

Nonautonomous Lotka–Volterra Systems with Delays

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The paper studies the general nonautonomous Lotka–Volterra multispecies systems with finite delays. The ultimate boundedness, permanence, global attractivity, and existence and uniqueness of strictly positive solutions, positive periodic solutions, and almost periodic solutions are obtained. These results are basically an extension of the known results for nonautonomous Lotka–Volterra multispecies systems without delay to systems with delay. © 2002 Elsevier Science (USA)

Key Words: nonautonomous Lotka–Volterra system; delay, permanence; global attractivity; strictly positive solution; periodic solution; almost periodic solution.

1. INTRODUCTION

In this paper we consider the following general nonautonomous Lotka–Volterra type multispecies systems with delays:

$$\frac{dx_i(t)}{dt} = x_i(t) \left[a_i(t) - b_i(t) x_i(t) - \sum_{j=1}^n c_{ij}(t) x_j(t - \tau_{ij}(t)) - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}(t, s) x_j(t+s) ds \right], \quad i = 1, 2, \dots, n, \quad (1)$$

where $x_i(t)$ represents the population density of the i th species at time t ; the functions $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, and $\tau_{ij}(t)$ ($i, j = 1, 2, \dots, n$) are defined on $R = (-\infty, +\infty)$ and are continuous for all $t \in R$; the functions $k_{ij}(t, s)$ ($i, j = 1, 2, \dots, n$) are defined on $R \times [-\sigma_{ij}, 0]$ and are continuous with respect to $t \in R$ and integrable with respect to $s \in [-\sigma_{ij}, 0]$ and σ_{ij} ($i, j = 1, 2, \dots, n$) are nonnegative constants.

When the delays $\tau_{ij}(t) \equiv 0$ and $\sigma_{ij} \equiv 0$ for all $t \in R$ and $i, j = 1, 2, \dots, n$, then system (1) will degenerate into the following nondelayed nonautonomous multispecies Lotka–Volterra system

$$\frac{dx_i(t)}{dt} = x_i(t) (a_i(t) - \sum_{j=1}^n b_{ij}(t) x_j(t)), \quad i = 1, 2, \dots, n, \quad (2)$$

where $b_{ii}(t) = b_i(t) + c_{ii}(t)$ and $b_{ij}(t) = c_{ij}(t)$ for $i, j = 1, 2, \dots, n$ and $i \neq j$.

As we well know, systems like (1) and (2) are very important mathematical models which describe multispecies population dynamics. The most basic and important questions to ask for these systems in the theory of mathematical ecology are the persistences, extinctions, global asymptotic behaviors, and existences of coexistence states (for example, the positive equilibrium, strictly positive solution, positive periodic solution, and almost periodic solution, etc.) of population (see [4, 6, 10, 17, 21]). In this domain there have been numerous works for the nonautonomous Lotka-Volterra and Kolmogorov type systems with delays. In particular, for the nonautonomous Lotka-Volterra type competitive and predator-prey systems with delays the above questions have been extensively studied. Many important results can be found in [1, 2, 7–11, 15, 17–20, 22, 25, 28–31, 33, 34] and references cited therein. In those works the method of Liapunov functions [3, 12], the theory of monotone semiflows generated by functional differential equations [26, 27], the fixed point theory [5, 16], and so on are extensively applied.

Recently, the nondelayed system (2) was studied by Redheffer [23, 24] and Tineo [32]. Under remarkably weak conditions (See conditions (a)–(e) in [23] and condition (0.2) in [32]) the boundedness, permanence, extinction, global attractivity, and existence of positive periodic solutions and almost periodic solutions are obtained (See Theorem 1 in [23], Sections 3 and 4 in [32]).

Motivated by the above works, in this paper we study the delayed system (1). As a basic extension of the main results given by Redheffer in [23] we will establish a series of sufficient conditions on the ultimate boundedness, permanence, global attractivity, and existence and uniqueness of strictly positive solutions, positive periodic solutions, and almost periodic solutions for the delayed system (1).

The organization of this paper is as follows. In the next section, the main assumptions and our major theorems are described. In Section 3, the proofs of the main theorems are contained. The main methods used in the proofs of the theorems are motivated by the papers [13, 23, 24, 31].

2. MAIN RESULTS

For any function $f(t)$ defined on R we denote $f^+(t) = \max\{0, f(t)\}$ and $f^-(t) = \min\{0, f(t)\}$. Let $\lambda(t)$ be a nonnegative continuous function defined on R . If for any interval sequence $\{[a_s, b_s]\}$, satisfying $a_s > 0$, $[a_s, b_s] \cap [a_j, b_j] = \emptyset$, and $b_s - a_s = b_j - a_j > 0$ for all $s, j = 1, 2, \dots$ and $s \neq j$, one has $\sum_{s=1}^{\infty} \int_{a_s}^{b_s} \lambda(t) dt = \infty$, then we call $\lambda(t) \in S_1$. If for any interval sequence $\{[a_s, b_s]\}$, satisfying $b_s < 0$, $[a_s, b_s] \cap [a_j, b_j] = \emptyset$, and

$b_s - a_s = b_j - a_j > 0$ for all $s, j = 1, 2, \dots$ and $s \neq j$, one has $\sum_{s=1}^{\infty} \int_{a_s}^{b_s} \lambda(t) dt = \infty$, then we call $\lambda(t) \in S_2$.

For system (1) we first introduce the following assumptions. Let $t_0 \in R$ be a fixed initial time.

(A₁) For each $1 \leq i, j \leq n$, $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, and $\tau_{ij}(t)$ are bounded on $[t_0, \infty)$ and $a_i(t)$, $b_i(t)$, and $\tau_{ij}(t)$ also are nonnegative for all $t \geq t_0$.

(A₂) For any $t \geq t_0$, $k_{ij}(t-s, s)$ ($i, j = 1, 2, \dots, n$) are integrable with respect to $s \in [-\sigma_{ij}, 0]$ and there is a nonnegative integrable function $h_0(s)$ on $(-\infty, 0]$ such that $|k_{ij}(t, s)| \leq h_0(s)$ for all $t \geq t_0$, $s \in [-\sigma_{ij}, 0]$, and $i, j = 1, 2, \dots, n$.

(A₃) There exist positive constants p_i ($i = 1, 2, \dots, n$) and α such that the functions

$$\beta_i(t) = b_i(t) p_i + \sum_{j=1}^n c_{ij}^-(t)(1+\alpha) p_j + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(t, s)(1+\alpha) p_j ds$$

satisfy $\beta_i(t) \geq a_i(t)$ and $\beta(t) = \min_{1 \leq i \leq n} \beta_i(t) > 0$ for all $t \geq t_0$, $\int_{t_0}^{\infty} \beta(t) dt = \infty$, and $\limsup_{t \rightarrow \infty} a_i(t)/\beta(t) \leq 1$ for $i = 1, 2, \dots, n$.

(A₄) There exist positive constants ω and ζ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$\int_t^{t+\omega} \left[a_i(u) - \sum_{j \neq i}^n c_{ij}^+(u) p_j - \sum_{j \neq i}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(u, s) p_j ds \right] du \geq \zeta,$$

where the constants p_i ($i = 1, 2, \dots, n$) are given in the assumption (A₃).

(A₅) $\tau_{ij}(t)$ ($i, j = 1, 2, \dots, n$) are continuously differentiable on $[t_0, \infty)$ and $\min_{t \geq t_0} \{1 - \dot{\tau}_{ij}(t)\} > 0$, where $\dot{\tau}_{ij}(t) = d\tau_{ij}(t)/dt$.

(A₆) There exist positive constants d_i ($i = 1, 2, \dots, n$) and a nonnegative continuous function $\delta(t)$ defined on $[t_0, \infty)$ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$d_i b_i(t) - \sum_{j=1}^n d_j \left(\frac{|c_{ji}(\psi_{ji}^{-1}(t))|}{1 - \dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} + \int_{-\sigma_{ji}}^0 |k_{ji}(t-s, s)| ds \right) \geq \delta(t),$$

where $\psi_{ji}^{-1}(t)$ is the inverse function of $\psi_{ji}(t) = t - \tau_{ji}(t)$.

(A₇) There exist positive constants h_i ($i = 1, 2, \dots, n$) and γ such that

$$b_i(t) h_i - \sum_{j=1}^n |c_{ij}(t)| h_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 |k_{ij}(t, s)| h_j ds \geq \gamma b_i(t)$$

for all $t \geq t_0$ and $i = 1, 2, \dots, n$.

Remark 1. Assumptions (A_1) , (A_2) and (A_5) are general and elementary because similar assumptions are required in many works on the qualitative analysis of nonautonomous Lotka-Volterra type systems with delays, for example [2, 11, 18–20, 28–31, 33]. Assumptions (A_3) , (A_4) , (A_6) , and (A_7) are important and substantive. Assumption (A_3) will ensure the ultimate boundedness of solutions of system (1) with the bound $p_i (i = 1, 2, \dots, n)$. Assumption (A_4) will ensure the permanence of solutions of system (1). Assumptions (A_6) and (A_7) shall ensure respectively the global attractivity of solutions for system (1).

Remark 2. When system (1) is competitive, i.e., $c_{ij}(t) \geq 0$ and $k_{ij}(t, s) \geq 0 (i, j = 1, 2, \dots, n)$ for all $t \in R$ and $s \in [-\sigma_{ij}, 0]$, then assumption (A_3) will become that there exist positive constants $p_i (i = 1, 2, \dots, n)$ such that $p_i b_i(t) > a_i(t)$ for all $t \geq t_0$ and $i = 1, 2, \dots, n$.

When system (1) is cooperative, i.e., $c_{ij}(t) \leq 0$ and $k_{ij}(t, s) \leq 0 (i, j = 1, 2, \dots, n)$ for all $t \in R$ and $s \in [-\sigma_{ij}, 0]$, then assumption (A_4) will become that there exist positive constants ω and ζ such that $\int_{t_0}^{t+\omega} a_i(s) ds \geq \zeta$ for all $t \geq t_0$ and $i = 1, 2, \dots, n$.

Remark 3. Corresponding to nondelayed system (2) the assumptions (A_1) – (A_7) will change into the following forms, where without loss of generality we suppose $c_{ii}(t) \equiv 0$.

(i) For each $1 \leq i, j \leq n$, $a_i(t)$ and $b_{ij}(t)$ are bounded on $[t_0, \infty)$ and $a_i(t)$, $b_{ii}(t)$ also are nonnegative for all $t \geq t_0$.

(ii) There exist positive constants $p_i (i = 1, 2, \dots, n)$ and α such that the functions

$$\beta_i(t) = b_{ii}(t) p_i + \sum_{j \neq i}^n b_{ij}^-(t) (1 + \alpha) p_j$$

satisfy $\beta_i(t) \geq a_i(t)$ and $\beta(t) = \min_{1 \leq i \leq n} \beta_i(t) > 0$ for all $t \geq t_0$, $\int_{t_0}^{\infty} \beta(t) dt = \infty$, and $\limsup_{t \rightarrow \infty} a_i(t) / \beta(t) \leq 1$ for $i = 1, 2, \dots, n$.

(iii) There exist positive constants ω and ζ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$\int_t^{t+\omega} \left[a_i(u) - \sum_{j \neq i}^n b_{ij}^+(t) p_j \right] du \geq \zeta,$$

where the constants $p_i (i = 1, 2, \dots, n)$ are given in the above.

(iv) There exist positive constants $d_i (i = 1, 2, \dots, n)$ and a nonnegative continuous function $\delta(t)$ defined on $[t_0, \infty)$ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$d_i b_{ii}(t) - \sum_{j \neq i}^n d_j |b_{ji}(t)| \geq \delta(t).$$

(v) There exist positive constants $h_i (i = 1, 2, \dots, n)$ and γ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$b_{ii}(t) h_i - \sum_{j \neq i}^n |b_{ij}(t)| h_j \geq \gamma b_{ii}(t).$$

Compared with the conditions in [23], we see that the boundedness of $a_i(t)$ and $b_{ij}(t)$ required in condition (i) is not assumed in [23] and condition (a) in [23] is decomposed in this paper into conditions (ii) and (iii). Obviously, condition (ii) is a little stronger than the right inequality of condition (a). However, condition (iii) is weaker than the left inequality of condition (a). In addition, conditions (iv) and (v) are corresponding to conditions (b) and

$$b_{ii} d_i - \sum_{j \neq i} |b_{ij}| d_j \geq b_{ii} \bar{d}_i$$

in [23], respectively. We also see that the conditions $\delta(t) \in S_1$ and $\delta(t) \in S_2$ are a little stronger than conditions (e) and (d) in [23], respectively.

Remark 4. Usually, assumption (A_6) shows that the coefficients of system (1) are provided with column diagonal dominance. Assumption (A_7) shows that the coefficients are provided with row diagonal dominance.

Let $\tau = \sup\{\tau_{ij}(t), \sigma_{ij}: t \geq t_0, i, j = 1, 2, \dots, n\}$. We define the phase space of system (1) to be the Banach space $C^n[-\tau, 0]$ of continuous functions $\phi(s): [-\tau, 0] \rightarrow R^n$ with the supremum norm defined by

$$\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)|,$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ and $|\phi(s)| = \sum_{i=1}^n |\phi_i(s)|$. By the fundamental theory of functional differential equations [3, 12, 17] we know that for any $\phi \in C^n[-\tau, 0]$ system (1) has a unique solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ satisfying the initial condition $x_{t_0}(\cdot, \phi) = \phi$.

Define the subset $C_+^n[-\tau, 0]$ of the space $C^n[-\tau, 0]$ as follows. If $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C_+^n[-\tau, 0]$, then $\phi_i(s) \geq 0$ and $\phi_i(0) > 0$ for all $s \in [-\tau, 0]$ and $i = 1, 2, \dots, n$. Motivated by the biological background of system (1), in this paper we will be concerned with only positive solutions

of system (1). Here, we say that a solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1) is positive if $x_i(t, \phi) > 0 (i = 1, 2, \dots, n)$ in its maximal interval of existence. It is not hard to prove that the solution $x(t, \phi)$ of system (1) is positive, if the initial function $\phi \in C_+^n[-\tau, 0]$. A solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of system (1) is said to be strictly positive, if the solution $x(t)$ is defined on $t \in R$ and satisfies $0 < \inf_{t \in R} x_i(t) \leq \sup_{t \in R} x_i(t) < \infty$ for $i = 1, 2, \dots, n$.

System (1) is said to be periodic with the period $\omega > 0$ if for each $1 \leq i, j \leq n$, $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, and $\tau_{ij}(t)$ are ω -periodic functions and $k_{ij}(t, s)$ also is an ω -periodic function with respect to t .

System (1) is said to be almost periodic if for each $1 \leq i, j \leq n$, $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, and $\tau_{ij}(t)$ are almost periodic functions and $k_{ij}(t, s)$ is a uniformly almost periodic function with respect to t .

We now state our main theorems in this paper.

THEOREM 1. *If assumptions (A_1) – (A_3) hold, then for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1),*

$$\limsup_{t \rightarrow \infty} x_i(t, \phi) \leq p_i \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 2. *If assumptions (A_1) – (A_4) hold, then there is a constant $m > 0$ such that, for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1),*

$$m \leq \liminf_{t \rightarrow \infty} x_i(t, \phi) \leq \limsup_{t \rightarrow \infty} x_i(t, \phi) \leq p_i \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 3. *If assumptions (A_1) – (A_4) hold on $t \in R$, then system (1) has a strictly positive solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ such that $x_i^*(t) \leq p_i (i = 1, 2, \dots, n)$ for all $t \in R$.*

THEOREM 4. *If assumptions (A_1) – (A_3) , (A_5) , and (A_6) hold and $\delta(t) \in S_1$, then for any two positive solutions $x(t, \phi_k) = (x_1(t, \phi_k), x_2(t, \phi_k), \dots, x_n(t, \phi_k)) (k = 1, 2)$ of system (1),*

$$\lim_{t \rightarrow \infty} (x_i(t, \phi_1) - x_i(t, \phi_2)) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 5. *If assumptions (A_1) – (A_6) hold on $t \in R$ and $\delta(t) \in S_2$, then system (1) has a unique strictly positive solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$. Moreover, if $\delta(t) \in S_1 \cap S_2$, then for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1) one has*

$$\lim_{t \rightarrow \infty} (x_i(t, \phi) - x_i^*(t)) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 6. *If system (1) is periodic with the period $\omega > 0$, and assumptions (A_1) – (A_4) hold, then system (1) has a positive ω -periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$.*

THEOREM 7. *If system (1) is periodic with the period $\omega > 0$, assumptions (A_1) – (A_6) hold on $t \in R$, $\delta(t)$ is ω -periodic, and $\delta(t) \in S_1$, then system (1) has a unique positive ω -periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ such that for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1)*

$$\lim_{t \rightarrow \infty} (x_i(t, \phi) - x_i^*(t)) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 8. *If system (1) is almost periodic, $\tau_{ij}(t) \equiv \tau_{ij}$ is constant for each $i, j = 1, 2, \dots, n$, assumptions (A_1) – (A_4) and (A_6) hold on $t \in R$, and $\delta(t) \equiv \delta$ is a positive constant, then system (1) has a unique strictly positive almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ such that for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1)*

$$\lim_{t \rightarrow \infty} (x_i(t, \phi) - x_i^*(t)) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 9. *If assumptions (A_1) – (A_5) and (A_7) hold, and system (1) has a solution $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$ satisfying $\lim_{t \rightarrow \infty} u_i(t) = q_i$ ($i = 1, 2, \dots, n$), where $q_i > 0$ is constant, then any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1) also satisfies*

$$\lim_{t \rightarrow \infty} x_i(t, \phi) = q_i \quad \text{for } i = 1, 2, \dots, n.$$

THEOREM 10. *If $b_i(t)$, $c_{ij}(t)$, and $\tau_{ij}(t)$ are constants, $k_{ij}(t, s) = k_{ij}(s)$ for $i, j = 1, 2, \dots, n$, and there are positive constants p_i , \bar{p}_i ($i = 1, 2, \dots, n$) and ρ such that for all $t \in R$*

$$(A_3^*) \quad b_i p_i + \sum_{j=1}^n c_{ij}^- p_j + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(s) p_j ds - \rho \geq a_i(t),$$

$$(A_4^*) \quad b_i \bar{p}_i - \sum_{j=1}^n c_{ij}^+ p_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(s) p_j ds + \rho \leq a_i(t),$$

then the conclusions of Theorems 1–9 remain true.

Remark 5. Theorem 1 shows that all positive solutions of system (1) are ultimately bounded with the bound $p = (p_1, p_2, \dots, p_n)$. Theorem 2 shows that system (1) is permanent. Theorems 3 and 6 give the existence of strictly positive solutions and positive ω -periodic solutions of system (1). Theorem 4 shows that system (1) is globally attractive. Theorems 5, 7, and

8 give the existence, uniqueness, and global attractivity of strictly positive solution, positive ω -periodic solution, and positive almost periodic solution of system (1), respectively. In addition, if system (1) has a positive equilibrium $q = (q_1, q_2, \dots, q_n)$, that is,

$$a_i(t) - b_i(t) q_i - \sum_{j=1}^n c_{ij}(t) q_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}(t, s) q_j ds \equiv 0$$

for all $t \in R$ and $i = 1, 2, \dots, n$, then Theorem 9 shows that the equilibrium q is globally attractive.

Remark 6. In [30], Teng and Chen have proved that in a periodic Kolmogorov type system with finite delays, if the system is permanent, then it has at least a positive periodic solution (see [Theorem 1, 30]). Therefore, we see that Theorem 6 is a direct corollary of Theorem 2 and the main results in [30]. In addition, we also see that Theorem 7 is a direct corollary of Theorems 4 and 6.

3. PROOFS OF THEOREMS

In this section we will give the proofs of the main results of this paper. We first have

Proof of Theorem 1. Let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ be any positive solution of system (1). Define the function as follows

$$V(t) = \max_{1 \leq i \leq n} \left\{ \frac{x_i(t, \phi)}{p_i} \right\}.$$

For any $t \geq t_0$, there is an index $i = i(t) \in \{1, 2, \dots, n\}$ such that $V(t) = x_i(t, \phi)/p_i$. Calculating the upper right derivative of $V(t)$ at time t , we have

$$\begin{aligned} D^+V(t) &= \frac{1}{p_i} \frac{dx_i(t, \phi)}{dt} \\ &\leq V(t) \left[a_i(t) - b_i(t) p_i V(t) - \sum_{j=1}^n c_{ij}^-(t) x_j(t - \tau_{ij}(t), \phi) \right. \\ &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(t, s) x_j(t + s, \phi) ds \right]. \end{aligned} \quad (3)$$

When $V(t+s) \leq (1+\alpha)V(t)$ for all $s \in [-\tau, 0]$, where the constant $\alpha > 0$ is given in assumption (A_3) , then we have $x_j(t+s, \phi)/p_j \leq (1+\alpha)V(t)$ for all $s \in [-\tau, 0]$ and $j = 1, 2, \dots, n$. Hence, from (3) we have

$$\begin{aligned}
D^+V(t) &\leq V(t) \left[a_i(t) - b_i(t) p_i V(t) - \sum_{j=1}^n c_{ij}^-(t)(1+\alpha) p_j V(t) \right. \\
&\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(t, s)(1+\alpha) p_j V(t) ds \right] \\
&= V(t)[a_i(t) - \beta_i(t) V(t)].
\end{aligned} \tag{4}$$

For any constant $\theta > 1$, by assumption (A_3) there is a sufficiently large $T > 0$ such that $a_i(t) \leq \theta \beta(t)$ for all $t \geq T$ and $i = 1, 2, \dots, n$. Hence, by (4) we have

$$D^+V(t) \leq \beta(t) V(t)(\theta - V(t)) \quad \text{for all } t \geq T. \tag{5}$$

Obviously, if $V(t) > \theta$ for all $t \geq T$, then we have $D^+V(t) < 0$. Therefore, employing the standard argument of the Liapunov–Razumikhin type theorems of the ultimate boundedness of functional differential equations with finite delay (see [3, 12, 17]), we can obtain $\limsup_{t \rightarrow \infty} V(t) \leq \theta$. Finally, the arbitrariness of θ implies $\limsup_{t \rightarrow \infty} V(t) \leq 1$. This shows $\limsup_{t \rightarrow \infty} x_i(t, \phi) \leq p_i$ for $i = 1, 2, \dots, n$. This completes the proof.

Proof of Theorem 2. For each $i = 1, 2, \dots, n$, we introduce the following functions

$$\lambda_i(t) = b_i(t) + \sum_{j=1}^n \left(c_{ij}^+(t) + \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s) ds \right)$$

and

$$\xi_i(t) = \sum_{j \neq i}^n c_{ij}^+(t) p_j + \sum_{j \neq i}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s) p_j ds.$$

By assumptions (A_1) , (A_2) , and (A_4) , we can choose the sufficiently small constant $\eta_0 > 0$ such that

$$\int_t^{t+\omega} [a_i(u) - \eta_0 \lambda_i(u) - \xi_i(u)] du > \frac{1}{2} \zeta \quad \text{for all } t \geq t_0, i = 1, 2, \dots, n. \tag{6}$$

Let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ be any positive solution of system (1). We first prove

$$\limsup_{t \rightarrow \infty} x_i(t, \phi) \geq \eta_0 \quad \text{for } i = 1, 2, \dots, n. \tag{7}$$

In fact, if $\limsup_{t \rightarrow \infty} x_l(t, \phi) < \eta_0$ for some $l \in \{1, 2, \dots, n\}$, then there is a $T_1 > t_0$ such that

$$x_l(t, \phi) < \eta_0 \quad \text{for all } t \geq T_1.$$

By Theorem 1, we have that there is a $T_2 \geq T_1$ such that

$$x_j(t, \phi) \leq p_j + \eta_0 \quad \text{for all } t \geq T_2, j = 1, 2, \dots, n.$$

Hence, for any $t \geq T_2 + \tau$, we have

$$\begin{aligned} \frac{dx_l(t, \phi)}{dt} &\geq x_l(t, \phi) \left[a_l(t) - b_l(t) x_l(t, \phi) - \sum_{j=1}^n c_{lj}^+(t) x_j(t - \tau_{lj}(t), \phi) \right. \\ &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{lj}}^0 k_{lj}^+(t, s) x_j(t + s, \phi) ds \right] \\ &\geq x_l(t, \phi) \left[a_l(t) - b_l(t) \eta_0 - c_{ll}^+(t) \eta_0 \right. \\ &\quad \left. - \sum_{j \neq l}^n c_{lj}^+(t) (p_j + \eta_0) - \int_{-\sigma_{jj}}^0 k_{ll}^+(t, s) \eta_0 ds \right. \\ &\quad \left. - \sum_{j \neq l}^n \int_{-\sigma_{lj}}^0 c_{lj}^+(t, s) (p_j + \eta_0) ds \right] \\ &= x_l(t, \phi) [a_l(t) - \eta_0 \lambda_l(t) - \xi_l(t)]. \end{aligned} \quad (8)$$

By integrating (8) from $T_2 + \tau$ to t , we obtain

$$x_l(t, \phi) \geq x_l(T_2 + \tau, \phi) \exp \int_{T_2 + \tau}^t [a_l(s) - \eta_0 \lambda_l(s) - \xi_l(s)] ds.$$

Obviously, inequality (6) implies $x_l(T_2 + \tau + N\omega, \phi) \rightarrow \infty$ as $N \rightarrow \infty$, which is a contradiction with the boundedness of the solution $x(t, \phi)$. Therefore, inequality (7) is true.

We now prove that there is a constant $m > 0$ such that

$$\liminf_{t \rightarrow \infty} x_i(t, \phi) \geq m \quad \text{for } i = 1, 2, \dots, n \quad (9)$$

for any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1). Let N_1 and N_2 be positive integers such that $N_1\omega > \tau$ and $\frac{1}{2}N_2\zeta > N_1\pi_0\omega$, where $\pi_0 = \max\{\pi_i : i = 1, 2, \dots, n\}$ and

$$\pi_i = \sup_{t \in R_+} \left\{ \eta_0 \lambda_i(t) + \xi_i(t) + c_{ii}^+(t) p_i + \int_{-\sigma_{ii}}^0 k_{ii}^+(t, s) p_i ds \right\} > 0.$$

By Theorem 1, without loss of generality, we can suppose

$$x_i(t, \phi) < p_i + \eta_0 \quad \text{for all } t \geq t_0, i = 1, 2, \dots, n. \quad (10)$$

Let $0 < \eta^* < \eta_0$ be a fixed constant. For each $i = 1, 2, \dots, n$, since $\limsup_{t \rightarrow \infty} x_i(t, \phi) \geq \eta_0$, then either $x_i(t, \phi) \geq \eta^*$ for all $t \geq t_0$ or there is a time sequence $t_0 + \tau \leq t_1 < t_2 < \dots < t_k < \dots$ with $t_k \rightarrow \infty$ as $k \rightarrow \infty$, such that

$$x_i(t, \phi) < \eta^* \quad \text{for all } t \in (t_{2k}, t_{2k+1}), k = 1, 2, \dots \quad (11)$$

and $x_i(t, \phi) \geq \eta^*$ otherwise. Suppose that there is an integer k such that $t_{2k+1} - t_{2k} > (N_1 + N_2) \omega$; then by (6), (10), and (11) we can obtain

$$\begin{aligned} & \int_{t_{2k}}^{t_{2k} + N_1 \omega} \left[a_i(t) - b_i(t) x_i(t, \phi) - \sum_{j=1}^n c_{ij}^+(t) x_j(t - \tau_{ij}(t), \phi) \right. \\ & \quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s) x_j(t + s, \phi) ds \right] dt \\ & \geq \int_{t_{2k}}^{t_{2k} + N_1 \omega} \left[a_i(t) - b_i(t)(p_i + \eta_0) - \sum_{j=1}^n c_{ij}^+(t)(p_j + \eta_0) \right. \\ & \quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s)(p_j + \eta_0) ds \right] dt \\ & \geq -\pi_i N_1 \omega \end{aligned}$$

and

$$\begin{aligned} & \int_{t_{2k} + N_1 \omega}^{t_{2k} + (N_1 + N_2) \omega} \left[a_i(t) - b_i(t) x_i(t, \phi) - \sum_{j=1}^n c_{ij}^+(t) x_j(t - \tau_{ij}(t), \phi) \right. \\ & \quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s) x_j(t + s, \phi) ds \right] dt \\ & \geq \int_{t_{2k} + N_1 \omega}^{t_{2k} + (N_1 + N_2) \omega} \left[a_i(t) - b_i(t) \eta^* - c_{ii}^+(t) \eta^* - \int_{-\sigma_{ii}}^0 k_{ii}^+(t, s) \eta^* ds \right. \\ & \quad \left. - \sum_{j \neq i}^n c_{ij}^+(t)(p_j + \eta_0) - \sum_{j \neq i}^n \int_{-\sigma_{ij}}^0 c_{ij}^+(t, s)(p_j + \eta_0) ds \right] dt \\ & \geq \int_{t_{2k} + N_1 \omega}^{t_{2k} + (N_1 + N_2) \omega} [a_i(t) - \eta_0 \lambda_i(t) - \xi_i(t)] dt \\ & \geq \frac{1}{2} \zeta N_2. \end{aligned}$$

Therefore, directly from system (1) we have

$$\begin{aligned}
 x_i(t_{2k} + (N_1 + N_2)\omega, \phi) &= x_i(t_{2k}, \phi) \exp \int_{t_{2k}}^{t_{2k} + (N_1 + N_2)\omega} \\
 &\quad \left[a_i(t) - b_i(t) x_i(t, \phi) - \sum_{j=1}^n c_{ij}(t) x_j(t - \tau_{ij}(t), \phi) \right. \\
 &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}(t, s) x_j(t + s, \phi) ds \right] dt \\
 &\geq \eta^* \exp \left\{ \int_{t_{2k}}^{t_{2k} + N_1\omega} + \int_{t_{2k} + N_1\omega}^{t_{2k} + (N_1 + N_2)\omega} \right. \\
 &\quad \left[a_i(t) - b_i(t) x_i(t, \phi) - \sum_{j=1}^n c_{ij}^+(t) x_j(t - \tau_{ij}(t), \phi) \right. \\
 &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(t, s) x_j(t + s, \phi) ds \right] dt \Big\} \\
 &\geq \eta^* \exp(-\pi_0 N_1 \omega + \tfrac{1}{2} N_2 \zeta) \\
 &> \eta^*.
 \end{aligned}$$

This leads to a contradiction in the inequality (11). Hence, we must have $t_{2k+1} - t_{2k} \leq (N_1 + N_2)\omega$ for all $k = 1, 2, \dots$. Consequently, we have

$$\begin{aligned}
 x_i(t, \phi) &\geq \eta^* \exp \int_{t_{2k}}^t \left[a_i(u) - b_i(u) x_i(u, \phi) - \sum_{j=1}^n c_{ij}^+(u) x_j(u - \tau_{ij}(u), \phi) \right. \\
 &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(u, s) x_j(u + s, \phi) ds \right] du \\
 &\geq \eta^* \exp \int_{t_{2k}}^t \left[a_i(u) - b_i(u)(p_i + \eta_0) - \sum_{j=1}^n c_{ij}^+(u)(p_j + \eta_0) \right. \\
 &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(u, s)(p_j + \eta_0) ds \right] du \\
 &\geq \eta^* \exp(-\pi_0(N_1 + N_2)\omega) \quad \text{for all } t \in (t_{2k}, t_{2k+1}).
 \end{aligned}$$

Let $m = \eta^* \exp(-\pi_0(N_1 + N_2)\omega)$; then we finally have inequality (8), and this completes the proof.

Proof of Theorem 3. We first prove the following claim.

Claim 1. For any positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1) with the initial function $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C_+^n[-\tau, 0]$, if $\phi_i(s) \leq p_i (i = 1, 2, \dots, n)$ for all $s \in [-\tau, 0]$, then $x_i(t, \phi) \leq p_i (i = 1, 2, \dots, n)$ for all $t \geq t_0$.

In fact, for any constant $\theta > 1$, from assumption (A_3) we can obtain

$$a_i(t) < \theta \beta_i(t) \quad \text{for all } t \geq t_0, i = 1, 2, \dots, n. \quad (12)$$

We now prove $x_i(t, \phi) < \theta p_i$ for all $t > t_0$ and $i = 1, 2, \dots, n$. If the conclusion is not true, then there is a $t^* > t_0$ such that for some $i \in \{1, 2, \dots, n\}$, $x_i(t^*, \phi) = \theta p_i$ and $x_j(t, \phi) \leq \theta p_j$ for all $t \in [t_0 - \tau, t^*]$ and $j = 1, 2, \dots, n$. Obviously, we have $dx_i(t^*, \phi)/dt \geq 0$. But, from (12) we obtain that

$$\begin{aligned} \frac{dx_i(t^*, \phi)}{dt} &\leq \theta p_i \left[a_i(t^*) - b_i(t^*) \theta p_i - \sum_{j=1}^n c_{ij}^-(t^*) \theta p_j \right. \\ &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(t^*, s) \theta p_j ds \right] < 0, \end{aligned}$$

a contradiction. The arbitrariness of θ implies that

$$x_i(t, \phi) \leq p_i \quad \text{for all } t \geq t_0, i = 1, 2, \dots, n.$$

For any initial time $t_0 \in R$, we consider the positive solution $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))$ of system (1) starting at $t = t_0$ with the initial function $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C_+^n[-\tau, 0]$ satisfying $\eta^* \leq \phi_i(s) \leq p_i (i = 1, 2, \dots, n)$ for all $s \in [-\tau, 0]$, where $0 < \eta^* < \eta_0$ and η_0 is chosen such that

$$\int_t^{t+\omega} [a_i(u) - \eta_0 \lambda_i(u) - \xi_i(u)] du > \frac{1}{2} \zeta \quad \text{for all } t \in R, i = 1, 2, \dots, n.$$

Let $\pi_0^* = \max\{\pi_i^*: i = 1, 2, \dots, n\}$ and

$$\pi_i^* = \sup_{t \in R} \left\{ \eta_0 \lambda_i(t) + \xi_i(t) + c_{ii}^+(t) p_i + \int_{-\sigma_{ii}}^0 k_{ii}^+(t, s) p_i ds \right\} > 0.$$

Let the positive integers N_1, N_2 be fixed such that $N_1 \omega > \tau$ and $\frac{1}{2} N_2 \zeta > N_1 \pi_0^* \omega$. From Claim 1 and a similar argument as in Theorem 2, we can obtain that

$$m^* \leq x_i(t, \phi) \leq p_i \quad \text{for all } t \geq t_0, i = 1, 2, \dots, n, \quad (13)$$

where $m^* = \eta^* \exp(-\pi_0^* (N_1 + N_2) \omega)$.

Choose the initial time $t_0 = -k (k = 1, 2, \dots)$; then we can obtain a sequence $\{x_k(t, \phi)\}$ of positive solutions of system (1) with the initial function $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ satisfying $\eta^* \leq \phi_i(s) \leq p_i (i = 1, 2, \dots, n)$ for all $s \in [-\tau, 0]$. It follows from (13) that the solutions $x_k(t, \phi) = (x_{1k}(t, \phi), x_{2k}(t, \phi), \dots, x_{nk}(t, \phi))$ satisfy

$$m^* \leq x_{ik}(t, \phi) \leq p_i, \quad i = 1, 2, \dots, n \quad (14)$$

for all $t \geq -k$ and $k = 1, 2, \dots$. Hence, for any integer $q > 0$, the sequence $\{x_k(t, \phi): k \geq q\}$ is uniformly bounded on interval $[-q, \infty)$. On the other hand, from assumptions (A_1) , (A_2) , and (14) we can obtain directly from system (1) that there is a constant $M_0 > 0$ such that

$$\left| \frac{dx_{ik}(t, \phi)}{dt} \right| \leq M_0$$

for all $t \geq -k, i = 1, 2, \dots, n$ and $k = 1, 2, \dots$. Consequently, the sequence $\{x_k(t, \phi): k \geq q\}$ also is equicontinuous on $[-q, \infty)$. Therefore, applying the Ascoli-Arzelà theorem we can assume that $\{x_k(t, \phi)\}$ uniformly converges to a function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ on any finite interval in R . Obviously, $x^*(t)$ is defined on the whole R , is continuous, and satisfies $m^* \leq x_i^*(t) \leq p_i (i = 1, 2, \dots, n)$ for all $t \in R$.

Finally, by a similar argument as in [Theorem 5, 31] we can obtain that $x^*(t)$ is a strictly positive solution of system (1). This completes the proof.

Proof of Theorem 4. We first introduce the following claim.

Claim 2. If $\delta(t) \in S_1$, $f(t)$ is uniformly continuous on $[0, \infty)$, and $\int_0^\infty \delta(t) |f(t)| dt < \infty$, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly, if $\delta(t) \in S_2$, $f(t)$ is uniformly continuous on $(-\infty, 0]$, and $\int_{-\infty}^0 \delta(t) |f(t)| dt < \infty$, then $f(t) \rightarrow 0$ as $t \rightarrow -\infty$.

In fact, if there is a time sequence $\{t_k\}$, $t_k > 0$ and $\lim_{k \rightarrow \infty} t_k = \infty$, such that $|f(t_k)| \geq r_0 > 0$ for all $k = 1, 2, \dots$, where r_0 is some constant, then by the uniform continuity of $f(t)$, there is a constant $\varepsilon > 0$ such that $|f(t)| \geq \frac{1}{2} r_0$ for all $t \in [t_k - \varepsilon, t_k + \varepsilon]$ and $k = 1, 2, \dots$. Without loss of generality, we can assume $[t_k - \varepsilon, t_k + \varepsilon] \cap [t_j - \varepsilon, t_j + \varepsilon] = \emptyset$ for $k \neq j$. Hence, by $\delta(t) \in S_1$ we have

$$\int_0^\infty \delta(t) |f(t)| dt \geq \sum_{k=1}^\infty \int_{t_k-\varepsilon}^{t_k+\varepsilon} \frac{1}{2} r_0 \delta(t) dt = \infty.$$

This leads to a contradiction to the conditions of the claim.

Let $x(t, \phi_k) = (x_1(t, \phi_k), x_2(t, \phi_k), \dots, x_n(t, \phi_k)) (k = 1, 2)$ be any two positive solutions of system (1). We use the function as follows

$$\begin{aligned}
V(t) = & \sum_{i=1}^n d_i \left[|\ln x_i(t, \phi_1) - \ln x_i(t, \phi_2)| \right. \\
& + \sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \frac{|c_{ij}(\psi_{ij}^{-1}(s))|}{1 - \dot{\tau}_{ij}(\psi_{ij}^{-1}(s))} |x_j(s, \phi_1) - x_j(s, \phi_2)| ds \\
& \left. + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \left(\int_{t+\theta}^t |k_{ij}(s-\theta, \theta)| |x_j(s, \phi_1) - x_j(s, \phi_2)| ds \right) d\theta \right],
\end{aligned}$$

similar forms of which have been used by Ahmad and Mohana Rao in [1] and Bereketoglu and Gyori in [2]. Assumptions (A_1) , (A_2) , and (A_5) imply that $V(t)$ is defined for all $t \geq t_0$. Calculating the upper right derivative of $V(t)$, we can obtain

$$\begin{aligned}
D^+V(t) \leq & \sum_{i=1}^n d_i \left[-b_i(t) |x_i(t, \phi_1) - x_i(t, \phi_2)| \right. \\
& + \sum_{j=1}^n \frac{|c_{ij}(\psi_{ij}^{-1}(s))|}{1 - \dot{\tau}_{ij}(\psi_{ij}^{-1}(s))} |x_j(t, \phi_1) - x_j(t, \phi_2)| \\
& \left. + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 |k_{ij}(t-\theta, \theta)| |x_j(t, \phi_1) - x_j(t, \phi_2)| d\theta \right] \\
\leq & -\delta(t) \sum_{i=1}^n |x_i(t, \phi_1) - x_i(t, \phi_2)| \tag{15}
\end{aligned}$$

for all $t \geq t_0$. Integrating (15) from t_0 to t , we obtain

$$\int_{t_0}^t \delta(s) \sum_{i=1}^n |x_i(s, \phi_1) - x_i(s, \phi_2)| ds \leq V(t_0) - V(t),$$

and consequently

$$\int_{t_0}^{\infty} \delta(t) |x_i(t, \phi_1) - x_i(t, \phi_2)| dt < \infty \quad \text{for } i = 1, 2, \dots, n. \tag{16}$$

By assumptions (A_1) , (A_2) , and (A_5) and the boundedness of $x(t, \phi_k)$ ($k = 1, 2$) on $[t_0, \infty)$, we obtain directly from system (1) that $|x_i(t, \phi_1) - x_i(t, \phi_2)|$ ($i = 1, 2, \dots, n$) are uniformly continuous on $[t_0, \infty)$. Hence, by Claim 2 and (16) it follows that

$$\lim_{t \rightarrow \infty} |x_i(t, \phi_1) - x_i(t, \phi_2)| = 0, \quad i = 1, 2, \dots, n.$$

This completes the proof of Theorem 4.

Proof of Theorem 5. The existence and global asymptotic stability of the strictly positive solutions of system (1) are given by Theorems 3 and 4. Here we only need to prove the uniqueness of the strictly positive solution. Let $x^{(k)}(t) = (x_1^{(k)}(t), x_2^{(k)}(t), \dots, x_n^{(k)}(t))$ ($k = 1, 2$) be two strictly positive solutions of system (1). Then there are positive constants m and M such that

$$m \leq x_i^{(1)}(t), x_i^{(2)}(t) \leq M \quad \text{for all } t \in R, i = 1, 2, \dots, n. \quad (17)$$

Define the function as in the proof of Theorem 4

$$\begin{aligned} V(t) = & \sum_{i=1}^n d_i \left[|\ln x_i^{(1)}(t) - \ln x_i^{(2)}(t)| \right. \\ & + \sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \frac{|c_{ij}(\psi_{ij}^{-1}(s))|}{1 - \dot{\tau}_{ij}(\psi_{ij}^{-1}(s))} |x_j^{(1)}(s) - x_j^{(2)}(s)| ds \\ & \left. + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \left(\int_{t+\theta}^t |k_{ij}(s-\theta, \theta)| |x_j^{(1)}(s) - x_j^{(2)}(s)| ds \right) d\theta \right]. \quad (18) \end{aligned}$$

Obviously, by (17), $V(t)$ is bounded on the whole R and $V(t) \geq 0$ for all $t \in R$. Calculating the upper right derivative of $V(t)$, then similar to (15), we can obtain

$$D^+V(t) \leq -\delta(t) \sum_{i=1}^n |x_i^{(1)}(t) - x_i^{(2)}(t)| \quad \text{for all } t \in R. \quad (19)$$

This implies that $V(t)$ is nonincreasing for all $t \in R$. For any $t, T \in R$ and $t \geq T$, integrating (19) from T to t we have

$$\sum_{i=1}^n \int_T^t \delta(s) |x_i^{(1)}(s) - x_i^{(2)}(s)| ds \leq V(T) - V(t).$$

Hence,

$$\int_{-\infty}^0 \delta(t) |x_i^{(1)}(t) - x_i^{(2)}(t)| dt < \infty, \quad \int_0^{\infty} \delta(t) |x_i^{(1)}(t) - x_i^{(2)}(t)| dt < \infty.$$

Consequently, by $\delta(t) \in S_1 \cap S_2$ and Claim 2 we have

$$x_i^{(1)}(t) - x_i^{(2)}(t) \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty, i = 1, 2, \dots, n.$$

Let the constant

$$Q = \sum_{i=1}^n d_i \left[\frac{1}{m} + \tau \sum_{j=1}^n \max_{t \in R} \frac{|c_{ij}(t)|}{1 - \tau_{ij}(t)} + \frac{1}{2} \tau^2 \sum_{j=1}^n \max_{(t,s) \in R \times [-\sigma_{ij}, 0]} |k_{ij}(t, s)| \right].$$

For any constant $\varepsilon > 0$ there is a sufficiently large $N(\varepsilon)$ such that

$$|x_i^{(1)}(t) - x_i^{(2)}(t)| < \frac{\varepsilon}{Q} \quad \text{for all } t \leq -N(\varepsilon), i = 1, 2, \dots, n.$$

Hence, when $t \leq -N(\varepsilon)$ we have directly from (18)

$$\begin{aligned} V(t) &\leq \sum_{i=1}^n d_i \left[\frac{1}{m} |x_i^{(1)}(t) - x_i^{(2)}(t)| \right. \\ &\quad + \sum_{j=1}^n \tau_{ij}(t) \max_{s \leq t} \left\{ \frac{|c_{ij}(\psi_{ij}^{-1}(s))|}{1 - \tau_{ij}(\psi_{ij}^{-1}(s))} |x_j^{(1)}(s) - x_j^{(2)}(s)| \right\} \\ &\quad + \sum_{j=1}^n \frac{1}{2} \sigma_{ij}^2 \max_{(t,s) \in R \times [-\sigma_{ij}, 0]} |k_{ij}(t, s)| \max_{s \leq t} \{|x_j^{(1)}(s) - x_j^{(2)}(s)|\} \\ &\leq Q \frac{\varepsilon}{Q} = \varepsilon. \end{aligned}$$

This shows $V(t) \rightarrow 0$ as $t \rightarrow -\infty$. So, $V(t) = 0$ for all $t \in R$ by the nonincrease of $V(t)$ on R , and hence $x^{(1)}(t) = x^{(2)}(t)$ for all $t \in R$. This completes the proof.

Proof of Theorem 8. For an almost periodic function $f(t)$ we denote by $H(f)$ the hull of $f(t)$.

When $\tau_{ij}(t) \equiv \tau_{ij}$ is constant for each $i, j = 1, 2, \dots, n$, then assumption (A_6) will become

(A_6^*) There exist positive constants $d_i (i = 1, 2, \dots, n)$ and δ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$d_i b_i(t) - \sum_{j=1}^n d_j \left(|c_{ji}(t + \tau_{ij})| + \int_{-\sigma_{ji}}^0 |k_{ji}(t - s, s)| ds \right) \geq \delta.$$

For any $a_i^* \in H(a_i)$, $b_i^* \in H(b_i)$, $c_{ij}^* \in H(c_{ij})$, and $k_{ij}^* \in H(k_{ij})$ we consider the following hull system of system (1)

$$\begin{aligned} \frac{dx_i(t)}{dt} = x_i(t) &\left[a_i^*(t) - b_i^*(t) x_i(t) - \sum_{j=1}^n c_{ij}^*(t) x_j(t - \tau_{ij}) \right. \\ &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^*(t, s) x_j(t + s) ds \right], \quad i = 1, 2, \dots, n. \end{aligned} \quad (20)$$

By the theory of almost periodic functions we easily prove that, if assumptions (A_1) – (A_4) and (A_6^*) hold on the whole R for the functions $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, and $k_{ij}(t, s)$ ($i, j = 1, 2, \dots, n$), then assumptions (A_1) – (A_4) and (A_6^*) also hold on the whole R for the functions $a_i^*(t)$, $b_i^*(t)$, $c_{ij}^*(t)$, and $k_{ij}^*(t, s)$ ($i, j = 1, 2, \dots, n$). In addition to these, the parameters p_i , d_i ($i = 1, 2, \dots, n$), α , ω , ζ , and δ can be chosen to be common. Therefore, from Theorem 5 we obtain that hull system (20) has a unique strictly positive solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$. We easily prove

$$m^* \leq x_i^*(t) \leq p_i \quad \text{for all } t \in R, i = 1, 2, \dots, n,$$

where the constant m^* is given in the proof of Theorem 3.

By Theorem 3.2 in [Chap. 3, 13] on the existence of almost periodic solutions, we can obtain that system (1) has a unique strictly positive almost periodic solution. The global attractivity is obtained from Theorem 4. This completes the proof.

Proof of Theorem 9. Here, we will give the proof of Theorem 9 by improving the method given by Redheffer in [24]. Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ be any positive solution of system (1). By Theorem 2, there exist positive constants m , M and $T_0 > t_0$ such that

$$m \leq x_i(t) \leq M \quad \text{for all } t \geq T_0, i = 1, 2, \dots, n. \quad (21)$$

Let $y_i(t) = x_i(t)/u_i(t) - 1$ ($i = 1, 2, \dots, n$); then for each $i = 1, 2, \dots, n$ we have

$$\begin{aligned} \frac{dy_i(t)}{dt} &= \frac{1}{u_i(t)} \frac{dx_i(t)}{dt} - \frac{x_i(t)}{u_i(t)^2} \frac{du_i(t)}{dt} \\ &= -\frac{x_i(t)}{u_i(t)} \left[b_i^*(t) y_i(t) + \sum_{j=1}^n c_{ij}^*(t) y_j(t - \tau_{ij}(t)) \right. \\ &\quad \left. + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^*(t, s) y_j(t + s) ds \right], \end{aligned} \quad (22)$$

where $b_i^*(t) = b_i(t) u_i(t)$, $c_{ij}^*(t) = c_{ij}(t) u_j(t - \tau_{ij}(t))$ and $k_{ij}^*(t, s) = k_{ij}(t, s) u_j(t + s)$ for each $i, j = 1, 2, \dots, n$.

By assumption (A_3) we obtain $b_i(t) > 0$ for all $t \geq t_0$ and $i = 1, 2, \dots, n$. From assumption (A_7) we have

$$h_i - \sum_{j=1}^n \frac{|c_{ij}(t)|}{b_i(t)} h_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \frac{|k_{ij}(t, s)|}{b_i(t)} h_j ds \geq \gamma$$

for all $t \geq t_0$ and $i = 1, 2, \dots, n$. This implies that the functions $|c_{ij}(t)|/b_i(t)$ and $\int_{-\sigma_{ij}}^0 |k_{ij}(t, s)| ds/b_i(t)$ are bounded on $[t_0, \infty)$ for $i, j = 1, 2, \dots, n$. Obviously, we have

$$h_i - \sum_{j=1}^n \frac{|c_{ij}^*(t)|}{b_i^*(t)} \frac{u_i(t)}{u_j(t - \tau_{ij}(t))} h_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \frac{|k_{ij}^*(t, s)|}{b_i^*(t)} \frac{u_i(t)}{u_j(t+s)} h_j ds \geq \gamma$$

for all $t \geq t_0$. From $\lim_{t \rightarrow \infty} u_i(t) = q_i > 0$ for $i = 1, 2, \dots, n$, we can obtain that there are sufficiently small constant $\mu > 0$ and sufficiently large $T_1 \geq T_0$ such that

$$h_i - \sum_{j=1}^n \frac{|c_{ij}^*(t)|}{b_i^*(t)} (1+\mu) \frac{q_i}{q_j} h_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 \frac{|k_{ij}^*(t, s)|}{b_i^*(t)} (1+\mu) \frac{q_i}{q_j} h_j ds \geq \frac{1}{2} \gamma$$

and

$$u_i(t) \leq 1 + q_i \quad (23)$$

for all $t \geq T_1$ and $i = 1, 2, \dots, n$. Consequently,

$$b_i^*(t) r_i - \sum_{j=1}^n |c_{ij}^*(t)| (1+\mu) r_j - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 |k_{ij}^*(t, s)| (1+\mu) r_j ds \geq \frac{1}{2} \gamma_i b_i^*(t) \quad (24)$$

for all $t \geq T_1$ and $i = 1, 2, \dots, n$, where $r_i = h_i/q_i$ and $\gamma_i = \gamma/q_i$.

We now use the Liapunov function

$$V(t) = \max_{1 \leq i \leq n} \left\{ \frac{|y_i(t)|}{r_i} \right\}.$$

For any $t \geq T_1$ there is an $i = i(t)$ such that $V(t) = r_i^{-1} |y_i(t)|$. Calculating the upper right derivative of $V(t)$ at time t , by (22) we have

$$\begin{aligned} D^+V(t) &= \text{sign}(y_i(t)) r_i^{-1} \frac{dy_i(t)}{dt} \\ &\leq r_i^{-1} \frac{x_i(t)}{u_i(t)} \left[-b_i^*(t) |y_i(t)| + \sum_{j=1}^n |c_{ij}^*(t)| |y_j(t - \tau_{ij}(t))| \right. \\ &\quad \left. + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 |k_{ij}^*(t, s)| |y_j(t+s)| ds \right]. \end{aligned} \quad (25)$$

When $V(t+s) \leq (1+\mu)V(t)$ for all $s \in [-\tau, 0]$, we have $r_j^{-1}|y_j(t+s)| \leq (1+\mu)V(t)$ for all $s \in [-\tau, 0]$ and $j = 1, 2, \dots, n$. Therefore, for any $t \geq T_1$ from (21) and (23)–(25) we have

$$\begin{aligned} D^+V(t) &\leq r_i^{-1} \frac{x_i(t)}{u_i(t)} \left[-r_i b_i^*(t) + \sum_{j=1}^n (1+\mu) r_j |c_{ij}^*(t)| \right. \\ &\quad \left. - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 (1+\mu) r_j |k_{ij}^*(t, s)| ds \right] V(t) \\ &\leq -\theta V(t), \end{aligned}$$

where $\theta = \frac{1}{2} \min_{1 \leq i \leq n} \{r_i^{-1}(1+q_i)^{-1}\} \gamma_i m$. Therefore, employing the standard argument of the Liapunov–Razumikhin type theorems of the asymptotic stability of functional differential equations with finite delay (see [3, 12, 17]), we can obtain $y_i(t) \rightarrow 0$ as $t \rightarrow \infty$ for $i = 1, 2, \dots, n$. This shows that $x_i(t) \rightarrow q_i$ as $t \rightarrow \infty$ for $i = 1, 2, \dots, n$. This completes the proof.

Proof of Theorem 10. From conditions (A_3^*) and (A_4^*) we easily verify that assumptions (A_3) and (A_4) hold, and

$$b_i p_i - \sum_{j=1}^n \left(|c_{ij}| + \int_{-\sigma_{ij}}^0 |k_{ij}(s)| ds \right) h_j \geq \bar{p}_i b_i, \quad i = 1, 2, \dots, n. \quad (26)$$

This shows that assumption (A_7) holds. Using Lemma 10 given by Redheffer in [23] and by (26), we can obtain that there are positive constants $d_i (i = 1, 2, \dots, n)$ and δ such that

$$d_i b_i - \sum_{j=1}^n \left(|c_{ji}| + \int_{-\sigma_{ji}}^0 |k_{ji}(s)| ds \right) d_j \geq \delta, \quad i = 1, 2, \dots, n.$$

This shows that assumption (A_6) holds. Therefore, the conclusions of Theorems 1–9 remain true. This completes the proof.

Last, we propose several remarks to conclude this paper.

Remark 7. From the proof of Theorem 1 we see that in Theorem 1 the boundedness of the functions $a_i(t)$, $b_i(t)$, and $c_{ij}(t) (i, j = 1, 2, \dots, n)$ can be removed. But, in the proofs of Theorems 2–9 we require the boundedness of the functions $a_i(t)$, $b_i(t)$, and $c_{ij}(t)$. However, in the main result Theorem 1 given in [23] the boundedness of the coefficients $a_i(t)$ and $b_{ij}(t) (i, j = 1, 2, \dots, n)$ is not assumed.

Remark 8. If we replace assumptions (A_3) and (A_4) by the following assumptions:

(A₃') There exist continuously differentiable functions $p_i(t)$ ($i = 1, 2, \dots, n$) bounded above and below by positive constants on $[t_0, \infty)$ and a constant $\alpha > 0$ such that the functions

$$\begin{aligned}\beta_i(t) = & b_i(t) p_i(t) + \sum_{j=1}^n c_{ij}^-(t)(1+\alpha) p_j(t - \tau_{ij}(t)) \\ & + \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}^-(t, s)(1+\alpha) p_j(t+s) ds + \frac{|\dot{p}_i(t)|}{p_i(t)}\end{aligned}$$

satisfy $\beta_i(t) \geq a_i(t)$ and $\beta(t) = \min_{1 \leq i \leq n} \beta_i(t) > 0$ for all $t \geq t_0$, $\int_{t_0}^{\infty} \beta(t) dt = \infty$ and $\limsup_{t \rightarrow \infty} a_i(t)/\beta(t) \leq 1$ for $i = 1, 2, \dots, n$.

(A₄') There exist positive constants ω and δ such that for all $t \geq t_0$ and $i = 1, 2, \dots, n$

$$\int_t^{t+\omega} \left[a_i(u) - \sum_{j \neq i}^n c_{ij}^+(u) p_j(u - \tau_{ij}(u)) - \sum_{j \neq i}^n \int_{-\sigma_{ij}}^0 k_{ij}^+(u, s) p_j(u+s) ds \right] du \geq \delta;$$

then the results similar to Theorems 1–9 will be obtained. For example, as an improvement of Theorem 1 we have that, if assumptions (A₁), (A₂), and (A₃') hold, then for any positive solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of system (1)

$$\limsup_{t \rightarrow \infty} \frac{x_i(t)}{p_i(t)} \leq 1, \quad i = 1, 2, \dots, n.$$

Remark 9. Theorem 9 shows that if system (1) has a positive solution with a limit as $t \rightarrow \infty$, then the row diagonal dominance of the coefficients guarantees the global attractivity of system (1). However, if this solution does not exist, then the Redheffer's counterexample in [24] shows that, even when system (1) is competitive, the row diagonal dominance cannot ensure the global attractivity of system (1). Therefore, to obtain the global attractivity of system (1), we need the column diagonal dominance; that is, introduction of assumption (A₆) is really necessary.

Remark 10. In Theorem 6, under assumptions (A₁)–(A₄) we can obtain the existence of a positive periodic solution for system (1). In Theorem 8, to obtain the existence of strictly positive almost periodic solutions for system (1) we require, in addition to assumptions (A₁)–(A₄), that assumption (A₆) also holds. Therefore, an important open problem is to prove the existence of strictly positive almost periodic solutions when system (1) only satisfies assumptions (A₁)–(A₄).

Remark 11. In this paper the delay $\tau = \max\{\tau_{ij}(t), \sigma_{ij}: t \in R, i, j = 1, 2, \dots, n\} < \infty$ is assumed. Therefore, an important open problem is, when $\tau = \infty$, particularly $\sigma_{ij} = \infty$ for some $1 \leq i, j \leq n$, whether the conclusions of this paper still can hold. Obviously, from the proof of Theorem 4 we easily see that, when $\tau_{ij}(t) (i, j = 1, 2, \dots, n)$ are bounded on R and $\sigma_{ij} = \infty$ for some $1 \leq i, j \leq n$, the conclusions of Theorem 4 still are true. In addition, the infinite delay Lotka-Volterra systems have been studied in [1, 2, 9, 14, 17, 22].

Remark 12. From assumptions (A_3) and (A_6) it follows that $b_i(t) > 0 (i = 1, 2, \dots, n)$ for all $t \in R$. If $b_i(t) \equiv 0 (i = 1, 2, \dots, n)$, then system (1) will become

$$\frac{dx_i(t)}{dt} = x_i(t) \left[a_i(t) - \sum_{j=1}^n c_{ij}(t) x_j(t - \tau_{ij}(t)) - \sum_{j=1}^n \int_{-\sigma_{ij}}^0 k_{ij}(t, s) x_j(t+s) ds \right], \quad i = 1, 2, \dots, n. \quad (27)$$

For system (27) whether the similar results can be established is apparently as yet an important open problem. The similar open problem is proposed by Kuang in [19]. For some special cases of system (27), for example, the constant coefficient case, competition, and predator-prey cases, etc., the global attractivity of positive equilibrium and permanence of solutions have been studied in [10, 14, 17, 28].

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