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Spreading speed for a KPP type reaction-diffusion system with heat losses and fast decaying initial data

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Abstract

In this work we consider a two-components reaction-diffusion system of KPP type with heat losses posed in a straight cylinder and equipped with fast decaying initial data. We derive a criteria condition – expressed in term of the sign of a suitable elliptic eigenvalue – for extinction and for propagation of the solutions. In the case of propagation we derive a spreading speed property and we obtain an asymptotic expansion in the large times for the location of the front, that is strongly related with the minimal wave speed of the travelling waves. We also obtain decay estimates for the solutions ahead the front.

MSC: 35K57; 35B40

Keywords: Spreading speed; Reaction-diffusion system of KPP type; Heat losses

1. Introduction and main results

Let Ω be a given bounded and smooth domain of $\mathbb{R}^N$, for some given integer $N \geq 1$. In this work we consider the following reaction-diffusion system, posed on the straight cylinder $\Sigma = \mathbb{R} \times \Omega$, for the unknown functions u and v:

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$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = -\beta(y)uv, \\ \\ \dfrac{\partial v}{\partial t} - \Delta v = \beta(y)uv - \mu(y)v. \end{cases} \tag{1.1}$$

This problem is posed for time $t > 0$ and for $(x, y) \in \Sigma$. The operator Δ stands for the standard Laplace operator on the cylinder Σ, namely $\Delta = \frac{\partial^2}{\partial x^2} + \Delta_y$ where Δ_y denotes the Laplace operator on the cross section Ω of the cylinder. Herein $\beta : \overline{\Omega} \to \mathbb{R}_+$ and $\mu : \overline{\Omega} \to \mathbb{R}_+$ are two given smooth functions while $D > 0$ is a given parameter.

This reaction-diffusion system is supplemented with the following boundary condition

$$\frac{\partial u}{\partial \nu} = 0,\ \frac{\partial v}{\partial \nu} = -\sigma(y)v,\ \text{for } t > 0,\ (x, y) \in \partial\Sigma = \mathbb{R} \times \partial\Omega. \tag{1.2}$$

In the above boundary conditions, ν denotes the outward unit normal vector to the boundary $\partial\Sigma = \mathbb{R} \times \partial\Omega$ and $\frac{\partial}{\partial \nu}$ denotes the normal derivative. And, $\sigma : \partial\Omega \to \mathbb{R}_+$ denotes a given smooth function.

In this work we are concerned with the quenching and spreading behaviour of the solutions of the above system of equations when it is supplemented by the following initial data:

$$u(0, x, y) = u_0(x, y) \equiv 1,\ \ v(0, x, y) = v_0(x, y), \tag{1.3}$$

wherein $v_0 : \overline{\Sigma} \to \mathbb{R}_+$ is a non-trivial smooth non-negative function with compact support.

The above problem arises in various applicative fields, such as in chemistry, the combustion theory and in epidemiology.

To see this, consider an – isothermal – auto-catalytic chemical reaction arising in the cylinder Σ of the form

$$A + B \xrightarrow{k_1} 2B.$$

Herein A, B denote chemical reactants while k_1 corresponds to the reaction rate. Then, setting u and v respectively the concentration of the chemical reactants A and B and assuming there is no chemical flux through the boundary of the cylinder Σ, the spatio-temporal evolution of u and v is given by System (1.1)-(1.2) with $\beta(y) = k_1$, $\mu(y) = 0$ and $\sigma(y) = 0$ while D denotes the normalized diffusion rate corresponding to the ratio between the molecular diffusion of A and B. Moreover observe that the initial conditions, as described in (1.3), mean that the reactor is initially uniformly filled with the substance A and a localized amount of the chemical B is added. We refer to Billingham and Needham in [4] for the existence of one-dimensional travelling waves for such a problem, but also for more general system modelling cubic auto-catalytic reactions (see also the work of Chen and Qi in [8] for results on the case of general order reaction). This problem on a cylinder with an additional shear flow has been considered by Hamel and Ryzhik in [22] and, the authors derived the existence of multi-dimensional travelling wave solutions. We also refer to Chen and Qi [7] who derived a spreading speed property for the one-dimensional problem in the case where $0 < D \leq 1$ and, they also provide a refined analysis of the location of the reaction front.

One may also consider two-step chemical reactions of the form

$$A + B \xrightarrow{k_1} 2B \text{ and } B \xrightarrow{k_2} C, \tag{1.4}$$

wherein A, B and C are chemical reactants while k_1 and k_2 correspond to the rate of the first and the second reaction respectively. By setting u and v respectively the concentration of the chemical reactants A and B and still assuming there is no chemical flux through the boundary of the cylinder Σ, the evolution of u and v is given by System (1.1)-(1.2) with $\beta(y) = k_1$, $\mu(y) = k_2$ and $\sigma(y) = 0$. Here again D denotes the normalized diffusion rate corresponding to the ratio of the diffusion rates of A and B. Note also that in the case where the section of the cylinder Ω is an heterogeneous medium, the rates of each reaction may depend on $y \in \overline{\Omega}$. This justifies why we consider, in this work, that the functions β and μ may possibly depend on y.

The above chemical reaction scheme, namely (1.4), also arises in population dynamics and more particularly in epidemiology. In that setting, the substances A, B and C respectively correspond to the densities of susceptible, infected and removed individuals and, the reaction rates k_1 and k_2 stand for the contamination and removed rates respectively. We refer to the original works of Kermack and McKendrick in [14–16] and to the monograph of Murray [27] for more details about this kinetic reaction scheme in the framework of epidemiology. The diffusion coefficient D denotes, in this context, the normalized diffusion rate of the populations, namely the ratio between the diffusivity rates of susceptible and infected individuals. Note that it can be different from 1 if the epidemic influences the spatial motion of the infected individuals. The existence of one-dimensional travelling waves for such a problem has been studied by Hosono and Ilyas in [25] (see also [13] for results on a similar system with age structure). Note also that if the medium is heterogeneous, it is relevant to assume that the functions β and μ may depend on y. Let us mention that the existence of multi-dimensional travelling waves – on a straight cylinder – for this model with heterogeneous functions β and μ has been studied by Giletti in [18] for arbitrary diffusion coefficient $D > 0$. We also refer to Ducrot and Giletti [12] for a study of the one-dimensional problem with $D = 0$ in a periodic environment and to [6] for the study of the spreading speed of a similar system with age structure. In this aforementioned work [12], the existence of waves and the large time convergence to wave profiles have been studied. In addition, let us mention that in this population dynamics framework, the Robin boundary condition ($\sigma(y) \not\equiv 0$) for the v-component may also be relevant. It can be interpreted as an infected removal rate at the boundary due to control policies for individuals reaching the boundary of the domain.

Let us finally mention that System (1.1) with $\beta(y) = 1$ and $\mu(y) = 0$ also describes the concentration of a chemical reactant A, u, and its temperature, v, of a one-step chemical reaction $A \longrightarrow B$; when the term v, arising in the product uv, is replaced by the Arrhenius reaction. Within this framework, system (1.1) corresponds to the usual thermo-diffusive reaction-diffusion system modelling flame propagation and, the diffusion coefficient, D, denotes the inverse of the so-called Lewis number, the ratio between the thermal and reactant diffusivity. We refer to Matkowsky and Sivashinsky in [26]. Moreover in that context, the Robin type boundary conditions for v describe the heat losses through the boundary of Σ.

Problem (1.1)-(1.2), posed on a straight cylinder with $\beta(y) = 1$, $\mu(y) = 0$ and $\sigma(y) = q > 0$ has also been widely considered in the literature. We refer to the work of Berestycki et al. in [3] where the authors study this problem with an addition shear flow. They derive several properties of the solutions including flame extinction, blow-off and propagation. While extinction and blow-off properties has been studied for rather general initial data, their results for propagation are concerned with initial temperature profile, namely v_0, that do not have a too fast decay at infinity, and roughly speaking when this decay rate is similar to the one of a travelling wave associated to a non-minimal (super-critical) wave speed. This does not cover the case where the function v_0 is compactly supported, which is the topic of the present manuscript. In the aforementioned work, the authors also prove the existence of multi-dimensional travelling wave solutions for this problem in the case $D = 1$. This result has been generalized for an arbitrary diffusion coefficient $D > 0$ by Hamel and Ryzhik in [21].

In this work, as mentioned above, we focus on the case where the initial temperature profile, namely v_0 in (1.3), has a very fast decay at infinity. Here it is assumed to be compactly supported. Roughly speaking we shall show that under some threshold condition to avoid flame extinction, the solution exhibits a propagating behaviour into the cylinder and, we shall furthermore provide refined information on the location of the reaction front, that is related to what we shall call the minimal wave speed c^* (see (1.6) below).

In order to describe the asymptotic behaviour of Problem (1.1)-(1.3) let us introduce, $\Lambda \in \mathbb{R}$, the principal eigenvalue of the elliptic problem on the bounded section Ω:

$$\begin{cases} \Lambda\varphi(y) = \Delta_y\varphi(y) + (\beta(y) - \mu(y))\,\varphi(y) \text{ in } \Omega, \\ \nabla\varphi(y)\cdot\overrightarrow{n}(y) = -\sigma(y)\varphi(y),\ \ y \in \partial\Omega, \\ \varphi \in C^0\left(\overline{\Omega}\right) \cap C^2(\Omega) \ \text{ with } \varphi > 0 \text{ on } \overline{\Omega}. \end{cases} \tag{1.5}$$

Here $\overrightarrow{n}(y)$ denotes the outward unit normal vector to $\partial\Omega$ at $y \in \partial\Omega$. We also consider $\varphi \in C^2\left(\overline{\Omega}\right)$, a principle eigenvector of the above spectral problem. Observe that Λ has the following variational expression:

$$-\Lambda = \inf_{\psi\in H^1(\Omega),\ \|\psi\|_2=1} \left\{ \int_\Omega |\nabla\psi|^2 \mathrm{d}y + \int_\Omega (\mu(y) - \beta(y))\psi^2 \mathrm{d}y + \int_{\partial\Omega} \sigma(y)\psi^2 S(\mathrm{d}y) \right\},$$

wherein $S(\mathrm{d}y)$ denotes the volume surface element of $\partial\Omega$.

The above defined number Λ will act as a threshold for the dynamical behaviour of (1.1)-(1.3). In order to describe our main results, we shall assume, throughout this work, that both production and heat losses are effective, in the following sense.

Assumption 1.1. We assume that the function $\beta : \overline{\Omega} \to \mathbb{R}_+$, $\mu : \overline{\Omega} \to \mathbb{R}_+$ and $\sigma : \partial\Omega \to \mathbb{R}_+$ satisfies

$$\|\beta\|_{L^\infty(\Omega)} > 0 \text{ and } \|\mu\|_{L^\infty(\Omega)} + \|\sigma\|_{L^\infty(\partial\Omega)} > 0.$$

Using the above assumption, our first result reads as the following uniform boundedness property.

Proposition 1.2 *(Uniform boundedness). Let Assumption 1.1 be satisfied. Then the solution pair, (u, v), of (1.1)-(1.3) is uniformly bounded in time.*

According to the author's knowledge, when Assumption 1.1 fails to hold true, and in particular when there is no heat loss (namely $\mu \equiv 0$ and $\sigma \equiv 0$), the boundedness for the solution mostly remains an open problem. It has been proved by Chen and Qi in [7] in the one-dimensional case and when the diffusion parameter D satisfies $0 < D \leq 1$. For a similar problem posed on the whole space $\mathbb{R}^N$, it has been proved by Herrero et al. [23] that the solutions are bounded in the case where $D \leq 1$, while for general value of the diffusion rate D the best upper-bound for the solutions has been obtained by Collet and Xin in [9] and, in that case, the v-component is – uniformly in space – less that $O(\ln(\ln t))$ as $t \to \infty$.

As mentioned above, the parameter Λ, the principal eigenvalue of (1.5), will act as a threshold for the asymptotic behaviour of the solution of (1.1)-(1.3). We now split our main results according to the sign of Λ. Roughly speaking, the condition $\Lambda \leq 0$ will lead us to extinction while the condition $\Lambda > 0$ will ensure the propagation of the local initial disturbance v_0.

Our next result is concerned with the case where $\Lambda \leq 0$. In that case, spatial propagation does not occur and the flame uniformly quenches as time becomes large. Our result reads as follows.

Theorem 1.3 *(Extinction). Let Assumption 1.1 be satisfied. Assume that $\Lambda \leq 0$. Then, if (u, v) denotes a solution of (1.1)-(1.3) then*

$$\lim_{t\to\infty} v(t, x, y) = 0 \textit{ uniformly for } (x, y) \in \overline{\Sigma}.$$

This result has been proved by Berestycki et al. in [3] when $\beta(y) = 1$, $\mu(y) = 0$ and $\sigma(y) = q > 0$ in the case where $\Lambda < 0$. Here we somehow extend this result by considering heterogeneous functions β, μ and σ but also the limit case $\Lambda = 0$. Here the result is stated for some specific initial data as described in (1.3). However, the proof of this result, given below, can easily be extended to the more general case where $0 \leq u_0 \leq 1$ and v_0 is a bounded function on $\overline{\Sigma}$.

We now assume that $\Lambda > 0$ and we define $c^* > 0$ by

$$c^* = 2\sqrt{\Lambda}. \tag{1.6}$$

This quantity c^* will be hereafter refereed to as the minimal wave speed (of (1.1)-(1.2)). It corresponds to the minimal wave speed for travelling wave solutions for all the specific situations mentioned above. Let us also mention that for all these problems, travelling wave solutions do exist for all wave speed $c \geq c^*$ and do not admit wave solution for $c < c^*$. Moreover, due to the heat losses, the travelling wave profiles exhibit a particular shape. The u-component of the waves – monotonically – connects the equilibrium $u = 1$ to an other (spatially homogeneous) equilibrium $\underline{u} > 0$ while the v-component of the waves has a pulse shape profile that connects the equilibrium $v = 0$ to itself.

In this context, $\Lambda > 0$, our main result, stated below, proves that when the heat losses are effective (see Assumption 1.1 above) the solution of Problem (1.1)-(1.3) spreads and propagates the local initial disturbance throughout the spatial domain with the speed c^* defined above and, similarly to the Fisher-KPP equation, the front is located at the abscissa $x(t) = c^*t - \frac{3}{c^*}\ln t$ as $t \to \infty$. We refer to [5,10,19] and the references therein for results on the Fisher-KPP scalar equation (see also [20] for similar results on the KPP equation in a periodic medium). We also refer to the work of Chen and Qi in [7] where a similar result has been obtained for the one-dimensional solution of Problem (1.1)-(1.2) without effective heat loss, namely with $\beta(y) = 1$, $\mu(y) = 0$ and $\sigma(y) = 0$, and when the diffusion coefficient satisfies $D \in (0, 1]$.

Now our spreading result reads as follows.

Theorem 1.4 *(Spreading). Let Assumption 1.1 be satisfied. Assume furthermore that $\Lambda > 0$. Let (u, v) denote the solution of Problem (1.1)-(1.3). Consider the function $\zeta = \zeta(t)$ given by*

$$\zeta(t) = c^* t - \frac{3}{c^*} \ln t \ \ \text{for } t > 0. \tag{1.7}$$

Then the following properties hold true.

(i) ***(Outer spreading for** u**)*** *There exist constants $M > 0$, $\gamma > 0$ and $x_0 > 0$ large enough such that the function u satisfies*

$$1 - Me^{-\gamma(|x|-\zeta(t))} \leq u(t, x, y) < 1,$$

for all $t > 0$ large enough, $|x| \geq x_0 + \zeta(t)$ and for any $y \in \overline{\Omega}$.

(ii) ***(Inner spreading for** u**)*** *One also has*

$$\limsup_{t \to \infty} \sup_{\substack{|x| \leq \zeta(t) \\ y \in \overline{\Omega}}} u(t, x, y) < 1.$$

(iii) ***(Propagation for** v **and decay estimates ahead the front)*** *There exist constants $K > 1$, $\varrho > 0$ and $x_0 > 0$ large enough such that*

$$K^{-1}(|x| - \zeta(t))\, e^{-\frac{c^*}{2}(|x|-\zeta(t))} \leq v(t, x, y),$$

for all time $t > 0$ large enough, $y \in \overline{\Omega}$ and $|x| \in \left[x_0 + \zeta(t), x_0 + \zeta(t) + \varrho\sqrt{t}\right]$ and,

$$v(t, x, y) \leq K(|x| - \zeta(t))\, e^{-\frac{c^*}{2}(|x|-\zeta(t))},$$

for all time $t > 0$ large enough, $y \in \overline{\Omega}$ and $|x| \in [x_0 + \zeta(t), \infty)$.

Remark 1.5. The above propagating result does not explicitly require Assumption 1.1, that is used here to ensure the boundedness of the solutions, as stated in Proposition 1.2. In particular, with no heat loose, namely $\|\mu\|_{L^\infty(\Omega)} + \|\sigma\|_{L^\infty(\partial\Omega)} = 0$, the bounded solutions (with initial conditions $u_0 \equiv 1$ and v_0 non-trivial and compactly supported) enjoy the same propagating behaviour. The boundedness of such solutions holds when $D \leq 1$, by using similar heat kernel estimates as in [7,23], while this remains an open question when $D > 1$.

As a special case of the above result, we recover the notion of spreading speed for the u-component in the spirit of the work of Aronson and Weinberger [1] for scalar reaction-diffusion equations. In our case this property reads as follows:

$$\lim_{t \to \infty} \sup_{\substack{|x| \geq ct \\ y \in \overline{\Omega}}} \{|1 - u(t, x, y)| + v(t, x, y)\} = 0, \ \forall c > c^*,$$

and,

$$\limsup_{t\to\infty} \sup_{\substack{|x|\le ct \\ y\in\overline{\Omega}}} u(t,x,y) < 1,\ \forall c \in \left[0, c^*\right).$$

As far as the inner propagating zone is concerned, namely for $|x| \le ct$ for some $0 < c < c^*$, we expect that the solution looks like a travelling wave profile associated with the minimal wave speed. This means that we expect that the solution, (u, v), satisfy, for all $0 < c < c^*$ and uniformly for $(x, y) \in [-ct, ct] \times \overline{\Omega}$

$$u(t,x,y) \approx \overline{u} \text{ and } v(t,x,y) \approx 0 \text{ for } t \gg 1,$$

for some constant $\overline{u} \in (0, u^*]$. Here $u^* > 0$ is defined as the unique solution of the equation $\Lambda(u^*) = 0$ where $\Lambda(s)$ denotes the principal eigenvalue of the following problem on Ω

$$\begin{cases} \Lambda(s)\psi(y) = \Delta_y \psi(y) + (s\beta(y) - \mu(y))\,\psi(y) \text{ in } \Omega, \\ \nabla\psi(y) \cdot \overrightarrow{n}(y) = -\sigma(y)\psi(y),\ y \in \partial\Omega. \end{cases}$$

At this stage, we are not able to prove the above inner propagating behaviour that remains an open problem.

This manuscript is organized as follows. In Section 2 we prove the uniform boundedness of the solutions as well the extinction of the solutions when $\Lambda \le 0$, namely Proposition 1.2 and Theorem 1.3. Section 3 is devoted to the derivation of preliminary estimates that will be crucially used in Section 4 for the proof of Theorem 1.4.

2. Proof of Proposition 1.2 and Theorem 1.3

This section is concerned with the proof of the boundedness of the solution of (1.1)-(1.3). We shall also prove Theorem 1.3.

2.1. Proof of Proposition 1.2

Now let us first observe that, since $v_0 \not\equiv 0$, one already knows, by applying the comparison principle to each equation separately, that

$$0 < u(t,x,y) < 1 \text{ and } v(t,x,y) > 0 \text{ for all } t > 0 \text{ and } z = (x,y) \in \overline{\Sigma}. \tag{2.8}$$

In order to prove Proposition 1.2 we need to show that the function v is also uniformly bounded. This point is discussed in the following proposition.

Proposition 2.1. *Under the same assumptions that the ones of Theorem 1.4, there exists some constant $\widehat{M} > 0$ such that*

$$v(t,x,y) \le \widehat{M} \text{ for all } t \ge 0 \text{ and } (x,y) \in \overline{\Sigma}.$$

Proof. In order to prove the above result (and thus Proposition 1.2), let us consider $\tilde{\lambda} \in \mathbb{R}$ and $\phi \in C\left(\overline{\Omega}\right)$ with $\phi > 0$, a principal eigenpair of the problem

$$\Delta_y \phi - \mu(y)\phi = -\tilde{\lambda}\phi \text{ in } \Omega \text{ with } \nabla\phi \cdot \overrightarrow{n} = -\sigma(y)\phi \text{ on } \partial\Omega.$$

Observe that, due to Assumption 1.1, one has $\tilde{\lambda} > 0$.

Consider the functions $U = U(t, x)$ and $V = V(t, x)$ defined by

$$U(t,x) = \int_\Omega u(t,x,y)\phi(y)\mathrm{d}y \text{ and } V(t,x) = \int_\Omega v(t,x,y)\phi(y)\mathrm{d}y.$$

Then the functions U and V satisfy

$$\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} + \tilde{\lambda} V = \int_\Omega \beta(y)uv\phi(y)\mathrm{d}y, \tag{2.9}$$

and,

$$\begin{aligned}\frac{\partial U}{\partial t} - D\frac{\partial^2 U}{\partial x^2} =& D\int_{\partial\Omega} \sigma(y)u\phi(y)S(\mathrm{d}y) - D\int_\Omega \left(\mu(y) - \tilde{\lambda}\right)u\phi(y)\mathrm{d}y \\ & - \int_\Omega \beta(y)uv\phi(y)\mathrm{d}y.\end{aligned}$$

As a consequence, since $u \le 1$, there exists some constant $\Theta > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}$, one has

$$\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} + \tilde{\lambda} V \le \Theta - \frac{\partial U}{\partial t} + D\frac{\partial^2 U}{\partial x^2}.$$

Now, for each $p \in [1, \infty]$, we consider the so-called locally uniform Lebesgue space $L^p_{\mathrm{u}}(\mathbb{R})$ defined by

$$L^p_{\mathrm{u}}(\mathbb{R}) = \left\{\psi \in L^p_{\mathrm{loc}}(\mathbb{R}) : \sup_{x\in\mathbb{R}} \|\psi\|_{L^p(x-1,x+1)} < \infty\right\},$$

that becomes a Banach space when endowed with the norm $\|\cdot\|_{L^p_u(\mathbb{R})}$ defined by

$$\|\psi\|_{L^p_{\mathrm{u}}(\mathbb{R})} = \sup_{x\in\mathbb{R}} \|\psi\|_{L^p(x-1,x+1)}, \ \forall \psi \in L^p_{\mathrm{u}}(\mathbb{R}).$$

We refer to the work of Arrieta et al. in [2] and the references cited therein for more details on these spaces and, for results on the heat kernel within this functional framework and in particular for the smoothing property (2.11) below.

Next, since U is bounded and $\tilde{\lambda} > 0$, Theorem 3.8 in [11] applies and ensures that for each $p > 1$ there exists some constant R_p such that for all $T > \tau > 0$ the following estimate holds true

$$\int_\tau^T \|V(t,.)\|_{L^p_{\rm u}(\mathbb{R})}^p \mathrm{d}t \le R_p^p\left[1+\|V(\tau,.)\|_{L^p_{\rm u}(\mathbb{R})}+(T-\tau)^{\frac{1}{p}}\right]^p.$$

From this estimate, applying Lemma 7 in [24], for each $p\in(1,\infty)$ there exist two constants $\Lambda_0(p)$ and $\Gamma_0(p)$ and a sequence $\{t_k\}_{k\ge 0}\subset[0,\infty)$ such that $t_0=0$ and for each $k\ge 0$:

(i) $1\le t_{k+1}-t_k\le\Lambda_0(p)$,
(ii) $\|V(t_k,.)\|_{L^p_u(\mathbb{R})}\le R_p+1$,
(iii) $\int_{t_k}^{t_{k+1}}\|V(t,.)\|^p_{L^p_{\rm u}(\mathbb{R})}\mathrm{d}t\le\Gamma_0(p)$.

Now we fix a value $p>\frac{3}{2}$ and, with the above notation, for all $t\in[t_k,t_{k+1}]$ with $k\ge 1$, (2.9) yields the following formulation

$$V(t,.)=e^{(t-t_{k-1})\left(\partial_x^2-\tilde\lambda\right)}V(t_{k-1},.)+\int_{t_{k-1}}^t e^{(t-s)\left(\partial_x^2-\tilde\lambda\right)}F(s,.)\mathrm{d}s, \tag{2.10}$$

wherein $\left\{e^{t\partial_x^2}\right\}_{t\ge 0}$ denotes the heat semigroup and the function $F(t,x)$ is defined by

$$F(t,x)=\int_\Omega \beta(y)uv\phi(y)\mathrm{d}y.$$

We now recall the following smoothing property of the heat semigroup in $L^p_{\rm u}(\mathbb{R})$:
for each $t>0$ one has $e^{t\partial_x^2}L^p_{\rm u}(\mathbb{R})\subset L^\infty(\mathbb{R})$ and, there exists some constant $K>0$ such that

$$\left\|e^{t\partial_x^2}\psi\right\|_{L^\infty(\mathbb{R})}\le K\left(1+t^{-\frac{1}{2p}}\right)\|\psi\|_{L^p_{\rm u}(\mathbb{R})},\ \forall t>0,\ \psi\in L^p_{\rm u}(\mathbb{R}). \tag{2.11}$$

Hence, we infer from (2.10) that, for all $t\in\left[t_k,t_{k+1}\right]$,

$$\begin{aligned}\|V(t,.)\|_{L^\infty(\mathbb{R})}\le& K\left(1+(t-t_{k-1})^{-\frac{1}{2p}}\right)\|V(t_{k-1},.)\|_{L^p_{\rm u}(\mathbb{R})}\\ &+\int_{t_{k-1}}^t K\left(1+(t-s)^{-\frac{1}{2p}}\right)\|F(s,.)\|_{L^p_{\rm u}(\mathbb{R})}\mathrm{d}s.\end{aligned}$$

Observe now that one also has

$$F(t,x)\le\|\beta\|_{L^\infty(\Omega)}V(t,x)\text{ and }\|F(t,.)\|_{L^p_{\rm u}(\mathbb{R})}\le\|\beta\|_{L^\infty(\Omega)}\|U(t,.)\|_{L^p_{\rm u}(\mathbb{R})}.$$

Hence, due to Hölder inequality, one obtains, by setting $q^{-1}=1-p^{-1}$, that for all $t\in[t_k,t_{k+1}]$

$$\begin{aligned}\|V(t,.)\|_{L^\infty(\mathbb{R})} \leq & 2K\left(R_p+1\right) \\ & + K\|\beta\|_{L^\infty(\Omega)}\left[\int_{t_{k-1}}^{t}\left(1+(t-s)^{-\frac{1}{2p}}\right)^q \mathrm{d}s\right]^{\frac{1}{q}}\left[\int_{t_{k-1}}^{t_{k+1}}\|V(s,.)\|^p_{L^p_{\mathrm{u}}(\mathbb{R})}\mathrm{d}s\right]^{\frac{1}{p}} \\ & \leq 2K\left(R_p+1\right) \\ & + (2\Gamma_0(p))^{\frac{1}{p}}K\|\beta\|_{L^\infty(\Omega)}\left[\int_0^{\Lambda_0(p)}\left(1+l^{-\frac{1}{2p}}\right)^q \mathrm{d}l\right]^{\frac{1}{q}}.\end{aligned}$$

Since $p>\frac{3}{2}$ then $\frac{q}{2p}<1$ and, the function $t\mapsto\|V(t,.)\|_{L^\infty(\mathbb{R})}$ is bounded on $[0,\infty)$. Finally, since $\phi>0$ on $\overline{\Omega}$, the application of the parabolic Harnack inequality to the v-equation in (1.1)-(1.2) completes the proof of the proposition. □

2.2. *Proof of Theorem 1.3*

Equipped with Proposition 1.2 we are able to complete the proof of Theorem 1.3. To that aim, recalling Assumption 1.1, we assume throughout this section that

$$\Lambda\leq 0. \tag{2.12}$$

In view of the boundedness property, stated in Proposition 1.2, and parabolic estimates, in order to complete the proof of Theorem 1.3 it is sufficient to prove the following claim.

Claim 2.2. *Any bounded entire solutions,* $(\widehat{u},\widehat{v})=(\widehat{u},\widehat{v})(t,x,y)$ *of (1.1)-(1.2) with* $0\leq\widehat{u}\leq 1$ *and* $\widehat{v}\geq 0$ *satisfies* $\widehat{v}\equiv 0$.

Proof. Consider $\varphi=\varphi(y)>0$ a principal eigenvector of (1.5). To prove this claim, set $\widehat{w}=\widehat{w}(t,x)$ the non-negative and bounded function defined by

$$\widehat{w}(t,x)=\int_\Omega \widehat{v}(t,x,y)\varphi(y)\mathrm{d}y,\ t\in\mathbb{R},\ x\in\mathbb{R}.$$

Consider a sequence $\{(t_n,x_n)\}_{n\geq 0}\subset\mathbb{R}\times\mathbb{R}$ such that

$$\lim_{n\to\infty}\widehat{w}(t_n,x_n)=\sup_{(t,w)\in\mathbb{R}\times\mathbb{R}}\widehat{w}(t,x),$$

as well as the sequences of functions u_n, v_n defined on $\mathbb{R}\times\overline{\Sigma}$ and w_n defined on $\mathbb{R}\times\mathbb{R}$ by

$$(u_n,v_n)(t,x,y)=(\widehat{u},\widehat{v})(t+t_n,x+x_n,y) \text{ and } w_n(t,x)=\widehat{w}(t+t_n,x+x_n).$$

Because of parabolic regularity, one may assume, possibly along a subsequence that is not re-labelled, that $(u_n,v_n)\to(u_\infty,v_\infty)$, as $n\to\infty$, locally uniformly for $(t,z)\in\mathbb{R}\times\overline{\Sigma}$ wherein the limit functions, u_∞ and v_∞, satisfy System (1.1)-(1.2) on $\mathbb{R} \times \Sigma$ together with $w_n \to w_\infty$ locally uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}$ and

$$\begin{aligned} w_\infty(t,x) &:= \int_\Omega v_\infty(t,x,y)\varphi(y)\mathrm{d}y, \text{ for } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ w_\infty(0,0) &= \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} w_\infty(t,x) = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} \widehat{w}(t,x). \end{aligned} \tag{2.13}$$

Let us prove that $w_\infty(t,x) \equiv 0$. Note that the function w_∞ satisfies

$$\frac{\partial w_\infty}{\partial t} - \frac{\partial^2 w_\infty}{\partial x^2} = \Lambda w_\infty + \int_\Omega \beta(y) v_\infty (u_\infty - 1)\varphi(y)\mathrm{d}y, \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.14}$$

Hence since $u_\infty \leq 1$ then

$$\frac{\partial w_\infty}{\partial t} - \frac{\partial^2 w_\infty}{\partial x^2} \leq \Lambda w_\infty, \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}.$$

As a first consequence, if $\Lambda < 0$ then $w_\infty(t,x) \equiv 0$.
It remains to consider the limit case $\Lambda = 0$. In order to prove that $w_\infty(t,x) \equiv 0$, we argue by contradiction by assuming that $w_\infty \geq 0$ and $w_\infty \not\equiv 0$. Note that this implies that $v_\infty \not\equiv 0$ and therefore $v_\infty > 0$ and, since $\beta \not\equiv 0$, $u_\infty < 1$. Next we infer from (2.14) that w_∞ satisfies

$$\frac{\partial w_\infty}{\partial t} - \frac{\partial^2 w_\infty}{\partial x^2} < 0, \ \forall (t,x) \in \mathbb{R}\times\mathbb{R},$$

and the strong maximum principle yields a contradiction with the second condition in (2.13) above.

Thus we conclude that $w_\infty(t,x) \equiv 0$. And, as a consequence, we obtain that

$$0 = w_\infty(0,0) = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} \widehat{w}(t,x).$$

Hence $\widehat{w} = 0$ and, since $\varphi > 0$, we also get $\widehat{v} = 0$, that completes the proof of the claim and thus, that of Theorem 1.3. □

3. Preliminary estimates

This section is devoted to the derivation of preliminary estimates that will be used to prove Theorem 1.4 in the next section. Our first concerned is related to the derivation of estimates for the solutions of the following one-dimensional non-autonomous linear equation posed on a half line:

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{\partial^2 z}{\partial x^2} - \left(c - \dfrac{3}{2(t+t_0)}\right)\dfrac{\partial z}{\partial x} - \left(\dfrac{c}{2}\right)^2 z = 0, \ t > 0, \ x > 0, \\ z(t,0) = 0, \ t > 0, \\ z(0,x) = z_0(x), \ x \geq 0, \end{cases} \tag{3.15}$$

wherein the initial data, z_0, is a non-trivial, non-negative, smooth and compactly supported function on $\mathbb{R}_+$ while $c > 0$ and $t_0 > 0$ are given constants. Our result reads as follows.

Proposition 3.1. *Let $c > 0$ be given and fixed. Then there exists $\tilde{t_0} > 0$ large enough such that, for all $t_0 \geq \tilde{t_0}$, the solution $z \equiv z(t,x)$ of (3.15) enjoys the following properties:*

(i) *For each $x > 0$, one has $\liminf_{t\to\infty} z(t,x) > 0$.*
(ii) *There exist some constant $C^+ > 0$ and some time $\bar{t} > 0$ large enough such that:*

$$z(t,x) \leq C^+ x \left(1 + \frac{1}{\sqrt{t+t_0}}\right) e^{-\frac{cx}{2}},\ \forall x \geq 0,\ t \geq \bar{t}. \tag{3.16}$$

The proof of Proposition 3.1 (i) can be found in [19] (see also [10]). The proof of (ii) is also mostly proved in the aforementioned works in the sense that (3.16) is proved for $x = O(\sqrt{t})$ and $t \gg 1$. However such a 'local' estimate will not be sufficient for the purpose of this work and we shall need a uniform estimate, as stated above. Similarly to [19,10], our proof of this uniform estimate makes use of self-similar variables and semigroup estimates.

In order to prove (ii) let us introduce some functional framework that will be used in its proof. We introduce the weight function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\rho(\xi) = \exp\left(\frac{\xi^2}{4}\right),$$

as well as the weighted space

$$H := L^2_\rho = \left\{\varphi \in L^2(0,\infty) : \sqrt{\rho}\varphi \in L^2(0,\infty)\right\},$$

endowed with the usual norm denoted by $\|.\|_{2,\rho}$ and defined by

$$\|\varphi\|_{2,\rho} = \|\sqrt{\rho}\varphi\|_{L^2(0,\infty)},\ \forall \varphi \in H.$$

It becomes a Hilbert space when endowed with the usual inner product

$$\langle u, v\rangle_\rho = \int_0^\infty \rho(\xi)u(\xi)v(\xi)\mathrm{d}\xi,\ \forall (u,v) \in H \times H.$$

We also introduce the weighted Sobolev spaces, for each integer $m \geq 1$,

$$H^m_\rho = \left\{u \in H^m(0,\infty) : \frac{\mathrm{d}^k u}{\mathrm{d}\xi^k} \in L^2_\rho,\ \forall k = 0,..,m\right\}.$$

Next let us consider the linear operator $\mathcal{A} : \mathrm{Dom}\,(\mathcal{A}) \subset H \to H$ defined by

$$\mathrm{Dom}\,(\mathcal{A}) = H^2_\rho \cap H^1_0(0,\infty),$$
$$\mathcal{A}\varphi = \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\rho\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right) + \varphi = \frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \frac{\xi}{2}\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} + \varphi,\ \forall \varphi \in \mathrm{Dom}\,(\mathcal{A}).$$

Then the following lemma holds true:

Lemma 3.2. *The linear operator* $\mathcal{A} : \mathrm{Dom}\,(\mathcal{A}) \subset H \to H$ *satisfies the following properties:*

(a) *It generates a strongly continuous analytic, compact and positive semigroup on* H *denoted by* $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$.

(b) *The operator* $-\mathcal{A}$ *is a self adjoint operator with the null space generated by the simple eigenvector* φ_0 *defined by*

$$\varphi_0(\xi) \equiv \left(2\sqrt{\pi}\right)^{-\frac{1}{2}} \xi e^{-\frac{\xi^2}{4}},\ \xi \geq 0.$$

The quadratic form associated to $-\mathcal{A}$, *denoted by* $\mathcal{Q} : \mathrm{Dom}\,(\mathcal{Q}) := H_0^1(0,\infty) \cap H_\rho^1 \to \mathbb{R}_+$ *and defined by*

$$\mathcal{Q}(\varphi) = \int_0^\infty \rho(\xi) \left[\left| \frac{\mathrm{d}\varphi(\xi)}{\mathrm{d}\xi} \right|^2 - \varphi^2(\xi) \right] \mathrm{d}\xi, \tag{3.17}$$

satisfies

$$\left|\left\langle \varphi, \varphi' \right\rangle_\rho\right| \leq \mathcal{Q}(\varphi) + \|\varphi\|_{2,\rho}^2,\ \forall \varphi \in \mathrm{Dom}\,(\mathcal{Q}), \tag{3.18}$$

and,

$$\mathcal{Q}(\varphi) \geq \|\varphi\|_{2,\rho}^2,\ \ \forall \varphi \in \mathrm{Dom}\,(\mathcal{Q}) \cap \langle \varphi_0 \rangle^{\perp}. \tag{3.19}$$

Herein we have set $\langle \varphi_0 \rangle^{\perp} = \{\psi \in H :\ \langle \psi, \varphi_0 \rangle_\rho = 0\}$.

(c) *The linear operator* $\mathcal{A}_s$, *defined as the part of* $\mathcal{A}$ *in* $H_s := \langle \varphi_0 \rangle^{\perp}$, *that is*

$$\begin{cases} \mathrm{Dom}\,(\mathcal{A}_s) = \{\varphi \in \mathrm{Dom}\,(\mathcal{A})\ \ \mathcal{A}\varphi \in H_s\}, \\ \mathcal{A}_s\varphi = \mathcal{A}\varphi,\ \ \forall \varphi \in \mathrm{Dom}\,(\mathcal{A}_s), \end{cases}$$

has a spectral bound smaller than -1 *and it enjoys the maximal parabolic regularity, in the sense that for each* $p \in (1,\infty)$ *there exists some constant* $M_p > 0$ *such that for each* $f \in L^p(0,\infty; H_s)$, *each* $t \geq 0$, *one has*

$$\left\| \int_0^t e^{(t-l)\mathcal{A}_s} f(l)\mathrm{d}l \right\|_{W^{1,p}(0,\infty;H_s)\cap L^p(0,\infty;\mathrm{Dom}\,(\mathcal{A}_s))} \leq M_p \|f\|_{L^p(0,\infty;H_s)}.$$

Here $\mathrm{Dom}\,(\mathcal{A}_s)$ *is endowed with the norm of the graph.*

(d) *The following estimates hold true for each* $\varphi \in \mathrm{Dom}\,(\mathcal{A})$:

$$\left\| \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} \right\|_{2,\rho} \leq \langle (I - \mathcal{A})\varphi, \varphi \rangle_\rho, \tag{3.20}$$

for each power $\beta \in \left(\frac{3}{4}, 1\right)$ *there exists some constant* $C_\beta > 0$ *such that*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\rho^{1/2}\varphi\right) \right\|_\infty \le C_\beta \left\| (-\mathcal{A}_s)^\beta \varphi \right\|_{2,\rho}, \ \forall \varphi \in \mathrm{Dom}\left((-\mathcal{A}_s)^\beta\right), \tag{3.21}$$

and, for each $\delta > 0$, $\alpha \in [0, 1]$ *there exists some constant* $M_\alpha(\delta) > 0$ *such that*

$$\left\| (-\mathcal{A}_s)^\alpha e^{t\mathcal{A}_s} \right\|_{\mathcal{L}(H_s)} \le M_\alpha(\delta) t^{-\alpha} e^{-(1-\delta)t}, \ \forall t > 0. \tag{3.22}$$

This Lemma can be found in [10] (see Lemma 2.4, Remark 2.5 and 2.6 in this paper).

Equipped with this lemma we are able to prove Proposition 3.1 (ii). To that aim let us fix $c > 0$ and $t_0 > 0$ such that $t_0 > \frac{9}{c^2}$ and consider the self-similar variables $\tau \ge 0$ and $\xi \ge 0$ defined by

$$\tau = \ln \frac{t + t_0}{t_0} \text{ and } \xi = \frac{x}{\sqrt{t + t_0}}.$$

Next set $\lambda = \frac{c}{2}$ and consider the function $w = w(\tau, \xi)$ defined by

$$w(\tau, \xi) = e^{-\frac{\tau}{2}} e^{\lambda x} z(t, x). \tag{3.23}$$

It satisfies the following equation

$$\begin{aligned} &\frac{\partial w}{\partial \tau} = \mathcal{A}w - \varepsilon e^{-\frac{\tau}{2}} \frac{\partial w}{\partial \xi}, \text{ for } \tau > 0, \ \xi > 0, \\ &w(\tau, 0) = 0 \text{ and } w(0, \xi) = w_0(\xi) := e^{\lambda x} z_0(x) = e^{\lambda t_0^{\frac{1}{2}} \xi} z_0\left(t_0^{\frac{1}{2}} \xi\right). \end{aligned} \tag{3.24}$$

Here we have set $\varepsilon := \frac{3}{c} t_0^{-\frac{1}{2}}$. Note that the condition $t_0 > \frac{9}{c^2}$ re-writes as $\varepsilon \in (0, 1)$.

Next in order to prove (ii) we will prove the following estimate for the function w.

Lemma 3.3. *Let* w *be the solution of (3.24). Then there exists some constant* $C > 0$ *such that for all* $\tau \ge 2$ *and* $\xi \ge 0$

$$w(\tau, \xi) \le C\xi \left\{ \frac{1}{\rho(\xi)} + \frac{e^{-\frac{\tau}{2}}}{\sqrt{\rho(\xi)}} \right\}.$$

Before going to the proof of this lemma, let us observe that the above estimate and (3.23) ensure that, for all t large enough (so that $\tau \ge 2$) and all $x \ge 0$, one has

$$\begin{aligned} z(t, x) &\le C e^{-\lambda x} e^{\frac{\tau}{2}} \xi \left[1 + e^{-\frac{\tau}{2}}\right] \\ &\le C e^{-\lambda x} \sqrt{\frac{t + t_0}{t_0}} \frac{x}{\sqrt{t + t_0}} \left[1 + \sqrt{\frac{t_0}{t + t_0}}\right]. \end{aligned}$$

Hence Proposition 3.1 (ii) follows.

It remains to prove Lemma 3.3.

Proof of Lemma 3.3. Throughout this proof $C > 0$ denotes any constant independent of ξ and τ that may change from place to place.

First multiplying (3.24) by ρw and integrating over $(0, \infty)$ leads us to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w\|_{2,\rho}^2 + \mathcal{Q}(w(\tau,.)) = -\varepsilon e^{-\frac{\tau}{2}}\int_0^\infty \rho w \frac{\partial w}{\partial \xi}\mathrm{d}\xi. \tag{3.25}$$

Next (3.18) yields, for all $\tau > 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w(\tau,.)\|_{2,\rho}^2 + \left(1 - \varepsilon e^{-\frac{\tau}{2}}\right)\mathcal{Q}(w(\tau,.)) \le \varepsilon e^{-\frac{\tau}{2}}\|w(\tau,.)\|_{2,\rho}^2.$$

Recalling that $\varepsilon \in (0, 1)$ and $\mathcal{Q} \ge 0$, integrating the above inequality ensures that there exists some constant $C > 0$ such that

$$\|w(\tau,.)\|_{2,\rho}^2 \le C, \ \forall \tau \ge 0. \tag{3.26}$$

Now we shall decompose the function w according to the orthogonal space splitting $H_c \oplus H_s$ by introducing the – orthogonal – spectral projectors

$$\Pi_c\varphi = \langle \varphi, \varphi_0\rangle_\rho\, \varphi_0 \text{ and } \Pi_s = I_H - \Pi_c,$$
$$H_c = \Pi_c(H) = \langle \varphi_0\rangle, \ \ H_s = \Pi_s(H) = \langle \varphi_0\rangle^\perp.$$

Hence we decompose the function w as follows

$$w(\tau,.) = w_c(\tau)\varphi_0 + w_s(\tau,.) \text{ with } w_c(\tau) := \langle w(\tau,.), \varphi_0\rangle_\rho\,, \ w_s(\tau,.) = \Pi_s w(\tau,.).$$

With this notation, note that (3.26) re-writes as

$$|w_c(\tau)|^2 + \|w_s(\tau,.)\|_{2,\rho}^2 \le C, \ \forall \tau \ge 0. \tag{3.27}$$

Next note that function w_s satisfies the equation

$$\begin{cases} \dfrac{\partial w_s}{\partial \tau} - \mathcal{A}_s w_s = -\varepsilon e^{-\frac{\tau}{2}}\Pi_s \dfrac{\partial w}{\partial \xi}(\tau,.), \ \tau > 0, \ \xi > 0, \\ w_s(0,.) = \Pi_s w_0. \end{cases} \tag{3.28}$$

Multiplying the above equation by ρw_s and integrating on $(0, \infty)$ yields, for all $\tau > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\|w_s\|_{2,\rho}^2 + \mathcal{Q}(w_s) = -\varepsilon e^{-\frac{\tau}{2}}\left\langle w_s, \Pi_s\frac{\partial w}{\partial \xi}\right\rangle_\rho.$$

Next observe that one also has

$$\left\langle w_s, \Pi_s\frac{\partial w}{\partial \xi}\right\rangle_\rho = w_c(\tau)\left\langle w_s, \varphi_0'\right\rangle_\rho + \left\langle w_s, \frac{\partial w_s}{\partial \xi}\right\rangle_\rho,$$

so that using (3.18) and (3.27) one gets, for some constant $C > 0$,

$$\left| \left\langle w_s, \Pi_s \frac{\partial w}{\partial \xi} \right\rangle_\rho \right| \le C + \mathcal{Q}(w_s), \ \forall \tau \ge 0.$$

As a consequence, one obtains that w_s satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \|w_s\|_{2,\rho}^2 + \left(1 - \varepsilon e^{-\frac{\tau}{2}}\right) \mathcal{Q}(w_s) \le \varepsilon C e^{-\frac{\tau}{2}}.$$

Hence we infer from (3.19) and the above inequality that there exists some constant still denoted by $C > 0$ such that

$$\|w_s(\tau, \cdot)\|_{2,\rho}^2 \le C e^{-\frac{\tau}{2}}, \ \forall \tau \ge 0. \tag{3.29}$$

Moreover one also gets

$$\int_0^\infty \mathcal{Q}(w_s(\tau, \cdot)) \mathrm{d}\tau \le C,$$

that implies, due to the above exponential bound for w_s, that

$$\int_0^\infty \left\| \frac{\partial w_s}{\partial \xi}(\tau, \cdot) \right\|_{2,\rho}^2 \mathrm{d}\tau \le C. \tag{3.30}$$

In order to provide more refined estimates for the function w_s we shall use the fractional powers of the linear operator $-\mathcal{A}_s$ and, for that purpose, we introduce, for each $\alpha \in (0, 1]$, the Banach space $H_s^\alpha := \mathrm{Dom}\,((-\mathcal{A}_s)^\alpha)$ endowed with the usual graph norm defined by

$$\|\varphi\|_\alpha := \left\| (-\mathcal{A}_s)^\alpha \varphi \right\|_{2,\rho}, \ \ \forall \varphi \in H_s^\alpha.$$

Note that, as a consequence of (3.29) and (3.30), we have

$$w_s \in L^4\left(0, \infty; H_s^{\frac{1}{2}}\right). \tag{3.31}$$

Now we make use of the constant variation formula to represent the function w_s as follows, for each $\tau \ge 0$ and $\tau_0 \ge 0$,

$$w_s(\tau + \tau_0, .) = e^{\tau \mathcal{A}_s} w_s(\tau_0, .) - \int_0^\tau \varepsilon e^{-\frac{l+\tau_0}{2}} e^{(\tau - l)\mathcal{A}_s} \Pi_s \frac{\partial w}{\partial \xi}(l + \tau_0, .)\mathrm{d}l. \tag{3.32}$$

In order to derive further estimates for the function w_s, we claim that there exists some constant $C > 0$ such that

$$\left\| \Pi_s \frac{\partial w}{\partial \xi}(\tau, .) \right\|_{2,\rho} \leq C \left[1 + \| w_s(\tau, .)\|_{\frac{1}{2}} \right], \ \ \forall \tau > 0. \tag{3.33}$$

To see this, observe that one has

$$\begin{aligned}\Pi_s \frac{\partial w}{\partial \xi}(\tau, \xi) =& w_c(\tau)\varphi_0'(\xi) + \frac{\partial w_s}{\partial \xi}(\tau, \xi) - w_c(\tau)\left\langle \varphi_0, \varphi_0' \right\rangle_\rho \varphi_0(\xi) \\ &+ \left\langle \frac{\partial w_s}{\partial \xi}(\tau, .), \varphi_0 \right\rangle_\rho \varphi_0(\xi),\end{aligned}$$

so that, since w_c is bounded (see (3.27)), one obtains that, for some constant $C > 0$, one has

$$\left\| \Pi_s \frac{\partial w}{\partial \xi}(\tau, .) \right\|_{2,\rho} \leq C \left[1 + \left\| \frac{\partial w_s}{\partial \xi}(\tau, .) \right\|_{2,\rho} \right], \ \forall \tau > 0.$$

Now, observe that, due to (3.20), there exists some constant $C > 0$ such that

$$\left\| \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} \right\|_{2,\rho} \leq C\|\varphi\|_{\frac{1}{2}} \text{ for all } \varphi \in H_s^{\frac{1}{2}},$$

and, the claim follows.

Now we chose $\delta = \frac{1}{4}$ and we apply (3.22) – with $\alpha = \frac{1}{2}$ – to the representation formula (3.32) starting at $\tau_0 = 0$. Hence, using (3.27), we obtain that there exists some constant $C > 0$ such that for all $\tau > 0$

$$\|w_s(\tau, .)\|_{\frac{1}{2}} \leq C \left[\frac{e^{-\frac{3\tau}{4}}}{\tau^{1/2}} + \int_0^\tau \frac{e^{-\frac{3\tau}{4} + \frac{3l}{4}}}{(\tau - l)^{\frac{1}{2}}} e^{-\frac{l}{2}} \left\| \Pi_s \frac{\partial w}{\partial \xi}(l, .) \right\|_{2,\rho} \mathrm{d}l \right].$$

Next using (3.33) we get, for all $\tau > 0$,

$$\|w_s(\tau, .)\|_{\frac{1}{2}} \leq C \left[\frac{e^{-\frac{3}{4}\tau}}{\tau^{\frac{1}{2}}} + e^{-\frac{\tau}{2}} + \int_0^\tau \frac{e^{-\frac{3}{4}\tau + \frac{3}{4}l}}{(\tau - l)^{\frac{1}{2}}} e^{-\frac{l}{2}} \|w_s(l, .)\|_{\frac{1}{2}} \mathrm{d}l \right].$$

Next we infer from Hölder inequality that for all $\tau > 0$:

$$\begin{aligned}\|w_s(\tau, .)\|_{\frac{1}{2}} \leq& C \left[\frac{e^{-\frac{3}{4}\tau}}{\tau^{\frac{1}{2}}} + e^{-\frac{\tau}{2}} \right] \\ &+ C \left[\int_0^\tau \frac{e^{-\frac{2\tau}{3} - \frac{l}{3}}}{l^{\frac{2}{3}}} \mathrm{d}l \right]^{\frac{3}{4}} \left[\int_0^\infty \|w_s(l, .)\|_{\frac{1}{2}}^4 \mathrm{d}l \right]^{\frac{1}{4}}.\end{aligned}$$

Hence, recalling (3.31), we obtain that $w_s \in L^\infty\left(1, \infty, H_s^{\frac{1}{2}}\right)$ and, (3.33) ensures that $\Pi_s \frac{\partial w}{\partial \xi} \in L^\infty(1, \infty, H_s)$.

We now make use of these estimates to bootstrap the above argument. To that aim we choose and fix $\beta \in \left(\frac{3}{4}, 1\right)$. Using once again the semigroup representation (see (3.32) starting at $\tau_0 = 1$) as well as (3.22) one gets, for each $\tau > 0$,

$$\begin{aligned}
\|w_s(1+\tau,.)\|_\beta &\le C\left[\frac{e^{-\frac{3}{4}\tau}}{\tau^\beta} + \int_0^\tau \frac{e^{-\frac{3}{4}\tau+\frac{3}{4}l}}{(\tau-l)^\beta} e^{-\frac{l}{2}} \left\|\Pi_s \frac{\partial w}{\partial \xi}(1+l,.)\right\|_{2,\rho} \mathrm{d}l\right] \\
&\le C\left[\frac{e^{-\frac{3}{4}\tau}}{\tau^\beta} + \int_0^\tau \frac{e^{-\frac{3}{4}\tau+\frac{3}{4}l}}{(\tau-l)^\beta} e^{-\frac{l}{2}} \mathrm{d}l\right] \\
&\le C\left[\frac{e^{-\frac{3}{4}\tau}}{\tau^\beta} + e^{-\frac{\tau}{2}}\right].
\end{aligned}$$

Finally (3.21) yields the existence of some constant $C > 1$ such that, for each $\tau \ge 2$ and $\xi \ge 0$, one has

$$\left|\frac{\partial}{\partial \xi}\left(\sqrt{\rho(\xi)}w_s\right)(\tau,\xi)\right| \le Ce^{-\frac{\tau}{2}},$$

and, the result follows by integrating the above inequality with respect to the ξ-variable. □

4. Proof of Theorem 1.4

The aim of this section is to prove Theorem 1.4. Hence throughout this section we assume that Assumption 1.1 is satisfied. We also assume that $\Lambda > 0$ and we recall the definition of the minimal wave speed c^* defined in (1.6). Next we set

$$\lambda^* = \frac{c^*}{2} = \sqrt{\Lambda}. \tag{4.34}$$

And, in this section we fix a function $\varphi = \varphi(y) > 0$ on $\overline{\Omega}$, a principal eigenvector of (1.5).

The strategy of this proof consists in a bootstrap argument. Estimate (2.8) provides a first upper-estimate for u, namely $u \le 1$, that will be used to derive, in a first step, a upper-estimate for v on a suitable domain. This upper-bound for v will be used, in a second step, to obtain a suitable lower-estimate for u. In a third step, the latter lower-estimate (for u) will allow us to obtain a lower-estimate for v. As consequence, the proof of Theorem 1.4 involves three main preliminary steps that are detailed below and a fourth and last step to complete the proof of the theorem.

First step: construction of an upper-estimate for v

In this first step, we provide a decay estimate of the function v ahead the front and we prove the right hand-side estimate for v stated in Theorem 1.4 (iii). Here we shall focus on the derivation of this estimate in the case where $x > 0$. The case where $x < 0$ can be handled similarly.

To reach this goal, we fix a value $a > 0$ and we choose and fix a smooth and compactly supported function $z_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$z_0(a) > 0 \text{ and } z_0(x+a)\varphi(y) \geq v_0(x,y),\ \forall (x,y) \in [0,\infty) \times \overline{\Omega}.$$

Here recall that the above conditions can be fulfilled since v_0 is compactly supported.
We consider Problem (3.15) with the parameter c^*, the initial data z_0 and a fixed value $t_0 > 0$ large enough such that

$$t_0 \geq \max\left(\tilde{t_0}, \frac{3}{c^{*2}}\right), \tag{4.35}$$

where $\tilde{t_0}$ is provided by Proposition 3.1. We denote by $z = z(t,x)$ the corresponding solution of Problem (3.15) and we consider the function $X = X(t)$ defined by

$$X(t) = c^* t - \frac{3}{c^*} \ln \frac{t+t_0}{t_0},\ t > 0. \tag{4.36}$$

Now we shall construct a super-solution for v on the domain $x > X(t)$ and $t > 0$. To proceed, consider the function $w = w(t,x,y)$ defined by

$$w(t,x,y) = v\left(t, x + X(t), y\right).$$

Next since $u \leq 1$, observe that it satisfies the following differential inequality

$$\begin{cases} L[w](t,x,y) \leq 0,\ \text{for } t > 0,\ (x,y) \in \Sigma_+ := \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial w}{\partial \nu} = -\sigma(y) w,\ \text{for } t > 0,\ (x,y) \in \partial\Sigma_+ = \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

wherein L denotes the parabolic differential operator on Σ_+ defined by

$$L = \frac{\partial}{\partial t} - \Delta - \left(c^* - \frac{3}{c^*(t+t_0)}\right)\frac{\partial}{\partial x} - \left(\beta(y) - \mu(y)\right).$$

Consider the function $\overline{w} = \overline{w}(t,x,y)$ defined by

$$\overline{w}(t,x,y) = z(t,x+a)\varphi(y),\ t \geq 0,\ x \geq 0,\ y \in \Omega.$$

Now first observe that, due to the choice of z_0, one already has

$$\overline{w}(0,x,y) = z_0(x+a)\varphi(y) \geq v_0(x,y)\, \forall x \geq 0,\ y \in \overline{\Omega}.$$

Next observe also that one has for all $t > 0$ and $(x,y) \in \Sigma_+$,

$$L\left[\overline{w}\right] = \varphi(y)\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \left(c^* - \frac{3}{c^*(t+t_0)}\right)\frac{\partial}{\partial x} - \left(\frac{c^*}{2}\right)^2\right] z(t,x+a) = 0$$

$$\frac{\partial \overline{w}}{\partial \nu} = -\sigma(y)\overline{w},\ t > 0,\ (x,y) \in \partial\Sigma_+.$$

Moreover, since $z_0(a) > 0$, Proposition 3.1 (i) ensures that $\inf_{t\geq 0} z(t,a) > 0$. Hence, due to Proposition 2.1, v, hence w, is bounded and there exists some constant $M > 1$ large enough such that

$$Mz(t,a)\varphi(y) \geq w(t,0,y),\ \forall t > 0,\ y \in \overline{\Omega}.$$

Thus the parabolic comparison principle applies and ensures that

$$v(t, x + X(t), y) \leq Mz(t, x+a)\varphi(y),\ \text{for all } t \geq 0,\ x \geq 0,\ y \in \overline{\Omega}.$$

Next, Proposition 3.1 (ii) ensures that there exists some constant $K > 0$ large enough such that

$$v(t, x + X(t), y) \leq K(x+1)e^{-\frac{c^* x}{2}},\ \text{for all } t \geq 0,\ x \geq 0,\ y \in \overline{\Omega}. \tag{4.37}$$

This provides a suitable decay estimate for the function v ahead the front and completes the proof of the right hand-side estimate stated in Theorem 1.4 (iii). This concludes our first step.

Second step: construction of a lower-estimate for u

In this second step we shall prove estimate (i) stated in Theorem 1.4. As in the previous step, we shall only focus on the case $x > 0$. The case $x < 0$ follows from the same construction. We now make use of (4.37) above to derive a lower-estimate for the function u on the set $x > X(t)$ and $t > 0$. To that aim, observe that the function u satisfies the following differential inequality

$$\begin{cases} M[u](t,x,y) \geq 0\ \forall (x,y) \in [X(t), \infty) \times \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ for } t > 0 \text{ and } (x,y) \in [X(t),\infty) \times \partial\Omega. \end{cases}$$

Herein M denotes the parabolic operator defined by

$$M = \frac{\partial}{\partial t} - D\Delta + g(x - X(t)),\ \text{with } g(x) = \|\beta\|_{L^\infty(\Omega)} K(x+1)e^{-\frac{c^* x}{2}}.$$

We are looking for a sub-solution on $t > 0$ and $x > X(t)$ as follows

$$\underline{u}(t,x) = 1 - \Gamma e^{-\alpha(x - X(t))},$$

where $\Gamma > 0$ and $\alpha > 0$ are constants that will be chosen below. To that aim, observe that one has

$$\begin{aligned} M[\underline{u}] &= \Gamma\left[-\alpha X'(t) + D\alpha^2\right] e^{-\alpha(x-X(t))} + g(x - X(t)) - \Gamma g(x - X(t))e^{-\alpha(x-X(t))} \\ &\leq e^{-\alpha(x-X(t))}\Bigg[\Gamma\left(-\alpha c^* + \frac{3\alpha}{c^* t_0} + D\alpha^2\right) \\ &\quad + \|\beta\|_{L^\infty(\Omega)} K\left((x - X(t)) + 1\right) e^{\left(\alpha - \frac{c^*}{2}\right)(x - X(t))}\Bigg]. \end{aligned}$$

Recalling the choice of t_0 in (4.35), there exists $\alpha \in \left(0, \frac{c^*}{2}\right)$ small enough such that

$$-\alpha c^* + \frac{3\alpha}{c^* t_0} + D\alpha^2 < 0.$$

Next choose $\Gamma > 1$ large enough such that

$$\Gamma\left(-\alpha c^* + \frac{3\alpha}{c^* t_0} + D\alpha^2\right) + \|\beta\|_{L^\infty(\Omega)} K \sup_{\zeta \geq 0}(\zeta + 1)e^{\left(\alpha - \frac{c^*}{2}\right)\zeta} < 0.$$

We conclude, with this choice of the parameters Γ and α, that $M[\underline{u}](t,x) < 0$ for any $t > 0$ and $x > X(t)$ while $\underline{u}(0,x) \leq u_0(x) = 1$ for all $x \geq X(0) = 0$ and $\underline{u}(t, X(t)) = 1 - \Gamma < 0$. Thus, recalling that $u \geq 0$, the parabolic comparison principle applies and ensures that

$$u(t,x,y) \geq \max\left(\underline{u}(t,x), 0\right), \ \forall t \geq 0, \ \forall (x,y) \in [X(t), \infty) \times \overline{\Omega}. \tag{4.38}$$

Recalling that $u < 1$ for $t > 0$, this completes the proof of Theorem 1.4 (i) and thus, our second step.

Third step: construction of a lower-estimate for v

In this third step, we derive a suitable lower-estimate for the function v in order to complete the lower-bound stated in Theorem 1.4 (iii). As above we only prove the estimate for the case $x > 0$. To that aim we shall make use of the second step above. Due to (4.38), let us consider some constant $K > 0$ such that

$$bK = 1,$$

and such that for all $t \geq 0$, $(x,y) \in \overline{\Sigma}$.

$$U(t,x,y) := u(t, x + X(t), y) \geq A(x) := \left(1 - \frac{K}{(1+x)^4}\right)^+, \tag{4.39}$$

wherein the exponent $+$ stands for the positive part. This lower-estimate for u will be more tractable than the exponential estimate derived above because of Lemma 4.1 below. Indeed this specific function will allow us to make use of explicit computations with simple functions. Moreover in order to slightly simplify the computation we assume without loose of generality that

$$b = 1 \text{ so that } K = 1. \tag{4.40}$$

Now in order to construct our lower-bound, consider the function $W(t,x,y)$ defined by

$$W(t,x,y) = \varphi(y)^{-1} v(t, x + X(t), y), \tag{4.41}$$

where $\varphi > 0$ denotes a principle eigenvector of (1.5). Recalling that Λ is defined in (1.5), the function $W > 0$ satisfies the equation

$$\frac{\partial W}{\partial t} - \Delta W - 2\frac{\nabla_y \varphi}{\varphi}\nabla_y W - \left(c^* - \frac{3}{c^*(t+t_0)}\right)\frac{\partial W}{\partial x} + [\beta(y)(1-U) - \Lambda] W = 0,$$

for $t > 0$ and $(x,y) \in \Sigma$ together with

$$\frac{\partial W}{\partial \nu} = 0 \text{ for } t > 0 \text{ and } (x, y) \in \mathbb{R} \times \partial\Omega.$$

Due to (4.39), recalling that $W > 0$ and $U \leq 1$ and by setting $b = \|\beta\|_{L^\infty(\Omega)}$, the function W satisfies the following differential inequality

$$\frac{\partial W}{\partial t} - \Delta W - 2\frac{\nabla_y \varphi}{\varphi}\nabla_y W - \left(c^* - \frac{3}{c^*(t+t_0)}\right)\frac{\partial W}{\partial x} + [b(1 - A(x)) - \Lambda]\, W \geq 0,$$

on $(0, \infty) \times \Sigma$ together with the homogeneous Neumann boundary condition on $(0, \infty) \times \partial\Sigma$.

From now on, we consider the parabolic operator L defined by

$$L = \frac{\partial}{\partial t} - \Delta - 2\frac{\nabla_y \varphi}{\varphi}\nabla_y - \left(c^* - \frac{3}{c^*(t+t_0)}\right)\frac{\partial}{\partial x} + [b(1 - A(x)) - \Lambda],$$

so that the above computations re-write as follows. The function W satisfies the following problem

$$\begin{cases} L[W](t, x, y) \geq 0, \text{ for } t > 0 \text{ and } (x, y) \in \Sigma, \\ \dfrac{\partial W}{\partial \nu} = 0 \text{ for } t > 0 \text{ and } (x, y) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{4.42}$$

Now to derive a suitable lower-estimate for the function v, we shall construct a sub-solution for the parabolic operator L. Our construction relies on the properties of the generalized principal eigenvector for the equation

$$-U_\sharp''(x) + (1 - A(x))\, U_\sharp(x) = 0, \; x \in \mathbb{R}, \tag{4.43}$$

and it is summarized in the following two lemmas.

Lemma 4.1. *Recalling (4.40) (namely $K = 1$), the function $U_\sharp \in C^2(\mathbb{R})$ defined by*

$$U_\sharp(x) = \begin{cases} e^x \text{ if } x \leq 0, \\ (x+1)\cosh\left(\frac{x}{x+1}\right) \text{ if } x \geq 0 \end{cases},$$

is a solution of the second order equation (4.43) satisfying the following properties $U_\sharp > 0$, $U_\sharp' > 0$. More specifically we have

$$\frac{U_\sharp'(x)}{U_\sharp(x)} = \frac{1}{1+x}\left[1 + \frac{1}{(x+1)}\tanh\left(\frac{x}{x+1}\right)\right], \; \forall x \geq 0,$$
$$\frac{U_\sharp'(x)}{U_\sharp(x)} = 1, \; \forall x \leq 0,$$

and there exists $x_0 > 1$ such that

$$0 \leq 1 - \frac{xU_\sharp'(x)}{U(x)} \leq \frac{1}{1+x} \leq \frac{1}{2}, \; \forall x \geq x_0.$$

Then using the function $U_\sharp$ presented in the above lemma, one may turn to the construction of a suitable sub-solution for the parabolic operator L. To that aim, we shall construct a sub-solution $\underline{W} = \underline{W}(t,x)$ of the form

$$\underline{W}(t,x) = e^{-\lambda^* x} U_\sharp(x) Z(t,x).$$

Using the above form, note that $\underline{W}$ is a sub-solution for L means that Z satisfies

$$\frac{\partial Z}{\partial t} \leq \frac{\partial^2 Z}{\partial x^2} + \frac{2U_\sharp'(x)}{U_\sharp(x)} \frac{\partial Z}{\partial x} - \frac{3}{2\lambda^*(t+t_0)} \left[\left(\frac{U_\sharp'(x)}{U_\sharp(x)} - \lambda^* \right) Z + \frac{\partial Z}{\partial x} \right]. \tag{4.44}$$

The idea of our construction comes from the work of Gallay in [17] where the asymptotic self-similar behaviour of the equation below is described

$$\frac{\partial Z}{\partial t} = \frac{\partial^2 Z}{\partial x^2} + \gamma(x) \frac{\partial Z}{\partial x},$$

wherein the advection term γ is an exponentially small perturbation at $x = \infty$ of γ_0 given by

$$\gamma_0(x) = \begin{cases} 2 \text{ if } x \leq 0, \\ \frac{2}{x+1} \text{ if } x \geq 0 \end{cases} .$$

In that case, Gallay proved in [17] that solution of the above problem equipped with suitable initial data decays as $t \to \infty$ like the 3-dimensional heat kernel. It more precisely looks like

$$t^{3/2} Z\left(t, t^{-\frac{1}{2}} x\right) \approx \Psi(t^{-\frac{1}{2}} x) \text{ as } t \to \infty,$$

with $\Psi(\xi) = 1$ if $\xi \leq 0$ and $e^{-\frac{\xi^2}{4}}$ if $\xi \geq 0$. Here to construct of sub-solution of (4.44), we shall work with the self-similar variables (τ, ξ) described below and we shall look for the sub-solution as a suitable perturbation of the function $\Psi = \Psi(\xi)$.

As mentioned above to perform our analysis we make use of the self-similar variables given by

$$\tau = \ln \frac{t + t_0}{t_0} \text{ and } \xi = \frac{x}{\sqrt{t+t_0}}.$$

To that aim, note that checking the condition $L\left[\underline{W}\right] \leq 0$ on a suitable set, re-writes by setting $Z(t,x) = \tilde{Z}(\tau, \xi)$ as

$$\widetilde{L}\left[\tilde{Z}\right](\tau,\xi) \leq 0, \tag{4.45}$$

wherein the operator $\widetilde{L}$ is given by

$$\widetilde{L} = \frac{\partial}{\partial \tau} - \mathcal{L} - \sqrt{t_0} e^{\frac{\tau}{2}} \frac{2U_\sharp'(x)}{U_\sharp(x)} \frac{\partial}{\partial \xi} + \frac{3}{2\lambda^*} \left[\left(\frac{U_\sharp'(x)}{U_\sharp(x)} - \lambda^* \right) + t_0^{-\frac{1}{2}} e^{-\frac{\tau}{2}} \frac{\partial}{\partial \xi} \right],$$

while $\mathcal{L}$ denotes the elliptic operator

$$\mathcal{L}=\frac{\partial^2}{\partial\xi^2}+\frac{\xi}{2}\frac{\partial}{\partial\xi}.$$

With these notations, the construction of our sub-solution is described in the next lemma.

Lemma 4.2. *There exist $\alpha_0>0$, $\beta_0>0$, γ_0 and $\tau_0>0$ large enough such that the function*

$$\tilde{Z}(\tau,\xi)=\begin{cases}\alpha(\tau)+\beta(\tau)\xi \text{ if } \xi\le 0\\ \left(\alpha(\tau)+\beta(\tau)\xi-\gamma(\tau)\xi^2\right)e^{-\frac{\xi^2}{4}} \text{ if } \xi\ge 0,\end{cases}$$

with

$$\alpha(\tau)=\frac{\alpha_0}{2}\left(1+e^{-\frac{\tau}{5}}\right),\ \beta(\tau)=\beta_0 e^{-\frac{\tau}{2}} \text{ and } \gamma(\tau)=\gamma_0 e^{-\frac{\tau}{2}},$$

is of class $C^1([0,\infty)\times\mathbb{R})$, $\partial_\xi^2\tilde{Z}\in L^\infty((0,\infty)\times\mathbb{R})$ and satisfies the following differential inequality

$$\widetilde{L}[\tilde{Z}](\tau,\xi)<0,\ \forall(\tau,\xi)\in\widetilde{\mathcal{U}},$$

wherein we have set

$$\widetilde{\mathcal{U}}=\left\{(\tau,\xi)\in[0,\infty)\times\mathbb{R}:\ \tau\ge\tau_0 \text{ and } \widetilde{Z}(\tau,\xi)\ge 0\right\}.$$

Before going to the proof of Lemma 4.2, that is postponed, let us first complete the proof of the lower-estimate for the function v as stated in Theorem 1.4 (iii). To that aim, let us first observe that from the above construction, the function

$$\underline{W}(t,x)=e^{-\lambda^* x}U_\sharp(x)Z(t,x) \text{ with } Z(t,x)=\widetilde{Z}(\tau,\xi),$$

satisfies $L[\underline{W}](t,x)<0$ on the set $\mathcal{U}$ given by

$$\mathcal{U}=\left\{(t,x)\in(0,\infty)\times\mathbb{R}:\ t\ge T,\ \ \varrho_*(\tau)\le\frac{x}{\sqrt{t+t_0}}\le\varrho^*(\tau)\right\},$$

with T is large enough such that $t\ge T\ \Leftrightarrow\ \tau=\ln\frac{t+t_0}{t_0}\ge\tau_0$ while

$$\varrho_*(\tau)=-\frac{\alpha(\tau)}{\beta(\tau)}<0 \text{ and } \varrho^*(\tau)=\frac{\gamma(\tau)+\sqrt{\beta(\tau)^2+4\alpha(\tau)\gamma(\tau)}}{2\gamma(\tau)}=O\left(e^{\frac{\tau}{4}}\right) \text{ as } \tau\to\infty.$$

Moreover $\underline{W}$ is zero on the lateral boundaries of the $\mathcal{U}$. Now, since $v>0$ (hence $W>0$), let us choose $\varepsilon>0$ small enough such that

$$W(T,x,y)\ge\varepsilon\underline{W}(T,x),\ \forall x\in\mathcal{U}\cap\{t=T\},\ \forall y\in\overline{\Omega}.$$

Next, recalling that $\underline{W}$ is independent of $y \in \Omega$ and that W satisfies (4.42), the parabolic comparison principle applies and ensures that

$$W(t,x,y) \geq \varepsilon \underline{W}(t,x),\ \forall ((t,x),y) \in \mathcal{U} \times \overline{\Omega}.$$

Recalling (4.41) this re-writes as

$$v(t,x+X(t),y) \geq \varepsilon\varphi(y)\underline{W}(t,x),\ \forall ((t,x),y) \in \mathcal{U} \times \overline{\Omega}.$$

Finally, since $U_\sharp(x) \geq M(1+x)$ for all $x \geq 0$ for some constant $M > 0$ and $\alpha(\tau) \geq \frac{\alpha_0}{2}$, the lower-bound in Theorem 1.4 (iii) follows by noting that the set $\mathcal{U}$ contains growing interval of the form $|x| = O(\sqrt{t})$ for all $t \gg 1$.

To complete this third step, it remains to prove Lemma 4.2.

Proof of Lemma 4.2. Here recall that, since $b = 1$ (see (4.40)) one has $\lambda^* \leq 1$ and choose $\alpha_0 > 0$, $\beta_0 > 0$ and $\gamma_0 > 0$ such that

$$\frac{3}{2\lambda^*}\left(1-\lambda^*\right)\alpha_0 + \alpha_0 < 2\sqrt{t_0}\beta_0 \tag{4.46}$$

and

$$\beta_0 > \frac{3\alpha_0}{2\lambda^*\sqrt{t_0}} + \frac{\alpha_0 x_0}{2\sqrt{t_0}}. \tag{4.47}$$

Herein x_0 is defined in Lemma 4.1.
Next note the function $\alpha = \alpha(\tau)$ satisfies

$$\frac{\alpha_0}{2} \leq \alpha(\tau) \leq \alpha_0,\ \forall \tau \geq 0.$$

For notational simplicity, in this proof we write $Z(\tau,\xi) = Z(t,x)$ instead of $\tilde{Z}(\tau,\xi)$. Next set $\varrho_*(\tau) < 0$ and $\varrho^*(\tau) > 0$ the solution of the equation $Z(\tau,\xi) = 0$, that is

$$\varrho_*(\tau) = -\frac{\alpha(\tau)}{\beta(\tau)} \text{ and } \varrho^*(\tau) = \frac{\gamma(\tau) + \sqrt{\beta(\tau)^2 + 4\alpha(\tau)\gamma(\tau)}}{2\gamma(\tau)} = O\left(e^{\frac{\tau}{4}}\right) \text{ as } \tau \to \infty.$$

We now study the quantity $\widetilde{L}[Z](\tau,\xi)$ for $\varrho_*(\tau) \leq \xi \leq \varrho^*(\tau)$ and $\tau \gg 1$ large enough and we split this analysis into three regions for the variable ξ.

For $\varrho_*(\tau) \leq \xi \leq 0$ and $\tau \gg 1$: Recalling that $\frac{U_\sharp'(x)}{U_\sharp(x)} = 1$ for $x \leq 0$, we have for all $\tau \geq 0$ and $\varrho_*(\tau) \leq \xi \leq 0$,

$$\begin{aligned}\widetilde{L}[Z](\tau,\xi)=&\alpha'(\tau)+\beta'(\tau)\xi-\frac{\xi}{2}\beta(\tau)\\&-\sqrt{t_0}e^{\frac{\tau}{2}}2\beta(\tau)+\frac{3}{2\lambda^*}\left[\left(1-\lambda^*\right)(\alpha+\beta\xi)+t_0^{-\frac{1}{2}}e^{-\frac{\tau}{2}}\beta(\tau)\right],\\=&-\frac{\alpha_0}{10}e^{-\frac{\tau}{5}}-\beta_0e^{-\frac{\tau}{2}}\xi-2\sqrt{t_0}\beta_0\\&+\frac{3}{2\lambda^*}\left[\left(1-\lambda^*\right)(\alpha(\tau)+\beta_0e^{-\frac{\tau}{2}}\xi)+t_0^{-\frac{1}{2}}\beta_0e^{-\tau}\right].\end{aligned}$$

Recalling that $\lambda^*\leq 1$ and $\alpha\leq\alpha_0$, we get for all $\tau\gg 1$ and $\varrho_*(\tau)\leq\xi\leq 0$

$$\widetilde{L}[Z](\tau,\xi)\leq-\frac{\alpha_0}{10}e^{-\frac{\tau}{5}}+O(e^{-\tau})-2\sqrt{t_0}\beta_0+\alpha_0+\frac{3\alpha_0}{2\lambda^*}\left(1-\lambda^*\right).$$

Now recalling (4.46) there exists τ_1 large enough such that

$$\widetilde{L}[Z](\tau,\xi)<0,\ \ \forall\tau\geq\tau_1,\ \forall\xi\in[\varrho_*(\tau),0].$$

For $0\leq\xi\leq\varrho^*(\tau)$, ξ close to 0 and $\tau\gg 1$: Let us introduce

$$e_0(\xi)=e^{-\frac{\xi^2}{4}}\text{ and }e_n(\xi)=\xi^ne_0(\xi),\ n\in\mathbb{N}\setminus\{0\},$$

and observe that one has

$$\mathcal{L}e_0=-\frac{1}{2}e_0,\ \mathcal{L}e_1=-e_1\text{ and }\mathcal{L}e_2=2e_0-\frac{3}{2}e_2,\tag{4.48}$$

so that one has

$$\left(\mathcal{L}+\frac{3}{2}\right)e_0=e_0,\ \left(\mathcal{L}+\frac{3}{2}\right)e_1=\frac{1}{2}e_1\text{ and }\left(\mathcal{L}+\frac{3}{2}\right)e_2=2e_0,$$

while

$$e_0'(\xi)=-\frac{\xi}{2}e_0(\xi),\ e_1'(\xi)=\left(1-\frac{\xi^2}{2}\right)e_0(\xi),\ e_2'(\xi)=\left(2\xi-\frac{\xi^3}{2}\right)e_0(\xi).$$

Hence we get

$$\begin{aligned}e_0^{-1}(\xi)\widetilde{L}[Z](\tau,\xi)=&\alpha'(\tau)+\beta'(\tau)\xi-\gamma'(\tau)\xi^2-[\alpha(\tau)+\beta(\tau)\frac{\xi}{2}-2\gamma(\tau)]\\&-\sqrt{t_0}e^{\frac{\tau}{2}}\frac{2U_\sharp'(x)}{U_\sharp(x)}\left[-\alpha(\tau)\frac{\xi}{2}+\beta(\tau)\left(1-\frac{\xi^2}{2}\right)-\gamma(\tau)\left(2\xi-\frac{\xi^3}{2}\right)\right]\\&+\frac{3}{2\lambda^*}\frac{U_\sharp'(x)}{U_\sharp(x)}[\alpha(\tau)+\beta(\tau)\xi-\gamma(\tau)\xi^2]\\&+\frac{3}{2\lambda^*}t_0^{-\frac{1}{2}}e^{-\frac{\tau}{2}}\left[-\alpha(\tau)\frac{\xi}{2}+\beta(\tau)\left(1-\frac{\xi^2}{2}\right)-\gamma(\tau)\left(2\xi-\frac{\xi^3}{2}\right)\right],\end{aligned}$$

that yields, uniformly for $0 \leq \xi \leq 1$ and all $\tau \gg 1$,

$$e_0^{-1}(\xi)\widetilde{L}[Z](\tau,\xi) \leq -\frac{\alpha_0}{10}e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{2}}) - \alpha(\tau) + \sqrt{t_0}e^{\frac{\tau}{2}}\frac{U_\sharp'(x)}{U_\sharp(x)}\alpha(\tau)\xi$$
$$- \sqrt{t_0}\frac{2U_\sharp'(x)}{U_\sharp(x)}\left\{\beta_0(1-\frac{\xi^2}{4}) - 2\xi\gamma_0 - \frac{3\alpha_0}{2\lambda^*\sqrt{t_0}}\right\}.$$

Recalling the definition of x_0 in Lemma 4.1 and that $x = \sqrt{t_0}e^{\frac{\tau}{2}}\xi$, this yields

$$-\alpha(\tau) + \sqrt{t_0}e^{\frac{\tau}{2}}\frac{U_\sharp'(x)}{U_\sharp(x)}\alpha(\tau)\xi = -\alpha(\tau)\left[1 - \frac{xU_\sharp'(x)}{U_\sharp(x)}\right] \leq 0,$$

as soon as $x \geq x_0$, while for $x \in [0, x_0]$ one has

$$-\alpha(\tau) + \sqrt{t_0}e^{\frac{\tau}{2}}\frac{U_\sharp'(x)}{U_\sharp(x)}\alpha(\tau)\xi \leq \frac{U_\sharp'(x)}{U_\sharp(x)}x_0.$$

As a consequence one gets, for all $\tau \gg 1$ and $0 \leq \xi \leq 1$,

$$e_0^{-1}(\xi)\widetilde{L}[Z](\tau,\xi) \leq -\frac{\alpha_0}{10}e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{2}})$$
$$- \sqrt{t_0}\frac{2U_\sharp'(x)}{U_\sharp(x)}\left\{-\frac{\alpha_0 x_0}{2\sqrt{t_0}} + \beta_0(1-\frac{\xi^2}{4}) - 2\xi\gamma_0 - \frac{3\alpha_0}{2\lambda^*\sqrt{t_0}}\right\}.$$

Recalling (4.47), fix $\xi_0 \in (0, 1]$ such that

$$\beta_0\left(1-\frac{\xi^2}{4}\right) - 2\xi\gamma_0 - \alpha_0\frac{3+x_0\lambda^*}{2\lambda^*\sqrt{t_0}} > 0, \ \forall \xi \in [0, \xi_0].$$

Thus coupling the above estimates ensures that there exists $\tau_2 > 0$ large enough such that

$$\widetilde{L}[Z](\tau,\xi) < 0, \ \forall \xi \in [0, \xi_0], \ \forall \tau \geq \tau_2.$$

For $\xi_0 \leq \xi \leq \varrho^*(\tau)$ and $\tau \gg 1$: Note that from the above computation, for $\tau \geq 0$ and $\xi_0 \leq \xi$, we have

$$e_0^{-1}(\xi)\widetilde{L}[Z](\tau,\xi) = -\frac{\alpha_0}{10}e^{-\frac{\tau}{5}} + \left[\frac{xU_\sharp'(x)}{U_\sharp(x)} - 1\right]\frac{\beta(\tau)}{2}\xi + 2\gamma(\tau)\left(1 + \frac{2xU_\sharp'(x)}{U_\sharp(x)}\right)$$
$$- (\alpha + \frac{\beta}{2}\xi)\left[1 - \frac{xU_\sharp'(x)}{U_\sharp(x)}\right] - \sqrt{t_0}e^{\frac{\tau}{2}}\beta\frac{2U_\sharp'(x)}{U_\sharp(x)} - \left[-\frac{1}{2} + \frac{xU_\sharp'(x)}{U_\sharp(x)}\right]\gamma(\tau)\xi^2$$
$$+ \frac{3}{2\lambda^*}\frac{U_\sharp'(x)}{U_\sharp(x)}[\alpha + \beta\xi - \gamma\xi^2]$$
$$+ \frac{3}{2\lambda^*}t_0^{-\frac{1}{2}}e^{-\frac{\tau}{2}}\left[-\frac{\alpha\xi}{2} + \beta\left(1 - \frac{\xi^2}{2}\right) - \gamma\left(2\xi - \frac{\xi^3}{2}\right)\right].$$

Hence recalling the definition of x_0 in Lemma 4.1, one has

$$\frac{1}{2} \leq x \frac{U_\sharp'(x)}{U_\sharp(x)} \leq 1, \ \forall x \geq x_0,$$

so that, since $x = \sqrt{t_0} e^{\frac{\tau}{2}} \xi \geq \sqrt{t_0} e^{\frac{\tau}{2}} \xi_0$, there exists $\tau_3 > 0$ large enough such that

$$\frac{1}{2} \leq x \frac{U_\sharp'(x)}{U_\sharp(x)} \leq 1, \ \forall \xi \geq \xi_0, \ \forall \tau \geq \tau_3.$$

We infer from the above estimates that, for all $\tau \gg 1$ and all $\xi_0 \leq \xi \leq \varrho^*(\tau) = O(e^{\tau/4})$ we have

$$\begin{aligned} e_0^{-1}(\xi) \widetilde{L}[Z](\tau, \xi) &\leq -\frac{\alpha_0}{10} e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{4}}) + \frac{3}{2\lambda^*} \frac{1}{x} [\alpha_0/2 + \beta(\tau)\xi] \\ &\leq -\frac{\alpha_0}{10} e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{4}}) + \frac{3}{2\lambda^*} \frac{1}{\sqrt{t_0} e^{\frac{\tau}{2}} \xi} [\alpha_0/2 + \beta(\tau)\xi] \\ &\leq -\frac{\alpha_0}{10} e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{4}}) + \frac{3}{2\lambda^*} t_0^{-1/2} e^{-\frac{\tau}{2}} \left[\frac{\alpha_0}{2\xi_0} + \beta_0 e^{-\frac{\tau}{2}} \right] \\ &\leq -\frac{\alpha_0}{10} e^{-\frac{\tau}{5}} + O(e^{-\frac{\tau}{4}}). \end{aligned}$$

Finally the study of the three regions provided above yields the expected estimate

$$\widetilde{L}[Z](\tau, \xi) < 0, \ \forall \tau \gg 1, \ \forall \xi \in [\varrho_*(\tau), \varrho^*(\tau)],$$

that completes the proof of the lemma. □

Fourth step: end of the proof of Theorem 1.4

Here recall that we have already proved, in the previous steps, statements (i) and (iii) in Theorem 1.4. Indeed (i) follows from the second step while (iii) follows from the firth and the third steps. To complete the proof of Theorem 1.4, it remains to check that (ii) also holds true. And, this point follows from the parabolic maximum principle applied to the u-equation. To see this, let us observe that (iii) ensures that

$$\liminf_{t \to \infty} \min_{y \in \overline{\Omega}} v\left(t, \pm\zeta(t), y\right) > 0. \tag{4.49}$$

Here ζ is the function introduced in (1.7). From this property, we claim that the following holds true:

$$\limsup_{t \to \infty} \sup_{y \in \overline{\Omega}} u\left(t, \zeta(t), y\right) < 1. \tag{4.50}$$

As a consequence, since $u(t,x,y)<1$ for all $t\geq 1$, $(x,y)\in\overline{\Sigma}$ and, since u satisfies

$$\begin{cases} \frac{\partial u}{\partial t} - D\Delta u < 0, \quad \text{for } t\geq 1 \text{ and } (x,y)\in[-\zeta(t),\zeta(t)]\times\Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{for } t\geq 1 \text{ and } (x,y)\in\partial\Sigma, \end{cases}$$

the parabolic maximum principle applies and ensures that for all $t\geq 1$ one has

$$\sup_{(x,y)\in[-\zeta(t),\zeta(t)]\times\overline{\Omega}} u(t,x,y) \leq \max\left(\sup_{(x,y)\in[-\zeta(1),\zeta(1)]\times\overline{\Omega}} u(1,x,y); \sup_{1\leq s\leq t,\ y\in\overline{\Omega}} u(s,\pm\zeta(s),y)\right).$$

Hence the result follows due to (4.50). This completes the proof of Theorem 1.4 (*ii*) and thus the proof of the theorem provided the proof of (4.50).

In order to prove (4.50), we argue by contradiction by assuming that there exist a sequence $\{t_n\}_{n\geq 0}$ tending to ∞ and a sequence $\{y_n\}_{n\geq 0}\subset\overline{\Omega}$ such that

$$u\left(t_n,\zeta(t_n),y_n\right)\to 1 \text{ as } n\to\infty.$$

Next consider the sequence of functions u_n and v_n defined by

$$(u_n,v_n)(t,x,y) = (u,v)\left(t+t_n, x+\zeta(t+t_n), y\right).$$

Note that one has $u_n(0,0,y_n)\to 1$ as $n\to\infty$. Furthermore the functions u_n and v_n satisfies

$$\begin{aligned} &\frac{\partial u_n}{\partial t} - \left(c^* - \frac{3}{c^*(t+t_n)}\right)\frac{\partial u_n}{\partial x} - D\Delta u_n = -\beta(y)u_n v_n, \\ &\frac{\partial v_n}{\partial t} - \left(c^* - \frac{3}{c^*(t+t_n)}\right)\frac{\partial v_n}{\partial x} - \Delta v_n = \beta(y)u_n v_n - \mu(y)v_n, \\ &\frac{\partial u_n}{\partial \nu} = \frac{\partial v_n}{\partial \nu} + \sigma(y)v_n = 0. \end{aligned}$$

Since the sequences $\{u_n\}$ and $\{v_n\}$ are uniformly bounded (see Proposition 1.2), possibly along a subsequence not relabelled, one may assume that

$$(u_n,v_n)(t,x,y)\to(u_\infty,v_\infty)(t,x,y) \text{ locally uniformly for } (t,(x,y))\in\mathbb{R}\times\overline{\Sigma}.$$

Hence, if we furthermore assume, up to a subsequence not relabelled, that $y_n\to y_\infty\in\overline{\Omega}$ then the limit function (u_∞,v_∞) satisfies the following problem for all $t\in\mathbb{R}$

$$\begin{aligned} &\frac{\partial u_\infty}{\partial t} - c^*\frac{\partial u_\infty}{\partial x} - D\Delta u_\infty = -\beta(y)u_\infty v_\infty \text{ on } \Sigma, \\ &\frac{\partial v_\infty}{\partial t} - c^*\frac{\partial v_\infty}{\partial x} - \Delta v_\infty = \beta(y)u_\infty v_\infty - \mu(y)v_\infty \text{ on } \Sigma, \\ &\frac{\partial u_\infty}{\partial \nu} = \frac{\partial v_\infty}{\partial \nu} + \sigma(y)v_\infty = 0 \text{ on } \partial\Sigma, \end{aligned}$$

together with $u_\infty(0,0,y_\infty)=1$, $u_\infty \le 1$ and $v_\infty(0,0,y)>0$ for all $y\in\overline{\Omega}$ (see (4.49)). Because of the Neumann boundary condition for u_∞, $y_\infty\in\Omega$, and we conclude from the strong maximum principle that $u_\infty(t,x,y)\equiv 1$. This implies that $\beta(y)v_\infty(t,x,y)=0$, for all $t\in\mathbb{R}$ and $(x,y)\in\overline{\Sigma}$. Since $\beta(y)\not\equiv 0$ then $v_\infty(0,0,y_0)=0$ for some $y_0\in\Omega$, that contradicts the above property of v_∞, namely $v_\infty(0,0,y)>0$ for all $y\in\overline{\Omega}$. This completes the proof of (4.50).

References

[1] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population dynamics, Adv. Math. 30 (1978) 33–76.
[2] J.M. Arrieta, J.W. Cholewa, T. Dlotko, A. Rodríguez-Bernal, Linear parabolic equations in locally uniform spaces, Math. Models Methods Appl. Sci. 14 (2004) 253–293.
[3] H. Berestycki, F. Hamel, A. Kiselev, L. Ryzhik, Quenching and propagation in KPP reaction-diffusion equations with a heat loss, Arch. Ration. Mech. Anal. 178 (2005) 57–80.
[4] J. Billingham, D. Needham, The development of traveling waves in a quadratic and cubic autocatalysis with unequal diffusion. I: permanent form traveling waves, Philos. Trans. R. Soc. Lond. A 334 (1991) 1–24.
[5] M. Bramson, Convergence of Solutions of the Kolmogorov Equation of Travelling Waves, Mem. AMS, 1983, 196 pp.
[6] C. Beaumont, J.B. Burie, A. Ducrot, P. Zongo, Propagation of Salmonella within an industrial hen house, SIAM J. Appl. Math. 72 (2012) 1113–1148.
[7] X. Chen, Y. Qi, Propagation of local disturbances in reaction diffusion systems modelling quadratic autocatalysis, SIAM J. Appl. Math. 69 (2008) 273–282.
[8] X. Chen, Y. Qi, Travelling waves of auto-catalytic chemical reaction of general order – an elliptic approach, J. Differ. Equ. 246 (2009) 3038–3057.
[9] P. Collet, J. Xin, Global existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive combustion systems on $\mathbb{R}^n$, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 23 (1996) 625–642.
[10] A. Ducrot, On the large time behaviour of the multi-dimensional Fisher-KPP equation with compactly supported initial data, Nonlinearity 28 (2015) 1043–1076.
[11] A. Ducrot, Spatial propagation for a two component reaction-diffusion system arising in population dynamics, J. Differ. Equ. 260 (2016) 8316–8357.
[12] A. Ducrot, T. Giletti, Convergence to a pulsating travelling wave for an epidemic reaction-diffusion system with non-diffusive susceptible population, J. Math. Biol. 69 (2014) 533–552.
[13] A. Ducrot, P. Magal, Travelling wave solutions for an infection-age structured model with diffusion, Proc. R. Soc. Edinb. A 139 (2009) 459–482.
[14] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. Ser. A 115 (1927) 700–721.
[15] W.O. Kermack, A.G. McKendrick, Contributions to the mathematical theory of epidemics, part II, Proc. R. Soc. Lond. B 138 (1932) 55–83.
[16] W.O. Kermack, A.G. McKendrick, Contributions to the mathematical theory of epidemics, part III, Proc. R. Soc. Lond. B 141 (1933) 94–112.
[17] T. Gallay, Local stability of critical fronts in nonlinear parabolic partial differential equations, Nonlinearity 7 (1994) 741–764.
[18] T. Giletti, KPP reaction-diffusion system with a nonlinear loss inside a cylinder, Nonlinearity 23 (2010) 2307–2332.
[19] F. Hamel, J. Nolen, J.M. Roquejoffre, L. Ryzhik, A short proof of the logarithmic Bramson correction in Fisher-KPP equations, Netw. Heterog. Media 8 (2013) 275–279.
[20] F. Hamel, J. Nolen, J.M. Roquejoffre, L. Ryzhik, The logarithmic delay of KPP fronts in a periodic medium, J. Eur. Math. Soc. 18 (2016) 465–505.
[21] F. Hamel, L. Ryzhik, Non-adiabatic KPP fronts with an arbitrary Lewis number, Nonlinearity 18 (2005) 2881–2902.
[22] F. Hamel, L. Ryzhik, Travelling waves for the thermodiffusive system with arbitrary Lewis numbers, Arch. Ration. Mech. Anal. 195 (2010) 923–952.
[23] M. Herrero, A. Lacey, J. Velázquez, Global existence for reaction-diffusion systems modelling ignition, Arch. Ration. Mech. Anal. 142 (1998) 219–251.
[24] S.I. Hollis, R.H. Martin, M. Pierre, Global existence and boundedness in reaction–diffusion systems, SIAM J. Math. Anal. 18 (1987) 744–761.

[25] Y. Hosono, B. Ilyas, Traveling waves for a simple diffusive epidemic model, Math. Models Methods Appl. Sci. 5 (1995) 935–966.
[26] B.J. Matkowsky, G.I. Sivashinsky, An asymptotic derivation of two models in flame theory associated with the constant density approximation, SIAM J. Appl. Math. 37 (1979) 686–699.
[27] J.D. Murray, Mathematical Biology, 3rd edn., Springer, Berlin, 2002.