\partial_z\rho^{n+1})|^2(\nabla\cdot u)dx-\int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z\rho^{n+1}))-u\cdot\nabla^k(\nabla(\partial_z\rho^{n+1}))\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad-\int_{\mathbb{T}^d}\partial_z\rho^{n+1}\nabla^k(\nabla\cdot u)\nabla^k(\partial_z\rho^{n+1})dx-\int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z\rho^{n+1}\nabla\cdot u)-\partial_z\rho^{n+1}\nabla^k(\nabla\cdot u)\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad-\int_{\mathbb{T}^d}\nabla^k(\nabla\rho)\cdot\partial_z u^n\nabla^k(\partial_z\rho^{n+1})dx-\int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z u^n\nabla\rho)-\partial_z u^n\cdot\nabla^k(\nabla\rho)\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad-\int_{\mathbb{T}^d}\rho\nabla^k(\nabla\cdot\partial_z u^n)\nabla^k(\partial_z\rho^{n+1})dx-\int_{\mathbb{T}^d}\Big[\nabla^k(\rho(\nabla\cdot\partial_z u^n))-\rho\nabla^k(\nabla\cdot\partial_z u^n)\Big]\nabla^k(\partial_z\rho^{n+1})dx,\\
&=:\sum_{i=1}^{8}\mathcal{I}_{1i}.
\end{aligned}
$$

Below, we estimate the terms $\mathcal{I}_{1i}$ separately as follows:

$\diamond$ (Estimates for $\mathcal{I}_{1i}$, $i=2,4,6,8$): We use the commutator estimate from Lemma 3.4 in [31] to obtain

$$
\begin{aligned}
\mathcal{I}_{12}&\le c\left(\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}+\|\nabla(\partial_z\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\right)\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}\\
&\le C\left(\|u\|_{H^{s-1}}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}+\|\partial_z\rho^{n+1}\|_{H^{s-1}}\|\nabla^k u\|_{L^2}\right)\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}\\
&\le C\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|_{H^{s-1}}^2,
\end{aligned}
$$

where c and C are positive random functions independent of n and we used the assumptions, Theorem 2.1 and the Sobolev embedding:

$$
\|\nabla u\|_{L^\infty}\le C\|u\|_{H^{\left[\frac{d}{2}\right]+1}}\le C\|u\|_{H^{s-1}}. \tag{3.6}
$$

For other terms, one uses the commutator estimate, (3.6), Theorem 2.1 and Young's inequality to get

$$
\begin{aligned}
\mathcal{I}_{14}&\le c\left(\|\nabla(\partial_z\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}+\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}\right)\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2},\\
&\le C\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|_{H^{s-1}}^2,\\
\mathcal{I}_{16}&\le c\left(\|\nabla(\partial_z u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2}\right)\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2},\\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}^2+\varepsilon^{3/2}),\\
\mathcal{I}_{18}&\le c\left(\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2}+\|\nabla\cdot(\partial_z u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2}\right)\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}\\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}^2+\varepsilon^{3/2}).
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{1i}$, $i = 1, 3, 5, 7$): By direct calculations, one easily obtains

$$\begin{aligned}
\mathcal{I}_{11} &\le \frac{\|\nabla \cdot u\|_{L^\infty}}{2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2}^2 \le \varepsilon^{1/2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2}^2, \\
\mathcal{I}_{13} &\le \|\partial_z \rho^{n+1}\|_{L^\infty} \|\nabla^{k+1} u\|_{L^2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2} \|\partial_z \rho^{n+1}\|_{H^{s-1}}^2 + \varepsilon^{3/2}), \\
\mathcal{I}_{15} &\le \|\nabla^{k+1} \rho\|_{L^2} \|\partial_z u^n\|_{L^\infty} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}), \\
\mathcal{I}_{17} &\le \|\rho\|_{L^\infty} \|\nabla^{k+1}(\partial_z u^n)\|_{L^2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2} \le C(\varepsilon^{1/2} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}$$

We combine all results for $\mathcal{I}_{1i}$'s to obtain

$$\frac{1}{2}\frac{\partial}{\partial t} \|\nabla^k(\partial_z \rho^{n+1})\|_{L^2}^2 \le C(\varepsilon^{1/2} \|\partial_z \rho^{n+1}\|_{H^{s-1}}^2 + \varepsilon^{3/2}). \tag{3.7}$$

Summing (3.7) over $1 \le k \le s-1$ and adding these to (3.5) yields

$$\frac{\partial}{\partial t} \|\partial_z \rho^{n+1}\|_{H^{s-1}}^2 \le C(\varepsilon^{1/2} \|\partial_z \rho^{n+1}\|_{H^{s-1}}^2 + \varepsilon^{3/2}).$$

Then, Grönwall's lemma and the smallness of ε yield the desired estimate:

$$\|\partial_z \rho^{n+1}\|_{H^{s-1}}^2 \le e^{\varepsilon^{1/2} CT} \|\partial_z \rho_0\|_{H^{s-1}}^2 + \varepsilon(e^{\varepsilon^{1/2} CT} - 1) < \frac{\varepsilon}{4}. \quad \square$$

Lemma 3.2. *Suppose that assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold and let $(\partial_z \rho^j, \partial_z u^j)$ be the j-th iterate satisfying the following assumptions: for each $z \in \Omega$,*

$$\max_{0 \le j \le n} \sup_{0 \le t \le T} \Big(\|\partial_z \rho^j(t,z)\|_{H^{s-1}} + \|\partial_z u^j(t,z)\|_{H^s} \Big) < \sqrt{\varepsilon(z)}.$$

Then for each $z \in \Omega$, there exists a unique $\partial_z u^{n+1} = \partial_z u^{n+1}(t,z) \in H^s$ satisfying relation $(3.3)_2$ and the following bound:

$$\sup_{0 \le t \le T} \|\partial_z u^{n+1}(t,z)\|_{H^s} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \textit{for each} \quad z \in \Omega.$$

Proof. Since the proof is similar to that of Lemma 3.1, we leave it to Appendix A. $\square$

Remark 3.1. From Lemmas 3.1 and 3.2, one can find out that if assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold, the induction on n yields that for every n and $z \in \Omega$:

$$\sup_{0 \le t \le T} \big(\|\partial_z \rho^n(t,z)\|_{H^{s-1}} + \|\partial_z u^n(t,z)\|_{H^s} \big) < \sqrt{\varepsilon(z)}.$$

Now, we provide estimates for the convergence of the sequence $(\partial_z \rho^n, \partial_z u^n)$ in $L^2 \times H^1$.

Lemma 3.3. *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold. Then, for each* $z \in \Omega$ *and* $n \in \mathbb{N}$,

$$\begin{aligned}
&\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(t,z)\|_{L^2}^2 + \|(\partial_z u^{n+1} - \partial_z u^n)(t,z)\|_{H^1}^2 \\
&\quad \le C(z)\Bigg(\int_0^t \Big(\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|_{H^1}^2\Big)ds \\
&\qquad + \int_0^t \|(\partial_z u^n - \partial_z u^{n-1})(s,z)\|_{H^1}^2 ds\Bigg),
\end{aligned}$$

where $C = C(z)$ *is a positive random function independent of* n.

Proof. It follows from $(3.3)_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z\rho^{n+1} - \partial_z\rho^n\|_{L^2}^2 \\
&= -\frac{1}{2}\int_{\mathbb{T}^d} \nabla|\partial_z\rho^{n+1} - \partial_z\rho^n|^2 \cdot u\, dx - \int_{\mathbb{T}^d} \nabla\cdot(\rho(\partial_z u^n - \partial_z u^{n-1}))(\partial_z\rho^{n+1} - \partial_z\rho^n)dx \\
&\le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\partial_z\rho^{n+1} - \partial_z\rho^n\|_{L^2}^2 + \|\rho\|_{W^{1,\infty}}\|\partial_z u^n - \partial_z u^{n-1}\|_{H^1}\|\partial_z\rho^{n+1} - \partial_z\rho^n\|_{L^2} \\
&\le C(\|\partial_z\rho^{n+1} - \partial_z\rho^n\|_{L^2}^2 + \|\partial_z u^n - \partial_z u^{n-1}\|_{H^1}^2).
\end{aligned}$$

We integrate the above relation to see

$$\begin{aligned}
&\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(t,z)\|_{L^2}^2 \\
&\quad \le C(z)\int_0^t \Big(\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z u^n - \partial_z u^{n-1})(s,z)\|_{H^1}^2\Big)ds.
\end{aligned} \tag{3.8}$$

Next, one uses $(3.3)_2$ to yield

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z u^{n+1} - \partial_z u^n\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} (\partial_z u^n - \partial_z u^{n-1})\cdot\nabla u\cdot(\partial_z u^{n+1} - \partial_z u^n)dx - \frac{1}{2}\int_{\mathbb{T}^d} u\cdot\nabla|\partial_z u^{n+1} - \partial_z u^n|^2 dx \\
&\quad + \int_{\mathbb{T}^{2d}} \psi(x-y)\partial_z\Big\{(u^n - u^{n-1})(y) - (u^n - u^{n-1})(x)\Big\}\rho(y)\partial_z(u^{n+1} - u^n)(x)dydx \\
&\quad + \int_{\mathbb{T}^{2d}} \psi(x-y)(u(y) - u(x))(\partial_z\rho^{n+1} - \partial_z\rho^n)(y)(\partial_z u^{n+1} - \partial_z u^n)(x)dydx \\
&\le \|\nabla u\|_{L^\infty}\|\partial_z u^n - \partial_z u^{n-1}\|_{L^2}\|\partial_z u^{n+1} - \partial_z u^n\|_{L^2} + \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\partial_z u^{n+1} - \partial_z u^n\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2\|\psi\|_s \|\rho\|_{L^2} \|\partial_z u^n - \partial_z u^{n-1}\|_{L^2} \|\partial_z u^{n+1} - \partial_z u^n\|_{L^2} \\
&+ 2\|\psi\|_s \|u\|_{L^2} \|\partial_z \rho^{n+1} - \partial_z \rho^n\|_{L^2} \|\partial_z u^{n+1} - \partial_z u^n\|_{L^2}.
\end{aligned}
$$

We use Young's inequality and integrate the above relation over $[0, t]$ to get

$$
\begin{aligned}
&\|(\partial_z u^{n+1} - \partial_z u^n)(t, z)\|_{L^2}^2 \\
&\quad \le C(z)\Bigg(\int_0^t \Big(\|(\partial_z \rho^{n+1} - \partial_z \rho^n)(s, z)\|_{L^2}^2 + \|(\partial_z u^{n+1} - \partial_z u^n)(s, z)\|_{L^2}^2 \Big) ds \\
&\qquad\qquad + \int_0^t \|(\partial_z u^n - \partial_z u^{n-1})(s, z)\|_{L^2}^2 ds \Bigg).
\end{aligned}
\tag{3.9}
$$

For the H^1-estimate for $(\partial_z u^{n+1} - \partial_z u^n)$, we use the Cauchy-Schwarz inequality to get

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla(\partial_z u^{n+1} - \partial_z u^n)\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \nabla((\partial_z u^n - \partial_z u^{n-1}) \cdot \nabla u) : \nabla(\partial_z u^{n+1} - \partial_z u^n) dx \\
&\quad - \int_{\mathbb{T}^d} \nabla(u \cdot \nabla(\partial_z u^{n+1} - \partial_z u^n)) : \nabla(\partial_z u^{n+1} - \partial_z u^n) dx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla\psi(x - y)\partial_z \Big\{ (u^n - u^{n-1})(y) - (u^n - u^{n-1})(x) \Big\} \rho(y) \nabla(\partial_z u^{n+1} - \partial_z u^n)(x) dy dx \\
&\quad - \int_{\mathbb{T}^{2d}} \psi(x - y)\nabla(\partial_z u^n - \partial_z u^{n-1})(x)\rho(y) : \nabla(\partial_z u^{n+1} - \partial_z u^n)(x) dy dx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla\{\psi(x - y)(u(y) - u(x))\}(\partial_z \rho^{n+1} - \partial_z \rho^n)(y) : \nabla(\partial_z u^{n+1} - \partial_z u^n)(x) dy dx \\
&\le \|\partial_z u^n - \partial_z u^{n-1}\|_{H^1} \|u\|_{W^{2,\infty}} \|\nabla(\partial_z u^{n+1} - \partial_z u^n)\|_{L^2} + \frac{\|\nabla \cdot u\|_{L^\infty}}{2} \|\nabla(\partial_z u^{n+1} - \partial_z u^n)\|_{L^2}^2 \\
&\quad + 2\|\psi\| \|\rho\|_{L^2} \|\partial_z u^n - \partial_z u^{n-1}\|_{H^1} \|\nabla(\partial_z u^{n+1} - \partial_z u^n)\|_{L^2} \\
&\quad + 2\|\psi\| \|u\|_{H^1} \|\partial_z \rho^{n+1} - \partial_z \rho^n\|_{L^2} \|\nabla(\partial_z u^{n+1} - \partial_z u^n)\|_{L^2}.
\end{aligned}
$$

Again, using Young's inequality and integration along $[0, t]$ give

$$\begin{aligned}
&\|\nabla(\partial_z u^{n+1} - \partial_z u^n)(t,z)\|_{L^2}^2 \\
&\quad \le C(z)\Bigg(\int_0^t \Big(\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z u^n - \partial_z u^{n-1})(s,z)\|_{H^1}^2\Big)\,ds \\
&\qquad\qquad + \int_0^t \|\nabla(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|_{L^2}^2 ds\Bigg).
\end{aligned} \tag{3.10}$$

Finally, one combines (3.8), (3.9) and (3.10) to yield the desired result. □

Now, we are ready to state our first result on the well-posedness of a global solution to (3.3).

Theorem 3.1. *Suppose that assumptions ($\mathcal{A}1$)-($\mathcal{A}2$) hold. Then for each $z \in \Omega$, there exists a unique solution $(\partial_z\rho(z), \partial_z u(z)) \in H^{s-1} \times H^s$ satisfying system (3.3) and uniform bound estimates:*

$$\sup_{0\le t\le T} (\|\partial_z\rho(t,z)\|_{H^{s-1}} + \|\partial_z u(t,z)\|_{H^s}) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z \in \Omega.$$

Proof. For each $n \in \mathbb{N}$ and $z \in \Omega$, define

$$\Delta_n(t,z) := \|\partial_z\rho^n - \partial_z\rho^{n-1}\|_{L^2}^2 + \|\partial_z u^n - \partial_z u^{n-1}\|_{H^1}^2.$$

We can deduce from Lemma 3.3 that for each $z \in \Omega$,

$$\Delta_{n+1}(t,z) \le C(z)\left(\int_0^t \Delta_{n+1}(s,z)ds + \int_0^t \Delta_n(s,z)ds\right), \quad t \in [0,T].$$

Then, the Grönwall-type lemma in [2] gives

$$\sup_{0\le t\le T}\Big(\|(\partial_z\rho^n - \partial_z\rho^{n-1})(t,z)\|_{L^2}^2 + \|(\partial_z u^n - \partial_z u^{n-1})(t,z)\|_{H^1}^2\Big) \le \frac{(C(z)T)^n}{n!}, \quad z \in \Omega.$$

This implies that $\{\partial_z\rho^n\}$ and $\{\partial_z u^n\}$ are Cauchy sequences in $C([0,T];L^2)$ and $C([0,T];H^1)$, respectively. From here, one can follow the proof of Theorem 3.1 in [21] to complete the proof. □

3.2. Higher-order z-variations

In this subsection, we consider higher-order z-variations, i.e. the case when $m \ge 2$ in (3.2), in order to complete the induction process on m. Similar to the case $m = 1$, we again construct a sequence of approximated solutions $(\partial_z^m\rho^{n+1}, \partial_z^m u^{n+1})$ to (3.2) as follows:

$$
\begin{aligned}
&\partial_t(\partial_z^m \rho^{n+1}) + \nabla\cdot(\partial_z^m \rho^{n+1} u) + \nabla\cdot(\rho\partial_z^m u^n) + \sum_{1\le l\le m-1}\binom{m}{l}\nabla\cdot(\partial_z^l\rho\partial_z^{m-l}u) = 0,\\
&\partial_t(\partial_z^m u^{n+1}) + \partial_z^m u^n\cdot\nabla u + u\cdot\nabla(\partial_z^m u^{n+1}) + \sum_{1\le l\le m-1}\binom{m}{l}\partial_z^l u\cdot\nabla(\partial_z^{m-1}u)\\
&\quad = \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta,\gamma\neq m}}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^d}\partial_z^\alpha\psi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\partial_z^\gamma\rho(y)dy\\
&\qquad + \int_{\mathbb{T}^d}\psi(x-y)(\partial_z^m u^n(y)-\partial_z^m u^n(x))\rho(y)dy\\
&\qquad + \int_{\mathbb{T}^d}\psi(x-y)(u(y)-u(x))\partial_z^m\rho^{n+1}(y)dy,\\
&(\partial_z^m\rho^0, \partial_z^m u^0) = (\partial_z^m\rho_0, \partial_z^m u_0),
\end{aligned}
\tag{3.11}
$$

subject to the initial data:

$$
(\partial_z^m\rho^{n+1}(0,z), \partial_z^m u^{n+1}(0,z)) = (\partial_z^m\rho_0(z), \partial_z^m u_0(z)).
$$

Similar to the previous subsection, we first show the uniform boundedness of the sequence $\{(\partial_z^m\rho^n, \partial_z^m u^n)\}_{n=0}^\infty$.

Lemma 3.4. *For $m\ge 2$ and $n\in\mathbb{N}$, suppose that the following conditions hold:*

(1) Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.
(2) For $l\le m-1$, the l-th z-variations $\{(\partial_z^l\rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:

$$
\max_{0\le l\le m-1}\sup_{0\le t\le T}\Big(\|\partial_z^l\rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}}\Big) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z\in\Omega.
$$

(3) The sequence approximating the m-th z-variation of the bulk velocity process satisfies the following boundedness condition:

$$
\max_{0\le j\le n}\sup_{0\le t\le T}\|\partial_z^m u^j(t,z)\|_{H^s} < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z\in\Omega.
$$

Then, there exists a unique $\partial_z^m\rho^{n+1}\in H^{s-m}$ which satisfies relation $(3.11)_1$ and the bound:

$$
\sup_{0\le t\le T}\|\partial_z^m\rho^{n+1}(t,z)\|_{H^{s-m}} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \textit{for each} \quad z\in\Omega.
$$

Proof. We split the estimates into zeroth-order and higher-order cases as follows:

• Step A (The zeroth-order estimates): We multiply (3.11)$_1$ by $\partial_z^m \rho^{n+1}$ and integrate the resulting relation over $\mathbb{T}^d$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z^m \rho^{n+1}\|_{L^2}^2 \\
&\quad = -\frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|\partial_z^m \rho^{n+1}|^2 dx - \int_{\mathbb{T}^d} \nabla\cdot(\rho\partial_z^m u^n)\partial_z^m \rho^{n+1} dx \\
&\qquad - \sum_{1\le l\le m-1}\binom{m}{l}\int_{\mathbb{T}^d} \nabla\cdot(\partial_z^l \rho\partial_z^{m-l} u)\partial_z^m \rho^{n+1} dx \\
&\quad \le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\partial_z^m \rho^{n+1}\|_{L^2}^2 + \|\rho\|_{W^{1,\infty}}\|\partial_z^m u^n\|_{H^1}\|\partial_z^m \rho^{n+1}\|_{L^2} \\
&\qquad + \sum_{1\le l\le m-1}\binom{m}{l}\|\partial_z^l \rho\|_{W^{1,\infty}}\|\partial_z^{m-l} u\|_{H^1}\|\partial_z^m \rho^{n+1}\|_{L^2} \\
&\quad \le C(\varepsilon^{1/2}\|\partial_z^m \rho^{n+1}\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}$$

where C is a positive constant independent of n and we used Young's inequality. Integrating the above relation along $[0,t]$ gives

$$\|\partial_z^m \rho^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z^m \rho^{n+1}(s)\|_{L^2}^2 ds + \varepsilon^{3/2}\right), \quad \text{for each } z\in\Omega. \tag{3.12}$$

• Step B (Higher-order estimates): For $1\le k\le s-m$, we apply ∇^k to (3.11)$_1$, multiply by $\nabla^k(\partial_z^m \rho^{n+1})$ and integrate the resulting relation over $\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\partial_z^m \rho^{n+1} u))\nabla^k(\partial_z^m \rho^{n+1}) dx - \int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\rho\partial_z^m u^n)\nabla^k(\partial_z^m \rho^{n+1})) dx \\
&\quad - \sum_{1\le l\le m-1}\binom{m}{l}\int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\partial_z^l \rho\partial_z^{m-l} u))\nabla^k(\partial_z^m \rho^{n+1}) dx \\
&=: \sum_{i=1}^3 \mathcal{I}_{2i}.
\end{aligned}$$

We separately estimate $\mathcal{I}_{2i}$'s as follows.

◇ (Estimate for $\mathcal{I}_{21}$): We use the commutator estimate, Sobolev embedding theorem, Cauchy-Schwarz inequality and Young's inequality to get

$$\begin{aligned}
\mathcal{I}_{21} &= \frac{1}{2}\int_{\mathbb{T}^d} |\nabla^k(\partial_z^m \rho^{n+1})|^2(\nabla\cdot u)dx \\
&\quad - \int_{\mathbb{T}^d} \Big[\nabla^k(u\cdot\nabla(\partial_z^m\rho^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z^m\rho^{n+1}))\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\quad - \int_{\mathbb{T}^d} \partial_z^m\rho^{n+1}\nabla^k(\nabla\cdot u)\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\quad - \int_{\mathbb{T}^d} \Big[\nabla^k(\partial_z^m\rho^{n+1}(\nabla\cdot u)) - \partial_z^m\rho^{n+1}\nabla^k(\nabla\cdot u)\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}^2 \\
&\quad + c\Big(\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} + \|\nabla(\partial_z^m\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\Big)\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\quad + \|\nabla^{k+1}u\|_{L^2}\|\partial_z^m\rho^{n+1}\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\quad + c\Big(\|\nabla(\partial_z^m\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}\Big)\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\partial_z^m\rho^{n+1}\|_{H^{s-m}}^2 + \varepsilon^{3/2}),
\end{aligned}$$

where c and C are positive random functions independent of n.

$\diamond$ (Estimate for $\mathcal{I}_{22}$): Similar to the previous case,

$$\begin{aligned}
\mathcal{I}_{22} &= -\int_{\mathbb{T}^d} \nabla^k(\nabla\rho)\cdot\partial_z^m u^n\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\quad - \int_{\mathbb{T}^d} \Big[\nabla^k(\partial_z^m u^n\cdot\nabla\rho) - \partial_z^m u^n\cdot\nabla^k(\nabla\rho)\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\quad - \int_{\mathbb{T}^d} \rho\nabla^k(\nabla\cdot\partial_z^m u^n)\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\quad - \int_{\mathbb{T}^d} \Big[\nabla^k(\rho(\nabla\cdot\partial_z^m u^n)) - \rho\nabla^k(\nabla\cdot\partial_z^m u^n)\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&\le \|\nabla^{k+1}\rho\|_{L^2}\|\partial_z^m u^n\|_{L^\infty}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\quad + c\Big(\|\nabla(\partial_z^m u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2}\Big)\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\quad + \|\rho\|_{L^\infty}\|\nabla^{k+1}(\partial_z^m u^n)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\quad + c\Big(\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2} + \|\nabla\cdot\partial_z^m u^n\|_{L^\infty}\|\nabla^k\rho\|_{L^2}\Big)\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}$$

where c and C are positive random functions independent of n.

$\diamond$ (Estimates for $\mathcal{I}_{23}$): By direct calculation,

$$
\begin{aligned}
\mathcal{I}_{23} = -\sum_{1\le l\le m-1}\binom{m}{l}\Bigg\{ & \int_{\mathbb{T}^d} \nabla^k(\nabla(\partial_z^l\rho))\cdot\partial_z^{m-l}u\nabla^k(\partial_z^m\rho^{n+1})dx \\
& + \int_{\mathbb{T}^d} [\nabla^k(\partial_z^{m-l}u\cdot\nabla(\partial_z^l\rho)) - \partial_z^{m-l}u\nabla^k(\nabla(\partial_z^l\rho))]\nabla^k(\partial_z^m\rho^{n+1})dx \\
& + \int_{\mathbb{T}^d} \partial_z^l\rho\nabla^k(\nabla\cdot\partial_z^{m-l}u)\nabla^k(\partial_z^m\rho^{n+1})dx \\
& + \int_{\mathbb{T}^d} [\nabla^k(\partial_z^l\rho(\nabla\cdot\partial_z^{m-l}u)) - \partial_z^l\rho\nabla^k(\nabla\cdot\partial_z^{m-l}u)]\nabla^k(\partial_z^m\rho^{n+1})dx\Bigg\} \\
\le C\sum_{1\le l\le m-1}\Bigg(& \|\nabla^{k+1}\partial_z^l\rho\|_{L^2}\|\partial_z^{m-l}u\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
& + \|\nabla(\partial_z^{m-l}u)\|_{L^\infty}\|\nabla^k(\partial_z^l\rho)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
& + \|\nabla(\partial_z^l\rho)\|_{L^\infty}\|\nabla^k(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
& + \|\partial_z^l\rho\|_{L^\infty}\|\nabla^{k+1}(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
& + \|\nabla(\partial_z^l\rho)\|_{L^\infty}\|\nabla^k(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
& + \|\nabla\cdot\partial_z^{m-l}u\|_{L^\infty}\|\nabla^k(\partial_z^l\rho)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}\Bigg) \\
\le C(\varepsilon^{1/2} & \|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where C is a positive random function independent of n.

Now, we gather all the results for $\mathcal{I}_{2i}$'s to yield that for each $z\in\Omega$,

$$
\frac{\partial}{\partial t}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}^2 \le C(\varepsilon^{1/2}\|\partial_z^m\rho^{n+1}\|_{H^{s-m}}^2 + \varepsilon^{3/2}). \tag{3.13}
$$

Summing (3.13) over $1\le k\le s-m$, integrating over $[0,t]$ and combining with (3.12) give

$$
\|\partial_z^m\rho^{n+1}\|_{H^{s-m}}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z^m\rho^{n+1}(s)\|_{H^{s-m}}^2 ds + \varepsilon^{3/2}\right).
$$

Finally, one can use Grönwall's lemma to obtain the desired result. $\square$

Lemma 3.5. *For $m\ge 2$ and $n\in\mathbb{N}$, suppose that the following conditions hold:*

(1) Assumptions ($\mathcal{A}$1)-($\mathcal{A}$2) hold.

(2) For $l \le m-1$, the l-th z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:

$$\max_{0\le l\le m-1} \sup_{0\le t\le T} \left(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z\in\Omega.$$

(3) The sequence approximating the m-th z-variation of the local mass and bulk velocity processes satisfies the following boundedness condition:

$$\max_{0\le j\le n} \sup_{0\le t\le T} \left(\|\partial_z^m \rho^j(t,z)\|_{H^{s-1}} + \|\partial_z^m u^j(t,z)\|_{H^s} \right) < \sqrt{\varepsilon(z)}.$$

Then for each $z\in\Omega$, there exists a unique $\partial_z^m u^{n+1} = \partial_z^m u^{n+1}(t,z) \in H^{s-m+1}$ satisfying relation $(3.11)_2$ and the bound:

$$\sup_{0\le t\le T} \|\partial_z^m u^{n+1}(t,z)\|_{H^{s-m+1}} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \textit{for each} \quad z\in\Omega.$$

Proof. We leave its proof to Appendix B. □

Remark 3.2. For $m\ge 2$ and $n\in\mathbb{N}$, suppose that the following conditions hold:

(1) Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.
(2) For $l\le m-1$, the l-th z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:

$$\max_{0\le l\le m-1} \sup_{0\le t\le T} \left(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) < \sqrt{\varepsilon(z)}, \quad \text{for each} \quad z\in\Omega.$$

Then, it follows from Lemmas 3.4 and 3.5, that for every $n, m\in\mathbb{N}$ and $z\in\Omega$:

$$\sup_{0\le t\le T} \left(\|\partial_z^m \rho^n(t,z)\|_{H^{s-m}} + \|\partial_z^m u^n(t,z)\|_{H^{s-m+1}} \right) < \sqrt{\varepsilon(z)}, \quad \text{for each} \quad z\in\Omega.$$

Now, we assert that the sequence is Cauchy under the induction hypothesis on m.

Lemma 3.6. *For $m\ge 2$ and $n\in\mathbb{N}$, suppose that the following conditions hold:*

(1) Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.
(2) For $l\le m-1$, the l-th z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:

$$\max_{0\le l\le m-1} \sup_{0\le t\le T} \left(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z\in\Omega.$$

Then, for each $z \in \Omega$,

$$
\begin{aligned}
&\|(\partial_z^m \rho^{n+1} - \partial_z^m \rho^n)(t,z)\|_{L^2}^2 + \|(\partial_z^m u^{n+1} - \partial_z^m u^n)(t,z)\|_{H^1}^2 \\
&\qquad \le C(z)\Bigg(\int_0^t \Big(\|(\partial_z^m \rho^{n+1} - \partial_z^m \rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z^m u^{n+1} - \partial_z u^n)(s,z)\|_{H^1}^2 \Big)\, ds \\
&\qquad\qquad + \int_0^t \|(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|_{H^1}^2 ds \Bigg),
\end{aligned}
$$

where $C = C(z)$ is a positive random function independent of n.

Proof. We can replace ∂_z in the proof of Lemma 3.3 by ∂_z^m to get the desired proof. The details will be omitted. $\square$

Finally, we are ready to present our result on the well-posedness.

Theorem 3.2. *Suppose that assumptions ($\mathcal{A}1$)-($\mathcal{A}2$) hold. Then for each $m \in \mathbb{N}$ and $z \in \Omega$, there exists a unique pair $(\partial_z^m \rho(z), \partial_z^m u(z)) \in H^{s-m} \times H^{s-m+1}$ satisfying system (3.2) and the following uniform bound estimates:*

$$
\sup_{0 \le t \le T} (\|\partial_z^m \rho(t,z)\|_{H^{s-m}} + \|\partial_z^m u(t,z)\|_{H^{s-m+1}}) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z \in \Omega.
$$

Proof. One can use induction on m, Lemma 3.6 and follow the proof of Theorem 3.1 to show that $\{(\partial_z^m \rho^n(z), \partial_z^m u^n(z))\}_{n=0}^\infty$ is a Cauchy sequence in $C([0,T]; L^2) \times C([0,T]; H^1)$ for each $z \in \Omega$. From here, we again refer to [21] to complete the rest of the proof. $\square$

4. The local sensitivity analysis for stability estimates

In this section, we conduct a local sensitivity analysis for the L^2-stability estimates of the solution processes to (1.1) and their z-variations.

4.1. Higher-order L^2-stability

In this subsection, we derive a higher-order L^2-stability estimate of solution processes to (1.1) which will be used in the L^2-stability of the z-variations. First, we begin with the L^2-stability estimate for the local mass processes.

Lemma 4.1. *Suppose that assumptions ($\mathcal{A}1$)-($\mathcal{A}2$) hold, and let (ρ, u) and $(\bar\rho, \bar u)$ be two classical solution processes to (1.1) corresponding to the initial data (ρ_0, u_0) and $(\bar\rho_0, \bar u_0)$, respectively. Then,*

$$
\frac{\partial}{\partial t}\|(\rho - \bar\rho)(t,z)\|_{H^m}^2 \le C(T,z)(\|(\rho - \bar\rho)(t,z)\|_{H^m}^2 + \|(u - \bar u)(t,z)\|_{H^{m+1}}^2),
$$

where $C = C(T,z)$ is a positive random function.

Proof. It follows from Theorem 2.1 that

$$(\rho, u), (\bar{\rho}, \bar{u}) \in H^s \times H^{s+1} \quad \text{with } s > \frac{d}{2} + m + 1.$$

Since the proof for the case $m = 0$ is analogous to the higher-order case, we only consider the higher-order estimates. So we first apply ∇^k to $(1.1)_1$ for $1 \le k \le m$ to get

$$\partial_t \nabla^k (\rho - \bar{\rho}) + \nabla^k \nabla \cdot ((\rho - \bar{\rho})\bar{u} + \rho(u - \bar{u})) = 0. \tag{4.1}$$

Then, we multiply (4.1) by $\nabla^k(\rho - \bar{\rho})$ and integrate the resulting relation over $\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\rho - \bar{\rho})\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})\nabla^k[\nabla \cdot (\rho(u - \bar{u})) + \nabla \cdot ((\rho - \bar{\rho})\bar{u})]dx \\
&\quad = -\sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla\rho) \cdot (\nabla^{k-r}(u - \bar{u}))]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[(\nabla^r\rho)(\nabla^{k-r}(\nabla \cdot (u - \bar{u})))]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla(\rho - \bar{\rho})) \cdot (\nabla^{k-r}\bar{u})]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[(\nabla^r(\rho - \bar{\rho}))(\nabla^{k-r}(\nabla \cdot \bar{u}))]dx \\
&\quad =: \sum_{i=1}^4 \mathcal{I}_{3i}.
\end{aligned}$$

Next, we estimate $\mathcal{I}_{3i}$'s one by one as follows:

$\diamond$ (Estimates for $\mathcal{I}_{31}$): We use the Sobolev embedding theorem to obtain

$$\begin{aligned}
\mathcal{I}_{31} &= -\sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla\rho) \cdot (\nabla^{k-r}(u - \bar{u}))]dx \\
&\le \sum_{0 \le r \le k} \binom{k}{r} \|\nabla^{r+1}\rho\|_{L^\infty} \|\nabla^k(\rho - \bar{\rho})\|_{L^2} \|\nabla^{k-r}(u - \bar{u})\|_{L^2} \\
&\le C \sum_{0 \le r \le k} \binom{k}{r} \|\rho\|_{H^s} \|\nabla^k(\rho - \bar{\rho})\|_{L^2} \|\nabla^{k-r}(u - \bar{u})\|_{L^2} \\
&\le C(T, z)(\|\nabla^k(\rho - \bar{\rho})\|_{L^2}^2 + \|u - \bar{u}\|_{H^k}^2).
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{32}$): Similarly,

$$\begin{aligned}\mathcal{I}_{32} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar{\rho})[(\nabla^r\rho)(\nabla^{k-r}(\nabla\cdot(u-\bar{u})))]dx\\ &\le \sum_{0\le r\le k}\binom{k}{r}\|\nabla^r\rho\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r+1}(u-\bar{u})\|_{L^2}\\ &\le C\sum_{0\le r\le k}\binom{k}{r}\|\rho\|_{H^{s-1}}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r+1}(u-\bar{u})\|_{L^2}\\ &\le C(T,z)(\|\nabla^k(\rho-\bar{\rho})\|_{L^2}^2+\|u-\bar{u}\|_{H^{k+1}}^2).\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{33}$): One has

$$\begin{aligned}\mathcal{I}_{33} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar{\rho})[\nabla^r(\nabla(\rho-\bar{\rho}))\cdot(\nabla^{k-r}\bar{u})]dx\\ &\le \sum_{0\le r\le k-1}\binom{k}{r}\|\nabla^{k-r}\bar{u}\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{r+1}(\rho-\bar{\rho})\|_{L^2}+\frac{1}{2}\int_{\mathbb{T}^d}|\nabla^k(\rho-\bar{\rho})|^2(\nabla\cdot\bar{u})dx\\ &\le C\sum_{0\le r\le k-1}\binom{k}{r}\|\bar{u}\|_{H^{s-1}}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{r+1}(\rho-\bar{\rho})\|_{L^2}+C\|\nabla^k(\rho-\bar{\rho})\|_{L^2}^2\|\bar{u}\|_{H^{s-m}}\\ &\le C(T,z)\|\rho-\bar{\rho}\|_{H^k}^2.\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{34}$): We have

$$\begin{aligned}\mathcal{I}_{34} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar{\rho})[(\nabla^r(\rho-\bar{\rho}))(\nabla^{k-r}(\nabla\cdot\bar{u}))]dx\\ &\le \sum_{0\le r\le k}\binom{k}{r}\|\nabla^{k-r+1}\bar{u}\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^r(\rho-\bar{\rho})\|_{L^2}\\ &\le C\sum_{0\le r\le k}\binom{k}{r}\|\bar{u}\|_{H^s}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^r(\rho-\bar{\rho})\|_{L^2}\\ &\le C(T,z)\|\rho-\bar{\rho}\|_{H^k}^2.\end{aligned}$$

By collecting all results for $\mathcal{I}_{3i}$'s, summing over $1\le k\le m$ and combining with lower-order estimates, one gets the desired estimate. $\square$

Next, we return to the L^2-stability of the bulk velocity processes.

Lemma 4.2. *Suppose that assumptions* $(\mathcal{A}1)$*-*$(\mathcal{A}2)$ *hold, and let* (ρ, u) *and* $(\bar\rho, \bar u)$ *be two classical solution processes to* (1.1) *corresponding to the initial data* (ρ_0, u_0) *and* $(\bar\rho_0, \bar u_0)$*, respectively. Then,*

$$\frac{\partial}{\partial t}\|(u-\bar u)(t,z)\|^2_{H^{m+1}} \le C(T,z)(\|(u-\bar u)(t,z)\|^2_{H^{m+1}} + \|(\rho-\bar\rho)(t,z)\|^2_{L^2}),$$

where $C = C(T,z)$ *is a positive random function.*

Proof. As in the proof of Lemma 4.1, we only consider the higher-order estimates. Applying ∇^k to $(1.1)_2$ for $1 \le k \le m+1$ gives

$$\begin{aligned}
&\partial_t \nabla^k (u-\bar u) + \nabla^k((u-\bar u)\cdot\nabla u) + \nabla^k(\bar u \cdot \nabla(u-\bar u)) \\
&= \nabla^k \int_{\mathbb{T}^d} \psi(x-y,z)\big[(u(y)-u(x))(\rho(y)-\bar\rho(y)) + \bar\rho(y)(u(y)-\bar u(y)) - \bar\rho(y)(u(x)-\bar u(x))\big]dy.
\end{aligned}$$

Then, we use commutator estimates, Sobolev embedding and Young's inequality to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(u-\bar u)\|^2_{L^2} \\
&\quad = -\int_{\mathbb{T}^d} \nabla^k[(u-\bar u)\cdot\nabla u]\nabla^k(u-\bar u)dx \\
&\qquad - \int_{\mathbb{T}^d} \nabla^k[\bar u\cdot\nabla(u-\bar u)]\nabla^k(u-\bar u)dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \nabla^k\left[\psi(x-y,z)(u(y)-u(x))(\rho(y)-\bar\rho(y))\right]\nabla^k(u(x)-\bar u(x))dydx \\
&\qquad + \int_{\mathbb{T}^{2d}} \nabla^k\left[\psi(x-y,z)\bar\rho(y)(u(y)-\bar u(y))\right]\nabla^k(u(x)-\bar u(x))dydx \\
&\qquad - \int_{\mathbb{T}^{2d}} \nabla^k\left[\psi(x-y,z)\bar\rho(y)(u(x)-\bar u(x))\right]\nabla^k(u(x)-\bar u(x))dydx \\
&\quad \le C\|u\|_{H^s}\|u-\bar u\|^2_{H^k} + +C\|\psi\|_s\|u\|_{H^s}(\|\rho-\bar\rho\|^2_{L^2} + \|\nabla^k(u-\bar u)\|^2_{L^2}) + C\|\psi\|_s\|\bar\rho\|_{L^2}\|u-\bar u\|^2_{H^k} \\
&\quad \le C(T,z)(\|u-\bar u\|^2_{H^k} + \|\rho-\bar\rho\|^2_{L^2}).
\end{aligned}$$

We sum the above relation over $1 \le k \le m+1$ and combine with lower-order estimates, which can be obtained analogously, to get the higher-order estimates. □

Finally, we combine Lemma 4.1 and Lemma 4.2 to derive our second main result as follows.

Theorem 4.1. *Suppose that assumptions* $(\mathcal{A}1)$*-*$(\mathcal{A}2)$ *hold, and let* (ρ, u) *and* $(\bar\rho, \bar u)$ *be two classical solution processes to* (1.1) *corresponding to the initial data* (ρ_0, u_0) *and* $(\bar\rho_0, \bar u_0)$*, respectively. Then, there exists a positive random function* $C(T, z)$ *such that*

$$\sup_{0\le t\le T}\left(\|(\rho-\bar\rho)(t,z)\|_{H^m}^2+\|(u-\bar u)(t,z)\|_{H^{m+1}}^2\right)$$
$$\le C(T,z)(\|(\rho_0-\bar\rho_0)(z)\|_{H^m}^2+\|(u_0-\bar u_0)(z)\|_{H^{m+1}}^2).$$

Proof. We combine Lemma 4.1 and Lemma 4.2 to get

$$\frac{\partial}{\partial t}\left(\|(\rho-\bar\rho)\|_{H^m}^2+\|(u-\bar u)\|_{H^{m+1}}^2\right)\le C(T,z)\left(\|(\rho-\bar\rho)\|_{H^m}^2+\|(u-\bar u)\|_{H^{m+1}}^2\right).$$

Here, one can use Grönwall's lemma to yield the desired result. □

4.2. L^2-stability estimates for z-variations

In this subsection, we discuss the L^2-stability estimates for the z-variations of the solution processes. First, we consider the L^2-stability estimates for the z-variations $\{\partial_z^l\rho\}$ of local mass processes.

Lemma 4.3. *Suppose that assumptions ($\mathcal{A}1$)-($\mathcal{A}2$) hold, and let (ρ,u) and $(\bar\rho,\bar u)$ be two solution processes to (1.1) corresponding to the initial data (ρ_0,u_0) and $(\bar\rho_0,\bar u_0)$, respectively. Then,*

$$\frac{\partial}{\partial t}\sum_{0\le l\le m}\|\partial_z^l(\rho-\bar\rho)\|_{H^{m-l}}^2\le C(T,z)\sum_{0\le l\le m}\left(\|\partial_z^l(\rho-\bar\rho)\|_{H^{m-l}}^2+\|\partial_z^l(u-\bar u)\|_{H^{m-l+1}}^2\right),$$

where $C=C(T,z)$ is a positive random function.

Proof. As in Lemma 4.1, we only consider the higher-order estimates. For $1\le k\le m-l$ and $1\le l\le m$, we apply $\nabla^k\partial_z^l$ to (1.1) to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\\
&=-\sum_{\substack{0\le r_1\le l\\0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u)\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\quad-\sum_{\substack{0\le r_1\le l\\0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\nabla^{r_2}\partial_z^{r_1}(\rho-\bar\rho)\nabla^{k-r_2}\nabla\cdot(\partial_z^{l-r_1}u)\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\quad-\sum_{\substack{0\le r_1\le l\\0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\nabla^{r_2}\nabla(\partial_z^{r_1}\bar\rho)\cdot\nabla^{k-r_2}\partial_z^{l-r_1}(u-\bar u)\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\quad-\sum_{\substack{0\le r_1\le l\\0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\nabla^{r_2}(\partial_z^{r_1}\bar\rho)\ \nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}(u-\bar u))\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&=:\sum_{i=1}^4\mathcal{I}_{4i}.
\end{aligned}$$

Below, we estimate $\mathcal{I}_{4i}$'s separately.

$\diamond$ (Estimates for $\mathcal{I}_{41}$): In this case, we have

$$
\begin{aligned}
\mathcal{I}_{41} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u))\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u))\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&\quad + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|\nabla^k(\partial_z^l(\rho-\bar\rho))|^2 dx \\
&\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{r_2+1}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}\|\nabla^{k-r_2}(\partial_z^{l-r_1}u)\|_{L^\infty}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2} \\
&\quad + \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2 \\
&\le C\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\partial_z^{l-r_1}u\|_{H^{s-l}}\left(\|\nabla^{r_2+1}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}^2 + \|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\right) \\
&\quad + C\|u\|_{H^{s-m}}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2 \\
&\le C(T,z)\sum_{0\le r\le l}\|\partial_z^r(\rho-\bar\rho)\|_{H^{m-r}}^2.
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{42}$): By direct calculation, one has

$$
\begin{aligned}
\mathcal{I}_{42} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}(\partial_z^{r_1}(\rho-\bar\rho))\nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}u))\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}\partial_z^{r_1}(\rho-\bar\rho)\nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}u))\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&\quad - \int_{\mathbb{T}^d} (\nabla\cdot u)|\nabla^k(\partial_z^l(\rho-\bar\rho))|^2 dx \\
&\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{r_2}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}\|\nabla^{k-r_2+1}\partial_z^{l-r_1}u\|_{L^\infty}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\nabla \cdot u\|_{L^\infty} \|\nabla^k (\partial_z^l (\rho - \bar{\rho}))\|_{L^2}^2 \\
&\le C \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k \\ (r_1,r_2)\ne(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\partial_z^{l-r_1} u\|_{H^{s-l}} \Big(\|\nabla^{r_2}(\partial_z^{r_1}(\rho - \bar{\rho}))\|_{L^2}^2 + \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2}^2 \Big) \\
&\quad + C\|u\|_{H^{s-m}} \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2}^2 \\
&\le C(T,z) \sum_{0\le r\le l} \|\partial_z^r(\rho - \bar{\rho})\|_{H^{m-r}}^2 .
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{43}$): Similarly, one gets

$$
\begin{aligned}
\mathcal{I}_{43} &= - \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}\nabla(\partial_z^{r_1}\bar{\rho}) \cdot \nabla^{k-r_2}(\partial_z^{l-r_1}(u - \bar{u})))\nabla^k(\partial_z^l(\rho - \bar{\rho}))dx \\
&\le \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{r_2+1}(\partial_z^{r_1}\bar{\rho})\|_{L^\infty} \|\nabla^{k-r_2}\partial_z^{l-r_1}(u - \bar{u})\|_{L^2} \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2} \\
&\le C \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\partial_z^{r_1}\bar{\rho}\|_{H^{s-l}} \|\nabla^{k-r_2}\partial_z^{l-r_1}(u - \bar{u})\|_{L^2} \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2} \\
&\le C(T,z) \sum_{0\le r\le l} \Big(\|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2}^2 + \|\partial_z^r(u - \bar{u})\|_{H^{m-r}}^2 \Big).
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{44}$): By direct estimates, one obtains

$$
\begin{aligned}
\mathcal{I}_{44} &= - \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \int_{\mathbb{T}^d} (\nabla^{r_2}(\partial_z^{r_1}\bar{\rho})\nabla^{k-r_2}\nabla \cdot (\partial_z^{l-r_1}(u - \bar{u})))\nabla^k(\partial_z^l(\rho - \bar{\rho}))dx \\
&\le \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{r_2}(\partial_z^{r_1}\bar{\rho})\|_{L^\infty} \|\nabla^{k-r_2+1}(\partial_z^{l-r_1}(u - \bar{u}))\|_{L^2} \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2} \\
&\le C \sum_{\substack{0\le r_1\le l \\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\partial_z^{r_1}\bar{\rho}\|_{H^{s-l-1}} \|\nabla^{k-r_2+1}(\partial_z^{l-r_1}(u - \bar{u}))\|_{L^2} \|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2} \\
&\le C(T,z) \sum_{0\le r\le l} \Big(\|\nabla^k(\partial_z^l(\rho - \bar{\rho}))\|_{L^2}^2 + \|\partial_z^r(u - \bar{u})\|_{H^{m-r+1}}^2 \Big).
\end{aligned}
$$

Finally, we collect all estimates for $\mathcal{I}_{4i}$'s, sum them over $1 \le k \le m - l$, $0 \le l \le m$ and add the zeroth-order estimate to get the desired result. $\square$

Next, we provide estimates for z-variations $\{\partial_z^l u\}$ of the bulk velocity processes.

Lemma 4.4. *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* (ρ, u) *and* $(\bar\rho, \bar u)$ *be two classical solution processes to* (1.1) *corresponding to the initial data* (ρ_0, u_0) *and* $(\bar\rho_0, \bar u_0)$*, respectively. Then, we have*

$$\frac{\partial}{\partial t} \sum_{0 \le l \le m} \|\partial_z^l (u - \bar u)\|_{H^{m-l+1}}^2 \le C(T, z) \sum_{0 \le l \le m} \left(\|\partial_z^l (\rho - \bar\rho)\|_{L^2}^2 + \|\partial_z^l (u - \bar u)\|_{H^{m-l+1}}^2 \right),$$

where $C = C(T, z)$ *is a positive random function.*

Proof. Since the proof will be straightforward and similar to that of Lemma 4.2, we leave its proof to Appendix C. □

Finally, we combine Lemma 4.3 with Lemma 4.4 and use Grönwall's lemma to deduce the following result.

Theorem 4.2. *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* (ρ, u) *and* $\bar\rho, \bar u$ *be two classical solution processes to* (2.4) *with initial data* (ρ_0, u_0) *and* $(\bar\rho_0, \bar u_0)$*, respectively. Then, there exists a positive random function* $C(T, z)$ *such that*

$$\begin{aligned} &\sup_{0 \le t \le T} \sum_{0 \le l \le m} \left(\|\partial_z^l (\rho - \bar\rho)(t, z)\|_{H^{m-l}}^2 + \|\partial_z^l (u - \bar u)(t, z)\|_{H^{m-l+1}}^2 \right) \\ &\qquad \le C(T, z) \sum_{0 \le l \le m} \left(\|\partial_z^l (\rho_0 - \bar\rho_0)(z)\|_{H^{m-l}}^2 + \|\partial_z^l (u_0 - \bar u_0)(z)\|_{H^{m-l+1}}^2 \right). \end{aligned}$$

5. A local sensitivity analysis for flocking estimate

In this section, we provide a local sensitivity analysis for the flocking behavior to system (1.1).

It follows from Proposition 2.1 that

$$\int_{\mathbb{T}^d} \rho(t, z) dx = \int_{\mathbb{T}^d} \rho_0(z) dx, \qquad \int_{\mathbb{T}^d} (\rho u)(t, z) dx = \int_{\mathbb{T}^d} (\rho_0 u_0)(z) dx, \quad t \ge 0, \ \ z \in \Omega.$$

Then, without loss of generality, we may assume that the average bulk velocity is zero:

$$u_c(t, z) := \frac{\int_{\mathbb{T}^d} \rho u dx}{\int_{\mathbb{T}^d} \rho dx} \equiv 0.$$

For a given z-variations $\{\partial_z^m u\}$, we introduce a family of flocking functionals $\mathcal{E}_m$:

$$\mathcal{E}_m(t, z) := \int_{\mathbb{T}^d} \rho |\partial_z^m u|^2 dx, \qquad m \ge 1,$$

where $\mathcal{E}_0(t, z)$ is defined in Theorem 2.1. Although the functionals $\mathcal{E}_m$ are not z-variations of a certain quantity, estimates for these functionals will be of our concern since they play a role in estimating $\|\partial_z^m u\|_{L^2}$. Based on the estimates for $\mathcal{E}_m$, we provide estimates for the exponential decay of $\|\partial_z^m u\|_{L^2}$ under the following *a priori* assumptions $(\mathcal{B})$: for an integer $s > \frac{d}{2} + m + 1$, $T \in (0, \infty)$ and each $z \in \Omega$,

$(\mathcal{B}1)$ The solution process (ρ, u) and their z-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$ satisfy the following uniform boundedness condition:

$$\max_{0\le l\le m} \sup_{t\in[0,T]} \Big(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \Big) \le \mathcal{U}(z),$$

for some positive random function $\mathcal{U} = \mathcal{U}(z)$.

$(\mathcal{B}2)$ The initial mass ρ_0 satisfies the non-vacuum condition:

$$\inf_{x\in\mathbb{T}^d} \rho_0(x,z) > 0, \quad \text{for each } z \in \Omega.$$

$(\mathcal{B}3)$ The communication weight function $\psi : \mathbb{T}^d \times \Omega \to \mathbb{R}$ is in $\mathcal{C}^{s+1}(\mathbb{T}^d \times \Omega)$ and satisfies symmetric, positive, boundedness conditions: for each $x, y \in \mathbb{T}^d$ and $z \in \Omega$,

$$\psi(x-y,z) = \psi(y-x,z), \quad \inf_{x\in\mathbb{T}^d} \psi(x,z) =: \psi_m(z) > \sup_{0\le t\le T} \left(\frac{\|(\nabla\cdot u)(t)\|_{L^\infty}}{2\|\rho_0\|_{L^1}} \right),$$

$$\|\psi\|_s := \max_{|\alpha|+|\beta|\le s+1} \sup_{(x,z)\in\mathbb{T}^d\times\Omega} |\partial_z^\alpha \partial_x^\beta \psi(x,z)| < \infty.$$

Remark 5.1. The lower bound assumption for ψ given in $(\mathcal{B}3)$ implies a sufficient condition for system (1.1) to exhibit the decay of the bulk velocity toward the average bulk velocity. To be precise, the condition means the alignment force is so strong that it surpasses the tendency of bulk velocity field to deviate from the mean velocity, and it will lead to the velocity alignment.

In the following lemma, we study the exponential decay of $\|u\|_{L^2}$.

Lemma 5.1. *Let (ρ, u) be a classical solution and suppose that the a priori assumptions $(\mathcal{B})$ hold for (ρ, u). Then we have*

$$\|u(t,z)\|_{L^2}^2 \le \mathcal{F}_0(z) e^{-2\tilde{\Lambda}(z)t}, \quad t \in [0,T],$$

where $\mathcal{F}_0(z)$ and $\tilde{\Lambda}(z)$ are positive random functions.

Proof. It follows from (1.1) that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^d} |u|^2 dx$$
$$= -\int_{\mathbb{T}^d} (u\cdot\nabla u)\cdot u\,dx + \int_{\mathbb{T}^d\times\mathbb{T}^d} \psi(x-y)(u(y)-u(x))\cdot u(x)\rho(y)\,dy\,dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|u|^2 dx + \int_{\mathbb{T}^d\times\mathbb{T}^d} \psi(x-y)(u(y)-u(x))\cdot u(x)\rho(y)dydx \\
&\le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|u\|_{L^2}^2 - \psi_m\|\rho_0\|_{L^1}\|u\|_{L^2}^2 + \psi_M\|\rho u\|_{L^1}\|u\|_{L^1} \\
&\le \left(-\psi_m\|\rho_0\|_{L^1} + \frac{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty}}{2} + \delta\right)\|u\|^2 + \frac{\|\psi\|_s^2}{4\delta}\|\rho\|_{L^2}\mathcal{E}_0(t,z),
\end{aligned}
$$

where $\mathcal{E}_0$ is the functional defined in (2.5) and the positive constant $\delta > 0$ satisfies the following relation:

$$\psi_m\|\rho_0\|_{L^1} - \frac{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty}}{2} - \delta > 0.$$

We let $\tilde{\Lambda}(z) := \psi_m\|\rho_0\|_{L^1} - \frac{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty}}{2} - \delta$. Then, we have

$$\frac{\partial}{\partial t}\|u\|_{L^2}^2 \le -2\tilde{\Lambda}(z)\|u\|_{L^2}^2 + \hat{F}_0(z)e^{-2\Lambda(z)t}, \tag{5.1}$$

where random functions Λ and $\hat{F}_0$ are given by the following relations:

$$\Lambda(z) := \psi_m\|\rho_0\|_{L^1}, \quad \hat{F}_0(z) := \frac{\|\psi\|_s^2}{2\delta}\mathcal{U}(z)\|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

Now, we apply Grönwall's lemma to (5.1) to obtain

$$\|u\|_{L^2}^2 \le \|u_0\|_{L^2}^2 e^{-2\tilde{\Lambda}(z)t} + \frac{\hat{F}_0(z)}{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty} + 2\delta}\left(e^{-2\tilde{\Lambda}(z)t} - e^{-2\Lambda(z)t}\right) \le \mathcal{F}_0(z)e^{-2\tilde{\Lambda}(z)t},$$

where $\mathcal{F}_0(z)$ is given by

$$\mathcal{F}_0(z) := \|u_0\|_{L^2}^2 + \frac{\hat{F}_0(z)}{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty} + 2\delta}.$$

This implies our desired result. □

Next, we derive the temporal decay estimates for functional $\mathcal{E}_m(t,z)$ and $\|\partial_z^m u\|$ based on the induction argument.

Theorem 5.1. *Suppose that a priori assumptions ($\mathcal{B}$) and the following induction hypotheses hold for* $0 \le l \le m-1$*:*

$$\mathcal{E}_l(t,z) \le E_l(z)e^{-\tilde{\Lambda}(z)t}, \quad \|\partial_z^l u\|_{L^2}^2 \le \mathcal{F}_l(z)e^{-\tilde{\Lambda}(z)t},$$

where $E_l(z)$ *and* $\mathcal{F}_l(z)$ *are positive random functions and* $\tilde{\Lambda}(z)$ *is given in Lemma 5.1. Then, there exists a positive random function* $E_m(z)$ *such that*

$$\mathcal{E}_m(t,z) \le E_m(z)e^{-\tilde{\Lambda}(z)t}.$$

Proof. It follows from (3.2) that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{E}_m(t,z)\\
&= \int_{\mathbb{T}^d} \partial_t\rho|\partial_z^m u|^2 dx + 2\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot\partial_t(\partial_z^m u)dx\\
&= -\int_{\mathbb{T}^d} \nabla\cdot(\rho u)|\partial_z^m u|^2 dx - 2\sum_{l=0}^{m}\binom{m}{l}\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx\\
&\quad + 2\sum_{\alpha+\beta+\gamma=m}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}} \partial_z^\alpha\psi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\partial_z^\gamma\rho(y)\rho(x)\cdot\partial_z^m u(x)dydx\\
&= -2\sum_{l=1}^{m}\binom{m}{l}\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx\\
&\quad + 2\sum_{\alpha+\beta+\gamma=m}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}} \partial_z^\alpha\psi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\partial_z^\gamma\rho(y)\rho(x)\cdot\partial_z^m u(x)dydx\\
&=: \mathcal{I}_{51}+\mathcal{I}_{52}.
\end{aligned}
$$

Next, we separately estimate $\mathcal{I}_{51}$ and $\mathcal{I}_{52}$ as follows:

$\diamond$ (Estimates for $\mathcal{I}_{51}$): First,

$$
\begin{aligned}
\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^m u\cdot\nabla)udx &= -\int_{\mathbb{T}^d} \nabla\cdot(\rho\partial_z^m u\otimes\partial_z^m u)\cdot udx\\
&\le \|\nabla\cdot(\rho\partial_z^m u\otimes\partial_z^m u)\|_{L^2}\|u\|_{L^2} \le \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)}e^{-\tilde{\Lambda}(z)t}.
\end{aligned}
$$

For $m=1$, one gets

$$
\mathcal{I}_{51} \le \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)}e^{-\tilde{\Lambda}(z)t}.
$$

For $m\ge 2$,

$$
\begin{aligned}
2\sum_{l=1}^{m-1}\binom{m}{l}\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx &\le 2\sum_{l=1}^{m-1}\binom{m}{l}\|\nabla(\partial_z^{l-r}u)\|_{L^\infty}\mathcal{E}_m(t,z)\mathcal{E}_l(t,z)\\
&\le \delta\mathcal{E}_m^2(t,z) + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)\mathcal{E}_l^2(t,z),
\end{aligned}
$$

where δ is the same as that in Lemma 5.1 and we used Young's inequality. Hence, we can obtain that if $m\ge 2$,

$$\mathcal{I}_{51} \leq \delta \mathcal{E}_m^2(t,z) + \frac{m-1}{\delta} \sum_{l=1}^{m-1} \binom{m}{l}^2 \mathcal{U}^2(z)\mathcal{E}_l^2(t,z) + \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)}e^{-\tilde{\Lambda}(z)t}.$$

$\diamond$ (Estimates for $\mathcal{I}_{52}$): By direct calculation, one has

$$\begin{aligned}
\mathcal{I}_{52} = 2 \int_{\mathbb{T}^{2d}} &\psi(x-y)(\partial_z^m u(y) - \partial_z^m u(x))\rho(y)\rho(x)\partial_z^m u(x)dydx \\
&+ 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \partial_z^\alpha \psi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x))\partial_z^\gamma \rho(y)\rho(x) \cdot \partial_z^m u(x)dydx \\
= -\int_{\mathbb{T}^{2d}} &\psi(x-y)|\partial_z^m u(y) - \partial_z^m u(x)|^2 \rho(y)\rho(x)dydx \\
&+ 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \partial_z^\alpha \psi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x))\partial_z^\gamma \rho(y)\rho(x) \cdot \partial_z^m u(x)dydx \\
\leq -2\psi_m &\|\rho_0\|_{L^1}\mathcal{E}_m(t,z) \\
&+ 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \|\psi\|_s \left(\|\partial_z^\beta u \; \partial_z^\gamma \rho\|_{L^1} \|\rho \partial_z^m u\|_{L^1} + \|\partial_z^\gamma \rho\|_{L^1} \|\rho \partial_z^\beta u \cdot \partial_z^m u\|_{L^1} \right) \\
\leq -2\psi_m &\|\rho_0\|_{L^1}\mathcal{E}_m(t,z) \\
&+ 4 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \|\psi\|_s \|\partial_z^\gamma \rho\|_{L^2} \sqrt{\|\rho\|}_{L^\infty} \|\partial_z^\beta u\|_{L^2} \sqrt{\mathcal{E}_m(t,z)} \\
\leq (-2\psi_m &\|\rho_0\|_{L^1} + \delta)\mathcal{E}_m(t,z) \\
&+ \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta \neq m}} \left(\frac{m!}{\alpha!\beta!\gamma!} \|\psi\|_s \|\partial_z^\gamma \rho\|_{L^2} \sqrt{\|\rho\|}_{L^\infty} \right)^2 \left(\frac{\frac{(m+1)(m+2)}{2} - 1}{\delta} \right) \mathcal{F}_\beta(z) e^{-\tilde{\Lambda}(z)t},
\end{aligned}$$

where we used

$$\sum_{\alpha+\beta+\gamma=m} 1 = \frac{(m+2)(m+1)}{2},$$

and Young's inequality. Therefore, we collect all results for $\mathcal{I}_{51}$ and $\mathcal{I}_{52}$ to yield

$$\frac{\partial}{\partial t}\mathcal{E}_m(t,z) \leq -2(\psi_m\|\rho_0\|_{L^1} - \delta)\mathcal{E}_m(t,z) + \hat{E}_m(z)e^{-\tilde{\Lambda}(z)t}, \tag{5.2}$$

where $\hat{E}_m(z)$ is given by

$$
\begin{aligned}
\hat{E}_1(z) &:= \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)} + \frac{4}{\delta}\|\psi\|^2\mathcal{U}^3(z)\mathcal{F}_0(z), \\
\hat{E}_m(z) &:= \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)} + \mathcal{E}_m^2(t,z) + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)E_l(z) \\
&\qquad + \frac{m^2+3m}{2\delta}\sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\left(\frac{m!}{\alpha!\beta!\gamma!}\|\psi\|_s\mathcal{U}^{3/2}(z)\right)^2\mathcal{F}_\beta(z), \quad m \geq 2.
\end{aligned}
$$

Now, we integrate (5.2) with respect to time to get

$$
\begin{aligned}
\mathcal{E}_m(t,z) &\leq \mathcal{E}_m(0,z)e^{-2(\psi_m\|\rho_0\|_{L^1}-\delta)t} + \frac{\hat{E}_m(z)}{2(\psi_m\|\rho_0\|_{L^1}-\delta)-\tilde{\Lambda}(z)}(e^{-\tilde{\Lambda}(z)t} - e^{-2(\psi_m\|\rho_0\|_{L^1}-\delta)t}) \\
&\leq E_m(z)e^{-\tilde{\Lambda}(z)t},
\end{aligned}
$$

where $E_m(z)$ is written as

$$
E_m(z) := \mathcal{E}_m(0,z) + \frac{\hat{E}_m(z)}{2(\psi_m\|\rho_0\|_{L^1}-\delta)-\tilde{\Lambda}(z)}.
$$

This gives our desired result. □

Finally, we provide all estimates for the L^2-decay of the z-variations toward the corresponding z-variations of the average bulk velocity process.

Theorem 5.2. *For a positive constant $T \in (0,\infty)$, let (ρ, u) be a classical solution process on $[0,T]$ and suppose that a priori assumptions $(\mathcal{B})$ hold. Moreover, assume the following induction hypotheses hold for $0 \leq l \leq m$ and $0 \leq p \leq m-1$:*

$$
\mathcal{E}_l(t,z) \leq E_l(z)e^{-\tilde{\Lambda}(z)t}, \quad \|\partial_z^p u\|_{L^2}^2 \leq \mathcal{F}_p(z)e^{-\tilde{\Lambda}(z)t},
$$

where $E_l(z)$ and $\mathcal{F}_p(z)$ are positive random functions. Then, there exists a positive random function $\mathcal{F}_m(z)$ such that

$$
\|\partial_z^m u\|_{L^2}^2 \leq \mathcal{F}_m(z)e^{-\tilde{\Lambda}(z)t}.
$$

Proof. We leave its detailed proof in Appendix D. □

6. Conclusion

In this paper, we have presented the local sensitivity analysis for the random hydrodynamic Cucker-Smale model describing the emergence of flocking in the ensemble of Cucker-Smale flocking particles. In authors' earlier works, we have dealt with particle and kinetic models for the CS flocking and derived quantitative estimates for the variations of the solutions in random space. We extend aforementioned quantitative estimates to the hydrodynamic CS model, e.g., the propagation of the z-variations of spatial and velocity process, where z is the random input variable, the L^2-stability and flocking estimates along the sample path. Thanks to the regularity analysis of the deterministic HCS model, we can lift good regularity estimates to the random solution process along the sample path.

As mentioned in Introduction, the synthesis of flocking dynamics and local sensitivity analysis is not that mature yet. In fact, there are many open questions, for example, the effect of uncertainties on the formation of multi-cluster flocking and extension of the local sensitivity to the initial and boundary problems in the context of flocking. In fact, as far as the authors know, the initial and boundary value problems are not well studied in the flocking problems even for the deterministic flocking models. We leave these interesting issues for future works.

Appendix A. Proof of Lemma 3.2

Similarly to Lemma 3.1, it suffices to provide the upper-bound estimates. Again, we split the cases into the zeroth-order and higher-order estimates.

• Step A (The zeroth-order estimates): We multiply (3.3)$_2$ by $\partial_z u^{n+1}$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z u^{n+1}\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \partial_z u^n \cdot \nabla u \cdot \partial_z u^{n+1} dx + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla \cdot u)|\partial_z u^{n+1}|^2 dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \partial_z \psi(x-y)(u(y)-u(x))\rho(y)\partial_z u^{n+1}(x) dy dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \psi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)\partial_z u^{n+1} u(x) dy dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \psi(x-y)(u(y)-u(x))\partial_z \rho^{n+1}(y)\partial_z u^{n+1}(x) dy dx \\
&\quad \le \|\nabla u\|_{L^\infty}\|\partial_z u^n\|_{L^2}\|\partial_z u^{n+1}\|_{L^2} + \|\nabla u\|_{L^\infty}\|\partial_z u^{n+1}\|_{L^2}^2 \\
&\qquad + 2\|\psi\|_s\|u\|_{L^2}\|\rho\|_{L^2}\|\|\partial_z u^{n+1}\|_{L^2} + 2\|\psi\|_s\|\partial_z u^n\|_{L^2}\|\rho\|_{L^2}\|\partial_z u^{n+1}\|_{L^2} \\
&\qquad + 2\|\psi\|_s\|u\|_{L^2}\|\partial_z \rho^{n+1}\|_{L^2}\|\partial_z u^{n+1}\|_{L^2} \\
&\quad \le C(\varepsilon^{1/2}\|\partial_z u^{n+1}\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned} \tag{A.1}$$

where C is a positive random function independent of n and we used the Sobolev embedding theorem, Young's inequality and Lemma 3.1. Now, we apply Grönwall's lemma for (A.1) to yield

$$\|\partial_z u^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z u^{n+1}(s,z)\|_{L^2}^2 ds + \varepsilon^{3/2}\right). \tag{A.2}$$

• Step B (Higher-order estimates): For $1 \le k \le s$, we apply ∇^k to (3.3)$_2$, multiply by $\nabla^k(\partial_z u^{n+1})$ and integrate the resulting relation over $\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \partial_z u^n \cdot \nabla(\nabla^k u)\cdot \nabla^k(\partial_z u^{n+1})dx \\
&\qquad -\int_{\mathbb{T}^d} \Big[\nabla^k(\partial_z u^n\cdot\nabla u) - \partial_z u^n\cdot\nabla^k(\nabla u)\Big]\nabla^k(\partial_z u^{n+1})dx + \frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z u^{n+1})|^2 dx \\
&\qquad -\int_{\mathbb{T}^d} \Big[\nabla^k(u\cdot\nabla(\partial_z u^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z u^{n+1})\Big]\nabla^k(\partial_z u^{n+1})dx \\
&\qquad +\int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\partial_z\psi(x-y)(u(y)-u(x))\rho(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\qquad +\int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\psi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\qquad +\int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\psi(x-y)(u(y)-u(x))\partial_z\rho^{n+1}(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\quad =: \sum_{i=1}^{7}\mathcal{I}_{6i}.
\end{aligned}$$

Here, we separately estimate $\mathcal{I}_{6i}$'s as follows.

◇ (Estimates for $\mathcal{I}_{6i}$, $i=1,2,3,4$): We use the Cauchy-Schwarz inequality, commutator estimates and Young's inequality to get

$$\begin{aligned}
\mathcal{I}_{61} &\le \|\partial_z u^n\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}), \\
\mathcal{I}_{62} &\le c\left(\|\nabla(\partial_z u^n)\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2}\right)\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}), \\
\mathcal{I}_{63} &\le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 \le \varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2, \\
\mathcal{I}_{64} &\le c\left(\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} + \|\nabla(\partial_z u^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\right)\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\le C\varepsilon^{1/2}\|\partial_z u^{n+1}\|_{H^s}^2.
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{6i}$, $i=5,6,7$): For $\mathcal{I}_{65}$, one gets

$$\begin{aligned}\mathcal{I}_{65} &= \int_{\mathbb{T}^{2d}} \nabla^k(\partial_z\psi(x-y))(u(y)-u(x))\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\quad - \sum_{0\le r\le k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r(\partial_z\psi(x-y))\nabla^{k-r}u(x)\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\le C\|\psi\|_s\|u\|_{H^k}\|\rho\|_{L^2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2+\varepsilon^{3/2}),\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Young's inequality.

For $\mathcal{I}_{66}$, we use the same arguments as $\mathcal{I}_{65}$ to get

$$\begin{aligned}\mathcal{I}_{66} &= \int_{\mathbb{T}^{2d}} \nabla^k\psi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\quad - \sum_{0\le r\le k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r\psi(x-y)\nabla^{k-r}(\partial_z u^n)(x)\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\le C\|\psi\|_s\|\rho\|_{L^2}\|\partial_z u^n\|_{H^k}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2+\varepsilon^{3/2}).\end{aligned}$$

For $\mathcal{I}_{67}$, we have

$$\begin{aligned}\mathcal{I}_{67} &= \int_{\mathbb{T}^{2d}} \nabla^k\psi(x-y)(u(y)-u(x))\partial_z\rho^{n+1}\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\quad - \sum_{0\le r\le k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r\psi(x-y)\nabla^{k-r}u(x)\partial_z\rho^{n+1}(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\ &\le C\|\psi\|_s\|\partial_z\rho^{n+1}\|_{L^2}\|u\|_{H^k}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2+\varepsilon^{3/2}).\end{aligned}$$

We gather all results for $\mathcal{I}_{6i}$'s, sum over $0\le k\le s$, integrate the resulting relation and combine with (A.2) to get

$$\|\partial_z u^{n+1}\|_{H^s}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z u^{n+1}(s,z)\|_{H^s}^2 ds + \varepsilon^{3/2}\right).$$

Finally, we use Grönwall's lemma to obtain the desired result.

Appendix B. Proof of Lemma 3.5

We split the estimates into the zeroth-order and the higher-order cases.

• Step A (The zeroth-order estimates): It follows from $(3.11)_2$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z^m u^{n+1}\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \partial_z^m u^n \cdot \nabla u \cdot \partial_z^m u^{n+1} dx + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla \cdot u)|\partial_z^m u^{n+1}|^2 dx \\
&\qquad - \sum_{1\le l\le m-1} \binom{m}{l} \int_{\mathbb{T}^d} \partial_z^l u \cdot \nabla(\partial_z^{m-l} u) \cdot \partial_z^m u^{n+1} dx \\
&\qquad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \partial_z^\alpha \psi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x))\partial_z^\gamma \rho(y)\partial_z^m u^{n+1}(x) dy dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \psi(x-y)(\partial_z^m u^n(y) - \partial_z^m u^n(x))\rho(y)\partial_z^m u^{n+1}(x) dy dx \\
&\qquad + \int_{\mathbb{T}^{2d}} \psi(x-y)(u(y) - u(x))\partial_z^m \rho^{n+1}(y)\partial_z^m u^{n+1}(x) dy dx \\
&\quad \le \|\nabla u\|_{L^\infty}\|\partial_z^m u^n\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} + \frac{1}{2}\|\nabla \cdot u\|_{L^\infty}\|\partial_z^m u^{n+1}\|_{L^2}^2 \\
&\qquad + C\sum_{1\le l\le m-1} \|\partial_z^l u\|_{L^\infty}\|\nabla(\partial_z^{m-l}u)\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} \\
&\qquad + C\sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m}} \|\partial_z^\beta u\|_{L^2}\|\partial_z^\gamma \rho\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} \\
&\qquad + 2\|\psi\|_s\|\rho\|_{L^2}\|\partial_z^m u^n\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} + 2\|\psi\|_s\|u\|_{L^2}\|\partial_z^m \rho^{n+1}\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} \\
&\quad \le C(\varepsilon^{1/2}\|\partial_z^m u^{n+1}\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality and Young's inequality.

• Step B (Higher-order estimates): For $1 \le k \le s-m+1$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \nabla^k(\partial_z^m u^n \cdot \nabla u)\nabla^k(\partial_z^m u^{n+1}) dx - \int_{\mathbb{T}^d} \nabla^k(u \cdot \nabla(\partial_z^m u^{n+1}))\nabla^k(\partial_z^m u^{n+1}) dx \\
&\quad - \sum_{1\le l\le m-1} \binom{m}{l} \int_{\mathbb{T}^d} \nabla^k(\partial_z^l u \cdot \nabla(\partial_z^{m-l} u))\nabla^k(\partial_z^m u^{n+1}) dx \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \nabla^k\Big\{\partial_z^\alpha \psi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x))\Big\}\partial_z^\gamma \rho(y)\nabla^k(\partial_z^m u^{n+1})(x) dy dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{T}^{2d}} \nabla^k\Big(\psi(x-y)(\partial_z^m u^n(y)-\partial_z^m u^n(x))\Big)\rho(y)\nabla^k(\partial_z^m u^{n+1})(x)dydx\\
&+\int_{\mathbb{T}^{2d}} \nabla^k\Big(\psi(x-y)(u(y)-u(x))\Big)\partial_z^m \rho^{n+1}(y)\nabla^k(\partial_z^m u^{n+1})(x)dydx\\
&=: \sum_{i=1}^{6}\mathcal{I}_{7i}.
\end{aligned}
$$

In the sequel, we estimate the terms $\mathcal{I}_{7i}$'s separately.

⋄ (Estimates for $\mathcal{I}_{71}$ and $\mathcal{I}_{72}$): For $\mathcal{I}_{71}$,

$$
\begin{aligned}
\mathcal{I}_{71} &= -\int_{\mathbb{T}^d} \nabla^k(\partial_z^m u^n)\cdot\nabla u\cdot\nabla^k(\partial_z^m u^{n+1})dx\\
&\quad -\int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z^m u^n\cdot\nabla u)-\partial_z^m u^n\cdot\nabla^k(\nabla u)\Big]\nabla^k(\partial_z^m u^{n+1})dx\\
&\le \|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}\\
&\quad + c\Big(\|\nabla(\partial_z^m u^n)\|_{L^\infty}\|\nabla^k u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2}\Big)\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}\\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2+\varepsilon^{3/2}),
\end{aligned}
$$

where c and C are positive random functions independent of n. Similarly,

$$
\begin{aligned}
\mathcal{I}_{72} &= \frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z^m u^{n+1})|^2dx\\
&\quad -\int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z^m u^{n+1}))-u\cdot\nabla^k(\nabla(\partial_z^m u^{n+1}))\Big]\cdot\nabla^k(\partial_z^m u^{n+1})dx\\
&\le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2\\
&\quad + c\Big(\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}+\|\nabla(\partial_z^m u^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\Big)\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}\\
&\le C(\varepsilon^{1/2}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2+\varepsilon^{3/2}),
\end{aligned}
$$

where c and C are positive random functions independent of n.

⋄ (Estimates for $\mathcal{I}_{73}$): One gets

$$
\mathcal{I}_{73} = -\sum_{1\le l\le m-1}\binom{m}{l}\Bigg\{\int_{\mathbb{T}^d}\partial_z^l u\cdot\nabla^k(\nabla(\partial_z^{m-l}u))\cdot\nabla^k(\partial_z^m u^{n+1})dx
$$

$$
\begin{aligned}
&\qquad + \int_{\mathbb{T}^d} \Big[\nabla^k (\partial_z^l u \cdot \nabla(\partial_z^{m-l} u)) - \partial_z^l u \cdot \nabla^k (\nabla (\partial_z^{m-l} u)) \Big] \cdot \nabla^k (\partial_z^m u^{n+1}) dx \Bigg\} \\
&\le C \sum_{1 \le l \le m-1} \Big(\|\partial_z^l u\|_{L^\infty} \|\nabla^{k+1}(\partial_z^{m-l} u)\|_{L^2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\qquad + \|\nabla(\partial_z^l u)\|_{L^\infty} \|\nabla^k(\partial_z^{m-l} u)\|_{L^2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\qquad + \|\nabla(\partial_z^{m-l} u)\|_{L^\infty} \|\nabla^k(\partial_z^l u)\|_{L^2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \Big) \\
&\le C(\varepsilon^{1/2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{74}$): By direct calculation,

$$
\begin{aligned}
\mathcal{I}_{74} &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \nabla^k \Big(\partial_z^\alpha \psi(x-y) \Big) (\partial_z^\beta u(y) - \partial_z^\beta u(x)) \partial_z^\gamma \rho(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&\quad - \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \neq m \\ 0 \le r \le k-1}} \frac{m!}{\alpha!\beta!\gamma!} \binom{k}{r} \int_{\mathbb{T}^{2d}} \nabla^r \big(\partial_z^\alpha \psi(x-y) \big) \nabla^{k-r} \big(\partial_z^\beta u(x) \big) \partial_z^\gamma \rho(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&\le \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \neq m}} \frac{m!}{\alpha!\beta!\gamma!} 2 \|\psi\|_s \|\partial_z^\gamma \rho\|_{L^2} \|\partial_z^\beta u\|_{L^2} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \neq m \\ 0 \le r \le k-1}} \frac{m!}{\alpha!\beta!\gamma!} \binom{k}{r} \|\psi\|_s \|\partial_z^\gamma \rho\|_{L^2} \|\nabla^{k-r}(\partial_z^\beta u)\|_{L^2} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{75}$): In this case, we get

$$
\begin{aligned}
\mathcal{I}_{75} &= \int_{\mathbb{T}^{2d}} \nabla^k \psi(x-y) (\partial_z^m u^n(y) - \partial_z^m u^n(x)) \rho(y) \nabla^k \big(\partial_z^m u^{n+1} \big)(x) dy dx \\
&\quad - \sum_{r=0}^{k-1} \binom{k}{r} \int_{\mathbb{T}^{2d}} \nabla^r \psi(x-y) \nabla^{k-r} \big(\partial_z^m u^n \big)(x) \rho(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&\le 2 \|\psi\|_s \|\rho\|_{L^2} \|\partial_z^m u^n\|_{L^2} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{r=0}^{k-1} \binom{k}{r} \|\psi\|_s \|\rho\|_{L^2} \|\nabla^{k-r}(\partial_z^m u^n)\|_{L^2} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2} \|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{76}$): One has

$$\begin{aligned}
\mathcal{I}_{76} &= \int\limits_{\mathbb{T}^{2d}} \nabla^k \psi(x-y)(u(y)-u(x))\partial_z^m \rho^{n+1}(y)\nabla^k(\partial_z^m u^{n+1})(x)dydx \\
&\quad - \sum_{r=0}^{k-1}\binom{k}{r}\int\limits_{\mathbb{T}^{2d}} \nabla^r \psi(x-y)\nabla^{k-r}u(x)\partial_z^m \rho^{n+1}(y)\nabla^k(\partial_z^m u^{n+1})(x)dydx \\
&\le 2\|\psi\|_s\|u\|_{L^2}\|\partial_z^m\rho^{n+1}\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{r=0}^{k-1}\binom{k}{r}\|\psi\|_s\|\nabla^{k-r}u\|_{L^2}\|\partial_z^m\rho\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}$$

Now, we gather all results for $\mathcal{I}_{7i}$'s, sum over $0 \le k \le s-m+1$ and combine with the zeroth-order estimate to yield that for each $z \in \Omega$,

$$\frac{\partial}{\partial t}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2 \le C(\varepsilon^{1/2}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2 + \varepsilon^{3/2}).$$

Thus, we integrate the above relation over $[0,t]$ and use Grönwall's lemma to get the desired result.

Appendix C. Proof of Lemma 4.4

We consider only higher-order estimates. For $1 \le k \le m-l+1$ and $1 \le l \le m$, we apply $\nabla^k\partial_z^l$ to (1.1) to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2}^2 \\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int\limits_{\mathbb{T}^d}\nabla^{r_2}(\partial_z^{r_1}(u-\bar{u}))\cdot\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}u))\cdot\nabla^k(\partial_z^l(u-\bar{u}))dx \\
&\quad -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int\limits_{\mathbb{T}^d}\nabla^{r_2}(\partial_z^{r_1}\bar{u})\cdot\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}(u-\bar{u}))\cdot\nabla^k(\partial_z^l(u-\bar{u}))dx \\
&\quad +\sum_{\alpha+\beta+\gamma=l}\frac{l!}{\alpha!\beta!\gamma!}\int\limits_{\mathbb{T}^{2d}}\Big[\nabla^k\Big(\partial_z^\alpha\psi\ \partial_z^\beta((u-\bar{u})(y)-(u-\bar{u})(x))\Big)\partial_z^\gamma\rho(y)\nabla^k(\partial_z^l(u-\bar{u}))(x) \\
&\qquad\qquad + \nabla^k\Big(\partial_z^\alpha\psi\ \partial_z^\beta(\bar{u}(y)-\bar{u}(x))\Big)\partial_z^\gamma(\rho-\bar{\rho})(y)\nabla^k(\partial_z^l(u-\bar{u}))(x)\Big]dydx \\
&=: \sum_{i=1}^4 \mathcal{I}_{8i}.
\end{aligned}$$

In the sequel, we estimate the terms $\mathcal{I}_{8i}$'s one by one.

◇ (Estimates for $\mathcal{I}_{81}$): One has

$$\begin{aligned}
\mathcal{I}_{81} &\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{r_2}(\partial_z^{r_1}(u-\bar u))\|_{L^2} \|\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}u))\|_{L^\infty} \|\nabla^k(\partial_z^l(u-\bar u)\|_{L^2} \\
&\le C \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \|\nabla^{r_2}(\partial_z^{r_1}(u-\bar u))\|_{L^2} \|\partial_z^{l-r_1}u\|_{H^{s-l+1}} \|\nabla^k(\partial_z^l(u-\bar u)\|_{L^2} \\
&\le C\sum_{r=0}^{l} \|\partial_z^r(u-\bar u)\|_{H^{m-r+1}}^2.
\end{aligned}$$

◇ (Estimates for $\mathcal{I}_{82}$): Similarly, we have

$$\begin{aligned}
\mathcal{I}_{82} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \int_{\mathbb{T}^d} \nabla^{r_2}(\partial_z^{r_1}\bar u)\cdot\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}(u-\bar u)))\cdot\nabla^k(\partial_z^l(u-\bar u))dx \\
&\quad + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot\bar u)|\nabla^k(\partial_z^l(u-\bar u))|^2 dx \\
&\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{k-r_2}(\partial_z^{l-r_1}\bar u)\|_{L^\infty} \|\nabla^{r_2+1}(\partial_z^{r_1}(u-\bar u))\|_{L^2} \|\nabla^k(\partial_z^l(u-\bar u)\|_{L^2} \\
&\quad + \frac{\|\nabla\cdot\bar u\|_{L^\infty}}{2} \|\nabla^k(\partial_z^l(u-\bar u))\|_{L^2}^2 \\
&\le C\sum_{r=0}^{l} \|\partial_z^r(u-\bar u)\|_{H^{m-r+1}}^2.
\end{aligned}$$

◇ (Estimates for $\mathcal{I}_{83}$ and $\mathcal{I}_{84}$): One gets

$$\begin{aligned}
\mathcal{I}_{83} &\le C \sum_{\substack{\alpha+\beta+\gamma=l\\ 0\le r\le k}} \|\nabla^{k-r}(\partial_z^\beta(u-\bar u))\|_{L^2} \|\partial_z^\gamma\rho\|_{L^2} \|\nabla^k(\partial_z^l(u-\bar u))\|_{L^2} \\
&\le C\sum_{0\le r\le l} \|\partial_z^r(u-\bar u)\|_{H^{m-r+1}}^2, \\
\mathcal{I}_{84} &\le C \sum_{\substack{\alpha+\beta+\gamma=l\\ 0\le r\le k}} \|\nabla^{k-r}(\partial_z^\beta\bar u)\|_{L^2} \|\partial_z^\gamma(\rho-\bar\rho)\|_{L^2} \|\nabla^k(\partial_z^l(u-\bar u))\|_{L^2} \\
&\le C\left(\|\nabla^k(\partial_z^l(u-\bar u))\|_{L^2}^2 + \sum_{0\le r\le l} \|\partial_z^r(\rho-\bar\rho)\|_{L^2}^2 \right).
\end{aligned}$$

Now, we combine all the estimates for $\mathcal{I}_{8i}$, sum over $1 \le k \le m-l+1$, $0 \le l \le m$ and add the zeroth-order estimate to get the desired result.

Appendix D. Proof of Theorem 5.2

It follows from (3.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^d} |\partial_z^m u|^2 dx \\
&= -\int_{\mathbb{T}^d} (\partial_z^m u \cdot \nabla u)\cdot \partial_z^m u dx - \int_{\mathbb{T}^d} (u \cdot \nabla(\partial_z^m u))\cdot \partial_z^m u dx \\
&\quad - \sum_{l=1}^{m-1}\binom{m}{l}\int_{\mathbb{T}^d} (\partial_z^l u \cdot \nabla(\partial_z^{m-l} u))\cdot \partial_z^m u dx \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}} \partial_z^\alpha \psi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x))\partial_z^\gamma \rho(y)\cdot \partial_z^m u(x) dy dx \\
&\quad + \int_{\mathbb{T}^{2d}} \psi(x-y)(\partial_z^m u(y) - \partial_z^m u(x))\rho(y)\cdot \partial_z^m u(x) dy dx =: \sum_{i=1}^{5} \mathcal{I}_{9i}.
\end{aligned}$$

Next, we estimate each $\mathcal{I}_{9i}$ separately.

$\diamond$ (Estimates for $\mathcal{I}_{91}$ and $\mathcal{I}_{92}$): One gets

$$\mathcal{I}_{91} = \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|\partial_z^m u|^2 dx \le \frac{\|\nabla\cdot u\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}}{2}\|\partial_z^m u\|_{L^2}^2,$$

$$\begin{aligned}
\mathcal{I}_{92} &\le \|\nabla(\partial_z^m u)\|_{L^\infty}\|u\|_{L^2}\|\partial_z^m u\|_{L^2} \le \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{\mathcal{U}(z)^2}{\delta}\|u\|_{L^2}^2 \\
&\le \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z)e^{-2\tilde{\Lambda}(z)t}.
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{93}$): Note that this term does not appear when $m=1$. If $m \ge 2$,

$$\begin{aligned}
\mathcal{I}_{93} &\le \sum_{l=1}^{m-1}\binom{m}{l}\|\nabla(\partial_z^{m-l} u)\|_{L^\infty}\|\partial_z^l u\|_{L^2}\|\partial_z^m u\|_{L^2} \\
&\le \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2 \mathcal{U}^2(z)\|\partial_z^l u\|_{L^2}^2
\end{aligned}$$

$$\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2 \mathcal{U}^2(z)\mathcal{F}_l(z)e^{-\tilde{\Lambda}(z)t}.$$

$\diamond$ (Estimates for $\mathcal{I}_{94}$ and $\mathcal{I}_{95}$): One has

$$\begin{aligned}
\mathcal{I}_{94} &\leq \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!}\|\psi\|_s \left(\|\partial_z^\beta u\, \partial_z^\gamma \rho\|_{L^1}\|\partial_z^m u\|_{L^1} + \|\partial_z^\gamma \rho\|_{L^1}\|\partial_z^\beta u\cdot \partial_z^m u\|_{L^1}\right)\\
&\leq 2\sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!}\|\psi\|_s \|\partial_z^\gamma \rho\|_{L^2}\|\partial_z^\beta u\|_{L^2}\|\partial_z^m u\|_{L^2}\\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + 4\sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}}\left(\|\psi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2 \frac{\frac{(m+1)(m+2)}{2}-1}{\delta}\|\partial_z^\beta u\|_{L^2}^2\\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}}\left(\|\psi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2 \frac{2m^2+6m}{\delta}\mathcal{F}_\beta(z)e^{-\tilde{\Lambda}(z)t},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{95} &= -\int_{\mathbb{T}^d}\psi(x-y)|\partial_z^m u(x)|^2\rho(y)dydx + \int_{\mathbb{T}^d}\psi(x-y)\partial_z^m u(y)\cdot\partial_z^m u(x)\rho(y)dydx\\
&\leq -\psi_m\|\rho_0\|_{L^1}\|\partial_z^m u\|_{L^2}^2 + \|\psi\|_s\|\rho\partial_z^m u\|_{L^1}\|\partial_z^m u\|_{L^1}\\
&\leq -\left(\psi_m\|\rho_0\|_{L^1} - \frac{\delta}{4}\right)\|\partial_z^m u\|_{L^2} + \frac{\|\psi\|_s^2\|\rho\|_{L^2}}{\delta}\mathcal{E}_m(t,z)\\
&\leq -\left(\psi_m\|\rho_0\|_{L^1} - \frac{\delta}{4}\right)\|\partial_z^m u\|_{L^2} + \frac{\|\psi\|_s^2\|\rho\|_{L^2}}{\delta}E_m(z)e^{-\tilde{\Lambda}(z)t}.
\end{aligned}$$

Finally, we gather the estimates for $\mathcal{I}_{9i}$ $(i=1,\cdots,5)$ to obtain

$$\frac{\partial}{\partial t}\|\partial_z^m u\|_{L^2}^2 \leq -2\tilde{\Lambda}(z)\|\partial_z^m u\|_{L^2}^2 + \hat{F}_m(z)e^{-\tilde{\Lambda}(z)t}, \tag{D.1}$$

where $\hat{F}_m(z)$ is given by

$$\begin{aligned}
\hat{F}_1(z) &:= \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z) + \frac{16}{\delta}\mathcal{U}^2(z)\mathcal{F}_0(z) + \frac{\|\psi\|_s^2\mathcal{U}(z)}{\delta}E_1(z),\\
\hat{F}_m(z) &:= \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z) + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)\mathcal{F}_l(z)\\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}}\left(\|\psi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2\frac{2m^2+6m}{\delta}\mathcal{F}_\beta + \frac{\|\psi\|_s^2\mathcal{U}(z)}{\delta}E_1(z), \quad m\geq 2.
\end{aligned}$$

We apply Grönwall's lemma for (D.1) to yield

$$\|\partial_z^m u\|_{L^2}^2 \le \|\partial_z^m u_0\|_{L^2}^2 e^{-2\tilde{\Lambda}(z)t} + \frac{\hat{F}_m(z)}{\tilde{\Lambda}(z)}(e^{-\tilde{\Lambda}(z)t} - e^{-2\tilde{\Lambda}(z)t}) \le \mathcal{F}_m(z) e^{-\tilde{\Lambda}(z)t},$$

where $\mathcal{F}_m(z)$ is defined as

$$\mathcal{F}_m(z) := \|\partial_z^m u_0\|_{L^2}^2 + \frac{\hat{F}_m(z)}{\tilde{\Lambda}(z)}.$$

This implies the desired estimate.

References

[1] G. Albi, L. Pareschi, M. Zanella, Uncertain quantification in control problems for flocking models, Math. Probl. Eng. (2015) 850124.
[2] L. Boudin, L. Desvillettes, C. Grandmont, A. Moussa, Global existence of solution for the coupled Vlasov and Navier-Stokes equations, Differ. Integral Equ. 22 (2009) 1247–1271.
[3] J.A. Carrillo, M. Fornasier, J. Rosado, G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, SIAM J. Math. Anal. 42 (2010) 218–236.
[4] J.A. Carrillo, M. Fornasier, G. Toscani, F. Vecil, Particle, kinetic, and hydrodynamic models of swarming, in: Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, in: Model. Simul. Sci. Eng. Technol., Birkhauser Boston, Inc., Boston, MA, 2010, pp. 297–336.
[5] J.A. Carrillo, L. Pareschi, M. Zanella, Particle based gPC methods for mean-field models of swarming with uncertainty, Commun. Comput. Phys. 25 (2019) 508–531.
[6] J. Cho, S.-Y. Ha, F. Huang, C. Jin, D. Ko, Emergence of bi-cluster flocking for the Cucker-Smale model, Math. Models Methods Appl. Sci. 26 (2016) 1191–1218.
[7] Y.-P. Choi, S.-Y. Ha, Z. Li, Emergent dynamics of the Cucker-Smale flocking model and its variants, in: N. Bellomo, P. Degond, E. Tadmor (Eds.), Active Particles, Volume 1 - Advances in Theory, Models, and Applications, in: Series: Modeling and Simulation in Science and Technology, Birkhauser-Springer, 2017.
[8] F. Cucker, J.-G. Dong, On flocks influenced by closest neighbors, Math. Models Methods Appl. Sci. 26 (2016) 2685–2708.
[9] F. Cucker, J.-G. Dong, A general collision-avoiding flocking framework, IEEE Trans. Autom. Control 56 (2011) 1124–1129.
[10] F. Cucker, S. Smale, Emergent behavior in flocks, IEEE Trans. Autom. Control 52 (2007) 852–862.
[11] P. Degond, S. Motsch, Large-scale dynamics of the Persistent Turing Walker model of fish behavior, J. Stat. Phys. 131 (2008) 989–1022.
[12] B. Despres, B. Perthame, Uncertainty propagation; Intrusive kinetic formulations of scalar conservation laws, SIAM/ASA J. Uncertain. Quantificat. 4 (2016) 980–1013.
[13] Z. Ding, S.-Y. Ha, S. Jin, A local sensitivity analysis in the Landau damping for the kinetic Kuramoto equation, submitted for publication.
[14] R. Duan, M. Fornasier, G. Toscani, A kinetic flocking model with diffusion, Commun. Math. Phys. 300 (2010) 95–145.
[15] S.-Y. Ha, F. Huang, Y. Wang, A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation, J. Differ. Equ. 257 (2014) 1333–1371.
[16] S.-Y. Ha, S. Jin, Local sensitivity analysis for the Cucker-Smale model with random inputs, Kinet. Relat. Models 11 (2018) 859–889.
[17] S.-Y. Ha, J. Jung, S. Jin, A local sensitivity analysis for the kinetic Cucker-Smale equation with random inputs, J. Differ. Equ. 265 (2018) 3618–3649.
[18] S.-Y. Ha, S. Jin, J. Jung, A local sensitivity analysis for the kinetic Kuramoto equation with random inputs, Netw. Heterog. Media 14 (2019) 317–340.
[19] S.-Y. Ha, S. Jin, J. Jung, Local sensitivity analysis for the Kuramoto model with random inputs in a large coupling regime, submitted for publication.

[20] S.-Y. Ha, J. Kim, X. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, Kinet. Relat. Models 11 (2018) 1157–1181.
[21] S.-Y. Ha, B. Kwon, M.-J. Kang, A hydrodynamic model for the interaction of Cucker-Smale particles and incompressible fluid, Math. Models Methods Appl. Sci. 24 (2014) 2311–2359.
[22] S.-Y. Ha, J.-G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, Commun. Math. Sci. 7 (2009) 297–325.
[23] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic description of flocking, Kinet. Relat. Models 1 (2008) 415–435.
[24] J. Hu, S. Jin, Uncertainty quantification for kinetic equations, in: S. Jin, L. Pareschi (Eds.), Uncertainty Quantification for Hyperbolic and Kinetic Equations, in: SEMA-SIMAI Springer Series, Springer, 2018.
[25] S. Jin, D. Xiu, X. Zhu, Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings, J. Comput. Phys. 289 (2015) 35–52.
[26] S. Jin, D. Xiu, X. Zhu, A well-balanced stochastic Galerkin method for scalar hyperbolic balance laws with random inputs, J. Sci. Comput. 67 (2016) 1198–1218.
[27] T. Karper, A. Mellet, K. Trivisa, Hydrodynamic limit of the kinetic Cucker-Smale model, Math. Models Methods Appl. Sci. 25 (2015) 131–163.
[28] T. Karper, A. Mellet, K. Trivisa, Existence of weak solutions to kinetic flocking models, SIAM J. Math. Anal. 45 (2013) 215–243.
[29] T. Karper, A. Mellet, K. Trivisa, On strong local alignment in the kinetic Cucker-Smale model, Springer Proc. Math. Stat. (2013).
[30] N.E. Leonard, D.A. Paley, F. Lekien, R. Sepulchre, D.M. Fratantoni, R.E. Davis, Collective motion, sensor networks and ocean sampling, Proc. IEEE 95 (2007) 48–74.
[31] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge Univ. Press, 2002.
[32] L. Mathelin, M.A. Hussaini, T.A. Zang, F. Bataille, Uncertainty propagation for turbulent, compressible flow in a quasi-1D nozzle using stochastic methods, AIAA J. 42 (2004) 1669–1676.
[33] S. Motsch, E. Tadmor, Heterophilious dynamics: enhanced consensus, SIAM Rev. 56 (2014) 577–621.
[34] S. Motsch, E. Tadmor, A new model for self-organized dynamics and its flocking behavior, J. Stat. Phys. 144 (2011) 923–947.
[35] J.M. Murphy, D.M. Sexton, D.N. Barnett, G.S. Jones, M.J. Webb, M. Collins, D.A. Stainforth, Quantification of modelling uncertainties in a large ensemble of climate change simulations, Nature 430 (2004) 768–772.
[36] D.A. Paley, N.E. Leonard, R. Sepulchre, D. Grunbaum, J.K. Parrish, Oscillator models and collective motion, IEEE Control Syst. Mag. 27 (2007) 89–105.
[37] L. Perea, P. Elosegui, G. Gómez, Extension of the Cucker-Smale control law to space flight formation, J. Guid. Control Dyn. 32 (2009) 527–537.
[38] P. Pettersson, A. Doostan, J. Nordström, On stability and monotonicity requirements of discretized stochastic conservation laws with random viscosity, Comput. Methods Appl. Mech. Eng. 258 (2013) 134–151.
[39] P. Pettersson, G. Iaccarino, J. Nordström, Numerical analysis of the Burgers' equation in the presence of uncertainty, J. Comput. Phys. 228 (2009) 8394–8412.
[40] P. Pettersson, G. Iaccarino, J. Nordström, A stochastic Galerkin method for the Euler equations with Roe variable transformation, J. Comput. Phys. 257 (2014) 481–500.
[41] G. Pöette, B. Després, D. Lucor, Uncertainty quantification for systems of conservation laws, J. Comput. Phys. 228 (2009) 2443–2467.
[42] P.J. Roache, Quantification of uncertainty in computational fluid dynamics, Annu. Rev. Fluid Mech. 29 (1997) 123–160.
[43] C.J. Roy, W.L. Oberkampf, A comprehensive framework for verification, validation, and uncertainty quantification in scientific computing, Comput. Methods Appl. Mech. Eng. 200 (2011) 2131–2144.
[44] S. Sankararaman, S. Mahadevan, Integration of model verification, validation, and calibration for uncertainty quantification in engineering systems, Reliab. Eng. Syst. Saf. 138 (2015) 194–209.
[45] R.C. Smith, Uncertainty quantification: Theory, Implementation, and Applications, vol. 12, SIAM, 2013.
[46] E. Tadmor, Mathematical aspects of self-organized dynamics: consensus, emergence of leaders, and social hydrodynamics, SIAM News 48 (2015).
[47] J. Toner, Y. Tu, Flocks, herds, and schools: a quantitative theory of flocking, Phys. Rev. E 58 (1998) 4828–4858.
[48] T. Vicsek, A. Zefeiris, Collective motion, Phys. Rep. 517 (2012) 71–140.
[49] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Schochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75 (1995) 1226–1229.