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Ion-acoustic shock in a collisional plasma

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Abstract

The paper is concerned with the propagation of ion-acoustic shock waves in a collision dominated plasma whose equations of motion are described by the one-dimensional isothermal Navier-Stokes-Poisson system for ions with the electron density determined by the Boltzmann relation. The main results include three parts: (a) We establish the existence and uniqueness of a small-amplitude smooth traveling wave by solving a 3-D ODE in terms of the center manifold theorem. (b) We study the shock structure in a specific asymptotic regime where the viscosity coefficient and the shock strength are proportional to ε and the Debye length is proportional to $(\delta\varepsilon)^{1/2}$ with two parameters ε and δ, and show that in the limit $\varepsilon \to 0$, shock profiles obtained in (a) can be approximated by the profiles of KdV-Burgers uniformly for $0 < \delta \leq \delta_0$ with some $\delta_0 > 0$. The proof is based on the suitable construction of the KdV-Burgers shock profiles together with the delicate analysis of a linearized variable coefficient system in exponentially weighted Sobolev spaces involving parameters ε and δ. (c) We also prove the large time asymptotic stability of traveling waves under suitably small smooth zero-mass perturbations. Note that the ions' temperature is allowed to be zero in parts (a) and (b), but necessarily required to be strictly positive in the proof of part (c).

MSC: 35Q35; 35C07; 35B35; 35B40

Keywords: Collisional plasma; Ion-acoustic shock; Navier-Stokes-Poisson equations; KdV-Burgers equations; Nonlinear stability

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Contents

1. Introduction

As a fundamental issue of great physical importance, the problem of shock structures in plasma has received considerable attention in the literature from both theoretical and experimental perspectives, e.g. [25,26,42,45]. In this paper, we carry out a mathematical study of the existence and stability of smooth shock profiles for the Navier-Stokes-Poisson system describing the dynamics of single-ions under the Maxwell-Boltzmann relation. Moreover, we prove that the obtained shock wave solution converges to the one of the KdV-Burgers equations in a specific asymptotic regime where the viscosity, the square of the Debye length as well as the amplitude of the shock are of the same order in a small parameter ε.

1.1. Equations of motion

In this paper, we study the existence, structure and stability of small-amplitude shock waves of the following one-dimensional Navier-Stokes-Poisson system for ions in $\{(t,x): t>0, x\in\mathbb{R}\}$:

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + Tn) = \mu\partial_{xx}u - n\partial_x\phi, \\ -\lambda^2\partial_{xx}\phi = n - e^{\phi}. \end{cases} \tag{1.1}$$

Here $T\geq 0$, $\mu>0$ and $\lambda>0$ are constants which stand for the absolute temperature, viscosity coefficient and Debye length, respectively. Particularly, when $T=0$, the momentum equation is pressureless and (1.1) is usually used to model the motion of cold plasma. The electric potential $\phi=\phi(t,x)$ is induced by the total charge of ions and electrons. In (1.1), we have assumed that the density of electrons n_e follow the Maxwell-Boltzmann relation $n_e=e^{\phi}$, which is a physical assumption according to the fact that lighter electrons get close to the equilibrium state at a much faster rate than heavier ions in plasma, cf. [4,29]. As shown in [18,21], it can be formally derived from the two-fluid model by taking the velocity of electrons to be zero. To solve (1.1), the initial data are given by

$$[n,u](0,x) = [n_0,u_0](x),$$

with

$$\lim_{x\to\pm\infty}[n_0,u_0](x)=[n_\pm,u_\pm].$$

The far-field data of ϕ are given by

$$\lim_{x\to\pm\infty}\phi(t,x)=\phi_\pm,$$

under the quasi-neutral condition $\phi_\pm=\log n_\pm$ at $x=\pm\infty$. In this paper, we are interested in the case where $n_+\neq n_-$, thus the electric potential ϕ connects two distinct constants $\phi_\pm$ at the far-fields $x=\pm\infty$. Without loss of generality, we assume that $[n_-,u_-]=[1,0]$ throughout the paper.

1.2. *Existence of shock profile*

One of main purposes of this paper is to construct a shock profile $[n,u,\phi](\xi)$, which is a smooth traveling wave solution to (1.1) satisfying

$$\lim_{x\to\pm\infty}[n,u,\phi](x)=[n_\pm,u_\pm,\phi_\pm]. \tag{1.2}$$

Here $\xi:=x-st$ and s is a speed of the shock. Therefore, the equations of $[n,u,\phi]$ are given by

$$\begin{cases} -s\dfrac{\mathrm{d}n}{\mathrm{d}\xi}+\dfrac{\mathrm{d}(nu)}{\mathrm{d}\xi}=0,\\ -s\dfrac{\mathrm{d}(nu)}{\mathrm{d}\xi}+\dfrac{\mathrm{d}(nu^2+Tn)}{\mathrm{d}\xi}=\mu\dfrac{\mathrm{d}^2u}{\mathrm{d}\xi^2}-n\dfrac{\mathrm{d}\phi}{\mathrm{d}\xi},\\ -\lambda^2\dfrac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=n-e^\phi. \end{cases} \tag{1.3}$$

We observe that, by $(1.3)_3$, the last term in $(1.3)_2$ can be rewritten as

$$-n\frac{\mathrm{d}\phi}{\mathrm{d}\xi}=\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\lambda^2}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2-e^\phi\right].$$

Hence, the system (1.3) is equivalent to

$$\begin{cases} -s\dfrac{\mathrm{d}n}{\mathrm{d}\xi}+\dfrac{\mathrm{d}(nu)}{\mathrm{d}\xi}=0,\\ -s\dfrac{\mathrm{d}(nu)}{\mathrm{d}\xi}+\dfrac{\mathrm{d}(nu^2+Tn)}{\mathrm{d}\xi}=\mu\dfrac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left[\dfrac{\lambda^2}{2}\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2-e^\phi\right],\\ -\lambda^2\dfrac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=n-e^\phi. \end{cases} \tag{1.4}$$

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Integrating the first two equations in (1.4) over $\mathbb{R}$ and using $\phi_\pm = \log n_\pm$, we have the following Rankine-Hugoniot condition

$$\begin{cases} -s(n_+ - n_-) + n_+u_+ - n_-u_- = 0, \\ -s(n_+u_+ - n_-u_-) + n_+u_+^2 - n_-u_-^2 + (T+1)(n_+ - n_-) = 0, \end{cases} \tag{1.5}$$

which gives the shock speed $s = \pm\sqrt{(T+1)n_+}$. In this paper, we only consider the 2-shock and take $s = \sqrt{(T+1)n_+}$ accordingly. In such case, the Lax shock condition is given by $\sqrt{(T+1)n_+} < s < \sqrt{T+1}$, that is,

$$n_+ < 1. \tag{1.6}$$

First of all, as for the existence of shock profile solutions to (1.1) with fixed constants $\mu > 0$ and $\lambda > 0$, we have the following result.

Theorem 1.1. *Let $T \geq 0$. For given data $[n_-, u_-]$ with $n_- > 0$, there exist positive constants $\hat{\varepsilon}_0$, $\bar{C}$ and $\underline{C}$, such that if $[n_+, u_+]$ satisfies (1.5), (1.6) and*

$$|n_+ - n_-| \leq \hat{\varepsilon}_0,$$

the problem (1.3) has a unique (up to a shift in ξ) solution $[\bar{n}, \bar{u}, \bar{\phi}]$ satisfying

$$\underline{C}\bar{\phi}_\xi \leq \bar{n}_\xi = \frac{\bar{n}^2\bar{u}_\xi}{n_-|u_- - s|} \leq \bar{C}\bar{\phi}_\xi < 0, \tag{1.7}$$

for any $\xi \in \mathbb{R}$. Moreover, by a suitable choice of the shift, the solution satisfies

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \left[\bar{n} - n_\pm, \bar{u} - u_\pm, \bar{\phi} - \phi_\pm\right](\xi) \right| \leq C_k |n_+ - n_-|^{k+1} e^{-\theta|(n_+ - n_-)\xi|} \tag{1.8}$$

for $\xi \lessgtr 0$ and $k = 0, 1, \cdots$, where each C_k and θ are generic positive constants.

Remark 1.2. We would emphasize that $T = 0$, corresponding to the pressureless or cold plasma, is allowed for the existence of smooth shock profiles. Also note that the self-consistent electric potential ϕ connects two distinct constants $\phi_\pm$ at the far-fields $x = \pm\infty$, and thus the profile of ϕ is nontrivial.

Remark 1.3. Formally, if one lets $\lambda = 0$ then (1.1) reduces to the isothermal Navier-Stokes equations of which even the large-amplitude shock waves can be constructed in terms of the classical phase space analysis method. However, in our case we have to be restricted to the consideration of small-amplitude shock waves, because the phase space analysis seems no longer useful due to the appearance of the second-order elliptic equation $(1.4)_3$ with the nonlinear term e^ϕ such that ϕ cannot be explicitly solved by n. Thus, the existence of large-amplitude shock waves is left open.

1.3. The KdV-Burgers approximation

When the Debye length $\lambda = 0$, the plasma becomes quasi-neutral and the system (1.1) reduces to the classical Navier-Stokes system of which the shock profile is governed by the scalar viscous Burgers equation in the small amplitude limit, as shown in [31,36]. In typical plasma, the parameter λ is small compared with the characteristic length of physical interest. Thus it would be interesting to understand how the internal structure of plasma shock waves relies on this small parameter. In the second part of the paper, we will show that, in a specific regime where the square of the Debye length, the viscosity as well as the amplitude of the shock are of the same order, the shock wave solution obtained in Theorem 1.1 is close to the one of the KdV-Burgers equation. This result is also in agreement with numerical experiments carried out in [14,42].

To be precise, we re-set up the problem (1.3) in the regime where both the viscosity coefficient μ and the Debye length λ are small and depend on a small parameter $\varepsilon > 0$ by

$$\mu = \varepsilon\bar{\mu}, \quad \lambda = \varepsilon^{1/2}\bar{\lambda} \tag{1.9}$$

for two positive constants $\bar{\mu}$ and $\bar{\lambda}$ of the same order as a typical length. We denote

$$\delta = \frac{\bar{\lambda}^2}{\bar{\mu}^2} \tag{1.10}$$

and introduce a scaled variable $z = \xi/\bar{\mu}$. The equations of the shock waves $[n_\varepsilon, u_\varepsilon, \phi_\varepsilon]$ are given by

$$\begin{cases} -s_\varepsilon n_\varepsilon' + (n_\varepsilon u_\varepsilon)' = 0, \\ -s_\varepsilon (n_\varepsilon u_\varepsilon)' + (n_\varepsilon u_\varepsilon^2 + T n_\varepsilon)' = \varepsilon u_\varepsilon'' - n_\varepsilon \phi_\varepsilon', \\ -\varepsilon\delta\phi_\varepsilon'' = n_\varepsilon - e^{\phi_\varepsilon}, \\ \lim\limits_{z\to\pm\infty} [n_\varepsilon(z), u_\varepsilon(z), \phi_\varepsilon(z)] = [n_{\varepsilon,\pm}, u_{\varepsilon,\pm}, \phi_{\varepsilon,\pm}], \end{cases} \tag{1.11}$$

where $'$ stands for the differential operator $\frac{\mathrm{d}}{\mathrm{d}z}$. Notice that the R-H condition (1.5) for the far fields remains the same in this formulation.

Formally, we assume to have the following expansion of $[n_\varepsilon, u_\varepsilon, \phi_\varepsilon]$ near the equilibrium $[n_{\varepsilon,-}, u_{\varepsilon,-}, \phi_{\varepsilon,-}] = [1, 0, 0]$:

$$\begin{cases} n_\varepsilon = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots, \\ u_\varepsilon = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \\ \phi_\varepsilon = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots. \end{cases} \tag{1.12}$$

To make the far fields compatible with the expansion in ε, the shock speed s_ε and the downstream constant equilibrium $[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}]$ are also supposed to have the following asymptotic expansion:

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$$
\begin{cases}
s_\varepsilon = \sqrt{T+1} + \varepsilon s_1 + \varepsilon^2 s_2 + \cdots, \\
n_{\varepsilon,+} = 1 + \varepsilon n_{1,+} + \varepsilon^2 n_{2,+} + \cdots, \\
u_{\varepsilon,+} = \varepsilon u_{1,+} + \varepsilon^2 u_{2,+} + \cdots, \\
\phi_{\varepsilon,+} = \varepsilon \phi_{1,+} + \varepsilon^2 \phi_{2,+} + \cdots.
\end{cases}
\tag{1.13}
$$

Here $\sqrt{T+1}$ in the first equation of (1.13) is equal to the acoustic speed of the second family of characteristic field at $[n, u] = [1, 0]$, and it is the exact leading term of the asymptotic expansion of s_ε as $\varepsilon \to 0$. Here we take $0 < s_\varepsilon < \sqrt{T+1}$ according to compressibility of the shock. For brevity, in what follows we shall make a simple choice of s_ε by

$$
s_\varepsilon = \sqrt{T+1} - \varepsilon. \tag{1.14}
$$

Upon substituting (1.14) into the R-H condition (1.5), the data $[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}]$ are parameterized in terms of ε as follows:

$$
\begin{cases}
n_{\varepsilon,+} = \dfrac{s_\varepsilon^2}{T+1} = 1 - \varepsilon\left(\dfrac{2}{\sqrt{T+1}} - \dfrac{\varepsilon}{T+1}\right), \\
u_{\varepsilon,+} = s_\varepsilon(1 - \dfrac{1}{n_{\varepsilon,+}}) = -\varepsilon\left(2 + \dfrac{\varepsilon}{\sqrt{T+1} - \varepsilon}\right), \\
\phi_{\varepsilon,+} = \log n_{\varepsilon,+} = -\varepsilon\left(\dfrac{2}{\sqrt{T+1}} - a_\varepsilon\right).
\end{cases}
\tag{1.15}
$$

Here a_ε in the last line of (1.15) is denoted by

$$
a_\varepsilon = \frac{1}{\varepsilon}\log\frac{s_\varepsilon^2}{T+1} + \frac{2}{\sqrt{T+1}}.
$$

By (1.14), it is straightforward to verify that

$$
a_\varepsilon = \frac{2}{\varepsilon}\log\left(1 - \frac{\varepsilon}{\sqrt{T+1}}\right) + \frac{2}{\sqrt{T+1}} = -\frac{\varepsilon}{T+1} - O(1)\varepsilon^2.
$$

Now we are in the position to derive the equations for $[n_1, u_1, \phi_1]$ corresponding to the first-order terms in (1.12). In fact, substituting (1.12) into (1.11) immediately yields, to the vanishing zeroth order in ε and the subsequent orders, the following hierarchy of equations:

$$
\varepsilon^1 : -\sqrt{T+1} n_1' + u_1' = 0, \tag{1.16}
$$

$$
-\sqrt{T+1} u_1' + T n_1' = -\phi_1', \tag{1.17}
$$

$$
n_1 - \phi_1 = 0, \tag{1.18}
$$

$$
\varepsilon^2 : -\sqrt{T+1} n_2' + u_2' + n_1' + (n_1 u_1)' = 0, \tag{1.19}
$$

$$
\begin{aligned}
-\sqrt{T+1} u_2' + T n_2 + u_1' - \sqrt{T+1}(n_1 u_1)' + 2u_1 u_1' \\
= -\phi_2' + u_1'' - n_1 \phi_1',
\end{aligned}
\tag{1.20}
$$

$$n_2 - \phi_2 - \frac{1}{2}\phi_1^2 = -\delta\phi_1'', \tag{1.21}$$

$$\varepsilon^3 : \cdots .$$

From (1.16), (1.17) and (1.18), we solve u_1 and ϕ_1 in terms of n_1 as

$$u_1 = \sqrt{T+1}n_1, \qquad \phi_1 = n_1. \tag{1.22}$$

To further derive the equation for n_1, we differentiate (1.21) with respect to z, then multiply (1.19) by $\sqrt{T+1}$, and further add these two resultant equations to (1.20), so it follows that

$$\sqrt{T+1}n_1' + u_1' + 2u_1u_1' - u_1'' + n_1\phi_1' - \phi_1\phi_1' + \delta\phi_1''' = 0. \tag{1.23}$$

Substituting (1.22) into (1.23), one derives the equation for n_1:

$$2\sqrt{T+1}n_1' + 2(T+1)n_1n_1' - \sqrt{T+1}n_1'' + \delta n_1''' = 0, \tag{1.24}$$

with the far fields

$$\lim_{z\to+\infty} n_1(z) := n_{1,+} = -\frac{2}{\sqrt{T+1}}, \qquad \lim_{z\to-\infty} n_1(z) := n_{1,-} = 0. \tag{1.25}$$

By (1.22), the equations of u_1 and ϕ_1 are respectively given by

$$\begin{cases} 2u_1' + 2u_1u_1' - u_1'' + \dfrac{\delta}{\sqrt{T+1}}u_1''' = 0, \\ \lim\limits_{z\to+\infty} u_1(z) := u_{1,+} = -2, \qquad \lim\limits_{z\to-\infty} u_1(z) := u_{1,-} = 0, \end{cases} \tag{1.26}$$

and

$$\begin{cases} \sqrt{T+1}\phi_1' + 2(T+1)\phi_1\phi_1' - \sqrt{T+1}\phi_1'' + \delta\phi_1''' = 0, \\ \lim\limits_{z\to+\infty} \phi_1(z) := \phi_{1,+} = -\dfrac{2}{\sqrt{T+1}}, \qquad \lim\limits_{z\to-\infty} \phi_1(z) := \phi_{1,-} = 0. \end{cases} \tag{1.27}$$

The existence of KdV-Burgers shock waves n_1, u_1 and ϕ_1 is guaranteed by the result in [3], provided that δ is suitably small. To make the paper self-contained, we will list the related results in Lemma 5.1 in the Appendix. From (1.25), (1.26) and (1.27), we note that the far field $[n_{1,+}, u_{1,+}, \phi_{1,+}]$ matches (1.15) at the first order of ε. So the shock profile solution to the problem (1.11) is expected to have the following form:

$$[n_\varepsilon, u_\varepsilon, \phi_\varepsilon] = [1,0,0] + \varepsilon[n_1, u_1, \phi_1] + o(\varepsilon).$$

Now we are ready to state the main result concerning the KdV-Burgers limit of the shock profiles.

Theorem 1.4. *Let $T \geq 0$ and $0 < \alpha < 2$. There exist two positive constants ε_0 and δ_0, such that if*

$$\varepsilon \in (0, \varepsilon_0), \quad \delta = \frac{\bar{\lambda}^2}{\bar{\mu}^2} \in (0, \delta_0), \tag{1.28}$$

and the far field $[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}]$ satisfies (1.15), then the shock-wave equation (1.11) admits a unique solution $[n_\varepsilon, u_\varepsilon, \phi_\varepsilon]$ satisfying

$$n_\varepsilon(0) = 1 + \varepsilon n_1(0), \tag{1.29}$$

and

$$\sup_z \left| e^{\alpha|z|} \frac{\mathrm{d}^k}{\mathrm{d}z^k} [n_\varepsilon - 1 - \varepsilon n_1, u_\varepsilon - \varepsilon u_1, \phi_\varepsilon - \varepsilon \phi_1](z) \right| \leq C_k \varepsilon^2, \tag{1.30}$$

for any integer $k \geq 0$, where each $C_k > 0$ is a generic constant independent of ε and δ.

Remark 1.5. Since the solutions to (1.11) are not unique due to the translation invariance of the shock waves, we add a constraint condition (1.29) to fix the phase.

Remark 1.6. As in Remark 1.2, we emphasize that $T = 0$ is allowed in the above theorem.

Remark 1.7. On the one hand, since the estimate (1.30) is uniform in δ, one can recover the classical viscous Burgers approximation to the shock profile of the Navier-Stokes equations for any fixed $\varepsilon \in (0, \varepsilon_0]$ by letting $\delta \to 0+$. On the other hand, we should point out that in a different regime where $\mu > 0$ is fixed and $\lambda \sim \varepsilon \to 0+$, the method developed in this part can be also applied to construct the plasma shock profile for NSP even with large amplitude, through the approximation of the compressible Navier-Stokes profile.

Remark 1.8. The assumption (1.28) implies that the dissipation dominates over the dispersion, which leads to a monotone structure of smooth shock waves, see Lemma 5.1. On the other hand, when δ is suitably large, the oscillatory traveling wave solutions can be not only computed numerically, cf. [14,42], but also constructed in a rigorous way, cf. [3] for KdV-Burgers equation and [44] for their stability. Thus it may be interesting to study whether one can construct, for large $\delta > 0$, the traveling waves of the Navier-Stokes-Poisson system with an oscillatory structure through an approximation by the KdV-Burgers equations. This will be left for future.

1.4. Dynamical stability of shock profile

We shall also investigate the large time asymptotic stability of the smooth traveling shock profile obtained in Theorem 1.1. For convenience, we formulate the Navier-Stokes-Poisson system in *Lagrangian* coordinates which reads

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + T\partial_x v^{-1} = \mu \partial_x (v^{-1}\partial_x u) - v^{-1}\partial_x \phi, \\ -\lambda^2 \partial_x (v^{-1}\partial_x \phi) = 1 - ve^{\phi}, \quad t > 0, x \in \mathbb{R}. \end{cases} \tag{1.31}$$

Here, $v = \frac{1}{n}$ is the specific volume. We keep on using variables t and x in the Lagrangian coordinates for brevity. Initial data are given by

$$[v, u](0, x) = [v_0, u_0](x) \to [v_\pm, u_\pm] \quad (x \to \pm\infty), \tag{1.32}$$

and the far fields of $\phi(t, x)$ are given by

$$\lim_{x\to\pm\infty} \phi(t, x) = \phi_\pm, \tag{1.33}$$

under the quasi-neutral condition $\phi_\pm = -\log v_\pm$ at $x = \pm\infty$. Similar to the case of Eulerian coordinates, we can also write the momentum equation into a conservative form. In fact, multiplying the third equation of (1.31) by $v^{-1}\partial_x\phi$ gives

$$v^{-1}\partial_x\phi = \big[-\lambda^2\partial_x(v^{-1}\partial_x\phi) + ve^\phi\big]v^{-1}\partial_x\phi = \partial_x\left[-\frac{\lambda^2}{2}(v^{-1}\partial_x\phi)^2 + e^\phi\right].$$

Substituting the above identity into the second equation of (1.31), the system for $[v, u, \phi]$ is rewritten as

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + (T+1)\partial_x v^{-1} = \mu\partial_x(v^{-1}\partial_x u) + \lambda^2\partial_x\left[\dfrac{(v^{-1}\partial_x\phi)^2}{2} - v^{-1}\partial_x(v^{-1}\partial_x\phi)\right], \\ -\lambda^2\partial_x(v^{-1}\partial_x\phi) = 1 - ve^\phi. \end{cases} \tag{1.34}$$

From (1.34), the R-H condition is given by

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) + (T+1)(v_+^{-1} - v_-^{-1}) = 0. \end{cases} \tag{1.35}$$

Similar as (1.6), we only consider 2-shock, and thus assume that $v_+ > v_-$.

For the Boltzmann equation, the equivalence of shock profile in the Eulerian coordinates and the one in Lagrangian coordinates has been shown in [22]. The case of the Navier-Stokes-Poisson system is similar. Therefore, we re-state the properties of the shock profile in the Lagrangian coordinates in terms of Theorem 1.1 as follows.

Proposition 1.9. *Let $T \ge 0$. For given $[v_-, u_-]$ with $v_- > 0$, there exist positive constants $\tilde{\varepsilon}$, $\tilde{C}_1$ and $\tilde{C}_2$, such that if $[v_+, u_+]$ satisfies (1.35) and*

$$v_+ - v_- \in (0, \tilde{\varepsilon}), \tag{1.36}$$

then (1.31) has a unique (up to a shift) shock profile solution $[\bar{v}, \bar{u}, \bar{\phi}](y)$ satisfying $[\bar{v}, \bar{u}, \bar{\phi}](y) \to [v_\pm, u_\pm, \phi_\pm]$ as $y \to \pm\infty$ and

$$s\bar{v}_y = -\bar{u}_y > 0, \quad -\tilde{C}_1\bar{v}_y \le \bar{\phi}_y \le -\tilde{C}_2\bar{v}_y, \tag{1.37}$$

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where $y = x - st$ and s is the shock speed. Moreover, by a suitable choice of the shift, the solution satisfies

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}y^k} \left[\bar{v} - v_\pm, \bar{u} - u_\pm, \bar{\phi} - \phi_\pm\right](y) \right| \le C_k |v_+ - v_-|^{k+1} e^{-\theta|(v_+ - v_-)y|},$$

for $y \gtrless 0$ and $k = 0, 1, \cdots$, where each C_k and θ are generic positive constants.

Now we introduce the following anti-derivative variables

$$\begin{cases} \Phi(t,y) = \displaystyle\int_{-\infty}^{y} (v(t,y') - \bar{v}(y'))\mathrm{d}y', \\ \Psi(t,y) = \displaystyle\int_{-\infty}^{y} (u(t,y') - \bar{u}(y'))\mathrm{d}y', \end{cases} \tag{1.38}$$

and set

$$\Phi_0(y) = \int_{-\infty}^{y} (v_0 - \bar{v})(y')\mathrm{d}y', \quad \Psi_0(y) = \int_{-\infty}^{y} (u_0 - \bar{u})(y')\mathrm{d}y'. \tag{1.39}$$

The initial data $[v_0, u_0]$ are assumed to satisfy that $[v_0 - \bar{v}, u_0 - \bar{u}] \in L^1 \cap H^1$ and $[\Phi_0, \Psi_0] \in L^2$. Notice that these assumptions imply that the initial perturbations are of zero integral:

$$\int_{-\infty}^{+\infty} (v_0 - \bar{v})(y)\mathrm{d}y = \int_{-\infty}^{+\infty} (u_0 - \bar{u})(y)\mathrm{d}y = 0. \tag{1.40}$$

Define the initial energy

$$E_0 \equiv \|[v_0 - \bar{v}, u_0 - \bar{u}]\|_{H^1}^2 + \|[\Phi_0, \Psi_0]\|_{L^2}^2,$$

the instant energy functional

$$\mathcal{E}(t) \equiv \|[\Phi, \Psi, \phi - \bar{\phi}](t)\|_{H^2}^2 \tag{1.41}$$

and the dissipation rate functional

$$\mathcal{D}(t) \equiv \|(s\bar{v}\bar{v}_y)^{\frac{1}{2}}\Psi(t)\|_{L^2}^2 + \|[\Phi_y, \phi_t](t)\|_{H^1}^2 + \|[\Psi_y, \phi - \bar{\phi}](t)\|_{H^2}^2. \tag{1.42}$$

The main result about the dynamical stability of shock profiles is stated as follows.

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Theorem 1.10. *Let $T > 0$. Assume (1.36) with $\tilde{\varepsilon}$ chosen to be further small enough. There exists a positive constant e_0 such that if $E_0 \le e_0$, the Cauchy problem (1.31) together with (1.32) and (1.33) admits a unique global-in-time solution $[v, u, \phi](t, y)$ satisfying*

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau)\mathrm{d}\tau \le C E_0, \tag{1.43}$$

for all $t \ge 0$. Moreover, the solution $[v, u, \phi]$ tends in large time to the shock profile in the following sense

$$\lim_{t\to\infty} \sup_{y\in\mathbb{R}} \left| \left[v(t, y) - \bar{v}(y), u(t, y) - \bar{u}(y), \phi(t, y) - \bar{\phi}(y) \right] \right| = 0. \tag{1.44}$$

Remark 1.11. The dynamical stability of the ion-acoustic shock profile in the case of $T = 0$ remains left, see (4.27) in the proof of Lemma 4.2.

Remark 1.12. Our stability result holds for any 1-D, small and zero excessive mass perturbations. However, a result of multi-dimensional stability is out of reach. The energy approach used in this paper relies heavily on the zero-mass condition (1.40), which can propagate due to the conservation laws. However, this is not the case for the multi-dimensional system. The excessive initial mass would cause a shift in the phase of shock profile, which is hard to be determined for general systems. We refer to [17,24,34] for deep discussions in this direction.

1.5. Literature

Now we review some related works on the problems considered in this paper. First, the shock structure problem is an important problem in both mathematical and physical community. The pioneer work on this problem can be traced back to 50's when Gilbarg [15] constructed a shock profile for Navier-Stokes system and established its limit behavior as both the viscosity and the heat conductivity vanish. Since then, there have been a huge amount of works on this problem for other physical models, cf. [1,2,5,6,10,12,33,48]. Our approach for the proof of Theorem 1.1 is based on the Center Manifold theory, which was first applied by Kopell-Howard [28] to construct the viscous shock profiles for the system with identity viscosity. Later on, it has been widely used in the study of shock profiles for more general hyperbolic balance laws. We refer to, for instance, Majda-Pego [38] for strictly parabolic systems, Majda-Ralston [39] for various of the numerical schemes, Yong-Zumbrun [49] for relaxation systems, and recently Liu-Yu [32] for Boltzmann equations. In the end we mention that a nice and detailed introduction to the history of the shock structure problem can be found in the book by Dafermos [9].

Concerning the stability of shock waves, it was first studied by Matsumura-Nishihara [41] for the isentropic Navier-Stokes equations and later extended by Kawashima-Matsumura [27] in the heat-conductive case. As for the shock waves of general systems of viscous conservation laws, Goodman [16] proved their stability by using a characteristic-weighted energy method. The proofs of the stability in [16,27,41] heavily rely on the zero-mass condition. If the initial perturbation has an non-zero integral, the stability of shock waves was proved by Szepessy-Xin [47] for systems of viscous conservation laws with strictly parabolic viscosity tensor and by Liu-Zeng [36] for compressible Navier-Stokes system. We also mention series of works [31,35,

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36,50] by Liu, Liu-Zeng and Yu on the Green's functions of the linearized equations around the shock waves. Based on the Green's functions, they proved the pointwise convergence to shock waves, provided that the amplitude of shock waves is suitably small. As for shock waves over a full range of physical parameters including the large amplitude shocks, Gardner-Zumbrun [11] carried out a spectral stability criteria, based on Evan function approach. We refer to [23,37,40, 51] for more applications of this approach.

Another interesting model is the Euler-Poisson system, which is the invicid system corresponding to (1.1). There are a huge number of literatures have been devoted to study this system. Among them, we would like to mention some works only related to the current work. Concerning the traveling waves of Euler-Poisson system, the first result we are aware of comes from [7], in which three different types of traveling waves, namely solitons, periodic solutions and shock waves with discontinuities were constructed. Moreover, the quasi-neutral limits of these traveling waves were investigated. The quasi-neutral limit in the whole spacial domain $\mathbb{R}^d$ was studied in Cordier-Grenier [8] by using pseudo-differential operator techniques. In the presence of physical boundary, the quasi-neutrality breaks down near the boundary and the plasma sheath forms. To our knowledge, the problem of plasma sheath was firstly formulated by Ha-Slemrond [20], in which the authors studied the dynamical behavior of sheath for planar spherically symmetric flows. By use of a weighted energy approach, Suzuki [46] proved the existence of stationary solutions related to sheath profile in half space and their stability. See also [43] for justification of Bohm criterion. In addition, there are works on the verification of boundary layer expansion for Euler-Poisson system, cf. [13]. In the end, we would like to mention some works on long-wavelength limit of the Euler-Poisson system. On this matter, we refer to [19] for KdV limit and [30] for Zakharov-Kuznetsov limit.

1.6. *Idea of the proof of main results*

We briefly state the ideas for the proof of Theorems 1.1, 1.4 and 1.10. As mentioned before, the proof of Theorem 1.1 is based on the center manifold theorem as in [38]. As for Theorem 1.10, the main efforts have been made to treat the extra effect of the self-consistent force on the energy estimates, compared to the case of the classical Navier-Stokes equations.

For the proof of Theorem 1.4, the key step is to obtain the uniform-in-ε estimates on the kth derivatives ($k \geq 2$) of the solution to the linearized remainder system (3.13). The difficulties come from the second order derivative term $\delta\phi''/\sqrt{T+1}$ on the right-hand side of the first equation of (3.13). Specifically, when estimating $\mathrm{d}^k n/\mathrm{d}z^k$, the trouble terms like

$$\left(\frac{\delta}{\sqrt{T+1}}\frac{\mathrm{d}^{k+1}\phi}{\mathrm{d}z^{k+1}}, w_\alpha^2\frac{\mathrm{d}^k n}{\mathrm{d}z^k}\right)$$

are hard to handle, due to the degeneracy of the Poisson equation when $\varepsilon \to 0$. To resolve them, we make an essential use of the structure of the Poisson equation. Indeed, our strategy is to use the Poisson equation to represent $\mathrm{d}^k n/\mathrm{d}z^k$ in terms of ϕ, which leads to a crucial cancellation in this inner product term. This strategy is also used in the later proof of Theorem 1.10. Unfortunately, for the dynamical stability problem, the principle part of the similar trouble term is involved in the energy functional $\mathcal{E}_1(t)$, see (4.25). And, the restriction $T > 0$ is essentially required to assure the positivity of $\mathcal{E}_1(t)$. This is the reason why the condition $T > 0$ is necessary in Theorem 1.10.

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1.7. Organization and notations

The rest of this paper is organized as follows. In Section 2, we shall prove Theorem 1.1 for the construction of the shock profile in terms of (1.4) for the fixed viscosity coefficient $\mu > 0$ and the fixed Debye length $\lambda > 0$. In Section 3, we shall prove Theorem 1.4 for the KdV-Burgers approximation to the shock profile. One key point there is to show Proposition 3.1 for the existence of solutions to the linear inhomogeneous problem, particularly to obtain the estimate (3.14). The smallness of $\delta = \bar{\lambda}^2/\bar{\mu}^2$ plays an essential role in the analysis. In Section 4, we shall prove Theorem 1.10 concerning the dynamical stability of the shock profile. In the Appendix, for completeness, we first list a lemma about the property of the KdV-Burgers shock profile, give the estimates on the error between $[n_1, u_1, \phi_1]$ and an approximation solution $[n_{1,\varepsilon}, u_{1,\varepsilon}, \phi_{1,\varepsilon}]$ defined in (3.1), write down the explicit formula of remainder r_2 and r_3 defined in (3.10), and in the end show Lemmas 4.4 and 4.5 related to the higher order energy estimates.

Notations. Throughout this paper, C denotes some generic positive (generally large) constant and c denotes some generic positive (generally small) constant, where both C and c may take different values in different places. $\|\cdot\|_{L^p}$ stands for the spacial L^p-norm ($1 \le p \le \infty$). Sometimes, we denote $(\cdot,\cdot)$ to be the inner product in L^2 for convenience. We also use H^k ($k \ge 0$) to denote the usual Sobolev space with respect to spacial variable.

2. Existence for shock profiles of small amplitude

In this section, we construct the shock profile solution

$$[\bar{n}, \bar{u}, \bar{\phi}](\xi)$$

to the system (1.1) for the fixed constant viscosity $\mu > 0$ and Debye length $\lambda > 0$. The starting point is to rewrite the Poisson equation as a first-order ODE system for $[\phi, \phi']$ so that the center manifold approach (cf. [28,38]) can be directly applied to treat the problem.

Proof of Theorem 1.1. Denote $Z = e^{\phi}$ and $W = \frac{d\phi}{d\xi}$. By solving u from the first equation in (1.4), we have $u = s(1 - n^{-1})$. Plugging it back into the second equation in (1.4) and then integrating the resultant equation from $-\infty$ to ξ, we obtain the following system

$$\begin{cases} \dfrac{dn}{d\xi} = \dfrac{n^2}{\mu s}\left[(T - s^2)(n-1) + \dfrac{s^2(n-1)^2}{n} - \dfrac{\lambda^2 W^2}{2} + Z - 1\right], \\ \dfrac{dZ}{d\xi} = ZW, \\ \dfrac{dW}{d\xi} = -\lambda^{-2}(n - Z), \end{cases} \tag{2.1}$$

with the far fields given by

$$\lim_{\xi \to \pm\infty} [n(\xi), Z(\xi), W(\xi)] = [n_\pm, Z_\pm, W_\pm] = [n_\pm, n_\pm, 0].$$

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For convenience, we introduce $U = [n, Z, W]$, $U_\pm = [n_\pm, Z_\pm, W_\pm]$ and write (2.1) in the form $\dot{U} = F(U)$, where $F(\cdot)$ denotes the vector field on the right hand of (2.1). Borrowing the idea from [38], we introduce the following extended ODE system

$$\begin{cases} \dot{n} = \dfrac{n^2}{\mu\tau}\left[(T-\tau^2)(n-1) + \dfrac{\tau^2(n-1)^2}{n} - \dfrac{\lambda^2 W^2}{2} + Z - 1\right], \\ \dot{Z} = ZW, \\ \dot{W} = -\lambda^{-2}(n-Z), \\ \dot{\tau} = 0. \end{cases} \tag{2.2}$$

One can see that $[n_\pm, Z_\pm, W_\pm, s]$ are the only two critical points of (2.2). Now we fix $[n_-, Z_-, W_-, \sqrt{T+1}]$ as a reference state and construct the center manifold of (2.2) around this reference state. To do so, we calculate the Jacobian of (2.2) at the critical point $[n_-, Z_-, W_-, \sqrt{T+1}]$ as

$$J = \begin{pmatrix} -\frac{1}{\mu\sqrt{T+1}} & \frac{1}{\mu\sqrt{T+1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\lambda^2} & \frac{1}{\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of J are given by

$$\begin{aligned} \sigma_1 &= -\frac{1}{2\mu\sqrt{T+1}} - \sqrt{\frac{1}{4\mu^2(T+1)} + \frac{1}{\lambda^2}} < 0, \\ \sigma_2 &= \sigma_3 = 0, \\ \sigma_4 &= -\frac{1}{2\mu\sqrt{T+1}} + \sqrt{\frac{1}{4\mu^2(T+1)} + \frac{1}{\lambda^2}} > 0. \end{aligned}$$

One has two eigenvectors associated with the zero eigenvalue:

$$\mathcal{R}_1 = (1, 1, 0, 0), \quad \mathcal{R}_2 = (0, 0, 0, 1).$$

Then using the *Centre Manifold Theorem* (e.g. Proposition 3.2 in [38]), we have the following 2-d manifold M^* which is invariant by the flow (2.2):

$$\begin{cases} U = U(\eta, \tau) = U_- + \eta[1, 1, 0] + \mathcal{H}(\eta, \tau), \\ \tau = \tau, \end{cases}$$

with $|\eta| + |\tau - \sqrt{T+1}| \le c$ for some constant $c > 0$. Here

$$\mathcal{H}(\eta, \tau) = [\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3](\eta, \tau)$$

is a higher order term satisfying

$$\mathcal{H}(0, \sqrt{T+1}) = \mathcal{H}_\eta(0, \sqrt{T+1}) = \mathcal{H}_\tau(0, \sqrt{T+1}) = 0. \tag{2.3}$$

If $|n_+ - 1|$ is small enough, the equilibria $[U_-, s]$ and $[U_+, s]$ are located in M^*. Therefore, there exists $\eta_+ < 0$ such that $U_+ = U_- + \eta_+[1, 1, 0] + \mathcal{H}(\eta_+, s)$. Here the sign of η_+ follows from the entropy condition (1.6). Pick up the following curve

$$U(\eta, \tau)|_{\tau=s} = [n(\eta), Z(\eta), W(\eta)] := U_- + \eta[1, 1, 0] + \mathcal{H}(\eta, s), \tag{2.4}$$

for $\eta \in [\eta_+, 0]$. It is straightforward to check that $U(\eta, s)$ is invariant by the flow (2.1), so that $\mathrm{d}U(\eta, s)/\mathrm{d}\eta$, the direction field of $U(\eta, s)$, is parallel to $F(U(\eta, s))$. Therefore, (2.1) admits a solution if and only if the following ODE

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = g(\eta), -\infty < \xi < +\infty, \\ \lim\limits_{\xi \to -\infty} \eta(\xi) = 0, \lim\limits_{\xi \to +\infty} \eta(\xi) = \eta_+, \end{cases} \tag{2.5}$$

induces a well-defined trajectory $\eta = \eta(\xi)$ for all $\xi \in \mathbb{R}$, where $g(\eta)$ is determined by

$$g(\eta) U_\eta(\eta, s) = F(U(\eta, s)). \tag{2.6}$$

In what follows we only focus on the existence of the solution to (2.5). Note that since $\eta_+ < 0$, the standard ODE theory shows that (2.5) has a smooth solution if and only if $g(\eta) < 0$ for all $\eta \in (\eta_+, 0)$. Since U_+ is the unique state near U_- such that $F(U_+) = 0$, it follows from (2.6) that $g(\eta)$ vanishes only at two end points of $[\eta_+, 0]$. To further show that $g(\eta)$ has the strictly negative sign in the open interval $(\eta_+, 0)$, we calculate $\dot{g}(0)$ by differentiating (2.6) and then taking the inner product of the resulting equation with $U_\eta(0, s)$ as

$$\begin{aligned} \dot{g}(0) &= \frac{U_\eta(0, s) dF(U_-) U_\eta(0, s)^T}{|U_\eta(0, s)|^2} \\ &= \frac{T+1}{2\mu s}(1 - n_+) + O\left(H_\eta(0, s)\right) \\ &= \frac{\sqrt{T+1}}{2\mu}(1 - n_+) + O\left((1 - n_+)^2\right) > 0, \end{aligned} \tag{2.7}$$

provided that $|n_+ - 1|$ small enough. Here (2.3) has been used in the last equality in (2.7). Therefore, it holds that $g(\eta) < 0$ in $(\eta_+, 0)$, so that (2.5) admits a smooth solution $\eta(\xi)$ for $\xi \in \mathbb{R}$. Let

$$[\bar{n}, \bar{u}, \bar{\phi}](\xi) := [n, s(1 - n^{-1}), \log Z]\,(\eta(\xi))\,,$$

where $n(\cdot)$ and $Z(\cdot)$ are defined in (2.4). It is straightforward to check that $[\bar{n}, \bar{u}, \bar{\phi}]$ solves (1.3) together with (1.2). The monotonicity (1.7) follows from the following computations:

$$
\begin{cases}
\dfrac{\mathrm{d}\bar{n}}{\mathrm{d}\xi} = \dot{n}(\eta(\xi))\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = \Big(1 + H^1_\eta(\eta(\xi), s)\Big)\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} \sim \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} < 0, \\
\dfrac{\mathrm{d}\bar{u}}{\mathrm{d}\xi} = \dfrac{s}{\bar{n}^2}\dfrac{\mathrm{d}\bar{n}}{\mathrm{d}\xi} < 0, \\
\dfrac{\mathrm{d}\bar{\phi}}{\mathrm{d}\xi} = Z^{-1}\dot{Z}(\eta(\xi))\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = Z^{-1}\{1 + H^2_\eta(\eta(\xi), s)\}\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} \sim \dfrac{\mathrm{d}\bar{n}}{\mathrm{d}\xi} < 0,
\end{cases}
$$

provided that $|n_+ - 1|$ is sufficiently small. Furthermore, to verify (1.8) for $k = 0$, it follows from (2.7) that

$$
|\eta(\xi)| \le C|\eta_+|e^{-\theta(1-n_+)|\xi|} \le C(1 - n_+)e^{-\theta(1-n_+)|\xi|}, \quad \xi < 0,
$$

with some constants $C > 0$ and $\theta > 0$ independent of ξ. Then it follows from (2.4) that

$$
\left|[\bar{n} - 1, \bar{u}, \bar{\phi}](\xi)\right| \le C(1 - n_+)e^{-\theta(1-n_+)|\xi|}, \quad \xi < 0.
$$

Similarly, one also has

$$
\left|[\bar{n} - n_+, \bar{u} - u_+, \bar{\phi} - \phi_+](\xi)\right| \le C(1 - n_+)e^{-\theta(1-n_+)|\xi|}, \quad \xi > 0.
$$

This then proves (1.8) for $k = 0$. Estimates on (1.8) with $k \ge 1$ for those high-order derivatives of $[\bar{n}, \bar{u}, \bar{\phi}]$ can be similarly obtained by differentiating (1.3), and the details of the proof are omitted for brevity. It is also straightforward to show the uniqueness (up to a shift) for the ODE system (1.3) together with (1.2). Therefore, we complete the proof of Theorem 1.1. □

3. KdV-Burgers approximation to shock profiles

In this section we shall prove Theorem 1.4 concerning the KdV-Burgers approximation of the smooth small-amplitude traveling shock profile under the scaling (1.9) provided that $\delta > 0$ given by (1.10) is small enough. We will first construct a suitable approximation solution and then establish some uniform estimates of the remainder.

3.1. Construction of an approximate solution

We start from the rescaled system (1.11), where we have chosen s_ε as in (1.14), the upstream equilibrium $[n_{\varepsilon,-}, u_{\varepsilon,-}, \phi_{\varepsilon,-}] = [1, 0, 0]$, and the downstream equilibrium $[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}]$ as in (1.15). Note that by comparing the far fields $n_{1,+}$, $u_{1,+}$, and $\phi_{1,+}$ of KdV-Burgers equations respectively given in (1.25), (1.26) and (1.27) to $[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}]$, one has

$$
\frac{1}{\varepsilon^2}\left\{[n_{\varepsilon,+}, u_{\varepsilon,+}, \phi_{\varepsilon,+}] - [1, 0, 0] - \varepsilon[n_{1,+}, u_{1,+}, \phi_{1,+}]\right\} = [\frac{1}{T+1}, -\frac{1}{\sqrt{T+1} - \varepsilon}, \frac{a_\varepsilon}{\varepsilon}]
$$

with

$$
\frac{a_\varepsilon}{\varepsilon} = -\frac{1}{T+1} - O(1)\varepsilon.
$$

Therefore, one can see that it may not be a good ansatz to directly take $[1,0,0]+\varepsilon[n_1,u_1,\phi_1]$ as the approximation of $[n_\varepsilon,u_\varepsilon,\phi_\varepsilon]$ up to the first order for making the energy estimates on remainders in L^2 setting, because their far-field data cannot be matched. To overcome this trouble, we introduce the modified first-order approximation $[n_{1,\varepsilon},u_{1,\varepsilon},\phi_{1,\varepsilon}]$ satisfying

$$\begin{cases} (2\sqrt{T+1}-\varepsilon)n'_{1,\varepsilon}+2(T+1)n_{1,\varepsilon}n'_{1,\varepsilon}-\sqrt{T+1}n''_{1,\varepsilon}+\delta n'''_{1,\varepsilon}=0, \\ \left(2+\dfrac{\varepsilon}{\sqrt{T+1}-\varepsilon}\right)u'_{1,\varepsilon}+2u_{1,\varepsilon}u'_{1,\varepsilon}-u''_{1,\varepsilon}+\dfrac{\delta}{\sqrt{T+1}}u'''_{1,\varepsilon}=0, \\ \left(2\sqrt{T+1}-a_\varepsilon(T+1)\right)\phi'_{1,\varepsilon}+2(T+1)\phi_{1,\varepsilon}\phi'_{1,\varepsilon}-\sqrt{T+1}\phi''_{1,\varepsilon}+\delta\phi'''_{1,\varepsilon}=0, \end{cases} \tag{3.1}$$

with the far-field data:

$$\lim_{z\to-\infty}[n_{1,\varepsilon},u_{1,\varepsilon},\phi_{1,\varepsilon}](z)=[0,0,0], \tag{3.2}$$

and

$$\begin{aligned} \lim_{z\to\infty}[n_{1,\varepsilon},u_{1,\varepsilon},\phi_{1,\varepsilon}](z) &= \frac{1}{\varepsilon}\{[n_{\varepsilon,+},u_{\varepsilon,+},\phi_{\varepsilon,+}]-[1,0,0]\} \\ &= [-\frac{2}{\sqrt{T+1}}+\frac{\varepsilon}{T+1},-2-\frac{\varepsilon}{\sqrt{T+1}-\varepsilon},-\frac{2}{\sqrt{T+1}}+a_\varepsilon], \end{aligned} \tag{3.3}$$

in terms of (1.15). Note that compared to (1.24), (1.26) and (1.27), we have modified the coefficients of $n'_{1,\varepsilon}$, $u'_{1,\varepsilon}$ and $\phi'_{1,\varepsilon}$ in (3.1), respectively, according to the far-field conditions (3.2) and (3.3). Moreover, since the shock profile is invariant under a spatial shift, we further set

$$n_{1,\varepsilon}(0)=n_1(0),\ u_{1,\varepsilon}(0)=u_1(0),\ \phi_{1,\varepsilon}(0)=\phi_1(0)$$

without loss of generality.

We seek for the shock profile solution in the form:

$$\begin{cases} n_\varepsilon=1+\varepsilon n_{1,\varepsilon}+\varepsilon^2 n_R=1+\varepsilon n_1+\varepsilon^2(n_2+n_R), \\ u_\varepsilon=\varepsilon u_{1,\varepsilon}+\varepsilon^2 u_R=\varepsilon u_1+\varepsilon^2(u_2+u_R), \\ \phi_\varepsilon=\varepsilon\phi_{1,\varepsilon}+\varepsilon^2\phi_R=\varepsilon\phi_1+\varepsilon^2(\phi_2+\phi_R), \end{cases} \tag{3.4}$$

where $[n_2,u_2,\phi_2]$ is defined by

$$[n_2,u_2,\phi_2]:=\varepsilon^{-1}[n_{1,\varepsilon}-n_1,u_{1,\varepsilon}-u_1,\phi_{1,\varepsilon}-\phi_1].$$

Note that $[n_2,u_2,\phi_2]$ is $O(1)$ in terms of Lemma 5.2 in the Appendix, and

$$\lim_{z\to\pm\infty}[n_R,u_R,\phi_R]=[0,0,0].$$

The key point is to establish uniform-in-ε estimates of the remainder $[n_R, u_R, \phi_R]$. For this purpose, we first derive the equation for $[n_R, u_R, \phi_R]$ from (1.11) as follows. In fact, integrating the first equation of (1.11) from $-\infty$ to z yields that

$$u_\varepsilon = \frac{s_\varepsilon(n_\varepsilon - 1)}{n_\varepsilon}. \tag{3.5}$$

Then from (3.5), one can solve u_R in terms of n_R as

$$u_R = n_\varepsilon^{-1}(s_\varepsilon - \varepsilon u_{1,\varepsilon})n_R + \varepsilon^{-1} n_\varepsilon^{-1}(s_\varepsilon n_{1,\varepsilon} - u_{1,\varepsilon} - \varepsilon n_{1,\varepsilon} u_{1,\varepsilon}). \tag{3.6}$$

From (1.22), we notice that the last term on the right hand side of (3.6)

$$\begin{aligned}&\varepsilon^{-1} n_\varepsilon^{-1}(s_\varepsilon n_{1,\varepsilon} - u_{1,\varepsilon} - \varepsilon n_{1,\varepsilon} u_{1,\varepsilon})\\ &\quad = \varepsilon^{-1} n_\varepsilon^{-1}\left(s_\varepsilon(n_{1,\varepsilon} - n_1) - (u_{1,\varepsilon} - u_1) - \varepsilon(n_1 + n_{1,\varepsilon} u_{1,\varepsilon})\right) = O(1),\end{aligned}$$

in terms of Lemma 5.2 in the Appendix. Similar for obtaining (1.4), $[n_\varepsilon, u_\varepsilon, \phi_\varepsilon]$ also satisfies the following system with two conservation laws:

$$\begin{cases} -s_\varepsilon n_\varepsilon' + (n_\varepsilon u_\varepsilon)' = 0, \\ -s_\varepsilon(n_\varepsilon u_\varepsilon)' + (n_\varepsilon u_\varepsilon^2 + T n_\varepsilon)' = \varepsilon u_\varepsilon'' + \left(\frac{1}{2}\varepsilon\delta(\phi_\varepsilon')^2 - e^{\phi_\varepsilon}\right)', \\ -\varepsilon\delta\phi_\varepsilon'' = n_\varepsilon - e^{\phi_\varepsilon}. \end{cases} \tag{3.7}$$

Substituting (3.5) into the second equation of (3.7) and integrating the resultant equation, one has

$$\varepsilon s_\varepsilon n_\varepsilon^{-2} n_\varepsilon' = (T + 1 - s_\varepsilon^2)(n_\varepsilon - 1) + s_\varepsilon^2 n_\varepsilon^{-1}(n_\varepsilon - 1)^2 - \frac{1}{2}\varepsilon\delta(\phi_\varepsilon')^2 + \varepsilon\delta\phi_\varepsilon''. \tag{3.8}$$

Here we have replaced e^{ϕ_ε} by $n_\varepsilon + \varepsilon\delta\phi_\varepsilon''$. Plugging (3.4) into equation (3.8) and the third equation of (3.7) simultaneously, one has the following system for $[n_R, \phi_R]$:

$$\begin{cases} n_R' = 2\left(1 + \sqrt{T+1}\, n_1(z)\right) n_R + \dfrac{\delta}{\sqrt{T+1}}\phi_R'' + r_1 + r_2 + r_3, \\ -\varepsilon\delta\phi_R'' = n_R - \phi_R + r_4 + r_5 + r_6, \end{cases} \tag{3.9}$$

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where the inhomogeneous terms r_i $(1 \le i \le 6)$ are given by

$$\begin{cases} r_1 := (\dfrac{1}{\sqrt{T+1}} + n_{1,\varepsilon})n'_{1,\varepsilon} + \dfrac{\delta(1+\varepsilon n_{1,\varepsilon})}{\varepsilon\sqrt{T+1}}\{(1+\varepsilon n_{1,\varepsilon})\phi''_{1,\varepsilon} - n''_{1,\varepsilon}\} \\ \qquad - \dfrac{\delta(1+\varepsilon n_{1,\varepsilon})^2}{2\sqrt{T+1}}(\phi'_{1,\varepsilon})^2, \\ r_2 = r_2[n_R, \phi_R] = O(\varepsilon)\{|n_R| + \delta|\phi''_R| + |\phi'_R| + |n'_R|\}, \\ r_3 = r_3[n_R, \phi_R] = O(\varepsilon)|n_R|^2 + O(\varepsilon^2)\Big\{|n_R|^2 \cdot \big[|n_R| + |\phi'_R| + \delta|\phi''_R|\big] \\ \qquad\qquad + |n_R| \cdot \big[|\phi'_R| + \delta|\phi''_R|\big] + |\phi'_R|^2 \cdot \big[1 + |n_R|^2\big]\Big\}, \\ r_4 = \varepsilon^{-2}(1 + \varepsilon n_{1,\varepsilon} - e^{\varepsilon\phi_{1,\varepsilon}}) = O(1), \\ r_5 = r_5[\phi_R] := (1 - e^{\varepsilon\phi_{1,\varepsilon}})\phi_R = O(\varepsilon)|\phi_R|, \\ r_6 = r_6[\phi_R] := \varepsilon^{-2}e^{\varepsilon\phi_{1,\varepsilon}}(1 - e^{\varepsilon^2\phi_R} + \varepsilon^2\phi_R) = O(\varepsilon^2)|\phi_R|^2. \end{cases} \tag{3.10}$$

For brevity of presentation, we put the explicit formulas of r_2 and r_3 into the Appendix, see (5.6) and (5.7) respectively. Moreover, from the phase condition (1.29), we have

$$n_R(0) = 0. \tag{3.11}$$

To prove Theorem 1.4, it suffices to study the existence of solutions to the ODE system (3.9) together with (3.11) for the remainders n_R and ϕ_R. Before doing that, we first define some function spaces which will be used later. Define a weight function

$$w_\alpha = w_\alpha(z) = \exp\left\{\alpha\sqrt{1+|z|^2}\right\},$$

for $\alpha > 0$, and for an integer $k \ge 0$ define the weighted Sobolev space

$$H^k_\alpha = \left\{ f = f(z) \in H^k \,\middle|\, w_\alpha \frac{\mathrm{d}^i f}{\mathrm{d}z^i} \in L^2,\ 0 \le i \le k \right\},$$

associated with the norm

$$\|f\|_{H^k_\alpha} = \left\{ \sum_{i=0}^{k} \left\| w_\alpha \frac{\mathrm{d}^i f}{\mathrm{d}z^i} \right\|^2_{L^2} \right\}^{1/2}.$$

For an integer $k \ge 2$ and $0 < \alpha < 2$, we also define the following solution space for the remainder equations (3.9):

$$\mathbf{X}_{\alpha,k} = \left\{ U(z) = [n(z), \phi(z)] \,\middle|\, n(0) = 0, \|n\|_{H^k_\alpha} + \|\phi\|_{H^{k+2}_\alpha} < \infty \right\} \tag{3.12}$$

with the norm

$$\|U\|_{\mathbf{X}_{\alpha,k}} \doteq \|[n,\phi]\|_{H^k_\alpha} + \sqrt{\varepsilon\delta}\left\|\frac{\mathrm{d}^{k+1}\phi}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha} + \varepsilon\delta\left\|\frac{\mathrm{d}^{k+2}\phi}{\mathrm{d}z^{k+2}}\right\|_{L^2_\alpha}.$$

3.2. Linear problem

First of all, we start from the following linear inhomogeneous problem

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}z} = A(z)n + \dfrac{\delta}{\sqrt{T+1}}\dfrac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + h_1, \\ -\varepsilon\delta\dfrac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = n - \phi + h_2, \\ n(0) = 0, \end{cases} \tag{3.13}$$

where $A(z) := 2[1 + \sqrt{T+1}n_1(z)]$. Recall the solution space $\mathbf{X}_{\alpha,k}$ in (3.12). The following result is concerned with the solvability and estimates of (3.13), which is a crucial step for further treating (3.9).

Proposition 3.1. *Let $k \ge 2$ be an arbitrary integer and $0 < \alpha < 2$. There exist positive constants ε_1 and δ_1 such that if $0 < \varepsilon \le \varepsilon_1$, $0 < \delta \le \delta_1$, and*

$$\|h_1\|_{H^{k-1}_\alpha} + \|h_2\|_{H^{k+1}_\alpha} < \infty,$$

then the linear ODE system (3.13) has a unique solution $U(z) = [n(z), \phi(z)]$ in $\mathbf{X}_{\alpha,k}$ satisfying the following estimate:

$$\|U\|_{\mathbf{X}_{\alpha,k}} \le C\|h_1\|_{H^{k-1}_\alpha} + C\|h_2\|_{H^k_\alpha} + C\delta\left\|\frac{\mathrm{d}^{k+1}h_2}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha}, \tag{3.14}$$

where $C > 0$ is a generic constant independent of ε and δ.

Proof. We divide the proof into three steps.

Step 1. In this step, we treat only the a priori estimates of solutions for the case $k = 2$, that is to prove that any smooth solution $U(z) = [n(z), \phi(z)]$ to the system (3.13) enjoys the estimate (3.14) with $k = 2$. First of all, we estimate n as follows. From the first equation of (3.13), we can represent n as

$$n(z) = \int_0^z e^{\int_{z'}^z A(\tau)\mathrm{d}\tau}\left[\frac{\delta\phi''}{\sqrt{T+1}} + h_1\right](z')\mathrm{d}z'. \tag{3.15}$$

It is straightforward to check that

$$\lim_{z\to+\infty} A(z) = -2 < 0, \qquad \lim_{z\to-\infty} A(z) = 2 > 0.$$

Then from (3.15), we have

$$\|n\|_{L^2_\alpha} \le C\delta\|\phi''\|_{L^2_\alpha} + C\|h_1\|_{L^2_\alpha}, \tag{3.16}$$

for $\alpha \in (0,2)$. Here we emphasize that the constant $C > 0$ is independent of ε and δ. Then, again from the first equation of (3.13), one has

$$\|n'\|_{L^2_\alpha} \le |A|_{L^\infty}\|n\|_{L^2_\alpha} + C\delta\|\phi''\|_{L^2_\alpha} + C\|h_1\|_{L^2_\alpha} \le C\delta\|\phi''\|_{L^2_\alpha} + C\|h_1\|_{L^2_\alpha}. \tag{3.17}$$

Next, we turn to estimate ϕ. Taking the inner product of the second equation of (3.13) with $w_\alpha^2\phi$, one has

$$\|\phi\|^2_{L^2_\alpha} = (\varepsilon\delta\phi'', w_\alpha^2\phi) + (n + h_2, w_\alpha^2\phi).$$

By Cauchy-Schwarz, the second inner product term is bounded as

$$|(n + h_2, w_\alpha^2\phi)| \le \eta\|\phi\|^2_{L^2_\alpha} + C_\eta\{\|n\|^2_{L^2_\alpha} + \|h_2\|^2_{L^2_\alpha}\},$$

with an arbitrary constant $0 < \eta < 1$ to be chosen later. As for the first inner product term, it holds from integration by parts and Cauchy-Schwarz that

$$\begin{aligned}(\varepsilon\delta\phi'', w_\alpha^2\phi) &= -\varepsilon\delta\|\phi'\|^2_{L^2_\alpha} + (\varepsilon\delta\phi', -2w_\alpha w_\alpha'\phi)\\ &\le -\varepsilon\delta\|\phi'\|^2_{L^2_\alpha} + \eta\varepsilon\delta\|\phi'\|^2_{L^2_\alpha} + C_\eta\varepsilon\delta\|\phi\|^2_{L^2_\alpha}.\end{aligned}$$

Therefore, by taking $\eta > 0$ suitably small, one has

$$\|\phi\|^2_{L^2_\alpha} + \varepsilon\delta\|\phi'\|^2_{L^2_\alpha} \le C\|n\|^2_{L^2_\alpha} + C\varepsilon\delta\|\phi\|^2_{L^2_\alpha} + C\|h_2\|^2_{L^2_\alpha}. \tag{3.18}$$

Similarly, taking the inner product of the second equation of (3.13) with $w_\alpha^2\phi''$ and integrating by parts, one has

$$\|\phi'\|^2_{L^2_\alpha} + \varepsilon\delta\|\phi''\|^2_{L^2_\alpha} \le C\{\|\phi\|^2_{L^2_\alpha} + \|n\|^2_{H^1_\alpha} + \|h_2\|^2_{H^1_\alpha}\}. \tag{3.19}$$

In what follows it is necessary to get the uniform-in-ε estimate for $\|[n'', \phi'']\|_{L^2_\alpha}$. Differentiating the first equation of (3.13) with respect to z and taking the inner product of the resultant equation with $w_\alpha^2 n''$, one has

$$\|n''\|^2_{L^2_\alpha} = \left(A'n + An' + h_1', w_\alpha^2 n''\right) + \delta\sqrt{T+1}^{-1}(\phi''', w_\alpha^2 n''). \tag{3.20}$$

By Cauchy-Schwarz, the first inner product term is bounded by

$$\eta\|n''\|^2_{L^2_\alpha} + C_\eta\{\|n\|^2_{H^1_\alpha} + \|h_1'\|_{L^2_\alpha}\}$$

with an arbitrary constant $0 < \eta < 1$ to be chosen later. To estimate the last term on the right-hand side of (3.20), we firstly differentiate the second equation of (3.13) twice and then solve n'' as

$$n''(z) = \phi''(z) - \varepsilon\delta\phi''''(z) - h_2''(z). \tag{3.21}$$

Thus, applying (3.21), one has

$$(\delta\phi''', w_\alpha^2 n'') = (\delta\phi''', w_\alpha^2\phi'') + (\delta\phi''', -\varepsilon\delta w_\alpha^2\phi'''') + (\delta\phi''', -w_\alpha^2 h_2''). \tag{3.22}$$

By integration by parts, the first term on the right is bounded as

$$|(\delta\phi''', w_\alpha^2\phi'')| = |(\delta\phi'', w_\alpha w_\alpha'\phi'')| \le C\delta\|\phi''\|_{L_\alpha^2}^2,$$

the second term is bounded as

$$|(\delta\phi''', -\varepsilon\delta w_\alpha^2\phi'''')| = \varepsilon\delta^2|(\phi''', w_\alpha w_\alpha'\phi''')| \le C\varepsilon\delta^2\|\phi'''\|_{L_\alpha^2}^2,$$

and the last term is bounded as

$$|(\delta\phi''', -w_\alpha^2 h_2'')| = |(\delta\phi'', w_\alpha\{w_\alpha h_2''' + 2w_\alpha' h_2''\})| \le \eta\|\phi''\|_{L_\alpha^2}^2 + C_\eta\delta^2\|h_2''\|_{H_\alpha^1}^2.$$

This completes all estimates on the right-hand side of (3.22). Plugging those estimates back to (3.20), one has

$$\|n''\|_{L_\alpha^2} \le C\{\|n\|_{H_\alpha^1} + (\sqrt{\delta} + \sqrt{\eta})\|\phi''\|_{L_\alpha^2} + \sqrt{\varepsilon}\delta\|\phi'''\|_{L_\alpha^2}\} + C\|h_1'\|_{L_\alpha^2} + C_\eta\delta\|h_2''\|_{H_\alpha^1}. \tag{3.23}$$

Here the constant $0 < \eta < 1$ can be chosen small enough. For the estimate of $\|\phi''\|_{L_\alpha^2}$, we take the inner product of (3.21) with $w_\alpha^2\phi''$, which yields that

$$\|\phi''\|_{L_\alpha^2}^2 = (\varepsilon\delta\phi'''', w_\alpha^2\phi'') + (h_2'', w_\alpha^2\phi'') + (n'', w_\alpha^2\phi''). \tag{3.24}$$

From integration by parts again, one has

$$\begin{aligned}(\varepsilon\delta\phi'''', w_\alpha^2\phi'') &= -\varepsilon\delta\|\phi'''\|_{L_\alpha^2}^2 + (-\varepsilon\delta\phi''', 2w_\alpha w_\alpha'\phi'')\\ &\le -\varepsilon\delta\|\phi'''\|_{L_\alpha^2}^2 + \eta\|\phi''\|_{L_\alpha^2}^2 + C_\eta\varepsilon^2\delta^2\|\phi'''\|_{L_\alpha^2}^2.\end{aligned}$$

Moreover, by Cauchy-Schwarz, the last two terms on the right-hand side of (3.24) are bounded by

$$\eta\|\phi''\|_{L_\alpha^2}^2 + C_\eta\{\|n''\|_{L_\alpha^2} + \|h_2''\|_{L_\alpha^2}^2\}.$$

Therefore, by collecting all estimates and taking $0<\eta<1$ suitably small, we have, from (3.24) that

$$\|\phi''\|_{L^2_\alpha}+\sqrt{\varepsilon\delta}\|\phi'''\|_{L^2_\alpha}\le C\varepsilon\delta\|\phi'''\|_{L^2_\alpha}+C\{\|n''\|_{L^2_\alpha}+\|h_2''\|_{L^2_\alpha}\}. \tag{3.25}$$

Furthermore, taking the inner product of (3.21) with $\varepsilon\delta\phi''''w_\alpha^2$, one has

$$\varepsilon\delta\|\phi''''\|_{L^2_\alpha}\le C\{\|\phi''\|_{L^2_\alpha}+\|n''\|_{L^2_\alpha}+\|h_2''\|_{L^2_\alpha}\}. \tag{3.26}$$

Finally, a suitable linear combination of (3.16), (3.17), (3.18), (3.19), (3.23), (3.25) and (3.26) yields that

$$\begin{aligned}
&\|[n,\phi]\|_{H^2_\alpha}+\sqrt{\varepsilon\delta}\|\phi'''\|_{L^2_\alpha}+\varepsilon\delta\|\phi''''\|_{L^2_\alpha}\\
&\le C(\delta+\sqrt{\delta})\|\phi''\|_{L^2_\alpha}+C\sqrt{\varepsilon\delta}(\sqrt{\delta}+\sqrt{\varepsilon\delta})\|\phi'''\|_{L^2_\alpha}\\
&\quad+C\{\|h_1\|_{H^1_\alpha}+\|h_2\|_{H^2_\alpha}+\delta\|h_2'''\|_{L^2_\alpha}\}.
\end{aligned} \tag{3.27}$$

Therefore, (3.14) with $k=2$ follows from (3.27) by taking $\delta>0$ and $\varepsilon>0$ suitably small.

Step 2. In this step, we use the induction argument to show that the estimates (3.14) is valid for any $k\ge 2$. Notice that (3.14) for $k=2$ has been proved in Step 1. Assume that this is valid for $k\ge 2$. Differentiating the first equation of (3.13) k-times with respect to z yields

$$\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}=\sum_{0\le k'\le k}\binom{k'}{k}\frac{\mathrm{d}^{k'}A}{\mathrm{d}z^{k'}}\cdot\frac{\mathrm{d}^{k-k'}n}{\mathrm{d}z^{k-k'}}+\frac{\delta}{\sqrt{T+1}}\frac{\mathrm{d}^{k+2}\phi}{\mathrm{d}z^{k+2}}+\frac{\mathrm{d}^k h_1}{\mathrm{d}z^k}. \tag{3.28}$$

Taking the inner product of (3.28) with $w_\alpha^2\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}$, we have

$$\begin{aligned}
\left\|\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\right\|^2_{L^2_\alpha}&=\sum_{0\le k'\le k}\binom{k'}{k}\left(\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\,,\,w_\alpha^2\frac{\mathrm{d}^{k'}A}{\mathrm{d}z^{k'}}\cdot\frac{\mathrm{d}^{k-k'}n}{\mathrm{d}z^{k-k'}}\right)+\left(\frac{\mathrm{d}^k h_1}{\mathrm{d}z^k}\,,\,w_\alpha^2\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\right)\\
&\quad+\left(\frac{\delta}{\sqrt{T+1}}\frac{\mathrm{d}^{k+2}\phi}{\mathrm{d}z^{k+2}}\,,\,w_\alpha^2\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\right):=J_1+J_2+J_3.
\end{aligned} \tag{3.29}$$

Using (5.2), the first inner product term on the right is bounded as

$$\begin{aligned}
|J_1|&\le C\sum_{0\le k'\le k}\binom{k'}{k}\left|\frac{\mathrm{d}^{k'}A}{\mathrm{d}z^{k'}}\right|_{L^\infty}\cdot\left\|\frac{\mathrm{d}^{k-k'}n}{\mathrm{d}z^{k-k'}}\right\|_{L^2_\alpha}\cdot\left\|\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha}\\
&\le\eta\left\|\frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}}\right\|^2_{L^2_\alpha}+C_{\eta,k}\|n\|^2_{H^k_\alpha}.
\end{aligned}$$

Here the positive constant $\eta > 0$ can be chosen to be arbitrarily small. By Cauchy-Schwarz, the second inner product is bounded as

$$|J_2| \le \eta \left\| \frac{\mathrm{d}^{k+1} n}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha}^2 + C_\eta \left\| \frac{\mathrm{d}^k h_1}{\mathrm{d}z^k} \right\|_{L^2_\alpha}^2 .$$

To further estimate J_3, in the similar way as before, we differentiate the second equation of (3.13) $k+1$ times and represent $\frac{\mathrm{d}^{k+1} n}{\mathrm{d}z^{k+1}}$ as

$$\frac{\mathrm{d}^{k+1} n}{\mathrm{d}z^{k+1}} = \frac{\mathrm{d}^{k+1} \phi}{\mathrm{d}z^{k+1}} - \varepsilon\delta \frac{\mathrm{d}^{k+3} \phi}{\mathrm{d}z^{k+3}} - \frac{\mathrm{d}^{k+1} h_2}{\mathrm{d}z^{k+1}}. \tag{3.30}$$

Substituting (3.30) into J_3, we have

$$\begin{aligned} J_3 =& \frac{\delta}{\sqrt{T+1}} \left(\frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \, , \, w_\alpha^2 \frac{\mathrm{d}^{k+1} \phi}{\mathrm{d}z^{k+1}} \right) + \frac{\varepsilon \delta^2}{\sqrt{T+1}} \left(- \frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \, , \, w_\alpha^2 \frac{\mathrm{d}^{k+3} \phi}{\mathrm{d}z^{k+3}} \right) \\ &+ \frac{\delta}{\sqrt{T+1}} \left(- \frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \, , \, w_\alpha^2 \frac{\mathrm{d}^{k+1} h_2}{\mathrm{d}z^{k+1}} \right). \end{aligned} \tag{3.31}$$

Then by integration by parts, it follows from (3.31) that

$$\begin{aligned} |J_3| \le& C(\delta + \eta) \left\| \frac{\mathrm{d}^{k+1} \phi}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha}^2 + C\varepsilon\delta^2 \left\| \frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha}^2 \\ &+ C_\eta \delta^2 \left(\left\| \frac{\mathrm{d}^{k+1} h_2}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha}^2 + \left\| \frac{\mathrm{d}^{k+2} h_2}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha}^2 \right). \end{aligned}$$

Substituting estimates of J_1 to J_3 into (3.29), we have, for any small $\eta > 0$, that

$$\begin{aligned} \left\| \frac{\mathrm{d}^{k+1} n}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} \le& C \|n\|_{H^k_\alpha} + C(\sqrt{\delta} + \sqrt{\eta}) \left\| \frac{\mathrm{d}^{k+1} \phi}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} + C\varepsilon^{1/2}\delta \left\| \frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha} \\ &+ C_\eta \left\| \frac{\mathrm{d}^k h_1}{\mathrm{d}z^k} \right\|_{L^2_\alpha} + C_\eta \delta \left(\left\| \frac{\mathrm{d}^{k+1} h_2}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} + \left\| \frac{\mathrm{d}^{k+2} h_2}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha} \right). \end{aligned} \tag{3.32}$$

By using the induction assumption, (3.32) implies that

$$\begin{aligned} \left\| \frac{\mathrm{d}^{k+1} n}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} \le& C(\sqrt{\delta} + \sqrt{\eta}) \left\| \frac{\mathrm{d}^{k+1} \phi}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} + C\varepsilon^{1/2}\delta \left\| \frac{\mathrm{d}^{k+2} \phi}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha} \\ &+ C_\eta \left\{ \|h_1\|_{H^k_\alpha} + \|h_2\|_{H^{k+1}_\alpha} + \delta \left\| \frac{\mathrm{d}^{k+2} h_2}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} \right\}. \end{aligned} \tag{3.33}$$

From (3.30), we obtain that

$$\left\| \frac{\mathrm{d}^{k+1}\phi}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} + \sqrt{\varepsilon\delta} \left\| \frac{\mathrm{d}^{k+2}\phi}{\mathrm{d}z^{k+2}} \right\|_{L^2_\alpha} + \varepsilon\delta \left\| \frac{\mathrm{d}^{k+3}\phi}{\mathrm{d}z^{k+3}} \right\|_{L^2_\alpha} \leq C \left\| \frac{\mathrm{d}^{k+1}n}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha} + C \left\| \frac{\mathrm{d}^{k+1}h_2}{\mathrm{d}z^{k+1}} \right\|_{L^2_\alpha}. \tag{3.34}$$

Therefore, (3.14) for $k+1$ follows from a suitable combination of (3.33) and (3.34) and taking both η and δ suitably small. This completes the proof of estimate (3.14).

Step 3. In this step, we construct the solution to (3.13) by using the approximation sequence $[n^{\varepsilon'}, \phi^{\varepsilon'}]$ in terms of solutions to the following ODE system:

$$\begin{cases} \dfrac{\mathrm{d}n^{\varepsilon'}}{\mathrm{d}z} = A(z)n^{\varepsilon'} + \dfrac{\varepsilon'\delta}{\sqrt{T+1}} \dfrac{\mathrm{d}^2\phi^{\varepsilon'}}{\mathrm{d}z^2} + h_1, \\ -\varepsilon\delta \dfrac{\mathrm{d}^2\phi^{\varepsilon'}}{\mathrm{d}z^2} = n^{\varepsilon'} - \phi^{\varepsilon'} + h_2, \\ n^{\varepsilon'}(0) = 0, \end{cases} \tag{3.35}$$

where $0 \leq \varepsilon' \leq 1$. Note that when $\varepsilon' = 1$, the system (3.35) is exactly (3.13) under consideration. In what follows let $L^{-1}_{\varepsilon',\varepsilon,\delta}$ formally denote the solution operator for the problem (3.35).

(i) Firstly we start with the case of $\varepsilon' = 0$. From the first equation of (3.35), one can solve $n^0(z)$ as

$$n^0(z) = \int_0^z e^{\int_y^z A(\tau)\mathrm{d}\tau} h_1(y)\mathrm{d}y.$$

Since it holds that

$$\lim_{z\to+\infty} A(z) = -2 < 0, \qquad \lim_{z\to-\infty} A(z) = 2 > 0,$$

one has

$$\|n^0\|_{H^k_\alpha} \leq C\|h_1\|_{H^{k-1}_\alpha} < \infty.$$

The existence of the solution ϕ^0 to the second equation of (3.35) in case of $\varepsilon' = 0$ can be shown by the Lax-Milgram Theorem and the H^{k+2}_α-regularity can be shown by using the H^k_α-estimate of n^0 and h_2. Here, the details of the proof are omitted for brevity. Therefore, the solution $U^0(z) = [n^0(z), \phi^0(z)]$ is well defined in the function space $\mathbf{X}_{\alpha,k}$. One can thereby use the similar argument in previous steps to deduce that the solution $U^0(z) = [n^0(z), \phi^0(z)]$ also satisfies the estimate (3.14). Hence the solution operator $L^{-1}_{0,\varepsilon,\delta}$ in $\mathbf{X}_{\alpha,k}$ has been constructed.

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(ii) Next, we construct the solution of (3.35) when $\varepsilon' > 0$ is small enough. For any $U = [n, \phi] \in \mathbf{X}_{\alpha,k}$, we introduce the linear mapping

$$T_{\varepsilon'} U := L^{-1}_{0,\varepsilon,\delta}[\frac{\varepsilon'\delta}{\sqrt{T+1}}\phi'' + h_1, h_2].$$

Then for any $U^1 = (\tilde{n}_1, \tilde{\phi}_1)$ and $U^2 = (\tilde{n}_2, \tilde{\phi}_2)$ in $\mathbf{X}_{\alpha,k}$, one has from (3.14) that

$$\|T_{\varepsilon'}(U^1 - U^2)\|_{\mathbf{X}_{\alpha,k}} \le C\varepsilon'\delta \|[\tilde{\phi}_1 - \tilde{\phi}_2]\|_{H^{k+1}_\alpha} \le C\varepsilon^{-1/2}\varepsilon' \|U^1 - U^2\|_{\mathbf{X}_{\alpha,k}}, \tag{3.36}$$

where the constant $C > 0$ is independent of δ, ε, and ε'. We now choose $\varepsilon'_0 = \varepsilon^{1/2}/2C$. Then it follows from (3.36) that $T_{\varepsilon'}$ is a contraction mapping on $\mathbf{X}_{\alpha,k}$ for any $0 < \varepsilon' \le \varepsilon'_0$. Thus $T_{\varepsilon'}$ has a unique fixed point $U^{\varepsilon'} := [n^{\varepsilon'}, \phi^{\varepsilon'}] \in \mathbf{X}_{\alpha,k}$. It is straightforward to check that $U^{\varepsilon'} = [n^{\varepsilon'}, \phi^{\varepsilon'}]$ is the solution to (3.35). Therefore, $L^{-1}_{\varepsilon',\varepsilon,\delta}$ is well-defined for any $0 < \varepsilon' \le \varepsilon'_0$. Moreover, the solution also satisfies the estimates in (3.14).

(iii) Lastly, we introduce

$$T_{\varepsilon'_0+\varepsilon'} U = T_{\varepsilon'_0+\varepsilon'}[n, \phi] = L^{-1}_{\varepsilon'_0,\varepsilon,\delta}[\varepsilon'\delta\phi'' + h_1, h_2].$$

Notice that by the uniform estimate (3.14), the upper bounds on the norm of the solution in terms of $L^{-1}_{\varepsilon',\varepsilon,\delta}$ is independent of ε and δ. Then by using the same argument as in (ii), one can show that $T_{\varepsilon'_0+\varepsilon'}$ is a contraction mapping on $\mathbf{X}_{\alpha,k}$ and thereby has a unique fixed point in $\mathbf{X}_{\alpha,k}$ for $0 < \varepsilon' \le \varepsilon'_0$. Therefore, the solution operator $L^{-1}_{2\varepsilon'_0,\varepsilon,\delta}$ has been constructed. Now we can repeat the same procedure and finally construct the solution operator $L^{-1}_{\varepsilon,\delta} \equiv L^{-1}_{1,\varepsilon,\delta}$ for the original problem (3.13) in $\mathbf{X}_{\alpha,k}$. The proof of Proposition 3.1 is then completed. □

3.3. *Justification of the approximation*

We have the following estimates on the remaining terms r_1 to r_6 given in (3.10).

Lemma 3.2. *Let $0 < \alpha < 2$. There exist positive constants ε_2 and δ_2 such that if $0 < \varepsilon \le \varepsilon_2$ and $0 < \delta \le \delta_2$, then the following estimates hold:*

$$\left|\frac{\mathrm{d}^k r_1}{\mathrm{d}z^k}(z)\right| \le C_k e^{-\alpha|z|}, \quad \left|\frac{\mathrm{d}^k r_4}{\mathrm{d}z^k}(z)\right| \le C_k e^{-\alpha|z|},$$

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for any integer $k \geq 0$ and any $z \in \mathbb{R}$, where each $C_k > 0$ is a generic constant independent of ε and δ. Moreover, we have

$$
\begin{aligned}
&\|r_1\|_{H_\alpha^{k-1}} \leq C_k, \\
&\|r_2[n,\phi]\|_{H_\alpha^{k-1}} \leq C_k\sqrt{\varepsilon}\|[n,\phi]\|_{\mathbf{X}_{\alpha,k}}, \\
&\|r_3\|_{H_\alpha^{k-1}} \leq C_k\varepsilon\|[n,\phi]\|^2_{\mathbf{X}_{\alpha,k}}\{1+\|[n,\phi]\|^2_{\mathbf{X}_{\alpha,k}}\}, \\
&\|r_4\|_{H_\alpha^{k}} \leq C_k, \\
&\|r_5[\phi]\|_{H_\alpha^{k}} \leq C_k\varepsilon\|\phi\|_{H_\alpha^{k}}, \\
&\|r_6[\phi]\|_{H_\alpha^{k}} \leq C_k\varepsilon^2 e^{\varepsilon^2\|\phi\|_{L^\infty}}\|\phi\|^2_{H_\alpha^{k}},
\end{aligned}
$$

for any integer $k \geq 2$, where each $C_k > 0$ is a generic constant independent of ε and δ.

Proof. We first consider r_4. It holds from (3.3) that

$$1+\varepsilon n_{1,\varepsilon}(\pm\infty)-e^{\varepsilon\phi_{1,\varepsilon}(\pm\infty)}=0.$$

Hence, by (3.10), one can rewrite r_4 as

$$r_4=\varepsilon^{-2}\left\{\varepsilon\left[n_{1,\varepsilon}-n_{1,\varepsilon}(\pm\infty)\right]-\left[e^{\varepsilon\phi_{1,\varepsilon}}-e^{\varepsilon\phi_{1,\varepsilon}(\pm\infty)}\right]\right\}.$$

Using (1.22), it further reduces to

$$
\begin{aligned}
r_4 =&\varepsilon^{-1}\left\{[n_{1,\varepsilon}-n_{1,\varepsilon}(\pm\infty)]-[n_1-n_{1,\pm}]\right\} \\
&-\varepsilon^{-1}\left\{[\phi_{1,\varepsilon}-\phi_{1,\varepsilon}(\pm\infty)]-[\phi_1-\phi_{1,\pm}]\right\} \\
&+O(1)|\phi_{1,\varepsilon}-\phi_{1,\varepsilon}(\pm\infty)|.
\end{aligned}
$$

Then we use Lemma 5.1 and Lemma 5.2 to conclude the desired estimates on r_4. Similarly, it holds that

$$
\begin{aligned}
r_1 =&\frac{\delta(1+\varepsilon n_{1,\varepsilon})}{\sqrt{T+1}}\varepsilon^{-1}[\phi''_{1,\varepsilon}-n''_{1,\varepsilon}]+(\frac{1}{\sqrt{T+1}}+n_{1,\varepsilon})n'_{1,\varepsilon} \\
&+\frac{\delta(1+\varepsilon n_{1,\varepsilon})n_{1,\varepsilon}\phi''_{1,\varepsilon}}{\sqrt{T+1}}-\frac{\delta(1+\varepsilon n_{1,\varepsilon})^2}{2\sqrt{T+1}}(\phi'_{1,\varepsilon})^2 \\
=&\frac{\delta(1+\varepsilon n_{1,\varepsilon})}{\sqrt{T+1}}\varepsilon^{-1}[\phi''_{1,\varepsilon}-\phi''_1-(n''_{1,\varepsilon}-n''_1)+\underbrace{\phi''_1-n''_1}_{=0\text{ by }(1.22)}]+(\frac{1}{\sqrt{T+1}}+n_{1,\varepsilon})n'_{1,\varepsilon} \\
&+\frac{\delta(1+\varepsilon n_{1,\varepsilon})n_{1,\varepsilon}\phi''_{1,\varepsilon}}{\sqrt{T+1}}-\frac{\delta(1+\varepsilon n_{1,\varepsilon})^2}{2\sqrt{T+1}}(\phi'_{1,\varepsilon})^2.
\end{aligned}
$$

Notice that each term on the right contains derivatives, so that all the right-hand terms and hence r_1 exponentially decay at $z \to \pm\infty$. Then the estimates on r_1 directly follow from Lemma 5.1

and Lemma 5.2. Estimates on other terms can be treated with the help of the Sobolev inequality; we omit the details of the proof for brevity. The proof of Lemma 3.2 is then complete. □

Now we are in position to prove Theorem 1.4. We start from the approximation sequence $U_i = [n_i, \phi_i]$ $(i = 0, 1, \cdots)$ in terms of

$$\begin{cases} n'_{i+1} = A(z)n_{i+1} + \dfrac{\delta}{\sqrt{T+1}}\phi''_{i+1} + r_1 + r_2[n_i, \phi_i] + r_3[n_i, \phi_i], \\ -\varepsilon\delta\phi''_{i+1} = n_{i+1} - \phi_{i+1} + r_4 + r_5[\phi_i] + r_6[\phi_i], \\ n_{i+1}(0) = 0,\, U_0 = [0, 0]. \end{cases}$$

Note that the existence of the sequence $\{U_i\}_{i\geq 0}$ is assured by Proposition 3.1. By induction, we claim to have the uniform bound of U_i as

$$\|U_i\|_{\mathbf{X}_{\alpha,k}} \leq K, \quad i = 0, 1, \cdots \tag{3.37}$$

for a suitably chosen constant $K > 0$ independent of ε, δ and i. Indeed, (3.37) is obviously true for $i = 0$, since $U_0 = [0, 0]$. To proceed, we assume that (3.37) is true up to $i \geq 0$. Applying Proposition 3.1 to $U = U_{i+1}$ with

$$h_1 = r_1 + r_2[n_i, \phi_i] + r_3[n_i, \phi_i], \quad h_2 = r_4 + r_5[\phi_i] + r_6[\phi_i],$$

and further using Lemma 3.2 to estimate the right-hand side of (3.14) as

$$\begin{aligned} \|h_1\|_{H^{k-1}_\alpha} &\leq \|r_1\|_{H^{k-1}_\alpha} + \|r_2\|_{H^{k-1}_\alpha} + \|r_3\|_{H^{k-1}_\alpha} \\ &\leq C_k + C_k\sqrt{\varepsilon}\|U_i\|_{\mathbf{X}_{\alpha,k}}\left(1 + \|U_i\|^3_{\mathbf{X}_{\alpha,k}}\right), \\ \|r_5[\phi_i]\|_{H^k_\alpha} + \delta\left\|\frac{\mathrm{d}^{k+1}r_5}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha} &\leq C_k\varepsilon\|\phi_i\|_{H^k_\alpha} + C_k\varepsilon\delta\left\|\frac{\mathrm{d}^{k+1}\phi_i}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha} \leq C_k\sqrt{\varepsilon}\|U_i\|_{\mathbf{X}_{\alpha,k}}, \\ \|r_6[\phi_i]\|_{H^k_\alpha} + \delta\left\|\frac{\mathrm{d}^{k+1}r_6}{\mathrm{d}z^{k+1}}\right\|_{L^2_\alpha} &\leq C_k\varepsilon^2 e^{\varepsilon^2\|\phi_i\|_{H^k_\alpha}}\left(\|\phi_i\|^2_{H^k_\alpha} + \delta\left\|\frac{\mathrm{d}^{k+1}\phi_i}{\mathrm{d}z^{k+1}}\right\|^2_{L^2_\alpha}\right) \\ &\leq C_k\varepsilon e^{\varepsilon^2\|\phi_i\|_{H^k_\alpha}}\|U_i\|^2_{\mathbf{X}_{\alpha,k}}, \end{aligned}$$

one can conclude that $\|U_{i+1}\|_{\mathbf{X}_{\alpha,k}}$ is bounded by

$$B + C\sqrt{\varepsilon}\|U_i\|_{\mathbf{X}_{\alpha,k}}\left\{1 + \|U_i\|_{\mathbf{X}_{\alpha,k}}\left(1 + e^{\varepsilon^2\|U_i\|_{\mathbf{X}_{\alpha,k}}}\right) + \|U_i\|^3_{\mathbf{X}_{\alpha,k}}\right\}, \tag{3.38}$$

for a generic constant $B > 0$ independent of ε, δ and i. In terms of the induction hypothesis, it follows from (3.38) that

$$\|U_{i+1}\|_{\mathbf{X}_{\alpha,k}} \leq K$$

by taking $K = 2B$ and $\varepsilon > 0$ small enough. This then proves (3.37).

By a similar argument, one can further show that the estimate

$$\|U_{i+1}-U_i\|_{\mathbf{X}_{\alpha,k}} \le \frac{1}{2}\|U_i-U_{i-1}\|_{\mathbf{X}_{\alpha,k}}$$

holds true for all $i \ge 1$, provided that $\varepsilon > 0$ is small enough. Thus, $\{U_i\}_{i\ge 0}$ is a Cauchy sequence in $\mathbf{X}_{\alpha,k}$, and hence there is a function $U \in \mathbf{X}_{\alpha,k}$ such that $U_i \to U$ as $i \to \infty$ in terms of the norm of $\mathbf{X}_{\alpha,k}$. It is straightforward to check that the limit function $U := [n_R, \phi_R]$ solves the problem (3.9) and satisfies

$$\|U\|_{\mathbf{X}_{\alpha,k}} \le K. \tag{3.39}$$

Once n_R is solved, u_R can be solved according to (3.6) and it follows that

$$\|u_R\|_{H^k_\alpha} \le C\|[n_R,\phi_R\|_{\mathbf{X}_{\alpha,k}} + C \le C(K+1). \tag{3.40}$$

Therefore, (1.30) is justified due to the uniform estimates (3.39) and (3.40) for the remaining terms $[n_R, u_R, \phi_R]$ and (5.5) for the correction terms $[n_2, u_2, \phi_2]$. The proof of Theorem 1.4 is complete. □

4. Dynamical stability of shock profiles

In this section we turn to the proof of Theorem 1.10 for the large time asymptotic stability of the smooth small-amplitude shock profile obtained in Theorem 1.1 under suitably small smooth perturbations. The proof is based on the anti-derivative technique and the elementary energy method. Compared to the classical result for the Navier-Stokes equations, the main difficulty is to treat the extra effect of the self-consistent force.

4.1. Reformulation

Recall the coordinate $(t, y) = (t, x - st)$. We define the perturbation around the shock profile $[\bar{v}, \bar{u}, \bar{\phi}](y)$ as

$$[\tilde{v}, \tilde{u}, \tilde{\phi}] := [v-\bar{v}, u-\bar{u}, \phi-\bar{\phi}].$$

Then by (1.31), $[\tilde{v}, \tilde{u}, \tilde{\phi}](t, y)$ satisfies

$$\begin{cases} \tilde{v}_t - s\tilde{v}_y - \tilde{u}_y = 0, \\ \tilde{u}_t - s\tilde{u}_y + T\left(\dfrac{1}{v}-\dfrac{1}{\bar{v}}\right)_y = \mu\left(\dfrac{u_y}{v}-\dfrac{\bar{u}_y}{\bar{v}}\right)_y - \left(\dfrac{\phi_y}{v}-\dfrac{\bar{\phi}_y}{\bar{v}}\right), \\ -\lambda^2\left(\dfrac{\phi_y}{v}-\dfrac{\bar{\phi}_y}{\bar{v}}\right)_y = \bar{v}e^{\bar{\phi}} - ve^{\phi}. \end{cases} \tag{4.1}$$

Similar to (1.34), the second equation of (4.1) can be rewritten as

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$$
\begin{aligned}
&\tilde{u}_t - s\tilde{u}_y + (T+1)\left(\frac{1}{v} - \frac{1}{\bar{v}}\right)_y - \mu\left(\frac{u_y}{v} - \frac{\bar{u}_y}{\bar{v}}\right)_y \\
&\qquad = \frac{\lambda^2}{2}\left[\left(\frac{\phi_y}{v}\right)^2 - \left(\frac{\bar{\phi}_y}{\bar{v}}\right)^2\right]_y - \lambda^2\left[\frac{1}{v}\left(\frac{\phi_y}{v}\right)_y - \frac{1}{\bar{v}}\left(\frac{\bar{\phi}_y}{\bar{v}}\right)_y\right]_y.
\end{aligned}
\tag{4.2}
$$

Recall (1.38) for a formal definition of $[\Phi, \Psi]$. Then, from (4.1) as well as (4.2), by formally taking integration of $[\tilde{v}, \tilde{u}](t, \cdot)$ from $-\infty$ to y, $[\Phi, \Psi, \phi](t, y)$ satisfies

$$
\begin{cases}
\Phi_t - s\Phi_y - \Psi_y = 0, \\
\Psi_t - s\Psi_y + (T+1)\left(\dfrac{1}{\bar{v}+\Phi_y} - \dfrac{1}{\bar{v}}\right) - \mu\left(\dfrac{\Psi_{yy}+\bar{u}_y}{\bar{v}+\Phi_y} - \dfrac{\bar{u}_y}{\bar{v}}\right) \\
\qquad = \dfrac{\lambda^2}{2}\left[\left(\dfrac{\phi_y}{v}\right)^2 - \left(\dfrac{\bar{\phi}_y}{\bar{v}}\right)^2\right] - \lambda^2\left[\dfrac{1}{v}\left(\dfrac{\phi_y}{v}\right)_y - \dfrac{1}{\bar{v}}\left(\dfrac{\bar{\phi}_y}{\bar{v}}\right)_y\right], \\
-\lambda^2\left(\dfrac{\phi_y}{v} - \dfrac{\bar{\phi}_y}{\bar{v}}\right)_y = \bar{v}e^{\bar{\phi}} - ve^{\phi},
\end{cases}
\tag{4.3}
$$

supplemented with the initial data of $[\Phi, \Psi]$ given in (1.39). We regard the Cauchy problem (4.3) and (1.39) on $[\Phi, \Psi, \phi](t, y)$ as an auxiliary problem for obtaining the existence of the original solution $[v, u, \phi](t, y)$ by defining

$$
[v, u] = [\bar{v}, \bar{u}] + [\Phi_y, \Psi_y], \tag{4.4}
$$

and the uniqueness of solutions $[v, u, \phi]$ in the prescribed function space can be independently proved. Thus there is actually no need to justify if the right-hand terms of (1.38) are well defined. Since it is a standard procedure, in what follows we will only focus on the existence of smooth solutions to the Cauchy problem (4.3) and (1.39) by the energy method.

4.2. A priori estimates

We are now devoted to obtaining the a priori estimates of solutions to the Cauchy problem (4.3) and (1.39).

Proposition 4.1. *Let $M > 0$ be an arbitrary constant and $[\Phi, \Psi, \tilde{\phi}]$ be a smooth solution to the Cauchy problem (4.3) on $[0, M]$ with initial data $[\Phi_0, \Psi_0] \in H^2$. There exist positive constants e_1 and $\tilde{\varepsilon}_1$ independent of M such that if*

$$
\sup_{0 \le t \le M} \|[\Phi, \Psi, \tilde{\phi}](t)\|_{H^2} \le e_1 \tag{4.5}
$$

and

$$
|v_+ - v_-| \le \tilde{\varepsilon}_1, \tag{4.6}
$$

then it holds that

$$\|[\Phi, \Psi, \tilde{\phi}](t)\|_{H^2}^2 + \int_0^t \|\sqrt{s\bar{v}\bar{v}_y}\Psi(\tau)\|_{L^2}^2 + \|[\Phi_y, \tilde{\phi}_t](\tau)\|_{H^1}^2 + \|[\Psi_y, \tilde{\phi}](\tau)\|_{H^2}^2 \mathrm{d}\tau \le C\|[\Phi_0, \Psi_0]\|_{H^2}^2, \tag{4.7}$$

for all $t \in [0, M]$.

We will devote the rest of this subsection to prove Proposition 4.1. Firstly we estimate the zero-order energy of $[\Phi, \Psi, \tilde{\phi}]$. For this, we rewrite (4.3) as follows:

$$\begin{cases} \Phi_t - s\Phi_y - \Psi_y = 0, \\ \Psi_t - s\Psi_y - \dfrac{(T+1)}{\bar{v}^2}\Phi_y - \dfrac{\mu}{\bar{v}}\Psi_{yy} = -\lambda^2\left[\dfrac{1}{v}\left(\dfrac{\phi_y}{v}\right)_y - \dfrac{1}{\bar{v}}\left(\dfrac{\bar{\phi}_y}{\bar{v}}\right)_y\right] + \mathcal{J}_1 + \mathcal{N}_1, \\ -\lambda^2\left(\dfrac{\phi_y}{v} - \dfrac{\bar{\phi}_y}{\bar{v}}\right)_y = -e^{\bar{\phi}}\Phi_y - \bar{v}e^{\bar{\phi}}\tilde{\phi} + \mathcal{N}_2, \end{cases} \tag{4.8}$$

where we have denoted

$$\mathcal{J}_1 \equiv \lambda^2\left(\frac{\bar{\phi}_y\tilde{\phi}_y}{v^2} - \frac{\bar{\phi}_y^2\Phi_y}{v^2\bar{v}}\right) - \frac{\mu\bar{u}_y\Phi_y}{v\bar{v}},$$

$$\mathcal{N}_1 \equiv \frac{-(T+1)\Phi_y^2}{\bar{v}^2 v} - \frac{\mu\Psi_{yy}\Phi_y}{v\bar{v}} + \frac{\lambda^2}{2}\left(\frac{\tilde{\phi}_y^2}{v^2} - \frac{\bar{\phi}_y^2\Phi_y^2}{v^2\bar{v}^2}\right),$$

$$\mathcal{N}_2 \equiv e^{\bar{\phi}}(1 - e^{\tilde{\phi}})\Phi_y + \bar{v}e^{\bar{\phi}}(1 - e^{-\tilde{\phi}} + \tilde{\phi}).$$

Lemma 4.2. *Under the assumptions of Proposition 4.1, it holds that*

$$\begin{aligned} &\|[\Phi, \Psi, \tilde{\phi}, \tilde{\phi}_y](t)\|_{L^2}^2 + \int_0^t \left\{\|\sqrt{s\bar{v}\bar{v}_y}\Psi(\tau)\|_{L^2}^2 + \|\Psi_y(\tau)\|_{L^2}^2\right\}\mathrm{d}\tau \\ &\le C\|[\Phi_0, \Psi_0]\|_{L^2}^2 + C\|\tilde{\phi}(0)\|_{H^1}^2 \\ &\quad + C(e_1 + \tilde{\varepsilon}_1)\int_0^t \left\{\|[\Phi_y, \Psi_y](\tau)\|_{H^1}^2 + \|\tilde{\phi}(\tau)\|_{H^2}^2 + \|\tilde{\phi}_t(\tau)\|_{H^1}^2\right\}\mathrm{d}\tau, \end{aligned} \tag{4.9}$$

for all $t \in [0, M]$.

Proof. Firstly, it holds from Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ as well as the a priori assumption (4.5) that

$$\|[\Phi, \Psi, \tilde{\phi}](t)\|_{L^\infty} + \|[\Phi_y, \Psi_y, \tilde{\phi}_y](t)\|_{L^\infty} \le Ce_1, \tag{4.10}$$

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with a generic constant $C > 0$. Then, for e_1 suitably small, in terms of (4.4) and $\tilde{v} = \Phi_y$, we have

$$\underline{V} \le v = \bar{v} + \tilde{v} \le \bar{V}, \tag{4.11}$$

for two positive constants $\underline{V}$, $\bar{V} > 0$. Multiplying the first and second equations of (4.8) by $(T+1)\Phi$ and $\bar{v}^2\Psi$ respectively and adding them up, we have

$$\begin{aligned}&\left\{\frac{T+1}{2}\Phi^2 + \frac{\bar{v}^2}{2}\Psi^2\right\}_t + \{\cdots\}_y + s\bar{v}\bar{v}_y\Psi^2 + \mu\bar{v}\Psi_y^2\\ &\qquad = -\mu\bar{v}_y\Psi\Psi_y + (\mathcal{J}_1 + \mathcal{N}_1)\bar{v}^2\Psi - \lambda^2\bar{v}^2\Psi\left[\frac{1}{v}\left(\frac{\phi_y}{v}\right)_y - \frac{1}{\bar{v}}\left(\frac{\bar{\phi}_y}{\bar{v}}\right)_y\right],\end{aligned} \tag{4.12}$$

where the second term $\{\cdots\}_y$ on the left stands for the total derivative term and will disappear after taking integration with respect to y. Note that the coefficient of Ψ^2 in the third term on the left is positive due to (1.37) for the compressibility of the shock profile. Now we estimate the right-hand side of (4.12) term by term. By Cauchy-Schwarz, the first term is bounded as

$$|\mu\bar{v}_y\Psi\Psi_y| \le \eta s\bar{v}\bar{v}_y\Psi^2 + C_\eta|v_+ - v_-|\cdot|\Psi_y|^2, \tag{4.13}$$

with an arbitrary constant $0 < \eta < 1$ to be chosen later. The second term on the right-hand side of (4.12) comes from inhomogeneous and nonlinear contributions. Therefore, it holds from (1.37), (4.10) and (4.11) that

$$\left|(\mathcal{J}_1 + \mathcal{N}_1)\bar{v}^2\Psi\right| \le \eta s\bar{v}\bar{v}_y\Psi^2 + \{C_\eta|v_+ - v_-| + C\sqrt{\mathcal{E}(t)}\}\left(\Phi_y^2 + \Psi_{yy}^2 + \tilde{\phi}_y^2\right). \tag{4.14}$$

Here and in the sequel we have used the notation $\mathcal{E}(t)$ given in (1.41). To estimate the last term on the right-hand side of (4.12), we first rewrite it as

$$\lambda^2\left[\frac{1}{v}\left(\frac{\phi_y}{v}\right)_y - \frac{1}{\bar{v}}\left(\frac{\bar{\phi}_y}{\bar{v}}\right)_y\right]\bar{v}^2\Psi = \lambda^2\left[\left(\frac{\phi_y}{v^2} - \frac{\bar{\phi}_y}{\bar{v}^2}\right)\bar{v}^2\Psi\right]_y - \lambda^2\tilde{\phi}_y\Psi_y + \mathcal{N}_3, \tag{4.15}$$

where we have denoted

$$\mathcal{N}_3 \equiv -\lambda^2\bar{v}^2\left(\frac{1}{v^2} - \frac{1}{\bar{v}^2}\right)\phi_y\Psi_y - 2\lambda^2\bar{v}\bar{v}_y\Psi\left(\frac{\phi_y}{v^2} - \frac{\bar{\phi}_y}{\bar{v}^2}\right) + \lambda^2\bar{v}^2\Psi\left(\frac{v_y\phi_y}{v^3} - \frac{\bar{v}_y\bar{\phi}_y}{\bar{v}^3}\right).$$

Using (4.10) and (4.11), it is straightforward to show that $\mathcal{N}_3$ is bounded by

$$|\mathcal{N}_3| \le \eta|\bar{v}_y|\Psi^2 + \{C_\eta|v_+ - v_-| + C\sqrt{\mathcal{E}(t)}\}\left(\Phi_y^2 + \Psi_y^2 + \tilde{\phi}_y^2 + \Phi_{yy}^2\right). \tag{4.16}$$

Next, substituting the first equation of (4.8) into the second term on the right-hand side of (4.15), one has

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$$
\begin{aligned}
-\lambda^2\tilde{\phi}_y\Psi_y &= -\lambda^2\tilde{\phi}_y\Phi_t + s\lambda^2\tilde{\phi}_y\Phi_y \\
&= -\lambda^2\left(\tilde{\phi}_y\Phi\right)_t + \lambda^2\tilde{\phi}_{yt}\Phi + s\lambda^2\tilde{\phi}_y\Phi_y \\
&= -\lambda^2\left(\tilde{\phi}_y\Phi\right)_t + \lambda^2\left(\tilde{\phi}_t\Phi\right)_y - \lambda^2\tilde{\phi}_t\Phi_y + s\lambda^2\tilde{\phi}_y\Phi_y.
\end{aligned} \tag{4.17}
$$

Now it remains to deal with the last two terms in the last line of (4.17). For this, one should utilize the Poisson equation. In fact, it follows from the third equation of (4.8) that

$$
\Phi_y = \lambda^2 e^{-\bar{\phi}}\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)_y - \bar{v}\tilde{\phi} + e^{-\bar{\phi}}\mathcal{N}_2. \tag{4.18}
$$

Then one has

$$
\begin{aligned}
-\lambda^2\tilde{\phi}_t\Phi_y &= -\lambda^4 e^{-\bar{\phi}}\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)_y\tilde{\phi}_t + \lambda^2\bar{v}\tilde{\phi}_t\tilde{\phi} - \lambda^2 e^{-\bar{\phi}}\tilde{\phi}_t\mathcal{N}_2 \\
&= \left[\frac{\lambda^2\bar{v}}{2}\tilde{\phi}^2 + \frac{\lambda^4 e^{-\bar{\phi}}}{2\bar{v}}\tilde{\phi}_y^2\right]_t - \left[\lambda^4 e^{-\bar{\phi}}\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)\tilde{\phi}_t\right]_y + \mathcal{I}_1,
\end{aligned} \tag{4.19}
$$

where

$$
\mathcal{I}_1 \equiv \lambda^4 e^{-\bar{\phi}}\left[\left(\frac{1}{v} - \frac{1}{\bar{v}}\right)\phi_y\tilde{\phi}_{ty} - \bar{\phi}_y\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)\tilde{\phi}_t\right] - \lambda^2 e^{-\bar{\phi}}\tilde{\phi}_t\mathcal{N}_2.
$$

Using (4.10) for $\tilde{\varepsilon}_1$ suitably small, it is straightforward to bound $\mathcal{I}_1$ as

$$
|\mathcal{I}_1| \le C\{|v_+ - v_-| + \sqrt{\mathcal{E}(t)}\}\left(|\tilde{\phi}|^2 + |\tilde{\phi}_t|^2 + |\tilde{\phi}_{ty}|^2 + |\Phi_y|^2\right). \tag{4.20}
$$

In the same way as before, the last term on the right-hand side of (4.17) can be computed as

$$
\begin{aligned}
s\lambda^2\tilde{\phi}_y\Phi_y &= s\lambda^4 e^{-\bar{\phi}}\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)_y\tilde{\phi}_y - s\lambda^2\bar{v}\tilde{\phi}\tilde{\phi}_y + s\lambda^2 e^{-\bar{\phi}}\mathcal{N}_2\tilde{\phi}_y \\
&= \left[s\lambda^4 e^{-\bar{\phi}}\left(\frac{\phi_y}{v} - \frac{\phi_y}{\bar{v}}\right)\tilde{\phi}_y + \frac{s\lambda^4 e^{-\bar{\phi}}\tilde{\phi}_y^2}{2\bar{v}} - \frac{s\lambda^2\bar{v}\tilde{\phi}^2}{2}\right]_y + \mathcal{I}_2,
\end{aligned} \tag{4.21}
$$

where

$$
\mathcal{I}_2 \equiv s\lambda^4 e^{-\bar{\phi}}\left\{\left(\frac{\bar{\phi}_y}{2\bar{v}} - \frac{\bar{v}_y}{2\bar{v}^2}\right)\tilde{\phi}_y^2 + \left(\frac{\phi_y}{v} - \frac{\phi_y}{\bar{v}}\right)\left(\bar{\phi}_y\tilde{\phi}_y - \tilde{\phi}_{yy}\right)\right\} + s\lambda^2 e^{-\bar{\phi}}\mathcal{N}_2\tilde{\phi}_y + \frac{s\lambda^2\bar{v}_y\tilde{\phi}^2}{2}.
$$

One can bound $\mathcal{I}_2$ as

$$
|\mathcal{I}_2| \le C\{|v_+ - v_-| + \sqrt{\mathcal{E}(t)}\}\left(|\tilde{\phi}|^2 + |\tilde{\phi}_y|^2 + |\tilde{\phi}_{yy}|^2 + |\Phi_y|^2\right). \tag{4.22}
$$

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In sum, by collecting all the above estimates (4.13), (4.14), (4.15), (4.16), (4.17), (4.19), (4.20), (4.21) and (4.22), it follows that

$$\begin{aligned}\text{The R.H.S of (4.12)} \leq & -\left\{\frac{\lambda^2\bar{v}\tilde{\phi}^2}{2}+\frac{\lambda^4 e^{-\bar{\phi}}\tilde{\phi}_y^2}{2\bar{v}}-\lambda^2\tilde{\phi}_y\Phi\right\}_t+\{\cdots\}_y+\eta s\bar{v}\bar{v}_y\Psi^2\\ &+\{C_\eta|v_+-v_-|+C\sqrt{\mathcal{E}(t)}\}\left\{\sum_{i=1}^{2}\left|\partial_y^i[\Phi,\Psi]\right|^2+\sum_{i=0}^{2}\left|\partial_y^i\tilde{\phi}\right|^2+\sum_{i=0}^{1}\left|\partial_y^i\tilde{\phi}_t\right|\right\},\end{aligned} \tag{4.23}$$

with an arbitrary constant $0<\eta<1$ to be chosen later. Substituting (4.23) into (4.12), integrating it with respect to y, and taking a suitably small constant $\eta>0$, one obtains that

$$\begin{aligned}&\frac{\mathrm{d}\mathcal{E}_1(t)}{\mathrm{d}t}+\int_{\mathbb{R}}\left(s\bar{v}\bar{v}_y\Psi^2+\mu\bar{v}\Psi_y^2\right)\mathrm{d}y\\ &\qquad\leq C\{|v_+-v_-|+\sqrt{\mathcal{E}(t)}\}\left(\|[\Phi_y,\Psi_y,\tilde{\phi}_t]\|_{H^1}^2+\|\tilde{\phi}\|_{H^2}^2\right),\end{aligned} \tag{4.24}$$

for all $t\in[0,M]$, where we have denoted

$$\mathcal{E}_1(t)\equiv\int_{\mathbb{R}}\left(\frac{\bar{v}^2\Psi^2}{2}+\frac{\lambda^2\bar{v}\tilde{\phi}^2}{2}+\frac{(T+1)\Phi^2}{2}+\frac{\lambda^4e^{-\bar{\phi}}\tilde{\phi}_y^2}{2\bar{v}}-\lambda^2\tilde{\phi}_y\Phi\right)\mathrm{d}y. \tag{4.25}$$

Finally, one can check that $\mathcal{E}_1(t)$ is a nonnegative energy functional. Indeed, by the Poisson equation, one has

$$\left|\bar{v}^{-1}e^{-\bar{\phi}}-1\right|=\left|\lambda^2\bar{v}^{-1}e^{-\bar{\phi}}(\bar{v}^{-1}\bar{\phi}_y)_y\right|\leq C|v_+-v_-|. \tag{4.26}$$

Then, due to (4.6) with $\tilde{\varepsilon}_1$ suitably small, the quadratic integrand of $\mathcal{E}_1(t)$ has a lower bound as

$$\frac{(T+1)\Phi^2}{2}+\frac{\lambda^4e^{-\bar{\phi}}\tilde{\phi}_y^2}{2\bar{v}}-\lambda^2\tilde{\phi}_y\Phi\geq c\left(|\Phi|^2+|\tilde{\phi}_y|^2\right), \tag{4.27}$$

for a generic constant $c>0$. Here we have essentially used the condition $T>0$. Therefore, (4.9) follows from integrating (4.24) over $[0,t]$. This completes the proof of Lemma 4.2. □

Next, we need to derive the dissipation terms $\|[\Phi_y,\tilde{\phi},\tilde{\phi}_y,\tilde{\phi}_{yy}]\|_{L^2}$ as well as the dissipation of $\tilde{\phi}$ in H^2.

Lemma 4.3. *Under the assumptions of Proposition 4.1, it holds that*

$$\|\Phi_y(t)\|_{L^2}^2+\int_0^t\left\{\|\Phi_y(\tau)\|_{L^2}^2+\|\tilde{\phi}(\tau)\|_{H^2}^2\right\}\mathrm{d}\tau$$

$$\le C\|[\Phi_{0y},\Psi_0]\|_{L^2}^2 + C(e_1+\tilde{\varepsilon}_1)\int_0^t \|[\Phi_{yy},\Psi_{yy}](\tau)\|_{L^2}^2 \mathrm{d}\tau$$

$$+ C\left(\|\Psi(t)\|_{L^2}^2 + \int_0^t \left\{\|\sqrt{s\bar{v}\bar{v}_y}\Psi(\tau)\|_{L^2}^2 + \|\Psi_y(\tau)\|_{L^2}^2\right\}\mathrm{d}\tau\right), \tag{4.28}$$

and

$$\|\tilde{\phi}(t)\|_{H^2}^2 \le C\|\Phi_y(t)\|_{L^2}^2 + C(e_1+\tilde{\varepsilon}_1)\|\Phi_{yy}(t)\|_{L^2}^2, \tag{4.29}$$

for all $t \in [0, M]$.

Proof. Differentiating the first equation of (4.3) with respect to y, one has

$$\Phi_{ty} - s\Phi_{yy} - \Psi_{yy} = 0. \tag{4.30}$$

Then, multiplying (4.30) and the second equation of (4.3) by Φ_y and $-\mu^{-1}\bar{v}\Phi_y$ respectively and adding them together, we have

$$\left(\frac{\Phi_y^2}{2}\right)_t - \left(\frac{s\Phi_y^2}{2}\right)_y + \sum_{j=3}^{8}\mathcal{I}_j = 0, \tag{4.31}$$

where $\mathcal{I}_j$ $(3 \le j \le 8)$ are defined as follows.

$$\mathcal{I}_3 \equiv -(T+1)\left(\frac{1}{\bar{v}+\Phi_y} - \frac{1}{\bar{v}}\right)\frac{\bar{v}\Phi_y}{\mu}, \quad \mathcal{I}_4 \equiv -\mu^{-1}(\Psi_t - s\Psi_y)\bar{v}\Phi_y,$$

$$\mathcal{I}_5 \equiv -\Psi_{yy}\Phi_y + \bar{v}\Phi_y\left(\frac{\Psi_{yy}+\bar{u}_y}{\bar{v}+\Phi_y} - \frac{\bar{u}_y}{\bar{v}}\right), \quad \mathcal{I}_6 \equiv \frac{\lambda^2\bar{v}\Phi_y}{2\mu}\left[\left(\frac{\phi_y}{v}\right)^2 - \left(\frac{\bar{\phi}_y}{\bar{v}}\right)^2\right],$$

$$\mathcal{I}_7 \equiv -\mu^{-1}\lambda^2\bar{v}(\bar{v}^{-1}\bar{\phi}_y)_y(\frac{1}{v} - \frac{1}{\bar{v}})\Phi_y, \quad \mathcal{I}_8 \equiv -\mu^{-1}\lambda^2 v^{-1}\bar{v}\Phi_y\left(\frac{\phi_y}{v} - \frac{\bar{\phi}_y}{\bar{v}}\right)_y.$$

Now we estimate $\mathcal{I}_3$ to $\mathcal{I}_8$ term by term. Firstly, it holds that

$$\mathcal{I}_3 = \frac{(T+1)\Phi_y^2}{\mu(\bar{v}+\Phi_y)} = \frac{(T+1)\Phi_y^2}{\mu\bar{v}} - \frac{(T+1)\Phi_y^3}{\mu(\bar{v}+\Phi_y)\bar{v}} \ge \frac{(T+1)\Phi_y^2}{\mu\bar{v}} - C\sqrt{\mathcal{E}(t)}\Phi_y^2.$$

For $\mathcal{I}_4$, it is straightforward to compute

$$\mathcal{I}_4 = -\left(\frac{\bar{v}\Phi_y\Psi}{\mu}\right)_t + \frac{\bar{v}\Phi_{yt}\Psi}{\mu} + \frac{s\bar{v}\Psi_y\Phi_y}{\mu}$$

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$$
\begin{aligned}
&= -\left(\frac{\bar{v}\Phi_y\Psi}{\mu}\right)_t + \frac{\bar{v}\Psi}{\mu}(s\Phi_{yy} + \Psi_{yy}) + \frac{s\bar{v}\Psi_y\Phi_y}{\mu} \\
&= -\left(\frac{\bar{v}\Phi_y\Psi}{\mu}\right)_t + \{\cdots\}_y - \left\{\frac{\bar{v}_y}{\mu}(s\Psi\Phi_y + \Psi\Psi_y) + \frac{\bar{v}\Psi_y^2}{\mu}\right\}.
\end{aligned}
\tag{4.32}
$$

By Cauchy-Schwarz, the last term of (4.32) is bounded by

$$
\eta\Phi_y^2 + C_\eta\{s\bar{v}\bar{v}_y\Psi^2 + \Psi_y^2\},
$$

where $\eta > 0$ can be small enough to be chosen later. As for $\mathcal{I}_5$ to $\mathcal{I}_7$, we have

$$
|\mathcal{I}_5| = \left|\left(\frac{\bar{v}}{v} - 1\right)\left(\Psi_{yy}\Phi_y + \bar{u}_y\Phi_y\right)\right| \le C\{\sqrt{\mathcal{E}(t)} + |v_+ - v_-|\}\{|\Psi_{yy}|^2 + |\Phi_y|^2\},
$$

and

$$
|\mathcal{I}_6| + |\mathcal{I}_7| \le C\{\sqrt{\mathcal{E}(t)} + |v_+ - v_-|\}\{|\Phi_y|^2 + |\tilde{\phi}_y|^2\}.
$$

Now it remains to estimate $\mathcal{I}_8$, which is delicate. Direct computations show that

$$
\mathcal{I}_8 = -\frac{\lambda^2\Phi_y\tilde{\phi}_{yy}}{\mu\bar{v}} + \frac{\lambda^2\bar{v}\Phi_y}{\mu v}\left\{\left(\frac{\phi_{yy}}{\bar{v}} - \frac{\phi_{yy}}{v}\right) - \frac{v}{\bar{v}}\left(\frac{\tilde{\phi}_{yy}}{v} - \frac{\tilde{\phi}_{yy}}{\bar{v}}\right) + \left(\frac{\phi_y v_y}{v^2} - \frac{\bar{\phi}_y\bar{v}_y}{\bar{v}^2}\right)\right\}.
\tag{4.33}
$$

The last term of (4.33) is bounded by

$$
C\{\sqrt{\mathcal{E}(t)} + |v_+ - v_-|\}\left\{|\Phi_y|^2 + |\Phi_{yy}|^2 + |\tilde{\phi}_y|^2 + |\tilde{\phi}_{yy}|^2\right\}.
$$

Note that the first term on the right-hand of (4.33) can not be cancelled by directly replacing Φ_y from (4.18) like what has been done earlier in (4.19). Indeed, one has to include some estimates on $\tilde{\phi}_{yy}$ simultaneously so that the quadratic form consisting of Φ_y and $\tilde{\phi}_{yy}$ is strictly positive. For this purpose, multiplying (4.18) by $-\lambda^2\tilde{\phi}_{yy}/\mu\bar{v}$, one has

$$
\begin{aligned}
&\frac{\lambda^4\tilde{\phi}_{yy}^2}{\mu\bar{v}} + \frac{\lambda^2\tilde{\phi}_y^2}{\mu} - \frac{\lambda^2\tilde{\phi}_{yy}\Phi_y}{\mu\bar{v}} - \left(\frac{\lambda^2\tilde{\phi}\tilde{\phi}_y}{\mu}\right)_y \\
&= \frac{\lambda^2 e^{-\bar{\phi}}\tilde{\phi}_{yy}}{\mu\bar{v}}\left\{\lambda^2\left(\frac{\phi_{yy}}{\bar{v}} - \frac{\phi_{yy}}{v}\right)\right. \\
&\qquad \left. + \lambda^2\left(\frac{\phi_y v_y}{v^2} - \frac{\bar{\phi}_y\bar{v}_y}{\bar{v}^2}\right) + \lambda^2(e^{\bar{\phi}} - \bar{v}^{-1})\tilde{\phi}_{yy} - \mathcal{N}_2\right\}.
\end{aligned}
\tag{4.34}
$$

Due to (4.26), the right-hand side of (4.34) is bounded by

$$
C\{|v_+ - v_-| + \sqrt{\mathcal{E}(t)}\}\left\{|\tilde{\phi}|^2 + |\tilde{\phi}_y|^2 + |\tilde{\phi}_{yy}|^2 + |\Phi_{yy}|^2 + |\Phi_y|^2\right\}.
$$

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By collecting all the above estimates for $\mathcal{I}_3$ to $\mathcal{I}_8$ as well as (4.34), we have, from (4.31) that

$$\left(\frac{\Phi_y^2}{2}-\frac{\bar{v}\Phi_y\Psi}{\mu}\right)_t+\left\{\frac{(T+1)\Phi_y^2}{\mu\bar{v}}+\frac{\lambda^4\tilde{\phi}_{yy}^2}{\mu\bar{v}}-\frac{2\lambda^2\tilde{\phi}_{yy}\Phi_y}{\mu\bar{v}}\right\}+\frac{\lambda^2\tilde{\phi}_y^2}{\mu}+\{\cdots\}_y$$
$$\le \eta|\Phi_y|^2+C_\eta\left(s\bar{v}\bar{v}_y\Psi^2+\Psi_y^2\right)$$
$$+C\{\sqrt{\mathcal{E}(t)}+|v_+-v_-|\}\left\{\sum_{i=0}^{2}|\partial_y^i\tilde{\phi}|^2+\sum_{i=1}^{2}|\partial_y^i\Phi|^2+|\Psi_{yy}|^2\right\}, \tag{4.35}$$

for an arbitrary constant $0<\eta<1$. Note that the quadratic term on the left-hand side has a lower bound as

$$\frac{\lambda^4\tilde{\phi}_{yy}^2}{\mu\bar{v}}+\frac{(T+1)\Phi_y^2}{\mu\bar{v}}-\frac{2\lambda^2\tilde{\phi}_{yy}\Phi_y}{\mu\bar{v}}\ge c\left(\frac{\lambda^4\tilde{\phi}_{yy}^2}{\mu\bar{v}}+\frac{(T+1)\Phi_y^2}{\mu\bar{v}}\right),$$

for a generic positive constant c. Therefore, integrating (4.35) with respect to y and taking $\eta>0$ suitably small, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\Phi_y(t)\|_{L^2}^2-\int_{\mathbb{R}}\frac{\bar{v}\Phi_y\Psi}{\mu}\mathrm{d}y\right)+\|\Phi_y\|_{L^2}^2+\|\tilde{\phi}_y\|_{H^1}^2$$
$$\le C\int_{\mathbb{R}}\left(s\bar{v}\bar{v}_y\Psi^2+\Psi_y^2\right)\mathrm{d}y$$
$$+C\{\sqrt{\mathcal{E}(t)}+|v_+-v_-|\}\left\{\|\Psi_{yy}\|_{L^2}^2+\|\Phi_y\|_{H^1}^2+\|\tilde{\phi}\|_{H^2}^2\right\}. \tag{4.36}$$

Moreover, multiplying the third equation of (4.8) by $\tilde{\phi}$, we obtain that

$$\bar{v}e^{\bar{\phi}}\tilde{\phi}^2+\frac{\tilde{\phi}_y^2}{v}+\{\cdots\}_y=-\lambda^2\bar{\phi}_y\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)\tilde{\phi}_y-e^{\bar{\phi}}\Phi_y\tilde{\phi}+\mathcal{N}_2\tilde{\phi}. \tag{4.37}$$

By integrating (4.37) with respect to y and using Cauchy-Schwarz, it follows that

$$\|\tilde{\phi}(t)\|_{H^1}^2\le C\|\Phi_y(t)\|_{L^2}^2+C\sqrt{\mathcal{E}(t)}\|\tilde{\phi}(t)\|_{L^2}^2. \tag{4.38}$$

Recall (4.5) and (4.6). Then, (4.28) follows from a suitable linear combination of (4.36) and (4.38) as well as letting e_1 and $\tilde{\varepsilon}_1$ be small enough. As to the H^2 estimate of $\tilde{\phi}$, we note that (4.34) gives

$$\|\tilde{\phi}_y(t)\|_{H^1}^2\le C\|\Phi_y(t)\|_{L^2}^2+C\{\sqrt{\mathcal{E}(t)}+|v_+-v_-|\}\{\|\Phi_y(t)\|_{H^1}^2+\|\tilde{\phi}\|_{H^2}^2\}. \tag{4.39}$$

Therefore, (4.29) follows from combining (4.39) and (4.38) and letting e_1 and $\tilde{\varepsilon}_1$ be further small enough. The proof for Lemma 4.3 is complete. □

Now we are prepared to derive the higher order energy estimates on $[\Phi, \Psi]$ in Lemma 4.4 and Lemma 4.5 whose proof will be postponed to Section 5.4 in Appendix. In fact, with $L_t^2 H_x^2$ estimate of $\tilde{\phi}$ on hand, one can regard the terms induced by the self-consistent force $\tilde{\phi}$ as the inhomogeneous sources.

Lemma 4.4. *Under the assumptions of Proposition 4.1, it holds that*

$$\|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_y(s)\|_{L^2}^2 \mathrm{d}s \le C\|\tilde{u}_0\|_{L^2}^2 + C\int_0^t \|[\Phi_y, \Psi_y, \tilde{\phi}_y](s)\|_{L^2}^2 \mathrm{d}s, \tag{4.40}$$

and

$$\|\tilde{u}_y(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_{yy}(s)\|_{L^2}^2 \mathrm{d}s \le C\|\tilde{u}_{0y}\|_{L^2}^2 + C\int_0^t \|[\tilde{v}, \tilde{v}_y, \tilde{u}_y, \tilde{\phi}_y](s)\|_{L^2}^2 \mathrm{d}s, \tag{4.41}$$

for all $t \in [0, M]$.

Next, we derive the energy dissipation term $\|\tilde{v}_y\|_{L^2}$.

Lemma 4.5. *Under the assumptions of Proposition 4.1, it holds that*

$$\begin{aligned}
&\|\tilde{v}_y(t)\|_{L^2}^2 + \int_0^t \|\tilde{v}_y(s)\|_{L^2}^2 \mathrm{d}s \\
&\le C\|[\tilde{v}_{0y}, \tilde{u}_0]\|_{L^2}^2 + C\|\tilde{u}(t)\|_{L^2}^2 + Ce_1\int_0^t \|\tilde{u}_{yy}(s)\|_{L^2}^2 \mathrm{d}s \\
&\quad + C\int_0^t \left\{\|\tilde{u}(s)\|_{H^1}^2 + \|\tilde{v}(s)\|_{L^2}^2 + \|\tilde{\phi}_y(s)\|_{L^2}^2\right\} \mathrm{d}s,
\end{aligned} \tag{4.42}$$

for all $t \in [0, M]$.

Finally, to close the a priori assumption (4.5), we need to estimate the time derivative $\tilde{\phi}_t$. In fact, we have the following

Lemma 4.6. *Under the assumptions of Proposition 4.1, it holds that*

$$\|\tilde{\phi}_t\|_{H^1}^2 \le C\{\|\tilde{u}_y\|_{L^2}^2 + \|\tilde{v}_y\|_{L^2}^2\}, \tag{4.43}$$

for all $t \in [0, M]$.

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Proof. Differentiating the third equation of (4.1) with respect to t and taking the inner product of the resultant equation with $\tilde{\phi}_t$, one has

$$
\begin{aligned}
(ve^{\phi}\tilde{\phi}_t,\tilde{\phi}_t)+(\lambda^2 v^{-1}\tilde{\phi}_{ty},\tilde{\phi}_{ty}) &= \left(\partial_y(-\lambda^2 v^{-2}\phi_y\tilde{v}_t),\tilde{\phi}_t\right)+(-e^{\phi}\tilde{v}_t,\tilde{\phi}_t)\\
&= \left(\lambda^2 v^{-2}\phi_y(s\tilde{v}_y+\tilde{u}_y),\tilde{\phi}_{ty}\right)+\left(-e^{\phi}(s\tilde{v}_y+\tilde{u}_y),\tilde{\phi}_t\right), \qquad (4.44)
\end{aligned}
$$

where the first equation of (4.1) has been used for obtaining the second equality. By Cauchy-Schwarz, it is straightforward to bound the right-hand side of (4.44) by

$$
\{\eta+C\sqrt{\mathcal{E}(t)}\}\|\tilde{\phi}_t\|_{H^1}^2+\{C_\eta+C|v_+-v_-|+C\sqrt{\mathcal{E}(t)}\}\{\|\tilde{u}_y\|_{L^2}^2+\|\tilde{v}_y\|_{L^2}^2\}.
$$

Recall (4.5) and (4.6). Thus, (4.43) follows by taking $\eta>0$ suitably small and also letting e_1 and $\tilde{\varepsilon}_1$ be small enough. The proof of Lemma 4.6 is then complete. □

Proof of Proposition 4.1. Letting positive constants e_1 and $\tilde{\varepsilon}_1$ be small enough, a suitable linear combination of all estimates (4.9), (4.28), (4.40), (4.41), (4.42) and (4.43) yields that

$$
\|[\Phi,\Psi](t)\|_{H^2}^2+\|\tilde{\phi}(t)\|_{H^1}^2+\int_0^t \mathcal{D}(s)\mathrm{d}s \le C\|[\Phi_0,\Psi_0]\|_{H^2}^2+C\|\tilde{\phi}(0,\cdot)\|_{H^1}^2, \qquad (4.45)
$$

where $\mathcal{D}(s)$ is defined in (1.42). Since e_1 and $\tilde{\varepsilon}_1$ can be further small enough, by (4.29) one has

$$
\|\tilde{\phi}(t)\|_{H^2}^2 \le C\|\Phi_y(t)\|_{H^1}^2, \qquad (4.46)
$$

for all $t\in[0,M]$. Then, from (4.45) together with (4.46), one has

$$
\|[\Phi,\Psi,\tilde{\phi}](t)\|_{H^2}^2+\int_0^t \mathcal{D}(s)\mathrm{d}s \le C\|[\Phi_0,\Psi_0]\|_{H^2}^2,
$$

which proves (4.7). Therefore, the proof of Proposition 4.1 is complete. □

4.3. Global existence and large time behavior

This part is devoted to proving Theorem 1.10. First, the local-in-time existence and uniqueness of solutions $[\Phi,\Psi,\tilde{\phi}]$ to the Cauchy problem on the system (4.3) with initial data $[\Phi_0,\Psi_0]$ can be obtained in a usual way; we omit the details by brevity. Furthermore, by a continuity argument, the global existence of the solution $[\Phi,\Psi,\tilde{\phi}]$ follows from the uniform a priori estimates obtained in Proposition 4.1. As a consequence, the solution to (4.1) with the corresponding initial data is given by $[\tilde{v},\tilde{u},\tilde{\phi}]=[\Phi_y,\Psi_y,\tilde{\phi}]$. As mentioned before, we also omit the proof of uniqueness for brevity. Therefore, it remains to show the large time behavior (1.44). To do this, we see from (4.1) as well as (1.43) that

$$\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|[\tilde{v},\tilde{u},\tilde{\phi}](t)\|_{L^2}^2\right|\mathrm{d}t \le 2\int_0^\infty \{|(\tilde{v},\tilde{v}_t)|+|(\tilde{u},\tilde{u}_t)|+|(\tilde{\phi},\tilde{\phi}_t)|\}\mathrm{d}t \le C\int_0^\infty \mathcal{D}(t)\mathrm{d}t \le CE_0.$$

Moreover, since it also holds that

$$\int_0^t \|[\tilde{v},\tilde{u},\tilde{\phi}](s)\|_{L^2}^2\mathrm{d}s \le C\int_0^\infty \mathcal{D}(t)\mathrm{d}t \le CE_0,$$

one can see that $\|[\tilde{v},\tilde{u},\tilde{\phi}](t)\|_{L^2}$ tends to zero as $t\to\infty$. Hence by Sobolev inequality, one has

$$\|[\tilde{v},\tilde{u},\tilde{\phi}](t)\|_{L^\infty} \le \sqrt{2}\|[\tilde{v},\tilde{u},\tilde{\phi}](t)\|_{L^2}^{1/2}\|[\tilde{v}_y,\tilde{u}_y,\tilde{\phi}_y](t)\|_{L^2}^{1/2} \le CE_0^{1/4}\|[\tilde{v},\tilde{u},\tilde{\phi}](t)\|_{L^2}^{1/2},$$

which goes to zero as $t\to\infty$. This proves (1.44). Therefore, the proof of Theorem 1.10 is complete. □

5. Appendix

5.1. KdV-Burgers shock profile

For suitably small $\delta>0$, the existence of monotone KdV-Burgers shock waves was proved by Bona and Schonbek [3]. For our use, one has to show the bounds on derivatives is independent of δ.

Lemma 5.1. *Let $0<\alpha<2$. If $\delta>0$ is suitably small, the equations (1.24), (1.26), and (1.27) have the smooth solutions n_1, u_1 and ϕ_1, respectively, which are unique up to a spatial shift and satisfy the following properties:*

$$n_1', u_1', \phi_1' < 0, \tag{5.1}$$

and

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}[n_1-n_{1,\pm}, u_1-u_{1,\pm}, \phi_1-\phi_{1,\pm}]\right| \le C_k e^{-\alpha|z|}, \quad z \gtrless 0, \tag{5.2}$$

for any integer $k\ge 0$, where each positive constant C_k is independent of δ.

Proof. The existence and uniqueness of the smooth shock profile with properties (5.1) have been proved in [3]. We only show (5.2). Integrating (1.24) from $-\infty$ to z, we have

$$2\sqrt{T+1}n_1 + (T+1)n_1^2 - \sqrt{T+1}n_1' + \delta n_1'' = 0. \tag{5.3}$$

Let $q=n_1'$. Then, (5.3) is equivalent to the following 1st-order ODE system for $[n_1,q]$:

$$\begin{cases} n_1' = q, \\ q' = \delta^{-1}\left\{\sqrt{T+1}q - (T+1)n_1^2 - 2\sqrt{T+1}n_1\right\}. \end{cases}$$

One can compute the Jacobian at the far fields $(0,0)$ and $(-\frac{2}{\sqrt{T+1}},0)$,

$$J_\pm = \begin{pmatrix} 0 & 1 \\ \pm 2\delta^{-1}\sqrt{T+1} & \delta^{-1}\sqrt{T+1} \end{pmatrix},$$

the eigenvalues

$$\lambda_{-,1} = \frac{\sqrt{T+1}-\sqrt{T+1-8\sqrt{T+1}\delta}}{2\delta}, \lambda_{-,2} = \frac{\sqrt{T+1}+\sqrt{T+1-8\sqrt{T+1}\delta}}{2\delta}$$

associated with J_- and the eigenvalues

$$\lambda_{+,1} = \frac{\sqrt{T+1}-\sqrt{T+1+8\sqrt{T+1}\delta}}{2\delta}, \lambda_{+,2} = \frac{\sqrt{T+1}+\sqrt{T+1+8\sqrt{T+1}\delta}}{2\delta}$$

associated with J_+. It is straightforward to check that

$$\lambda_{-,1} > 0,\ \lambda_{-,2} > 0,\ \lambda_{+,1} < 0,\ \lambda_{+,2} > 0.$$

Hence we have

$$\lim_{z\to-\infty} \frac{q}{n_1} = \lambda_{-,1} = \frac{4\sqrt{T+1}}{\sqrt{T+1}+\sqrt{T+1-8\sqrt{T+1}\delta}} = 2 + O(\delta),$$
$$\lim_{z\to+\infty} \frac{q}{n_1+\frac{2}{\sqrt{T+1}}} = \lambda_{+,1} = \frac{-4\sqrt{T+1}}{\sqrt{T+1}+\sqrt{T+1-8\sqrt{T+1}\delta}} = -2 + O(\delta).$$

This implies that, for any $0 < \alpha < 2$,

$$|n_1 - n_{1,\pm}| \le Ce^{-\alpha|z|}, \quad z \gtrless 0, \tag{5.4}$$

provided that $\delta > 0$ is suitably small. Next, we estimate the derivatives of n_1. Taking the inner product of (5.3) with $w_\alpha^2 n_1'$ gives

$$\sqrt{T+1}\|n_1'\|^2_{L^2_\alpha} = (\delta n_1'', w_\alpha^2 n_1') + (T+1)\left(n_1(n_1+\frac{2}{\sqrt{T+1}}), w_\alpha^2 n_1'\right).$$

From integration by parts, the first inner product term is equal to

$$-\delta(n_1', w_\alpha w_\alpha' n_1') \le C\delta\|n_1'\|^2_{L^2_\alpha}.$$

By Cauchy-Schwarz, the second one is bounded by

$$\eta\|n_1'\|^2_{L^2_\alpha} + C_\eta\left\|n_1(n_1+\frac{2}{\sqrt{T+1}})\right\|^2_{L^2_\alpha},$$

for $\eta > 0$. Therefore, by taking both $\eta > 0$ and $\delta > 0$ suitably small, we have

$$\|n_1'\|_{L^2_\alpha} \le C \left\| n_1(n_1 + \frac{2}{\sqrt{T+1}}) \right\|_{L^2_\alpha} \le C.$$

Here we have used the exponential decay property (5.4) in the last inequality. The higher-order derivatives can be treated similarly. The proof of Lemma 5.1 is complete. □

5.2. Error estimates

The following result gives the estimates on errors between the first-order approximation $[n_1, u_1, \phi_1]$ and the modified one $[n_{1,\varepsilon}, u_{1,\varepsilon}, \phi_{1,\varepsilon}]$ defined in (3.1). It can be shown by the same energy method as the one used for proving Lemma 5.1. So the proof is omitted for brevity.

Lemma 5.2. *Let $0 < \alpha < 2$. Assume that both $\varepsilon > 0$ and $\delta > 0$ are suitably small. For any integer $k \ge 0$, there exists a constant $C_{k,\alpha} > 0$ independent of δ and ε such that*

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} [n_{1,\varepsilon} - n_{1,\varepsilon}(\pm\infty) - (n_1 - n_{1,\pm})] \right| \le C_{k,\alpha} \varepsilon e^{-\alpha|z|},$$
$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} [u_{1,\varepsilon} - u_{1,\varepsilon}(\pm\infty) - (u_1 - u_{1,\pm})] \right| \le C_{k,\alpha} \varepsilon e^{-\alpha|z|},$$
$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} [\phi_{1,\varepsilon} - \phi_{1,\varepsilon}(\pm\infty) - (\phi_1 - \phi_{1,\pm})] \right| \le C_{k,\alpha} \varepsilon e^{-\alpha|z|},$$

for $z \lesseqgtr 0$. Moreover, let $[n_2, u_2, \phi_2] := \varepsilon^{-1}[n_{1,\varepsilon} - n_1, u_{1,\varepsilon} - u_1, \phi_{1,\varepsilon} - \phi_1]$, then it holds that

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} [n_2, u_2, \phi_2](z) \right| \le C_{k,\alpha}, \quad z \in \mathbb{R}. \tag{5.5}$$

5.3. Explicit formulas of r_2 and r_3

For completeness, we write down the explicit formulas of r_2 and r_3 as

$$\begin{aligned} r_2 =& \frac{\varepsilon n_R}{\sqrt{T+1}} \Big\{ 2(T+1)\varepsilon^{-1}(n_{1,\varepsilon} - n_1) - 1 + 2(2\sqrt{T+1} - \varepsilon) n_{1,\varepsilon} \\ & + 3(T+1) n_{1,\varepsilon}^2 + 2\delta(1 + \varepsilon n_{1,\varepsilon}) \phi_{1,\varepsilon}'' - \varepsilon\delta(1+\varepsilon n_{1,\varepsilon})(\phi_{1,\varepsilon}')^2 \Big\} \\ & + \frac{\varepsilon\delta(2n_{1,\varepsilon} + \varepsilon n_{1,\varepsilon}^2)}{\sqrt{T+1}} \phi_R'' + \frac{\varepsilon n_R'}{\sqrt{T+1}} - \frac{\varepsilon\delta\phi_{1,\varepsilon}'(1+\varepsilon n_{1,\varepsilon})^2}{\sqrt{T+1}} \phi_R', \end{aligned} \tag{5.6}$$

and

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$$
\begin{aligned}
r_3 =& \frac{\varepsilon[(T+1)(2+3\varepsilon n_{1,\varepsilon}) - s^2 + \varepsilon^2\delta\phi''_{1,\varepsilon} - \frac{\delta}{2}\varepsilon^3(\phi'_{1,\varepsilon})^2]n_R^2}{\sqrt{T+1}} \\
&+ (T+1)^{3/2}\varepsilon^3 n_R^3 + \frac{2\varepsilon^2\delta(1+\varepsilon n_{1,\varepsilon})n_R\phi''_R}{\sqrt{T+1}} + \frac{\varepsilon^4\delta n_R^2\phi''_R}{\sqrt{T+1}} \\
&- \frac{\varepsilon^3\delta\phi'_{1,\varepsilon}\phi'_R[2n_R(1+\varepsilon n_{1,\varepsilon})+\varepsilon^2 n_R^2]}{\sqrt{T+1}} - \frac{\varepsilon^2\delta(\phi'_R)^2(1+\varepsilon n_{1,\varepsilon}+\varepsilon^2 n_R)^2}{2\sqrt{T+1}},
\end{aligned}
\tag{5.7}
$$

respectively.

5.4. Higher order energy estimates

We give the detailed proof of Lemma 4.4 and Lemma 4.5 as follows.

Proof of Lemma 4.4. Taking the inner product of the second equation of (4.1) with $\tilde{u}$ with respect to y over $\mathbb{R}$, one has

$$
(\tilde{u}_t - s\tilde{u}_y, \tilde{u}) + \left(T\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)_y - \mu\left(\frac{u_y}{v}-\frac{\bar{u}_y}{v}\right)_y, \tilde{u}\right) - \left(\frac{\phi_y}{v}-\frac{\bar{\phi}_y}{\bar{v}}, \tilde{u}\right) = 0. \tag{5.8}
$$

We estimate the left-hand inner products term by term. The first term is equal to $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}(t)\|_{L^2}^2$. From integration by parts, the second term is computed as

$$
\begin{aligned}
\left(T\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)_y - \mu\left(\frac{u_y}{v}-\frac{\bar{u}_y}{v}\right)_y, \tilde{u}\right) &= \left(-T\left(\frac{1}{v}-\frac{1}{\bar{v}}\right) + \mu\left(\frac{u_y}{v}-\frac{\bar{u}_y}{v}\right), \tilde{u}_y\right) \\
&= \left(\mu v^{-1}\tilde{u}_y, \tilde{u}_y\right) + \left((-T+\mu\bar{u}_y)\left(\frac{1}{v}-\frac{1}{\bar{v}}\right), \tilde{u}_y\right),
\end{aligned}
$$

where the first term in the last line above is a good one and the second term is bounded by $\eta\|\tilde{u}_y\|_{L^2}^2 + C_\eta\|\Phi_y\|_{L^2}^2$ with an arbitrary constant $0<\eta<1$. The third term on the left-hand side of (5.8) is bounded by $C\{\|\Phi_y\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{\phi}_y\|_{L^2}^2\}$. Plugging these estimates back into (5.8) and letting $0<\eta<1$ be suitably small, one has

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}(t)\|_{L^2}^2 + c\|\tilde{u}_y\|_{L^2}^2 \le C\left(\|\Phi_y\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{\phi}_y\|_{L^2}^2\right). \tag{5.9}
$$

Then (4.40) follows from integrating (5.9) over $[0,t]$.

Next, we show (4.41). Taking the inner product of the second equation of (4.1) with $-\tilde{u}_{yy}$ with respect to y over $\mathbb{R}$ gives that

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$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}_y\|_{L^2}^2 + \underbrace{\left(T\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)_y, -\tilde{u}_{yy}\right)}_{\mathcal{I}_9} + \underbrace{\left(\mu\left(\frac{u_y}{v}-\frac{\bar{u}_y}{\bar{v}}\right)_y, \tilde{u}_{yy}\right)}_{\mathcal{I}_{10}}$$

$$+\underbrace{\left(\frac{\phi_y}{v}-\frac{\bar{\phi}_y}{\bar{v}}, \tilde{u}_{yy}\right)}_{\mathcal{I}_{11}} = 0. \tag{5.10}$$

The inner product terms $\mathcal{I}_9$, $\mathcal{I}_{10}$ and $\mathcal{I}_{11}$ above are computed as follows. By Cauchy-Schwarz, $\mathcal{I}_9$ and $\mathcal{I}_{11}$ can be bounded respectively as

$$|\mathcal{I}_9| = \left|T\left(\frac{-v_y}{v^2}+\frac{\bar{v}_y}{\bar{v}^2}, -\tilde{u}_{yy}\right)\right| \le \eta\|\tilde{u}_{yy}\|_{L^2}^2 + C_\eta(\|\tilde{v}\|_{L^2}^2 + \|\tilde{v}_y\|_{L^2}^2),$$

and

$$|\mathcal{I}_{11}| \le \eta\|\tilde{u}_{yy}\|_{L^2}^2 + C_\eta(\|\Phi_y\|_{L^2}^2 + \|\tilde{\phi}_y\|_{L^2}^2),$$

with an arbitrary constant $0<\eta<1$. As to $\mathcal{I}_{10}$, we rewrite it as

$$\mathcal{I}_{10} = \left(\mu v^{-1}\tilde{u}_{yy}, \tilde{u}_{yy}\right) - \left(\mu v^{-2}\tilde{v}_y\tilde{u}_y, \tilde{u}_{yy}\right) + \left(\mu\bar{u}_{yy}\left(\frac{1}{v}-\frac{1}{\bar{v}}\right), \tilde{u}_{yy}\right)$$
$$+\left(\mu\bar{u}_y\bar{v}_y\left(\frac{1}{\bar{v}^2}-\frac{1}{v^2}\right), \tilde{u}_{yy}\right) - \left(\mu\frac{\bar{v}_y\tilde{u}_y+\bar{u}_y\tilde{v}_y}{v^2}, \tilde{u}_{yy}\right). \tag{5.11}$$

On the right-hand side of (5.11), the first term is a good one, and the second term is bounded as

$$\begin{aligned}\left|\left(v^{-2}\tilde{v}_y\tilde{u}_y, \tilde{u}_{yy}\right)\right| &\le C\|\tilde{u}_y\|_{L^\infty}\|\tilde{v}_y\|_{L^2}\|\tilde{u}_{yy}\|_{L^2}\\ &\le C\|\tilde{u}_y\|_{L^2}^{\frac{1}{2}}\|\tilde{v}_y\|_{L^2}^{\frac{1}{4}}\|\tilde{u}_{yy}\|_{L^2}^{\frac{3}{2}}\|\tilde{v}_y\|_{L^2}^{\frac{3}{4}}\\ &\le C\|\tilde{v}_y\|_{L^2}\|\tilde{u}_y\|_{L^2}^2 + C\|\tilde{v}_y\|_{L^2}\|\tilde{u}_{yy}\|_{L^2}^2\\ &\le C\sqrt{\mathcal{E}(t)}\left(\|\tilde{u}_y\|_{L^2}^2 + \|\tilde{u}_{yy}\|_{L^2}^2\right),\end{aligned}$$

where we have used the Sobolev inequality in the second line and Young's inequality in the third line. Also, the last three terms on the right-hand side of (5.11) are bounded by

$$C|v_+-v_-|\left\{\|\tilde{u}_y\|_{L^2}^2 + \|\tilde{u}_{yy}\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2 + \|\tilde{v}_y\|_{L^2}^2\right\}.$$

Plugging those estimates on $\mathcal{I}_9$ to $\mathcal{I}_{11}$ back into (5.10) and taking $\eta>0$ suitably small, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}_y\|_{L^2}^2 + c\|\tilde{u}_{yy}\|_{L^2}^2 \le C\|[\tilde{v}, \tilde{v}_y, \tilde{u}_y, \tilde{\phi}_y]\|_{L^2}^2 + C\{|v_+-v_-| + \sqrt{\mathcal{E}(t)}\}\|\tilde{u}_{yy}\|_{L^2}^2. \tag{5.12}$$

Recall (4.5) and (4.6). Then (4.41) follows from integrating (5.12) over $[0,t]$ and letting e_1 and $\tilde{\varepsilon}_1$ be small enough. The proof of Lemma 4.4 is complete. □

Proof of Lemma 4.5. By taking the inner products of the first and second equations of (4.1) with $-\mu\tilde{v}_{yy}$ and $-v\tilde{v}_y$ respectively and adding the resultant equations together, we obtain that

$$\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}_y\|_{L^2}^2 + \underbrace{(\tilde{u}_t - s\tilde{u}_y, -v\tilde{v}_y)}_{\mathcal{I}_{12}} + \underbrace{\left(T\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)_y, -v\tilde{v}_y\right)}_{\mathcal{I}_{13}}$$
$$+\underbrace{(\mu\tilde{u}_y, \tilde{v}_{yy}) + \left(\left(\frac{\mu u_y}{v}-\frac{\mu\bar{u}_y}{\bar{v}}\right)_y, v\tilde{v}_y\right)}_{\mathcal{I}_{14}} - \underbrace{\left(\frac{\phi_y}{v}-\frac{\bar{\phi}_y}{\bar{v}}, v\tilde{v}_y\right)}_{\mathcal{I}_{15}} = 0. \quad (5.13)$$

We estimate terms $\mathcal{I}_{12}$ to $\mathcal{I}_{15}$ as follows. Firstly, $\mathcal{I}_{12}$ is computed as

$$\begin{aligned}\mathcal{I}_{12} &= \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, -v\tilde{v}_y) + (\tilde{u}, \tilde{v}_t\tilde{v}_y) + (\tilde{u}, v\tilde{v}_{ty}) + (s\tilde{u}_y, v\tilde{v}_y)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, -v\tilde{v}_y) + (\tilde{u}, -\tilde{v}_y\tilde{v}_t) + (\tilde{u}_y, -v\tilde{v}_t) + (s\tilde{u}_y, v\tilde{v}_y).\end{aligned}$$

Replacing $\tilde{v}_t$ by the first equation of (4.1), $\mathcal{I}_{12}$ is further equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, -v\tilde{v}_y) + (\tilde{u}, -\tilde{v}_y(s\tilde{v}_y+\tilde{u}_y)) + (\tilde{u}_y, -v(s\tilde{v}_y+\tilde{u}_y)) + s(\tilde{u}_y, v\tilde{v}_y),$$

where the last three terms are bounded by $\eta\|\tilde{v}_y\|_{L^2}^2 + C_\eta\{\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}_y\|_{L^2}^2\}$ with an arbitrary constant $0<\eta<1$. As to $\mathcal{I}_{13}$, it follows that

$$\mathcal{I}_{13} = \left(-\frac{Tv_y}{v^2}+\frac{T\bar{v}_y}{\bar{v}^2}, -v\tilde{v}_y\right) = (Tv^{-1}\tilde{v}_y, \tilde{v}_y) + \left(T\bar{v}_y\left(\frac{1}{\bar{v}^2}-\frac{1}{v^2}\right), -v\tilde{v}_y\right),$$

where the first term on the right is good and the second inner product is bounded by $\eta\|\tilde{v}_y\|_{L^2}^2 + C_\eta\|\tilde{v}\|_{L^2}^2$ with an arbitrary constant $0<\eta<1$. For $\mathcal{I}_{14}$, one has

$$\begin{aligned}\mathcal{I}_{14} &= \left(\frac{\mu u_{yy}}{v}-\frac{\mu\bar{u}_{yy}}{\bar{v}}, v\tilde{v}_y\right) + (\mu\tilde{u}_y, \tilde{v}_{yy}) + \left(\frac{-\mu u_y v_y}{v^2}+\frac{\mu\bar{u}_y\bar{v}_y}{\bar{v}^2}, v\tilde{v}_y\right)\\ &= \left(\mu\bar{u}_{yy}(v^{-1}-\bar{v}^{-1}), v\tilde{v}_y\right) + \left(\frac{-\mu u_y v_y}{v^2}+\frac{\mu\bar{u}_y\bar{v}_y}{\bar{v}^2}, v\tilde{v}_y\right).\end{aligned}$$

By Cauchy-Schwarz, the first inner product term on the right is bounded as

$$\left|\left(\mu\bar{u}_{yy}(v^{-1}-\bar{v}^{-1}), v\tilde{v}_y\right)\right| \le \eta\|\tilde{v}_y\|_{L^2}^2 + C_\eta\|\tilde{v}\|_{L^2}^2.$$

And the second one is computed as

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$$\left(\frac{-\mu u_y v_y}{v^2}+\frac{\mu\bar{u}_y\bar{v}_y}{\bar{v}^2}, v\tilde{v}_y\right)=\left(-\mu v^{-1}(\bar{u}_y\tilde{v}_y+\bar{v}_y\tilde{u}_y), \tilde{v}_y\right)$$
$$-\left(\mu\bar{u}_y\bar{v}_y\left(\frac{1}{v^2}-\frac{1}{\bar{v}^2}\right), v\tilde{v}_y\right)-(\mu v^{-1}\tilde{v}_y\tilde{u}_y, \tilde{v}_y). \quad (5.14)$$

On the right-hand side of (5.14), the last inner product term is bounded as

$$\begin{aligned}|(\mu v^{-1}\tilde{v}_y\tilde{u}_y, \tilde{v}_y)| &\le \|\tilde{u}_y\|_{L^\infty}\|\tilde{v}_y\|_{L^2}^2 \le C\|\tilde{u}_y\|_{H^1}\|\tilde{v}_y\|_{L^2}^2\\ &\le C\|\tilde{v}_y\|_{L^2}\|\tilde{u}_y\|_{H^1}^2+\|\tilde{v}_y\|_{L^2}^3 \le C\sqrt{\mathcal{E}(t)}\{\|\tilde{u}_y\|_{H^1}^2+\|\tilde{v}_y\|_{L^2}^2\},\end{aligned}$$

and the rest terms are bounded by $C\{|v_+-v_-|+\sqrt{\mathcal{E}(t)}\}\{\|\tilde{u}_y\|_{L^2}^2+\|\tilde{v}\|_{H^1}^2\}$. Thus, it follows from the above estimates that

$$|\mathcal{I}_{14}| \le \eta\|\tilde{v}_y\|_{L^2}^2+C_\eta\|\tilde{v}\|_{L^2}^2+C\{|v_+-v_-|+\sqrt{\mathcal{E}(t)}\}\{\|\tilde{u}_y\|_{H^1}^2+\|\tilde{v}_y\|_{L^2}^2\},$$

with an arbitrary constant $0<\eta<1$. For $\mathcal{I}_{15}$, it holds by Cauchy-Schwarz that

$$|\mathcal{I}_{15}| \le \eta\|\tilde{v}_y\|_{L^2}^2+C_\eta\{\|\tilde{v}\|_{L^2}^2+\|\tilde{\phi}_y\|_{L^2}^2\}.$$

Plugging those estimates on $\mathcal{I}_{12}$ to $\mathcal{I}_{15}$ back into (5.13) and letting $\eta>0$ be chosen suitably small, we obtain that

$$\frac{d}{dt}\left\{\frac{\mu}{2}\|\tilde{v}_y\|_{L^2}^2+(\tilde{u}, -v\tilde{v}_y)\right\}+c\|\tilde{v}_y\|_{L^2}^2 \le C\{\|\tilde{u}\|_{H^1}^2+\|\tilde{v}\|_{L^2}^2+\|\tilde{\phi}_y\|_{L^2}^2\}$$
$$+C\{|v_+-v_-|+\sqrt{\mathcal{E}(t)}\}\{\|\tilde{v}_y\|_{L^2}^2+\|\tilde{u}_{yy}\|_{L^2}^2\}. \quad (5.15)$$

Recall (4.5) and (4.6). Then (4.42) follows from integrating (5.15) over $[0,t]$ and letting e_1 and $\tilde{\varepsilon}_1$ be small enough. The proof of Lemma 4.5 is complete. □

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