J. Differential Equations 254 (2013) 141–178

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Unique continuation property and control for the Benjamin–Bona–Mahony equation on a periodic domain

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ARTICLE INFO

Article history:
Received 20 February 2012
Revised 23 July 2012
Available online 30 August 2012

MSC:
35Q53
93B05
93D15

Keywords:
Unique continuation property
Benjamin–Bona–Mahony equation
Korteweg–de Vries equation
Moving point control
Exact controllability
Stabilization

ABSTRACT

We consider the Benjamin–Bona–Mahony (BBM) equation on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We prove a Unique Continuation Property (UCP) for small data in $H^1(\mathbb{T})$ with nonnegative zero means. Next we extend the UCP to certain BBM-like equations, including the equal width wave equation and the KdV–BBM equation. Applications to the stabilization of the above equations are given. In particular, we show that when an internal control acting on a moving interval is applied in the BBM equation, then a semiglobal exponential stabilization can be derived in $H^s(\mathbb{T})$ for any $s \geqslant 1$. Furthermore, we prove that the BBM equation with a moving control is also locally exactly controllable in $H^s(\mathbb{T})$ for any $s \geqslant 0$ and globally exactly controllable in $H^s(\mathbb{T})$ for any $s \geqslant 1$ in a sufficiently large time depending on the H^s-norms of the initial and terminal states.

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1. Introduction

We are concerned here with the Benjamin–Bona–Mahony (BBM) equation

$$u_t - u_{txx} + u_x + uu_x = 0 \tag{1.1}$$

that was proposed in [2] as an alternative to the Korteweg–de Vries (KdV) equation

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http://dx.doi.org/10.1016/j.jde.2012.08.014

$$u_t + u_{xxx} + u_x + uu_x = 0 \tag{1.2}$$

as a model for the propagation of one-dimensional, unidirectional small amplitude long waves in nonlinear dispersive media. In the context of shallow-water waves, $u = u(x,t)$ represents the displacement of the water surface at location x and time t. In this paper, we shall assume that $x \in \mathbb{R}$ or $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus). Eq. (1.1) is often obtained from (1.2) in the derivation of the surface equation by noticing that, in the considered regime, $u_x \sim -u_t$, so that $u_{xxx} \sim -u_{txx}$. The dispersive term $-u_{txx}$ has a strong smoothing effect, thanks to which the well-posedness theory of (1.1) is dramatically easier than that of (1.2) (see [2,3,42] and the references therein). Numerics often involve the BBM equation, or the KdV–BBM equation (see below), because of the regularization provided by the term $-u_{txx}$. On the other hand, (1.1) is not integrable and it has only three invariants of motion [13,33].

In this paper, we investigate the Unique Continuation Property (UCP) of BBM and its applications to a control problem for (1.1). We say that the UCP holds in some class X of functions if, given any nonempty open set $\omega \subset \mathbb{T}$, the only solution $u \in X$ of (1.1) fulfilling

$$u(x,t) = 0 \quad \text{for } (x,t) \in \omega \times (0,T),$$

is the trivial one $u \equiv 0$. Such a property is very important in Control Theory, as it is equivalent to the approximate controllability for linear PDE, and it is involved in the classical uniqueness/compactness approach in the proof of the stability for a PDE with a localized damping. The UCP is usually proved with the aid of some Carleman estimate (see e.g. [45]). The UCP for KdV was established in [47] by the inverse scattering approach, in [11,39,45] by means of Carleman estimates, and in [4] by a perturbative approach and Fourier analysis. For BBM, the study of the UCP is only at its early age. The main reason is that both $x = \text{const}$ and $t = \text{const}$ are characteristic lines for (1.1). Thus, the Cauchy problem in the UCP (assuming e.g. that $u = 0$ for $x \leqslant 0$, and solving BBM for $x \geqslant 0$) is characteristic, which prevents from applying Holmgren's theorem, even for the linearized equation. The Carleman approach for the UCP of BBM was developed in [8] and in [46]. Unfortunately, Theorems 3.1–3.4 in [8] are not correct without further assumptions, as noticed in [49]. On the other hand, the UCP in [46] for the BBM-like equation

$$u_x - u_{txx} = p(x,t)u_x + q(x,t)u, \quad x \in (0,1),\ t \in (0,T),$$

where $p \in L^\infty(0,T;L^\infty(0,1))$ and $q \in L^\infty(0,T;L^2(0,1))$, requires $u(1,t) = u_x(1,t) = 0$ for $t \in (0,T)$ and

$$u(x,0) = 0 \quad \text{for } x \in (0,1). \tag{1.3}$$

(Note, however, that nothing is required for $u(0,t)$.) Because of (1.3), such a UCP cannot be used for the stabilization problem. More can be said for a linearized BBM equation with potential functions depending only on x. It was proved in [30] that the only solution $u \in C([0,T], H^1(0,1))$ of the linearized BBM equation

$$u_t - u_{txx} + u_x = 0, \quad x \in (0,1),\ t \in (0,T), \tag{1.4}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T) \tag{1.5}$$

fulfilling $u_x(1,t) = 0$ for all $t \in (0,T)$ is the trivial one $u \equiv 0$. It is worth noticing that the proof of that result strongly used the fact that the solutions of (1.4)–(1.5) are *analytic in time*. On the other hand, several difficult UCP results based on spectral analysis are given in [48,49] for the system

$$u_t - u_{txx} = \big[\alpha(x)u\big]_x + \beta(x)u, \quad x \in (0,1),\ t \in (0,T), \tag{1.6}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{1.7}$$

As noticed in [49], the UCP fails for (1.6)–(1.7) whenever both α and β vanish on some open set $\omega \subset \mathbb{T}$, so that the UCP depends not only on the regularity of the functions α and β, but also on their zero sets. Bourgain's approach [4] for the UCP of KdV or of the nonlinear Schrödinger equation (NLS) is based on the fact that the Fourier transform of a compactly supported function extends to an entire function of exponential type. The proof of the UCP in [4] rests on estimates at high frequencies using the intuitive property that the nonlinear term in Duhamel formula is perturbative. As noticed in [29], that argument does not seem to be applicable to BBM. Actually, if we follow Bourgain's idea for the linearized BBM equation

$$u_t - u_{txx} + u_x = 0 \tag{1.8}$$

on $\mathbb{R}$, and assume that some solution u vanishes for $|x| > L$ and $t \in (0,T)$, then its Fourier transform in x, denoted by $\hat{u}(\xi,t)$, is readily found to be

$$\hat{u}(\xi,t) = \exp\left(\frac{-it\xi}{\xi^2+1}\right)\hat{u}(\xi,0), \quad \xi \in \mathbb{R},\ t \in (0,T).$$

The consideration of high frequencies is useless here. By analytic continuation, the above equation still holds for all $\xi = \xi_1 + i\xi_2 \in \mathbb{C} \setminus \{\pm i\}$. Picking any $t > 0$, $\xi_1 = 0$ and letting $\xi_2 \to 1^-$, we readily infer that $\partial_\xi^n \hat{u}(i,0) = 0$ for all $n \geqslant 0$, so that $\hat{u}(\cdot,0) \equiv 0$ and hence $u \equiv 0$. Note that

$$\partial_\xi^n \hat{u}(i,t) = \int_{-\infty}^{\infty} u(x,t)(-ix)^n e^x\,dx, \tag{1.9}$$

and that it can be shown by induction on n that all the moments $M_n(t) = \int_{-\infty}^{\infty} u(x,t)x^n e^x\,dx$ vanish on $(0,T)$, so that $u \equiv 0$. Unfortunately, we cannot modify the above argument to deal with the UCP for the full BBM equation, as the nonlinear term has no reason to be perturbative at the "small" frequencies $\xi = \pm i$. We point out that a moment approach, inspired by the paper [7] that was concerned with the UCP for the Camassa–Holm equation (see also [14]), was nevertheless applied in [29] to prove the UCP for the Kadomtsev–Petviashvili (KP)–BBM-II equation.

In this paper, we shall apply the moment approach to prove the UCP for a generalized BBM equation

$$u_t - u_{txx} + \left[f(u)\right]_x = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and *nonnegative*. The choice $f(u) = u^2/2$ gives the so-called Morrison–Meiss–Carey (MMC) equation (also called *equal width wave equation*, see [13,32]). Incorporating a localized damping in the above equation, we obtain the equation

$$u_t - u_{txx} + \left[f(u)\right]_x + a(x)u = 0, \quad x \in \mathbb{T},$$

whose solutions are proved to tend weakly to 0 in $H^1(\mathbb{T})$ as $t \to \infty$. Note that similar results were proved in [19] with a boundary dissipation.

Bourgain's approach, in its complex analytic original form, can be used to derive the UCP for the following BBM-like equation

$$u_t - u_{txx} + u_x + (u * u)_x = 0$$

in which the (nonlocal) term $(u * u)_x$ is substituted to the classical nonlinear term uu_x in BBM.

For the original BBM equation (1.1), we shall derive a UCP for solutions issuing from initial data that are small enough in $H^1(\mathbb{T})$ and with nonnegative mean values. The proof, which is very reminiscent of La Salle invariance principle, will combine the analyticity in time of solutions of BBM, the existence of three invariants of motion, and the use of some appropriate Lyapunov function.

The second part of this work is concerned with the controllability of the BBM equation. Consider first the linearized BBM equation with a control force

$$u_t - u_{txx} + u_x = a(x)h(x,t), \tag{1.10}$$

where a is supported in some subset of $\mathbb{T}$ and h stands for the control input. It was proved in [30,49] that (1.10) is *approximatively controllable* in $H^1(\mathbb{T})$. It turns out that (1.10) is *not exactly controllable* in $H^1(\mathbb{T})$ [30]. This is in sharp contrast with the good control properties of other dispersive equations (on periodic domains, see e.g. [22,44] for KdV, [9,20,21,40,41] for the nonlinear Schrödinger equation, [24,25] for the Benjamin–Ono equation, [31] for Boussinesq system, and [12] for Camassa–Holm equation). The bad control properties of (1.10) come from the existence of a limit point in the spectrum. Such a phenomenon was noticed in [43] for the beam equation with internal damping, in [23] for the plate equation with internal damping, in [30] for the linearized BBM equation, and more recently in [38] for the wave equation with structural damping.

It is by now classical that an "intermediate" equation between (1.1) and (1.2) can be derived from (1.1) by working in a moving frame $x = -ct$ with $c \in \mathbb{R} \setminus \{0\}$. Indeed, letting

$$v(x,t) = u(x - ct, t) \tag{1.11}$$

we readily see that (1.1) is transformed into the following KdV–BBM equation

$$v_t + (c+1)v_x - cv_{xxx} - v_{txx} + vv_x = 0. \tag{1.12}$$

It is then reasonable to expect the control properties of (1.12) to be better than those of (1.1), thanks to the KdV term $-cv_{xxx}$ in (1.12). We shall prove that Eq. (1.12) with a forcing term $a(x)k(x,t)$ supported in (any given) subdomain is locally exactly controllable in $H^s(\mathbb{T})$ for any $s \geqslant 1$ in time $T > (2\pi)/|c|$. Going back to the original variables, it means that the equation

$$u_t + u_x - u_{txx} + uu_x = a(x+ct)h(x,t) \tag{1.13}$$

with a moving distributed control is exactly controllable in $H^s(\mathbb{T})$ for any $s \geqslant 1$ in (sufficiently) large time. Actually, the control time is chosen in such a way that the support of the control, which is moving at the constant velocity c, can visit all the domain $\mathbb{T}$. Using the same idea, it has been proved recently in [28] that the wave equation with structural damping is null controllable in large time when controlled with a moving distributed control.

The concept of moving point control was introduced by J.L. Lions in [26] for the wave equation. One important motivation for this kind of control is that the exact controllability of the wave equation with a pointwise control and Dirichlet boundary conditions fails if the point is a zero of some eigenfunction of the Dirichlet Laplacian, while it holds when the point is moving under some conditions easy to check (see e.g. [5]). The controllability of the wave equation (resp. of the heat equation) with a moving point control was investigated in [5,17,26] (resp. in [6,18]).

Thus, the appearance of the KdV term $-cv_{xxx}$ in (1.12) results in much better control properties. We shall see that

(i) there is no limit point in the spectrum of the linearized KdV–BBM equation, which is of "hyperbolic" type;
(ii) a UCP for the full KdV–BBM equation can be derived from Carleman estimates for a system of coupled elliptic–hyperbolic equations.

It follows that one can expect a semiglobal exponential stability when applying a localized damping with a moving support. We will see that this is indeed the case. Combining the local exact controllability to the semiglobal exponential stability result, we obtain the following theorem which is the main result of the paper.

Theorem 1.1. *Assume that $a \in C^\infty(\mathbb{T})$ with $a \neq 0$ is given and that $c \in \mathbb{R} \setminus \{0\}$. Let $s \geqslant 1$ and $R > 0$ be given. Then there exists a time $T = T(s, R) > 2\pi/|c|$ such that for any $u_0, u_T \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} \leqslant R, \qquad \|u_T\|_{H^s} \leqslant R, \tag{1.14}$$

there exists a control $h \in L^2(0, T; H^{s-2}(\mathbb{T}))$ such that the solution $u \in C([0, T]; H^s(\mathbb{T}))$ of

$$u_t - u_{txx} + u_x + uu_x = a(x + ct)h(x, t), \quad x \in \mathbb{T},\ t \in (0, T),$$
$$u(x, 0) = u_0(x), \quad x \in \mathbb{T}$$

satisfies

$$u(x, T) = u_T(x), \quad x \in \mathbb{T}.$$

The paper is scheduled as follows. In Section 2 we recall some useful facts (global well-posedness, invariants of motion, time analyticity) about BBM. In Section 3 we establish the UCP for BBM. In Section 4 we prove the UCP for other BBM-like equations, including the MMC equation and the BBM equation with a nonlocal term. Section 5 is concerned with the UCP for the KdV–BBM equation. The KdV–BBM equation is first split into a coupled system of an elliptic equation and a transport equation. Next, we prove some Carleman estimates with the same singular weights for both the elliptic and the hyperbolic equations, and we derive the UCP for KdV–BBM by combining these Carleman estimates with a regularization process. Those results are used in Section 6 to prove the exact controllability of KdV–BBM and the semiglobal exponential stability of the same equation with a localized damping term.

2. Well-posedness, analyticity in time and invariants of motion

Throughout the paper, for any $s \geqslant 0$, $H^s(\mathbb{T})$ denotes the Sobolev space

$$H^s(\mathbb{T}) = \left\{u : \mathbb{T} \to \mathbb{R};\ \|u\|_{H^s} := \left\|\left(1 - \partial_x^2\right)^{\frac{s}{2}} u\right\|_{L^2(\mathbb{T})} < \infty\right\}.$$

Its dual is denoted by $H^{-s}(\mathbb{T})$.

Let us consider the initial value problem (IVP)

$$u_t - u_{txx} + u_x + uu_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{2.1}$$
$$u(x, 0) = u_0(x). \tag{2.2}$$

Let $A = -(1 - \partial_x^2)^{-1}\partial_x \in \mathcal{L}(H^s(\mathbb{T}), H^{s+1}(\mathbb{T}))$ (for any $s \in \mathbb{R}$) and $W(t) = e^{tA}$ for $t \in \mathbb{R}$. We put (2.1)–(2.2) in its integral form

$$u(t) = W(t)u_0 + \int_0^t W(t - s)A\left(u^2/2\right)(s)\, ds. \tag{2.3}$$

For $s \geqslant 0$ and $T > 0$, let

$$X_T^s = C\big([-T, T]; H^s(\mathbb{T})\big).$$

Note that for $u \in X_T^s$, u solves (2.1) in $\mathcal{D}'(-T, T; H^{s-2}(\mathbb{T}))$ and (2.2) if, and only if, it fulfills (2.3) for all $t \in [-T, T]$. The following result will be used thereafter.

Theorem 2.1. *(See [3,42].) Let $s \geqslant 0$, $u_0 \in H^s(\mathbb{T})$ and $T > 0$. Then there exists a unique solution $u \in X_T^s$ of (2.1)–(2.2) (or, equivalently, (2.3)). Furthermore, for any $R > 0$, the map $u_0 \mapsto u$ is real analytic from $B_R(H^s(\mathbb{T}))$ into X_T^s.*

Some additional properties are collected in the following

Proposition 2.2. *For $u_0 \in H^1(\mathbb{T})$, the solution $u(t)$ of the IVP (2.1)–(2.2) satisfies $u \in C^\omega(\mathbb{R}; H^1(\mathbb{T}))$. Moreover the three integral terms $\int_{\mathbb{T}} u\,dx$, $\int_{\mathbb{T}}(u^2 + u_x^2)\,dx$ and $\int_T (u^3 + 3u^2)\,dx$ are invariants of motion (i.e., they remain constant over time).*

Proof. Let us begin with the invariants of motion. For $u_0 \in H^1(\mathbb{T})$, $u \in X_T^1$ for all $T > 0$, hence

$$u_t = -\big(1 - \partial_x^2\big)^{-1}\partial_x\left(u + \frac{u^2}{2}\right) \in X_T^2.$$

Therefore, all the terms in (2.1) belong to X_T^0. Scaling in (2.1) by 1 (resp. by u) yields after some integrations by parts

$$\frac{d}{dt}\int_{\mathbb{T}} u\,dx = 0 \qquad \left(\text{resp. } \frac{d}{dt}\int_{\mathbb{T}}\big(u^2 + u_x^2\big)\,dx = 0\right).$$

For the last invariant of motion, we notice (following [33]) that

$$\left(\frac{1}{3}(u+1)^3\right)_t - \left(u_t^2 - u_{xt}^2 + (u+1)^2 u_{xt} - \frac{1}{4}(u+1)^4\right)_x = 0.$$

Integrating on $\mathbb{T}$ yields $(d/dt)\int_T (u+1)^3 dx = 0$. Since $(d/dt)\int_{\mathbb{T}}(3u+1)\,dx = 0$, we infer that

$$\frac{d}{dt}\int_{\mathbb{T}}\big(u^3 + 3u^2\big)\,dx = 0.$$

Let us now prove that $u \in C^\omega(\mathbb{R}; H^1(\mathbb{T}))$. Since $u \in C^1(\mathbb{R}; H^1(\mathbb{T}))$, it is sufficient to check that for any $u_0 \in H^1(\mathbb{T})$ there are some numbers $b > 0$, $M > 0$, and some sequence $(u_n)_{n \geqslant 1}$ in $H^1(\mathbb{T})$ with

$$\|u_n\|_{H^1} \leqslant \frac{M}{b^n}, \quad n \geqslant 0, \tag{2.4}$$

such that

$$u(t) = \sum_{n \geqslant 0} t^n u_n, \quad t \in (-b, b). \tag{2.5}$$

Note that the convergence of the series in (2.5) holds in $H^1(\mathbb{T})$ uniformly on $[-rb, rb]$ for each $r<1$. Actually, we prove that u can be extended as an analytic function from $D_b := \{z \in \mathbb{C};\ |z|<b\}$ into the space $H^1_{\mathbb{C}}(\mathbb{T}) := H^1(\mathbb{T};\mathbb{C})$, endowed with the Euclidean norm

$$\left\| \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} \right\|_{H^1} = \left(\sum_{k\in\mathbb{Z}} \bigl(1+|k|^2\bigr) |\hat{u}_k|^2 \right)^{\frac{1}{2}}.$$

We adapt the classical proof of the analyticity of the flow for an ODE with an analytic vector field (see e.g. [15]) to our infinite dimensional framework. For $u \in H^1_{\mathbb{C}}(\mathbb{T})$, let $Au = -(1-\partial_x^2)^{-1}\partial_x u$ and $f(u) = A(u+u^2)$. Since $|k| \leqslant (k^2+1)/2$ for all $k\in\mathbb{Z}$, $\|A\|_{\mathcal{L}(H^1_{\mathbb{C}}(\mathbb{T}))} \leqslant 1/2$. Pick a positive constant C_1 such that

$$\bigl\|u^2\bigr\|_{H^1} \leqslant C_1 \|u\|^2_{H^1} \quad \text{for all } u \in H^1_{\mathbb{C}}(\mathbb{T}).$$

We define by induction on q a sequence (u^q) of analytic functions from $\mathbb{C}$ to $H^1_{\mathbb{C}}(\mathbb{T})$ which will converge uniformly on D_T, for $T>0$ small enough, to a solution of the integral equation

$$u(z) = u_0 + \int\limits_{[0,z]} f\bigl(u(\zeta)\bigr)\,d\zeta = u_0 + \int\limits_0^1 f\bigl(u(sz)\bigr) z\,ds.$$

Let

$$\begin{aligned} &u^0(z) = u_0, \quad \text{for } z\in\mathbb{C}, \\ &u^{q+1}(z) = u_0 + \int\limits_{[0,z]} f\bigl(u^q(\zeta)\bigr)\,d\zeta, \quad \text{for } q \geqslant 0,\ z\in\mathbb{C}. \end{aligned}$$

Claim 1. *$u^q(z) = \sum_{n\geqslant 0} z^n v_n^q$ for all $z\in\mathbb{C}$ and some sequence (v_n^q) in $H^1_{\mathbb{C}}(\mathbb{T})$ with*

$$\bigl\|v_n^q\bigr\|_{H^1} \leqslant \frac{M(q,b)}{b^n} \quad \textit{for all } q,n\in\mathbb{N},\ b>0.$$

The proof of Claim 1 is done by induction on $q \geqslant 0$. The result is clear for $q=0$ with $M(0,b) = \|u_0\|_{H^1}$, since $v_0^0 = u_0$ and $v_n^0 = 0$ for $n \geqslant 1$. Assume that Claim 1 is proved for some $q \geqslant 0$. Then, for any $r\in(0,1)$ and any $b>0$,

$$\bigl\|z^n v_n^q\bigr\|_{H^1} \leqslant M(q,b) r^n \quad \text{for } |z| \leqslant rb,$$

so that the series $\sum_{n\geqslant 0} z^n v_n^q$ converges absolutely in $H^1_{\mathbb{C}}(\mathbb{T})$ uniformly for $z \in \overline{D_{rb}}$. The same holds true for the series $\sum_{n\geqslant 0} z^n (\sum_{0\leqslant l\leqslant n} v_l^q v_{n-l}^q)$. It follows that

$$f\bigl(u^q(\zeta)\bigr) = A\left(\sum_{n\geqslant 0} \zeta^n v_n^q + \sum_{n\geqslant 0} \zeta^n \left(\sum_{0\leqslant l\leqslant n} v_l^q v_{n-l}^q \right) \right)$$

converges uniformly for $\zeta \in \overline{D_{rb}}$. Thus

$$
\begin{aligned}
u^{q+1}(z) &= u_0 + \int_{[0,z]} \sum_{n \geqslant 0} \zeta^n A\left(v_n^q + \sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q\right) d\zeta \\
&= \sum_{n \geqslant 0} z^n v_n^{q+1}
\end{aligned}
$$

where

$$
\begin{aligned}
v_0^{q+1} &= u_0, \\
v_n^{q+1} &= \frac{1}{n} A\left(v_{n-1}^q + \sum_{0 \leqslant l \leqslant n-1} v_l^q v_{n-1-l}^q\right) \quad \text{for } n \geqslant 1.
\end{aligned}
$$

It follows that for $n \geqslant 1$

$$
\left\| v_n^{q+1} \right\|_{H^1} \leqslant \frac{\|A\|}{n}\left(\frac{M(q,b)}{b^{n-1}} + nC_1 \frac{M^2(q,b)}{b^{n-1}}\right) \leqslant \frac{M(q+1,b)}{b^n}
$$

with

$$
M(q+1,b) := \sup\left\{\|u_0\|_{H^1}, b\|A\|\left(M(q,b) + C_1 M^2(q,b)\right)\right\}.
$$

Claim 1 is proved.

Claim 2. *Let $T := (2\|A\|(1+4C_1\|u_0\|_{H^1}))^{-1}$. Then $\|u^q - u\|_{L^\infty(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T}))} \to 0$ as $q \to \infty$ for some $u \in C(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T}))$.*

Let $Z_T = C(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T}))$ be endowed with the norm $|||v||| = \sup_{|z| \leqslant T} \|v(z)\|_{H^1}$. Let $R > 0$, and for $v \in B_R := \{v \in Z_T;\ |||v||| \leqslant R\}$, let

$$
(\Gamma v)(z) = u_0 + \int_{[0,z]} f\left(v(\zeta)\right) d\zeta.
$$

Then

$$
\begin{aligned}
|||\Gamma v||| &\leqslant \|u_0\|_{H^1} + T\|A\|\left(|||v||| + C_1 |||v|||^2\right) \leqslant \|u_0\|_{H^1} + T\|A\|\left(R + C_1 R^2\right), \\
|||\Gamma v_1 - \Gamma v_2||| &\leqslant T\|A\|\left(|||v_1 - v_2||| + \left|\left|\left|v_1^2 - v_2^2\right|\right|\right|\right) \leqslant T\|A\|(1+2C_1 R)|||v_1 - v_2|||.
\end{aligned}
$$

Pick $R = 2\|u_0\|_{H^1}$ and $T = (2\|A\|(1+2C_1 R))^{-1}$. Then Γ contracts in B_R. The sequence (u^q), which is given by Picard iteration scheme, has a limit u in Z_T which fulfills

$$
u(z) = u_0 + \int_{[0,z]} f\left(u(\zeta)\right) d\zeta, \quad |z| \leqslant T.
$$

In particular, $u \in C^1([-T,T]; H^1(\mathbb{T}))$ (the $u^q(z)$ being real-valued for $z \in \mathbb{R}$) and it satisfies $u_t = f(u)$ on $[-T,T]$ together with $u(0) = u_0$; that is, u solves (2.1)–(2.2) in the class $C^1([-T,T]; H^1(\mathbb{T})) \subset X_T^1$.

Claim 3. *$u(z) = \sum_{n \geqslant 0} z^n v_n$ for $|z| < T$, where $v_n = \lim_{q \to \infty} v_n^q$ for each $n \geqslant 0$.*

From Claim 1, we infer that for all $n \geqslant 1$

$$v_n^q = \frac{1}{2\pi i} \int\limits_{|z|=T} z^{-n-1} u^q(z)\, dz,$$

hence

$$\left\| v_n^p - v_n^q \right\|_{H^1} \leqslant T^{-n} \left|\left|\left| u^p - u^q \right|\right|\right|.$$

From Claim 2, we infer that (v_n^q) is a Cauchy sequence in $H^1_{\mathbb{C}}(\mathbb{T})$. Let v_n denote its limit in $H^1_{\mathbb{C}}(\mathbb{T})$. Note that

$$\left\| v_n - v_n^q \right\|_{H^1} \leqslant T^{-n} \left|\left|\left| u - u^q \right|\right|\right|,$$

and hence the series $\sum_{n \geqslant 0} z^n v_n$ is convergent for $|z| < T$. Therefore, for $|z| \leqslant rT$ with $r < 1$,

$$\left\| \sum_{n \geqslant 0} z^n \big(v_n - v_n^q \big) \right\|_{H^1} \leqslant (1-r)^{-1} \left|\left|\left| u - u^q \right|\right|\right|,$$

and hence $u^q(z) = \sum_{n \geqslant 0} z^n v_n^q \to \sum_{n \geqslant 0} z^n v_n$ in Z_{rT} as $q \to \infty$. It follows that

$$u(z) = \sum_{n \geqslant 0} z^n v_n \quad \text{for } |z| < T.$$

The proof of Proposition 2.2 is complete. $\square$

3. Unique continuation property for BBM

In this section we prove a UCP for the BBM equation for small solutions with nonnegative mean values.

Theorem 3.1. *Let $u_0 \in H^1(\mathbb{T})$ be such that*

$$\int\limits_{\mathbb{T}} u_0(x)\, dx \geqslant 0, \tag{3.1}$$

and

$$\|u_0\|_{L^\infty(\mathbb{T})} < 3. \tag{3.2}$$

Assume that the solution u of (2.1)–(2.2) satisfies

$$u(x,t) = 0 \quad \textit{for all } (x,t) \in \omega \times (0,T), \tag{3.3}$$

where $\omega \subset \mathbb{T}$ is a nonempty open set and $T > 0$. Then $u_0 = 0$, and hence $u \equiv 0$.

Proof. Using a system of coordinates, we may identify $\mathbb{T}$ to $[0,2\pi)$ in such a way that $\omega \supset [0,\varepsilon) \cup (2\pi-\varepsilon, 2\pi)$ for some $\varepsilon > 0$. (This is possible whenever we take the origin of the coordinates inside ω.) Since $u \in C^\omega(\mathbb{R}; H^1(\mathbb{T}))$ by Proposition 2.2, we have that $u(x,\cdot) \in C^\omega(\mathbb{R})$ for all $x \in \mathbb{T}$. (3.3) gives then that

$$u(x,t) = 0 \quad \text{for } (x,t) \in \omega \times \mathbb{R}. \tag{3.4}$$

Introduce the function

$$v(x,t) = \int_0^x u(y,t)\,dy.$$

Then $v \in C^\omega(\mathbb{R}; H^2(0,2\pi))$ and v satisfies

$$v_t - v_{txx} + v_x + \frac{u^2}{2} = 0, \quad x \in (0,2\pi), \tag{3.5}$$

as it may be seen by integrating (2.1) on $(0,x)$. Let

$$I(t) = \int_0^{2\pi} v(x,t)\,dx.$$

Note that $I \in C^\omega(\mathbb{R})$. Integrating (3.5) on $(0,2\pi)$ gives with (3.1)

$$I_t = -\int_0^{2\pi} u_0(x)\,dx - \frac{1}{2}\int_0^{2\pi} \left|u(x,t)\right|^2 dx \leqslant 0.$$

Since $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ for all $t \in \mathbb{R}$, $v \in L^\infty(\mathbb{R}, H^2(0,2\pi))$ and $I \in L^\infty(\mathbb{R})$. It follows that the function I has a finite limit as $t \to \infty$, that we denote by l. From the boundedness of $\|u(t)\|_{H^1(\mathbb{T})}$ for $t \in \mathbb{R}$, we infer the existence of a sequence $t_n \nearrow +\infty$ such that

$$u(t_n) \rightharpoonup \tilde{u}_0 \quad \text{in } H^1(\mathbb{T}) \tag{3.6}$$

for some $\tilde{u}_0 \in H^1(\mathbb{T})$. Let $\tilde{u}$ denote the solution of the IVP for BBM corresponding to the initial data $\tilde{u}_0$; that is, $\tilde{u}$ solves

$$\begin{aligned} &\tilde{u}_t - \tilde{u}_{txx} + \tilde{u}_x + \tilde{u}\tilde{u}_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ &\tilde{u}(x,0) = \tilde{u}_0(x). \end{aligned}$$

Pick any $s \in (1/2,1)$. As $u(t_n) \to \tilde{u}_0$ strongly in $H^s(\mathbb{T})$, we infer from Theorem 2.1 that

$$u(t_n + \cdot) \to \tilde{u} \quad \text{in } C\big([0,1]; H^s(\mathbb{T})\big). \tag{3.7}$$

It follows from (3.4), (3.7) and the fact that $\tilde{u} \in C^\omega(\mathbb{R}, H^1(\mathbb{T}))$ that

$$\tilde{u}(x,t) = 0 \quad \text{for } (x,t) \in \omega \times \mathbb{R}.$$

On the other hand, $\int_0^{2\pi} \tilde{u}_0(x)\,dx = \int_0^{2\pi} u_0(x)\,dx$ from (3.6) and the invariance of $\int_0^{2\pi} u(x,t)\,dx$. Let $\tilde{v}(x,t) = \int_0^x \tilde{u}(y,t)\,dy$ and $\tilde{I}(t) = \int_0^{2\pi} \tilde{v}(x,t)\,dx$. Then we still have that

$$\tilde{I}_t = -\int_0^{2\pi} u_0(x)\,dx - \frac{1}{2}\int_0^{2\pi} \left|\tilde{u}(x,t)\right|^2 dx \leqslant 0. \tag{3.8}$$

But we infer from (3.7) that

$$I(t_n) \to \tilde{I}(0), \qquad I(t_n+1) \to \tilde{I}(1).$$

Since

$$\lim_{n\to\infty} I(t_n) = \lim_{n\to\infty} I(t_n+1) = l,$$

we have that $\tilde{I}(0) = \tilde{I}(1)$. Combined with (3.8), this yields

$$\tilde{u}(x,t) = 0, \quad (x,t) \in \mathbb{T} \times [0,1].$$

In particular, $\tilde{u}_0 = 0$. From (3.6), we infer that

$$\int_0^{2\pi} \big(u^3(x,t_n) + 3u^2(x,t_n)\big)\,dx \to 0 \quad \text{as } n \to \infty.$$

As $\int_0^{2\pi} (u^3 + 3u^2)\,dx$ is a conserved quantity, we infer that

$$\int_0^{2\pi} \big(3 + u_0(x)\big)\left|u_0(x)\right|^2 dx = 0,$$

which, combined with (3.2), yields $u_0 = 0$. □

Remark 3.2. Note that Theorem 3.1 is false if the assumptions $u_0 \in H^1(\mathbb{T})$ and (3.1) are removed. Indeed, if $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ is defined for $x \in \mathbb{T} \sim (0, 2\pi)$ and $t \in \mathbb{R}$ by

$$u(x,t) = u_0(x) = \begin{cases} -2 & \text{if } |x-\pi| \leqslant \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} < |x-\pi| < \pi, \end{cases}$$

then (2.1) and (2.2) are satisfied, although $u \not\equiv 0$.

4. Unique continuation property for BBM-like equations

We shall consider BBM-like equations with different nonlinear terms. We first consider a generalized BBM equation without drift term, and next a BBM-like equation with a nonlocal bilinear term.

4.1. Generalized BBM equation without drift term

We consider the following generalized BBM equation

$$u_t - u_{txx} + \big[f(u)\big]_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{4.1}$$

$$u(x,0) = u_0(x), \tag{4.2}$$

where $f \in C^1(\mathbb{R})$, $f(u) \geqslant 0$ for all $u \in \mathbb{R}$, and the only solution $u \in (-\delta,\delta)$ of $f(u) = 0$ is $u = 0$, for some number $\delta > 0$. That class of BBM-like equations includes the Morrison–Meiss–Carey equation

$$u_t - u_{txx} + uu_x = 0$$

for $f(u) = u^2/2$. Note that the global well-posedness of (4.1)–(4.2) in $H^1(\mathbb{T})$ can easily be derived from the contraction mapping theorem and the conservation of the H^1-norm. It turns out that the UCP can be derived in a straight way and without any additional assumption on the initial data.

Theorem 4.1. *Let f be as above, and let ω be a nonempty open set in $\mathbb{T}$. Let $u_0 \in H^1(\mathbb{T})$ be such that the solution u of (4.1)–(4.2) satisfies $u(x,t) = 0$ for $(x,t) \in \omega \times (0,T)$ for some $T > 0$. Then $u_0 = 0$.*

Proof. Once again, we can assume without loss of generality that $\omega = [0,\varepsilon) \cup (2\pi - \varepsilon, 2\pi)$. The prolongation of u by 0 on $(\mathbb{R} \setminus (0,2\pi)) \times (0,T)$, still denoted by u, satisfies

$$u_t - u_{txx} + \big[f(u)\big]_x = 0, \quad x \in \mathbb{R},\ t \in (0,T), \tag{4.3}$$

$$u(x,t) = 0, \quad x \notin (\varepsilon, 2\pi - \varepsilon),\ t \in (0,T), \tag{4.4}$$

$$u \in C\big([0,T]; H^1(\mathbb{R})\big), \quad u_t \in C\big([0,T]; H^2(\mathbb{R})\big). \tag{4.5}$$

Scaling in (4.3) by e^x yields for $t \in (0,T)$

$$\int_{-\infty}^{\infty} f\big(u(x,t)\big)e^x\,dx = 0,$$

for $\int_{-\infty}^{\infty} u_{txx}e^x\,dx = \int_{-\infty}^{\infty} u_t e^x\,dx$ by two integrations by parts. Since f is nonnegative, this yields

$$f\big(u(x,t)\big) = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,T).$$

Since u is continuous and it vanishes for $x \notin (\varepsilon, 2\pi - \varepsilon)$, we infer from the assumptions about f that $u \equiv 0$. $\square$

Pick any nonnegative function $a \in C^\infty(\mathbb{T})$ with $\omega := \{x \in \mathbb{T};\ a(x) > 0\}$ nonempty. We are interested in the stability properties of the system

$$u_t - u_{txx} + \big[f(u)\big]_x + a(x)u = 0, \quad x \in \mathbb{T},\ t \geqslant 0, \tag{4.6}$$

$$u(x,0) = u_0(x), \tag{4.7}$$

where f is as above. The following weak stability result holds.

Corollary 4.2. *Let $u_0 \in H^1(\mathbb{T})$. Then (4.6)–(4.7) admits a unique solution $u \in C([0,T]; H^1(\mathbb{T}))$ for all $T > 0$. Furthermore, $u(t) \to 0$ weakly in $H^1(\mathbb{T})$, hence strongly in $H^s(\mathbb{T})$ for $s < 1$, as $t \to +\infty$.*

Proof. The local well-posedness in $H^s(\mathbb{T})$ for any $s > 1/2$ is derived from the contraction mapping theorem in much the same way as for Theorem 2.1. The global well-posedness in $H^1(\mathbb{T})$ follows at once from the energy identity

$$\big\|u(T)\big\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2\int_0^T\int_{\mathbb{T}} a(x)\big|u(x,t)\big|^2\,dx\,dt = 0, \tag{4.8}$$

obtained by scaling each term in (4.6) by u. On the other hand, still from the application of the contraction mapping theorem, given any $s > 1/2$, any $\rho > 0$ and any $u_0, v_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s(\mathbb{T})} \leqslant \rho$, $\|v_0\|_{H^s(\mathbb{T})} \leqslant \rho$, there is some time $T = T(s,\rho) > 0$ such that the solutions u and v of (4.6)–(4.7) corresponding to the initial data u_0 and v_0, respectively, fulfill

$$\|u - v\|_{C([0,T];H^s(\mathbb{T}))} \leqslant 2\|u_0 - v_0\|_{H^s(\mathbb{T})}. \tag{4.9}$$

Pick any initial data $u_0 \in H^1(\mathbb{T})$, any $s \in (1/2, 1)$, and let $\rho = \|u_0\|_{H^1(\mathbb{T})}$ and $T = T(s,\rho)$. Note that $\|u(t)\|_{H^1}$ is nonincreasing by (4.8), hence it has a nonnegative limit l as $t \to \infty$. Let v_0 be in the ω-limit set of $(u(t))_{t\geqslant 0}$ in $H^1(\mathbb{T})$ for the weak topology; that is, for some sequence $t_n \to \infty$ we have $u(t_n) \to v_0$ weakly in $H^1(\mathbb{T})$. Extracting a subsequence if needed, we may assume that $t_{n+1} - t_n \geqslant T$ for all n. From (4.8) we infer that

$$\lim_{n\to\infty}\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}} a(x)\big|u(x,t)\big|^2\,dx\,dt = 0. \tag{4.10}$$

Since $u(t_n) \to v_0$ (strongly) in $H^s(\mathbb{T})$, and $\|u(t_n)\|_{H^s(\mathbb{T})} \leqslant \|u(t_n)\|_{H^1(\mathbb{T})} \leqslant \rho$, we have from (4.9) that

$$u(t_n + \cdot) \to v \quad \text{in } C\big([0,T];H^s(\mathbb{T})\big) \quad \text{as } n \to \infty, \tag{4.11}$$

where $v = v(x,t)$ denotes the solution of

$$\begin{aligned} &v_t - v_{txx} + \big[f(v)\big]_x + a(x)v = 0, \quad x \in \mathbb{T},\ t \geqslant 0, \\ &v(x,0) = v_0(x). \end{aligned}$$

Note that $v \in C([0,T];H^1(\mathbb{T}))$ for $v_0 \in H^1(\mathbb{T})$. (4.10) combined with (4.11) yields

$$\int_0^T\int_{\mathbb{T}} a(x)\big|v(x,t)\big|^2\,dx\,dt = 0,$$

so that $av \equiv 0$. By Theorem 4.1, $v_0 = 0$ and hence, as $t \to \infty$,

$$\begin{aligned} &u(t) \to 0 \quad \text{weakly in } H^1(\mathbb{T}), \\ &u(t) \to 0 \quad \text{strongly in } H^s(\mathbb{T}) \text{ for } s < 1. \qquad \square \end{aligned}$$

4.2. A BBM-like equation with a nonlocal bilinear term

Here, we consider a BBM-type equation with the drift term, but with a nonlocal bilinear term given by a convolution, namely

$$u_t - u_{txx} + u_x + \lambda(u * u)_x = 0, \quad x \in \mathbb{R}, \tag{4.12}$$

where $\lambda \in \mathbb{R}$ is a constant and

$$(u * v)(x) = \int_{-\infty}^{\infty} u(x-y)v(y)\,dy \quad \text{for } x \in \mathbb{R}.$$

A UCP can be derived without any restriction on the initial data.

Theorem 4.3. *Assume that $\lambda \neq 0$. Let $u \in C^1([0,T]; H^1(\mathbb{R}))$ be a solution of (4.12) such that*

$$u(x,t) = 0 \quad \textit{for } |x| > L,\ t \in (0,T). \tag{4.13}$$

Then $u \equiv 0$.

Proof. Taking the Fourier transform of each term in (4.12) yields

$$\left(1+\xi^2\right)\hat{u}_t = -i\xi\left(\hat{u} + \lambda\hat{u}^2\right), \quad \xi \in \mathbb{R},\ t \in (0,T). \tag{4.14}$$

Note that, for each $t \in (0,T)$, $\hat{u}(\cdot,t)$ and $\hat{u}_t(\cdot,t)$ may be extended to $\mathbb{C}$ as entire functions of exponential type at most L. Furthermore, (4.14) is still true for $\xi \in \mathbb{C}$ and $t \in (0,T)$ by analytic continuation. To prove that $u \equiv 0$, it is sufficient to check that

$$\partial_\xi^k \hat{u}(i,t) = 0 \quad \forall k \in \mathbb{N},\ \forall t \in (0,T). \tag{4.15}$$

Let us prove (4.15) by induction on k. First, we see that (4.14) gives that either

$$\hat{u}(i,t) = 0 \quad \forall t \in (0,T), \tag{4.16}$$

or

$$\hat{u}(i,t) = -\lambda^{-1} \quad \forall t \in (0,T). \tag{4.17}$$

Derivating with respect to ξ in (4.14) yields (the upper script denoting the order of derivation in ξ)

$$\begin{aligned} &2\xi\hat{u}_t(\xi,t) + \left(1+\xi^2\right)\hat{u}_t^{(1)}(\xi,t) \\ &\quad = -i\hat{u}(\xi,t)\left(1+\lambda\hat{u}(\xi,t)\right) - i\xi\hat{u}^{(1)}(\xi,t)\left(1+2\lambda\hat{u}(\xi,t)\right). \end{aligned} \tag{4.18}$$

Note that $\hat{u}_t(i,t) = 0$ if either (4.16) or (4.17) hold. Combined with (4.18), this gives

$$\hat{u}^{(1)}(i,t) = 0, \quad t \in (0,T).$$

Assume now that, for some $k \geqslant 2$,

$$\hat{u}^{(l)}(i,t)=0 \quad \text{for } t\in(0,T) \text{ and any } l\in\{1,\dots,k-1\}. \tag{4.19}$$

Derivating k times with respect to ξ in (4.14) yields

$$\begin{aligned}&\left(1+\xi^2\right)\hat{u}_t^{(k)}+2k\xi\hat{u}_t^{(k-1)}+k(k-1)\hat{u}_t^{(k-2)}\\&\quad=-i\xi\left(\hat{u}^{(k)}+\lambda\sum_{l=0}^{k}C_k^l\hat{u}^{(l)}\hat{u}^{(k-l)}\right)-ik\left(\hat{u}^{(k-1)}+\lambda\sum_{l=0}^{k-1}C_{k-1}^l\hat{u}^{(l)}\hat{u}^{(k-1-l)}\right).\end{aligned} \tag{4.20}$$

From (4.19) and (4.20) we infer that

$$\hat{u}^{(k)}(i,t)\left(1+2\lambda\hat{u}(i,t)\right)=0.$$

Combined with (4.16) and (4.17), this yields

$$\hat{u}^{(k)}(i,t)=0.$$

Thus

$$\hat{u}^{(k)}(i,t)=0 \quad \forall k\geqslant 1. \tag{4.21}$$

(4.17) and (4.21) would imply

$$\hat{u}(\xi,t)=-\lambda^{-1} \quad \forall\xi\in\mathbb{C},$$

which contradicts the fact that $\hat{u}(\cdot,t)\in L^2(\mathbb{R})$. Thus (4.16) holds and $u\equiv 0$. $\square$

5. Unique continuation property for the KdV–BBM equation

In this section we prove some UCP for the following KdV–BBM equation

$$u_t-u_{txx}-cu_{xxx}+qu_x=0, \quad x\in\mathbb{T},\ t\in(0,T), \tag{5.1}$$

where $q\in L^\infty(0,T;L^\infty(\mathbb{T}))$ is a given potential function and $c\neq 0$ is a given real constant. The UCP obtained here will be used in the next section to obtain a semiglobal exponential stabilization result for BBM with a moving damping.

Theorem 5.1. *Let $c\in\mathbb{R}\setminus\{0\}$, $T>2\pi/|c|$, and $q\in L^\infty(0,T;L^\infty(\mathbb{T}))$. Let $\omega\subset\mathbb{T}$ be a nonempty open set. Let $u\in L^2(0,T;H^2(\mathbb{T}))\cup L^\infty(0,T;H^1(\mathbb{T}))$ satisfying (5.1) and*

$$u(x,t)=0 \quad \text{for a.e. } (x,t)\in\omega\times(0,T). \tag{5.2}$$

Then $u\equiv 0$ in $\mathbb{T}\times(0,T)$.

Proof. Assume first that

$$u\in L^2\left(0,T;H^2(\mathbb{T})\right). \tag{5.3}$$

Let $w=u-u_{xx}\in L^2(0,T;L^2(\mathbb{T}))$. Then (u,w) solves the following system

$$u - u_{xx} = w, \tag{5.4}$$

$$w_t + cw_x = (c - q)u_x. \tag{5.5}$$

We shall establish some Carleman estimates for the elliptic equation (5.4) and the transport equation (5.5) with the "same weights", and combine both Carleman estimates into a single one for (5.1). We refer the reader to [50] for a similar analysis for a coupled system of elliptic–hyperbolic equations, and to [1,10] for coupled systems of parabolic–hyperbolic equations.

Remark 5.2. There is a finite speed propagation for KdV–BBM: assuming for simplicity that $q(x) = c$ for all $x \in \mathbb{T}$, where $c > 0$ is given, and that $\omega = (2\pi - \varepsilon, 2\pi)$ for a small $\epsilon > 0$, then the UCP fails in time $T \leqslant (2\pi - 2\varepsilon)/c$. Indeed, picking any nontrivial initial state $u_0 \in C_0^\infty(0, \varepsilon)$, we easily see that the solution (u, w) of (5.4)–(5.5) reads $u(x, t) = u_0(x - ct)$, $w = w_0(x - ct)$ where $w_0 = (1 - \partial_x^2)u_0$. Then $u(x, t) = 0$ for $(x, t) \in \omega \times (0, (2\pi - 2\epsilon)/c)$, although $u \not\equiv 0$. Hence, the condition $T > 2\pi/|c|$ in Theorem 5.1 is sharp. The necessity of $T > 2\pi/|c|$ in all the control results for KdV–BBM is also clear from the value of the spectral gap deduced from (6.6) when q is constant. Note that, by contrast, there is an infinite speed propagation for both BBM and KdV (see [30,36,39]).

Introduce a few notations. We identify $\mathbb{T}$ with $[0, 2\pi)$ by choosing a system of coordinates. Without loss of generality, we can assume that $c > 0$ (the case $c < 0$ being similar), and that $\omega = (2\pi - \eta, 2\pi + \eta) \sim [0, \eta) \cup (2\pi - \eta, 2\pi)$ for some $\eta \in (0, \pi)$ (by choosing the origin of the coordinates inside ω). Assume given a time T fulfilling

$$T > \frac{2\pi}{c}. \tag{5.6}$$

Pick some numbers $\delta > 0$ and $\rho \in (0, 1)$ such that

$$\rho c T > 2\pi + \delta \tag{5.7}$$

and a function $\psi \in C^\infty([0, 2\pi])$ such that

$$\psi(x) = |x + \delta|^2 \quad \text{for } x \in [\eta/2, 2\pi - \eta/2], \tag{5.8}$$

$$\frac{d^k\psi}{dx^k}(0) = \frac{d^k\psi}{dx^k}(2\pi) \quad \text{for } k = 1, 2, 3, \tag{5.9}$$

$$2\delta \leqslant \frac{d\psi}{dx}(x) \leqslant 2(2\pi + \delta) \quad \text{for } x \in [0, 2\pi]. \tag{5.10}$$

Introduce the function $\varphi \in C^\infty([0, 2\pi] \times \mathbb{R})$ defined by

$$\varphi(x, t) = \psi(x) - \rho c^2 t^2. \tag{5.11}$$

Then the following Carleman estimate for (5.1) will be derived.

Proposition 5.3. *Let ω, c and T be as above. Then there exists some positive numbers s_2 and C_2 such that for all $s \geqslant s_2$ and all $u \in L^2(0, T; H^2(\mathbb{T}))$ satisfying (5.1), we have*

$$\int_0^T\int_{\mathbb{T}} \big[s|u_{xx}|^2+s|u_x|^2+s^3|u|^2\big]e^{2s\varphi}\,dx\,dt+s\int_{\mathbb{T}}\big[|u-u_{xx}|^2e^{2s\varphi}\big]_{|t=0}\,dx$$
$$\leqslant C_2\int_0^T\int_{\omega}\big[s|u_{xx}|^2+s^3|u|^2\big]e^{2s\varphi}\,dx\,dt. \tag{5.12}$$

Note that the Carleman estimate (5.12) yields at once the *observability inequality*

$$\big\|u(\cdot,0)\big\|^2_{H^2(\mathbb{T})}\leqslant C\int_0^T\big\|u(\cdot,t)\big\|^2_{H^2(\omega)}\,dt. \tag{5.13}$$

The proof of Proposition 5.3 is outlined as follows. In the first step, we prove a Carleman estimate for the elliptic equation (5.4) with the weight $e^{s\psi}$. In the second step, we prove a Carleman estimate for the transport equation (5.5) with the weight $e^{s\varphi}$. Note that we are concerned here with *global* Carleman estimates with weights suitably chosen in the control region, so that our results do not seem to be direct consequences of the *local* Carleman estimates in [16]. In the last step, we combine the two above Carleman estimates into a single one to obtain (5.12).

Step 1. Carleman estimate for the elliptic equation.

Lemma 5.4. *There exist $s_0\geqslant 1$ and $C_0>0$ such that for all $s\geqslant s_0$ and all $u\in H^2(\mathbb{T})$, the following holds*

$$\int_{\mathbb{T}}\big[s|u_x|^2+s^3|u|^2\big]e^{2s\psi}\,dx\leqslant C_0\bigg(\int_{\mathbb{T}}|u_{xx}|^2e^{2s\psi}\,dx+\int_{\omega}s^3|u|^2e^{2s\psi}\,dx\bigg). \tag{5.14}$$

Proof. Let $v=e^{s\psi}u$ and $P=\partial_x^2$. Then

$$e^{s\psi}Pu=e^{s\psi}P\big(e^{-s\psi}v\big)=P_sv+P_av$$

where

$$P_sv=(s\psi_x)^2v+v_{xx}, \tag{5.15}$$
$$P_av=-2s\psi_xv_x-s\psi_{xx}v \tag{5.16}$$

denote the (formal) self-adjoint and skew-adjoint parts of $e^{s\psi}P(e^{-s\psi}\cdot)$, respectively. It follows that

$$\big\|e^{s\psi}Pu\big\|^2=\|P_sv\|^2+\|P_av\|^2+2(P_sv,P_av)$$

where $(f,g)=\int_{\mathbb{T}}fg\,dx$, and $\|f\|^2=(f,f)$. Then

$$(P_sv,P_av)=\big((s\psi_x)^2v,-2s\psi_xv_x\big)+\big((s\psi_x)^2v,-s\psi_{xx}v\big)$$
$$+(v_{xx},-2s\psi_xv_x)+(v_{xx},-s\psi_{xx}v)=:I_1+I_2+I_3+I_4.$$

After some integrations by parts in x, we obtain with (5.9) that

$$I_1 = 3\int_{\mathbb{T}} (s\psi_x)^2 s\psi_{xx} v^2\,dx,$$

$$I_3 = \int_{\mathbb{T}} s\psi_{xx} v_x^2\,dx,$$

$$I_4 = \int_{\mathbb{T}} v_x(s\psi_{xxx} v + s\psi_{xx} v_x)\,dx = -\int_{\mathbb{T}} s\psi_{xxxx}\frac{v^2}{2}\,dx + \int_{\mathbb{T}} s\psi_{xx} v_x^2\,dx.$$

Therefore

$$\left\|e^{s\psi} Pu\right\|^2 = \|P_s v\|^2 + \|P_a v\|^2 + \int_{\mathbb{T}} \left[4(s\psi_x)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2\,dx + \int_{\mathbb{T}} (4s\psi_{xx}) v_x^2\,dx.$$

From (5.8), we infer that there exist some numbers $s_0 \geqslant 1$, $K > 0$ and $K' > 0$ such that for all $s \geqslant s_0$

$$4(s\psi_x)^2 s\psi_{xx} - s\psi_{xxxx} \geqslant Ks^3 \quad \text{for } (x,t) \in (\eta/2, 2\pi - \eta/2) \times (0,T),$$
$$4s\psi_{xx} \geqslant Ks \quad \text{for } (x,t) \in (\eta/2, 2\pi - \eta/2) \times (0,T),$$

while, setting $\omega_0 = [0, \eta/2) \cup (2\pi - \eta/2, 2\pi)$,

$$\left|4(s\psi_x)^2 s\psi_{xx} - s\psi_{xxxx}\right| \leqslant K's^3 \quad \text{for } (x,t) \in \omega_0 \times (0,T),$$
$$|4s\psi_{xx}| \leqslant K's \quad \text{for } (x,t) \in \omega_0 \times (0,T).$$

We conclude that for $s \geqslant s_0$ and some constant $C > 0$

$$\|P_s v\|^2 + \int_{\mathbb{T}} \left[s|v_x|^2 + s^3|v|^2\right] dx \leqslant C\left(\left\|e^{s\psi} Pu\right\|^2 + \int_{\omega_0} \left[s|v_x|^2 + s^3|v|^2\right] dx\right). \tag{5.17}$$

Next we show that $\int_{\mathbb{T}} s^{-1}|v_{xx}|^2\,dx$ is also less than the r.h.s. of (5.17). We have

$$\begin{aligned}\int_{\mathbb{T}} s^{-1}|v_{xx}|^2\,dx &\leqslant \int_{\mathbb{T}} s^{-1}\left|P_s v - (s\psi_x)^2 v\right|^2 dx \\ &\leqslant 2\int_{\mathbb{T}} s^{-1}\left(|P_s v|^2 + |s\psi_x|^4|v|^2\right) dx \\ &\leqslant C\left(s^{-1}\|P_s v\|^2 + \int_{\mathbb{T}} s^3|v|^2\,dx\right).\end{aligned}$$

Combined with (5.17), this gives

$$\int_{\mathbb{T}} \left\{s^{-1}|v_{xx}|^2 + s|v_x|^2 + s^3|v|^2\right\} dx \leqslant C\left(\left\|e^{s\psi} Pu\right\|^2 + \int_{\omega_0} s^3|v|^2\,dx + \int_{\omega_0} s|v_x|^2\,dx\right) \tag{5.18}$$

where C does not depend on s and v. Finally, we show that we can drop the last term in the r.h.s. of (5.18). Let $\xi \in C_0^\infty(\omega)$ with $0 \leqslant \xi \leqslant 1$ and $\xi(x) = 1$ for $x \in \omega_0$. Then

$$\begin{aligned}\int_{\omega_0} |v_x|^2\,dx &\leqslant \int_\omega \xi |v_x|^2\,dx \\ &\leqslant -\int_\omega (\xi_x v_x + \xi v_{xx}) v\,dx \\ &\leqslant \frac{1}{2}\int_\omega \xi_{xx} v^2\,dx - \int_\omega \xi v_{xx} v\,dx\end{aligned}$$

so that

$$2\int_{\omega_0} s|v_x|^2 \leqslant \|\xi_{xx}\|_{L^\infty(\mathbb{T})} \int_\omega s|v|^2\,dx + \kappa \int_\omega s^{-1}|v_{xx}|^2\,dx + \kappa^{-1}\int_\omega s^3|v|^2\,dx \tag{5.19}$$

where $\kappa > 0$ is a constant that can be chosen as small as desired. Combining (5.18) and (5.19) with κ small enough gives for $s \geqslant s_0$ (with a possibly increased value of s_0) and some constant C that does not depend on s and v

$$\int_{\mathbb{T}} \left\{s^{-1}|v_{xx}|^2 + s|v_x|^2 + s^3|v|^2\right\} dx \leqslant C\left(\left\|e^{s\psi} Pu\right\|^2 + \int_\omega s^3|v|^2\,dx\right). \tag{5.20}$$

Replacing v by $e^{s\psi}u$ in (5.20) gives at once (5.14). The proof of Lemma 5.4 is complete. □

Step 2. Carleman estimate for the transport equation.

Lemma 5.5. *There exist $s_1 \geqslant s_0$ and $C_1 > 0$ such that for all $s \geqslant s_1$ and all $w \in L^2(\mathbb{T} \times (0,T))$ with $w_t + cw_x \in L^2(\mathbb{T} \times (0,T))$, the following holds*

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}} s|w|^2 e^{2s\varphi}\,dx\,dt + \int_{\mathbb{T}} s\left[|w|^2 e^{2s\varphi}\right]_{|t=0} dx + \int_{\mathbb{T}} s\left[|w|^2 e^{2s\varphi}\right]_{|t=T} dx \\ &\quad\leqslant C_1\left(\int_0^T\int_{\mathbb{T}} |w_t + cw_x|^2 e^{2s\varphi}\,dx\,dt + \int_0^T\int_\omega s|w|^2 e^{2s\varphi}\,dx\,dt\right).\end{aligned} \tag{5.21}$$

Proof. We first assume that $w \in H^1(\mathbb{T} \times (0,T))$. Let $v = e^{s\varphi}w$ and $P = \partial_t + c\partial_x$. Then

$$\begin{aligned}e^{s\varphi}Pw &= e^{s\varphi}P\left(e^{-s\varphi}v\right) \\ &= (-s\varphi_t v - cs\varphi_x v) + (v_t + cv_x) \\ &=: P_s v + P_a v.\end{aligned}$$

It follows that

$$\left\|e^{s\varphi}Pw\right\|^2_{L^2(\mathbb{T}\times(0,T))} = \|P_s v\|^2_{L^2(\mathbb{T}\times(0,T))} + \|P_a v\|^2_{L^2(\mathbb{T}\times(0,T))} + 2(P_s v, P_a v)_{L^2(\mathbb{T}\times(0,T))}. \quad (5.22)$$

After some integrations by parts in t and x in the last term in (5.22), we obtain

$$2(P_s v, P_a v)_{L^2(\mathbb{T}\times(0,T))} = \int_0^T \int_{\mathbb{T}} s\big(\varphi_{tt} + 2c\varphi_{xt} + c^2\varphi_{xx}\big)v^2\,dx\,dt - \int_{\mathbb{T}} s(\varphi_t + c\varphi_x)v^2\Big|_0^T\,dx - \int_0^T cs(\varphi_t + c\varphi_x)v^2\Big|_0^{2\pi}\,dt. \quad (5.23)$$

Using (5.9), (5.11) and the fact that $v(0,t) = v(2\pi,t)$, we notice that the last term in (5.23) is null. From (5.7)–(5.11), we infer that

$$\varphi_{tt} + 2c\varphi_{xt} + c^2\varphi_{xx} = 2(1-\rho)c^2 > 0 \quad \text{for } x \in (\eta/2, 2\pi - \eta/2),\ t \in (0,T),$$
$$-(\varphi_t + c\varphi_x) \geqslant 2c(\rho c T - 2\pi - \delta) > 0 \quad \text{for } x \in (0,2\pi),\ t = T,$$
$$\varphi_t + c\varphi_x \geqslant 2c\delta > 0 \quad \text{for } x \in (0,2\pi),\ t = 0.$$

Thus

$$\int_0^T \int_{\mathbb{T}} s|v|^2\,dx\,dt + \int_{\mathbb{T}} s\big(|v|^2_{|t=0} + |v|^2_{|t=T}\big)\,dx \leqslant C\left(\int_0^T \int_{\mathbb{T}} \left|e^{s\varphi}Pw\right|^2 dx\,dt + \int_0^T \int_{\omega} s|v|^2\,dx\,dt\right),$$

which gives at once (5.21) by replacing v by $e^{s\varphi}w$. The proof of Lemma 5.5 is achieved when $w \in H^1(\mathbb{T}\times(0,T))$.

We now claim that Lemma 5.5 is still true when w and $f := w_t + cw_x$ are in $L^2(0,T;L^2(\mathbb{T}))$. Indeed, in that case $w \in C([0,T];L^2(\mathbb{T}))$, and if (w_0^n) and (f^n) are two sequences in $H^1(\mathbb{T})$ and $L^2(0,T;H^1(\mathbb{T}))$, respectively, such that

$$w_0^n \to w(0) \quad \text{in } L^2(\mathbb{T}),$$
$$f^n \to f \quad \text{in } L^2\big(0,T;L^2(\mathbb{T})\big),$$

then the solution $w^n \in C([0,T];H^1(\mathbb{T}))$ of

$$w_t^n + cw_x^n = f^n,$$
$$w^n(0) = w_0^n$$

satisfies $w^n \in H^1(\mathbb{T}\times(0,T))$ and $w^n \to w$ in $C([0,T];L^2(\mathbb{T}))$, so that we can apply (5.21) to w^n and next pass to the limit $n \to \infty$ in (5.21). The proof of Lemma 5.5 is complete. □

Let us complete the proof of Proposition 5.3. Let $u \in L^2(0,T;H^2(\mathbb{T}))$ satisfy (5.1), and let $w = u - u_{xx} \in L^2(0,T;L^2(\mathbb{T}))$. Then $w_t + cw_x = (c-q)u_x \in L^2(0,T;L^2(\mathbb{T}))$. Combining (5.4), (5.5), (5.14) (multiplied by $e^{-2s\rho c^2 t^2}$ and next integrated over $(0,T)$), and (5.21), we obtain for $s \geqslant s_1$ that

$$\int_0^T \int_{\mathbb{T}} \left[s|u_x|^2 + s^3|u|^2 + s|u-u_{xx}|^2\right]e^{2s\varphi}\,dx\,dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2 e^{2s\varphi}\right]_{|t=0}\,dx$$
$$\leqslant C\left(\int_0^T \int_{\mathbb{T}} \left[|u_{xx}|^2 + |(c-q)u_x|^2\right]e^{2s\varphi}\,dx\,dt + \int_0^T \int_{\omega} \left[s|u-u_{xx}|^2 + s^3|u|^2\right]e^{2s\varphi}\,dx\,dt\right). \quad (5.24)$$

Then choosing $s_2 \geqslant s_1$ and $C_2 > C$ large enough, we obtain (5.12) for any $s \geqslant s_2$ and any $u \in L^2(0,T;H^2(\mathbb{T}))$ satisfying (5.1). □

We are now in a position to prove Theorem 5.1. Pick any function u fulfilling (5.1) and (5.2). If $u \in L^2(0,T;H^2(\mathbb{T}))$, then it follows from (5.12) that $u \equiv 0$ in $\mathbb{T}\times(0,T)$.

Assume now that $u \in L^\infty(0,T;H^1(\mathbb{T}))$. We proceed as in [39]. Since u and $w := u - u_{xx}$ are not regular enough to apply Lemmas 5.4 and 5.5, we smooth them by using some convolution in time. For any function $v = v(x,t)$ and any number $h > 0$, we set

$$v^{[h]}(x,t) = \frac{1}{h}\int_t^{t+h} v(x,s)\,ds.$$

Recall that if $v \in L^p(0,T;V)$, where $1 \leqslant p \leqslant +\infty$ and V denotes any Banach space, then $v^{[h]} \in W^{1,p}(0,T-h;V)$, $\|v^{[h]}\|_{L^p(0,T-h;V)} \leqslant \|v\|_{L^p(0,T;V)}$, and for $p<\infty$ and $T'<T$

$$v^{[h]} \to v \quad \text{in } L^p\big(0,T';V\big) \quad \text{as } h \to 0.$$

In the sequel, $v_t^{[h]}$ denotes $(v^{[h]})_t$, $v_x^{[h]}$ denotes $(v^{[h]})_x$, etc. Assume again that $c>0$. Pick any $T' \in (\frac{2\pi}{c},T)$, any pair (ρ,δ) such that (5.7) still holds with T replaced by T', and define the functions ψ and φ as above. Then for any positive number $h < h_0 := T - T'$, $u^{[h]} \in W^{1,\infty}(0,T';H^1(\mathbb{T}))$, and it solves

$$u_t^{[h]} - u_{txx}^{[h]} - cu_{xxx}^{[h]} + (qu_x)^{[h]} = 0 \quad \text{in } \mathcal{D}'\big(0,T';H^{-2}(\mathbb{T})\big), \quad (5.25)$$
$$u^{[h]}(x,t) = 0, \quad (x,t) \in \omega \times \big(0,T'\big). \quad (5.26)$$

From (5.25), we infer that

$$u_{xxx}^{[h]} = c^{-1}\big(u_t^{[h]} - u_{txx}^{[h]} + (qu_x)^{[h]}\big) \in L^\infty\big(0,T';H^{-1}(\mathbb{T})\big),$$

hence

$$u^{[h]} \in L^\infty\big(0,T';H^2(\mathbb{T})\big). \quad (5.27)$$

This yields, with (5.4)–(5.5),

$$w^{[h]} = u^{[h]} - u_{xx}^{[h]} \in L^\infty\big(0,T';L^2(\mathbb{T})\big), \quad (5.28)$$
$$w_t^{[h]} + cw_x^{[h]} = \big((c-q)u_x\big)^{[h]} \in W^{1,\infty}\big(0,T;L^2(\mathbb{T})\big). \quad (5.29)$$

From (5.26)–(5.29) and Lemmas 5.4 and 5.5, we infer that there exist some constants $s_1 > 0$ and $C_1 > 0$ such that for all $s \geqslant s_1$ and all $h \in (0,h_0)$, we have

$$\begin{aligned}
&\int_0^{T'}\int_{\mathbb{T}} \left(s\left|u_x^{[h]}\right|^2 + s^3\left|u^{[h]}\right|^2 + s\left|w^{[h]}\right|^2\right)e^{2s\varphi}\,dx\,dt \\
&\quad\leqslant C_1\int_0^{T'}\int_{\mathbb{T}} \left(\left|u^{[h]}\right|^2 + \left|w^{[h]}\right|^2 + \left|\bigl((c-q)u_x\bigr)^{[h]}\right|^2\right)e^{2s\varphi}\,dx\,dt \\
&\quad\leqslant C_1\int_0^{T'}\int_{\mathbb{T}} \left(\left|u^{[h]}\right|^2 + \left|w^{[h]}\right|^2 + 2\left|(c-q)u_x^{[h]}\right|^2\right. \\
&\qquad\left. + 2\left|\bigl((c-q)u_x\bigr)^{[h]} - (c-q)u_x^{[h]}\right|^2\right)e^{2s\varphi}\,dx\,dt. \qquad (5.30)
\end{aligned}$$

Comparing the powers of s in (5.30), we obtain that for $s \geqslant s_3 > s_1$, $h \in (0, h_0)$ and some constant $C_3 > C_1$ (that does not depend on s, h)

$$\begin{aligned}
&\int_0^{T'}\int_{\mathbb{T}} \left(s\left|u_x^{[h]}\right|^2 + s^3\left|u^{[h]}\right|^2 + s\left|w^{[h]}\right|^2\right)e^{2s\varphi}\,dx\,dt \\
&\quad\leqslant C_3\int_0^{T'}\int_{\mathbb{T}} \left|\bigl((c-q)u_x\bigr)^{[h]} - (c-q)u_x^{[h]}\right|^2 e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

Fix s to the value s_3, and let $h \to 0$. We claim that

$$\int_0^{T'}\int_{\mathbb{T}} \left|\bigl((c-q)u_x\bigr)^{[h]} - (c-q)u_x^{[h]}\right|^2 e^{2s_3\varphi}\,dx\,dt \to 0 \quad \text{as } h \to 0.$$

Indeed, as $h \to 0$,

$$\begin{aligned}
&\bigl((c-q)u_x\bigr)^{[h]} \to (c-q)u_x \quad \text{in } L^2\bigl(0,T';L^2(\mathbb{T})\bigr), \\
&(c-q)u_x^{[h]} \to (c-q)u_x \quad \text{in } L^2\bigl(0,T';L^2(\mathbb{T})\bigr),
\end{aligned}$$

while $e^{2s_3\varphi} \in L^\infty(\mathbb{T}\times(0,T'))$. Therefore,

$$\int_0^{T'}\int_{\mathbb{T}} \left|u^{[h]}\right|^2 e^{2s_3\varphi}\,dx\,dt \to 0 \quad \text{as } h \to 0.$$

On the other hand, $u^{[h]} \to u$ in $L^2(0,T';L^2(\mathbb{T}))$, hence

$$\int_0^{T'}\int_{\mathbb{T}} \left|u^{[h]}\right|^2 e^{2s_3\varphi}\,dx\,dt \to \int_0^{T'}\int_{\mathbb{T}} |u|^2 e^{2s_3\varphi}\,dx\,dt$$

as $h \to 0$. We conclude that $u \equiv 0$ in $\mathbb{T}\times(0,T')$. As T' may be taken arbitrarily close to T, we infer that $u \equiv 0$ in $\mathbb{T}\times(0,T)$, as desired. The proof of Theorem 5.1 is complete. $\square$

6. Control and stabilization of the KdV–BBM equation

In this section we are concerned with the control properties of the system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x + uu_x = a(x)h, \quad x \in \mathbb{T},\ t \geqslant 0, \tag{6.1}$$

$$u(x,0) = u_0(x), \tag{6.2}$$

where $c \in \mathbb{R} \setminus \{0\}$ and $a \in C^\infty(\mathbb{T})$ is a given nonzero function. Let

$$\omega = \{x \in \mathbb{T};\ a(x) \neq 0\} \neq \emptyset. \tag{6.3}$$

6.1. Exact controllability

The first result is a local controllability result in large time.

Theorem 6.1. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$, $s \geqslant 0$ and $T > 2\pi/|c|$. Then there exists a $\delta > 0$ such that for any $u_0, u_T \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} + \|u_T\|_{H^s} < \delta,$$

one can find a control input $h \in L^2(0,T;H^{s-2}(\mathbb{T}))$ such that the system (6.1)–(6.2) admits a unique solution $u \in C([0,T],H^s(\mathbb{T}))$ satisfying $u(\cdot,T) = u_T$.

Proof. The result is first proved for the linearized equation, and next extended to the nonlinear one by a fixed-point argument.

Step 1. Exact controllability of the linearized system.

We first consider the exact controllability of the linearized system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x = a(x)h, \tag{6.4}$$

$$u(x,0) = u_0(x), \tag{6.5}$$

in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. Let $A = (1-\partial_x^2)^{-1}(c\partial_x^3 - (c+1)\partial_x)$ with domain $D(A) = H^{s+1}(\mathbb{T}) \subset H^s(\mathbb{T})$. The operator A generates a group of isometries $\{W(t)\}_{t\in\mathbb{R}}$ in $H^s(\mathbb{T})$, with

$$W(t)v = \sum_{k=-\infty}^{\infty} e^{-it\frac{ck^3+(c+1)k}{k^2+1}} \hat{v}_k e^{ikx} \tag{6.6}$$

for any

$$v = \sum_{k=-\infty}^{\infty} \hat{v}_k e^{ikx} \in H^s(\mathbb{T}).$$

The system (6.4)–(6.5) may be cast into the following integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)h(\tau)\right]d\tau.$$

We proceed as in [31]. Take $h(x,t)$ in (6.4) to have the following form

$$h(x,t) = a(x) \sum_{j=-\infty}^{\infty} f_j q_j(t) e^{ijx} \tag{6.7}$$

where f_j and $q_j(t)$ are to be determined later. Then the solution u of Eq. (6.4) can be written as

$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx}$$

where $\hat{u}_k(t)$ solves

$$\frac{d}{dt}\hat{u}_k(t) + ik\sigma(k)\hat{u}_k(t) = \frac{1}{1+k^2} \sum_{j=-\infty}^{\infty} f_j q_j(t) m_{j,k} \tag{6.8}$$

with $\sigma(k) = \frac{ck^2+c+1}{1+k^2}$, and

$$m_{j,k} = \frac{1}{2\pi} \int_{\mathbb{T}} a^2(x) e^{i(j-k)x}\, dx.$$

Thus

$$\hat{u}_k(T) - e^{-ik\sigma(k)T}\hat{u}_k(0) = \frac{1}{1+k^2} \sum_{j=-\infty}^{\infty} f_j m_{j,k} \int_0^T e^{-ik\sigma(k)(T-\tau)} q_j(\tau)\, d\tau$$

or

$$\hat{u}_k(T) e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2} \sum_{j=-\infty}^{\infty} f_j m_{j,k} \int_0^T e^{ik\sigma(k)\tau} q_j(\tau)\, d\tau.$$

It may occur that the eigenvalues

$$\lambda_k = ik\sigma(k), \quad k \in \mathbb{Z}$$

are not all different. If we count only the distinct values, we obtain the sequence $(\lambda_k)_{k\in\mathbb{I}}$, where $\mathbb{I} \subset \mathbb{Z}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1, k_2 \in \mathbb{I}$ with $k_1 \neq k_2$. For each $k_1 \in \mathbb{Z}$ set

$$I(k_1) = \{k \in \mathbb{Z};\ k\sigma(k) = k_1\sigma(k_1)\}$$

and $m(k_1) = |I(k_1)|$ (the number of elements in $I(k_1)$). Clearly, there exists some integer k^* such that $k \in \mathbb{I}$ if $|k| > k^*$. Thus there are only finite many integers in $\mathbb{I}$, say k_j, $j = 1, \dots, n$, such that one can find another integer $k \neq k_j$ with $\lambda_k = \lambda_{k_j}$. Let

$$\mathbb{I}_j = \{k \in \mathbb{Z};\ k \neq k_j,\ \lambda_k = \lambda_{k_j}\}, \quad j = 1, 2, \dots, n.$$

Then

$$\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \cdots \cup \mathbb{I}_n.$$

Note that $\mathbb{I}_j$ contains at most two integers, for $m(k_j) \leqslant 3$. We write

$$\mathbb{I}_j = \{k_{j,1}, k_{j,m(k_j)-1}\}, \quad j = 1, 2, \ldots, n$$

and rewrite k_j as $k_{j,0}$. Let

$$p_k(t) := e^{-ik\sigma(k)t}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Then the set

$$\mathcal{P} := \left\{ p_k(t);\ k \in \mathbb{I} \right\}$$

forms a Riesz basis for its closed span, $\mathcal{P}_T$, in $L^2(0, T)$ if

$$T > \frac{2\pi}{|c|}.$$

Let $\mathcal{L} := \{q_j(t);\ j \in \mathbb{I}\}$ be the unique dual Riesz basis for $\mathcal{P}$ in $\mathcal{P}_T$; that is, the functions in $\mathcal{L}$ are the unique elements of $\mathcal{P}_T$ such that

$$\int_0^T q_j(t) \overline{p_k(t)}\, dt = \delta_{kj}, \quad j, k \in \mathbb{I}.$$

In addition, we choose

$$q_k = q_{k_j} \quad \text{if } k \in \mathbb{I}_j.$$

For such choice of $q_j(t)$, we have then, for any $k \in \mathbb{Z}$,

$$\hat{u}_k(T) e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2} f_k m_{k,k} \quad \text{if } k \in \mathbb{I} \setminus \{k_1, \ldots, k_n\}; \tag{6.9}$$

$$\hat{u}_{k_{j,q}}(T) e^{ik_j\sigma(k_j)T} - \hat{u}_{k_{j,q}}(0) = \frac{1}{1+k_{j,q}^2} \sum_{l=0}^{m(k_j)-1} f_{k_{j,l}} m_{k_{j,l},k_{j,q}}$$
$$\text{if } k = k_{j,q},\ j = 1, \ldots, n,\ q = 0, \ldots, m(k_j) - 1. \tag{6.10}$$

It is well known that for any finite set $\mathcal{J} \subset \mathbb{Z}$, the Gram matrix $A_{\mathcal{J}} = (m_{p,q})_{p,q \in \mathcal{J}}$ is definite positive, hence invertible. It follows that the system (6.9)–(6.10) admits a unique solution $\vec{f}(\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$. Since

$$m_{k,k} = \frac{1}{2\pi} \int_{\mathbb{T}} a^2(x)\, dx =: \mu \neq 0,$$

we have that

$$f_k = \frac{1+k^2}{\mu}\big(\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0)\big) \quad \text{for } |k| > k^*.$$

Note that

$$\begin{aligned}
\|h\|^2_{L^2(0,T;H^{s-2}(\mathbb{T}))} &= \int_0^T \bigg\| a(x) \sum_{j=-\infty}^{\infty} f_j q_j(t) e^{ijx} \bigg\|^2_{H^{s-2}} dt \\
&\leqslant C \int_0^T \sum_{j=-\infty}^{\infty} \big(1+j^2\big)^{s-2} \big|f_j q_j(t)\big|^2 \\
&\leqslant C \sum_{j=-\infty}^{\infty} \big(1+j^2\big)^{s-2} |f_j|^2 \\
&\leqslant C\big(\big\|u(0)\big\|^2_{H^s} + \big\|u(T)\big\|^2_{H^s}\big).
\end{aligned}$$

This analysis leads us to the following controllability result for the linear system (6.4)–(6.5).

Proposition 6.2. *Let $s \in \mathbb{R}$ and $T > \frac{2\pi}{|c|}$ be given. For any $u_0, u_T \in H^s(\mathbb{T})$, there exists a control $h \in L^2(0,T;H^{s-2}(\mathbb{T}))$ such that the system (6.4)–(6.5) admits a unique solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfying*

$$u(x,T) = u_T(x).$$

Moreover, there exists a constant $C > 0$ depending only on s and T such that

$$\|h\|_{L^2(0,T;H^{s-2}(\mathbb{T}))} \leqslant C\big(\|u_0\|_{H^s} + \|u_T\|_{H^s}\big).$$

Introduce the (bounded) operator $\Phi : H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to L^2(0,T;H^{s-2}(\mathbb{T}))$ defined by

$$\Phi(u_0,u_T)(t) = h(t),$$

where h is given by (6.7) and $\vec{f}$ is the solution of (6.9)–(6.10) with $(\hat{u}_0)_k$ and $(\hat{u}_T)_k$ substituted to $\hat{u}_k(0)$ and $\hat{u}_k(T)$, respectively.

Then $h = \Phi(u_0,u_T)$ is a control driving the solution u of (6.4)–(6.5) from u_0 at $t=0$ to u_T at $t=T$.

STEP 2. LOCAL EXACT CONTROLLABILITY OF THE BBM EQUATION.

We proceed as in [36]. Pick any time $T > 2\pi/|c|$, and any $u_0, u_T \in H^s(\mathbb{T})$ $(s \geqslant 0)$ satisfying

$$\|u_0\|_{H^s} \leqslant \delta, \qquad \|u_T\|_{H^s} \leqslant \delta$$

with δ to be determined. For any $u \in C([0,T];H^s(\mathbb{T}))$, we set

$$\omega(u) = -\int_0^T W(T-\tau)\big(1-\partial_x^2\big)^{-1}(uu_x)(\tau)\,d\tau.$$

Then

$$\left\|\omega(u)-\omega(v)\right\|_{H^s}\leqslant CT\|u+v\|_{L^\infty(0,T;H^s(\mathbb{T}))}\|u-v\|_{L^\infty(0,T;H^s(\mathbb{T}))}.$$

Furthermore,

$$W(t)u_0+\int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0,u_T-\omega(u)\right)-uu_x\right](\tau)\,d\tau$$
$$=\begin{cases}u_0 & \text{if } t=0,\\ \omega(u)+(u_T-\omega(u))=u_T & \text{if } t=T.\end{cases}$$

We are led to consider the nonlinear map

$$\Gamma(u)=W(t)u_0+\int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0,u_T-\omega(u)\right)-uu_x\right](\tau)\,d\tau.$$

The proof of Theorem 6.1 will be complete if we can show that the map Γ has a fixed point in some closed ball of the space $C([0,T];H^s(\mathbb{T}))$. For any $R>0$, let

$$B_R=\left\{u\in C\left([0,T];H^s(\mathbb{T})\right);\ \|u\|_{C([0,T];H^s(\mathbb{T}))}\leqslant R\right\}.$$

From the above calculations, we see that there exist two positive constants C_1,C_2 (depending on s and T, but not on R, $\|u_0\|_{H^s}$ or $\|u_T\|_{H^s}$) such that for all $u,v\in B_R$

$$\left\|\Gamma(u)\right\|_{C([0,T];H^s(\mathbb{T}))}\leqslant C_1\left(\|u_0\|_{H^s}+\|u_T\|_{H^s}\right)+C_2R^2,$$
$$\left\|\Gamma(u)-\Gamma(v)\right\|_{C([0,T];H^s(\mathbb{T}))}\leqslant C_2R\|u-v\|_{C([0,T];H^s(\mathbb{T}))}.$$

Picking $R=(2C_2)^{-1}$ and $\delta=(8C_1C_2)^{-1}$, we obtain for u_0,u_T satisfying

$$\|u_0\|_{H^s}\leqslant\delta,\qquad \|u_T\|_{H^s}\leqslant\delta$$

and $u,v\in B_R$ that

$$\left\|\Gamma(u)\right\|_{C([0,T];H^s(\mathbb{T}))}\leqslant R, \tag{6.11}$$

$$\left\|\Gamma(u)-\Gamma(v)\right\|_{C([0,T];H^s(\mathbb{T}))}\leqslant\frac{1}{2}\|u-v\|_{C([0,T];H^s(\mathbb{T}))}. \tag{6.12}$$

It follows from the contraction mapping theorem that Γ has a unique fixed point u in B_R. Then u satisfies (6.1)–(6.2) with $h=\Phi(u_0,u_T-\omega(u))$ and $u(T)=u_T$, as desired. The proof of Theorem 6.1 is complete. □

6.2. Exponential stabilizability

We are now concerned with the stabilization of (6.1)–(6.2) with a feedback law $h = h(u)$. To guess the expression of h, it is convenient to write the linearized system (6.4)–(6.5) as

$$u_t = Au + Bk, \tag{6.13}$$

$$u(0) = u_0 \tag{6.14}$$

where $k(t) = (1 - \partial_x^2)^{-1} h(t) \in L^2(0, T; H^s(\mathbb{T}))$ is the new control input, and

$$B = \left(1 - \partial_x^2\right)^{-1} a\left(1 - \partial_x^2\right) \in \mathcal{L}\left(H^s(\mathbb{T})\right). \tag{6.15}$$

We already noticed that A is skew-adjoint in $H^s(\mathbb{T})$, and that (6.13)–(6.14) is exactly controllable in $H^s(\mathbb{T})$ (with some control functions $k \in L^2(0, T; H^s(\mathbb{T}))$) for any $s \geqslant 0$. If we choose the simple feedback law

$$k = -B^{*,s} u, \tag{6.16}$$

the resulting closed-loop system

$$u_t = Au - BB^{*,s} u, \tag{6.17}$$

$$u(0) = u_0 \tag{6.18}$$

is exponentially stable in $H^s(\mathbb{T})$ (see e.g. [27,37]). In (6.16), $B^{*,s}$ denotes the adjoint of B in $\mathcal{L}(H^s(\mathbb{T}))$. Easy computations show that

$$B^{*,s} u = \left(1 - \partial_x^2\right)^{1-s} a\left(1 - \partial_x^2\right)^{s-1} u. \tag{6.19}$$

In particular

$$B^{*,1} u = au.$$

Let $\tilde{A} = A - BB^{*,1}$, where $(BB^{*,1})u = (1 - \partial_x^2)^{-1}[a(1 - \partial_x^2)(au)]$. Since $BB^{*,1} \in \mathcal{L}(H^s(\mathbb{T}))$ and A is skew-adjoint in $H^s(\mathbb{T})$, $\tilde{A}$ is the infinitesimal generator of a group $\{W_a(t)\}_{t \in \mathbb{R}}$ on $H^s(\mathbb{T})$ (see e.g. [34, Theorem 1.1, p. 76]). We first show that the closed-loop system (6.17)–(6.18) is exponentially stable in $H^s(\mathbb{T})$ for all $s \geqslant 1$.

Lemma 6.3. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$. Then there exists a constant $\gamma > 0$ such that for any $s \geqslant 1$, one can find a constant $C_s > 0$ for which the following holds for all $u_0 \in H^s(\mathbb{T})$*

$$\left\| W_a(t) u_0 \right\|_{H^s} \leqslant C_s e^{-\gamma t} \| u_0 \|_{H^s} \quad \textit{for all } t \geqslant 0. \tag{6.20}$$

Proof. (6.20) is well known for $s = 1$ (see e.g. [27]). Assume that it is true for some $s \in \mathbb{N}^*$, and pick any $u_0 \in H^{s+1}(\mathbb{T})$. Let $v_0 = \tilde{A} u_0 \in H^s(\mathbb{T})$. Then

$$\left\| W_a(t) v_0 \right\|_{H^s} \leqslant C_s e^{-\gamma t} \| v_0 \|_{H^s}.$$

Clearly,

$$W_a(t) v_0 = \tilde{A} W_a(t) u_0 = A W_a(t) u_0 - BB^{*,1} W_a(t) u_0,$$

hence

$$\left\|AW_a(t)u_0\right\|_{H^s} \leqslant \left\|W_a(t)v_0\right\|_{H^s} + \left\|BB^{*,1}\right\|_{\mathcal{L}(H^s)}\left\|W_a(t)u_0\right\|_{H^s} \leqslant Ce^{-\gamma t}\|u_0\|_{H^{s+1}}.$$

Therefore

$$\left\|W_a(t)u_0\right\|_{H^{s+1}} \leqslant C_{s+1}e^{-\gamma t}\|u_0\|_{H^{s+1}},$$

as desired. The estimate (6.20) is thus proved for any $s \in \mathbb{N}^*$. It may be extended to any $s \in [1, +\infty)$ by interpolation. □

Plugging the feedback law $k = -B^{*,1}u = -au$ in the nonlinear equation gives the following closed-loop system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x + uu_x = -a\big(1-\partial_x^2\big)[au], \tag{6.21}$$

$$u(x,0) = u_0(x). \tag{6.22}$$

We first show that the system (6.21)–(6.22) is globally well-posed in the space $H^s(\mathbb{T})$ for any $s \geqslant 0$.

Theorem 6.4. *Let $s \geqslant 0$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the system (6.21)–(6.22) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$.*

The following bilinear estimate from [42] will be very helpful.

Lemma 6.5. *Let $w \in H^r(\mathbb{T})$ and $v \in H^{r'}(\mathbb{T})$ with $0 \leqslant r \leqslant s$, $0 \leqslant r' \leqslant s$ and $0 \leqslant 2s - r - r' < \frac{1}{4}$. Then*

$$\left\|\big(1-\partial_x^2\big)^{-1}\partial_x(wv)\right\|_{H^s} \leqslant c_{r,r',s}\|w\|_{H^r}\|v\|_{H^{r'}}.$$

In particular, if $w \in H^r(\mathbb{T})$ and $v \in H^s(\mathbb{T})$ with $0 \leqslant r \leqslant s < r + \frac{1}{4}$, then

$$\left\|\big(1-\partial_x^2\big)^{-1}\partial_x(wv)\right\|_{H^s} \leqslant c_{r,s}\|w\|_{H^r}\|v\|_{H^s}.$$

Proof of Theorem 6.4. The proof is divided in three steps.

Step 1. The system is locally well-posed in the space $H^s(\mathbb{T})$:

Let $s \geqslant 0$ and $R > 0$ be given. There exists a T^* depending only on s and R such that for any $u_0 \in H^s(\mathbb{T})$ with

$$\|u_0\|_{H^s} \leqslant R,$$

the system (6.21)–(6.22) admits a unique solution $u \in C([0,T^*]; H^s(\mathbb{T}))$. Moreover, $T^* \to \infty$ as $R \to 0$.

Rewrite (6.21)–(6.22) in its integral form

$$u(t) = W_a(t)u_0 - \int_0^t W_a(t-\tau)\big(1-\partial_x^2\big)^{-1}(uu_x)(\tau)\,d\tau. \tag{6.23}$$

For given $\theta > 0$, define a map Γ on $C([0,\theta]; H^s(\mathbb{T}))$ by

$$\Gamma(v) = W_a(t)u_0 - \int_0^t W_a(t-\tau)\big(1-\partial_x^2\big)^{-1}(vv_x)(\tau)\,d\tau$$

for any $v \in C([0,\theta]; H^s(\mathbb{T}))$. Note that, according to Lemma 6.3 and Lemma 6.5,

$$\big\|W_a(t)u_0\big\|_{C([0,\theta];H^s(\mathbb{T}))} \leqslant C_s\|u_0\|_{H^s},$$

and

$$\begin{aligned}\left\|\int_0^t W_a(t-\tau)\big(1-\partial_x^2\big)^{-1}(vv_x)(\tau)\,d\tau\right\|_{C([0,\theta];H^s(\mathbb{T}))} &\leqslant C_s\theta \sup_{0\leqslant t\leqslant\theta}\big\|\big(1-\partial_x^2\big)^{-1}(vv_x)(t)\big\|_{H^s}\\ &\leqslant \frac{C_s c_{s,s}}{2}\theta\|v\|^2_{C([0,\theta];H^s(\mathbb{T}))}.\end{aligned}$$

Thus, for given $R > 0$ and $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R$, one can choose $T^* = [2c_{s,s}(1+C_s)R]^{-1}$ such that Γ is a contraction mapping in the ball

$$B := \big\{v \in C\big(\big[0,T^*\big]; H^s(\mathbb{T})\big);\ \|v\|_{C([0,T^*];H^s(\mathbb{T}))} \leqslant 2C_sR\big\}$$

whose fixed point u is the desired solution.

Step 2. The system is globally well-posed in the space $H^s(\mathbb{T})$ for any $s \geqslant 1$.

To this end, it suffices to establish the following global *a priori* estimate for smooth solutions of the system (6.21)–(6.22):

Let $s \geqslant 1$ and $T > 0$ be given. There exists a continuous nondecreasing function

$$\alpha_{s,T} : \mathbb{R}^+ \to \mathbb{R}^+$$

such that any smooth solution u of the system (6.21)–(6.22) *satisfies*

$$\sup_{0\leqslant t\leqslant T}\big\|u(\cdot,t)\big\|_{H^s} \leqslant \alpha_{s,T}\big(\|u_0\|_{H^s}\big). \tag{6.24}$$

Estimate (6.24) holds obviously when $s = 1$ because of the energy identity

$$\big\|u(t)\big\|^2_{H^1} - \|u_0\|^2_{H^1} = -2\int_0^t \big\|au(\tau)\big\|^2_{H^1}\,d\tau \quad \forall t \geqslant 0.$$

When $1 < s \leqslant s_1 := 1 + \frac{1}{8}$, applying Lemma 6.3 and Lemma 6.5 to (6.23) yields that for any $0 < t \leqslant T$,

$$\begin{aligned}\big\|u(\cdot,t)\big\|_{H^s} &\leqslant C_s\|u_0\|_{H^s} + \frac{C_s c_{1,s}}{2}\int_0^t \big\|u(\cdot,\tau)\big\|_{H^1}\big\|u(\cdot,\tau)\big\|_{H^s}\,d\tau \\ &\leqslant C\|u_0\|_{H^s} + C\alpha_{1,T}\big(\|u_0\|_{H^1}\big)\int_0^t \big\|u(\cdot,\tau)\big\|_{H^s}\,d\tau.\end{aligned}$$

Estimate (6.24) for $1 < s \leqslant s_1$ follows by using Gronwall's lemma. Similarly, for $s_1 < s \leqslant s_2 := 1 + \frac{2}{8}$,

$$\begin{aligned}\big\|u(\cdot,t)\big\|_{H^s} &\leqslant C_s\|u_0\|_{H^s} + \frac{C_s c_{s_1,s}}{2}\int_0^t \big\|u(\cdot,\tau)\big\|_{H^{s_1}}\big\|u(\cdot,\tau)\big\|_{H^s}\,d\tau \\ &\leqslant C\|u_0\|_{H^s} + C\alpha_{s_1,T}\big(\|u_0\|_{H^{s_1}}\big)\int_0^t \big\|u(\cdot,\tau)\big\|_{H^s}\,d\tau.\end{aligned}$$

Estimate (6.24) thus holds for $1 < s \leqslant s_2$. Continuing this argument, we can show that the estimate (6.24) holds for $1 < s \leqslant s_k := 1 + \frac{k}{8}$ for any $k \geqslant 1$.

Step 3. *The system* (6.21)–(6.22) *is globally well-posed in the space* $H^s(\mathbb{T})$ *for any* $0 \leqslant s < 1$.

To see it is true, as in [42], we decompose any $u_0 \in H^s(\mathbb{T})$ as

$$u_0 = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} = \sum_{|k|\leqslant k_0} + \sum_{|k|>k_0} =: w_0 + v_0$$

with $v_0 \in H^s(\mathbb{T})$ satisfying

$$\|v_0\|_{H^s} \leqslant \delta$$

for some small $\delta > 0$ to be chosen, and $w_0 \in H^1(\mathbb{T})$. Consider the following two initial value problems

$$\begin{cases} v_t - v_{txx} - cv_{xxx} + (c+1)v_x + vv_x = -a\big(1-\partial_x^2\big)[av], \\ v(x,0) = v_0(x) \end{cases} \tag{6.25}$$

and

$$\begin{cases} w_t - w_{txx} - cw_{xxx} + (c+1)w_x + ww_x + (vw)_x = -a\big(1-\partial_x^2\big)[aw], \\ w(x,0) = w_0(x). \end{cases} \tag{6.26}$$

By the local well-posedness established in Step 1, for given $T > 0$, if δ is small enough, then (6.25) admits a unique solution $v \in C([0,T]; H^s(\mathbb{T}))$. For (6.26), with $v \in C([0,T]; H^s(\mathbb{T}))$, by using Lemma 6.3, the estimate

$$\big\|\big(1-\partial_x^2\big)^{-1}\partial_x(wv)\big\|_{H^1} \leqslant C\|wv\|_{L^2} \leqslant C\|w\|_{H^1}\|v\|_{H^s}$$

and the contraction mapping principle, one can show first that it is locally well-posed in the space $H^1(\mathbb{T})$. Then, for any smooth solution w of (6.26) it holds that

$$\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|_{H^1}^2 - \int_{\mathbb{T}} v(x,t)w(x,t)w_x(x,t)\,dx = -\|a(\cdot)w(\cdot,t)\|_{H^1}^2,$$

which implies that

$$\|w(\cdot,t)\|_{H^1}^2 \leqslant \|w_0\|_{H^1}^2 \exp\left(C\int_0^t \|v(\cdot,\tau)\|_{L^2}\,d\tau\right)$$

for any $t \geqslant 0$. The above estimate can be extended to any $w_0 \in H^1(\mathbb{T})$ by a density argument. Consequently, for $w_0 \in H^1(\mathbb{T})$ and $v \in C([0,T]; H^s(\mathbb{T}))$, (6.26) admits a unique solution $w \in C([0,T]; H^1(\mathbb{T}))$. Thus $u = w + v \in C([0,T]; H^s(\mathbb{T}))$ is the desired solution of system (6.21)–(6.22). The proof of Theorem 6.4 is complete. $\square$

Next we show that the system (6.21)–(6.22) is locally exponentially stable in $H^s(\mathbb{T})$ for any $s \geqslant 1$.

Proposition 6.6. *Let $s \geqslant 1$ be given and $\gamma > 0$ be as given in Lemma 6.3. Then there exist two numbers $\delta > 0$ and C_s' depending only on s such that for any $u_0 \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} \leqslant \delta,$$

the corresponding solution u of the system (6.21)–(6.22) satisfies

$$\|u(\cdot,t)\|_{H^s} \leqslant C_s' e^{-\gamma t}\|u_0\|_{H^s} \quad \forall t \geqslant 0.$$

Proof. We proceed as in [35]. As in the proof of Theorem 6.4, rewrite the system (6.21)–(6.22) in its integral form

$$u(t) = W_a(t)u_0 - \frac{1}{2}\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right)(\tau)\,d\tau$$

and consider the map

$$\Gamma(v) := W_a(t)u_0 - \frac{1}{2}\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(v^2\right)(\tau)\,d\tau.$$

For given $s \geqslant 1$, by Lemma 6.3 and Lemma 6.5, there exists a constant $C_s > 0$ such that

$$\begin{aligned}
\|\Gamma(v)(\cdot,t)\|_{H^s} &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2}\int_0^t e^{-\gamma(t-\tau)}\|v(\cdot,\tau)\|_{H^s}^2\,d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2}\sup_{0\leqslant\tau\leqslant t}\|e^{\gamma\tau}v(\cdot,\tau)\|_{H^s}^2 \int_0^t e^{-\gamma(t+\tau)}\,d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma}e^{-\gamma t}\left(1-e^{-\gamma t}\right)\sup_{0\leqslant\tau\leqslant t}\|e^{\gamma\tau}v(\cdot,\tau)\|_{H^s}^2
\end{aligned}$$

for any $t \geqslant 0$. Let us introduce the Banach space

$$Y_s := \Big\{ v \in C\big([0,\infty); H^s(\mathbb{T})\big) \colon \|v\|_{Y_s} := \sup_{0 \leqslant t < \infty} \big\| e^{\gamma t} v(\cdot,t) \big\|_{H^s} < \infty \Big\}.$$

For any $v \in Y_s$,

$$\big\| \Gamma(v) \big\|_{Y_s} \leqslant C_s \|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma} \|v\|_{Y_s}^2.$$

Choose

$$\delta = \frac{\gamma}{4C_s^2 c_{s,s}}, \qquad R = 2C_s \delta.$$

Then, if $\|u_0\| \leqslant \delta$, for any $v \in Y_s$ with $\|v\|_{Y_s} \leqslant R$,

$$\big\| \Gamma(v) \big\|_{Y_s} \leqslant C_s \delta + \frac{C_s c_{s,s}}{2\gamma} (2C_s \delta) R \leqslant R.$$

Moreover, for any $v_1, v_2 \in Y_s$ with $\|v_1\|_{Y_s} \leqslant R$ and $\|v_2\|_{Y_s} \leqslant R$,

$$\big\| \Gamma(v_1) - \Gamma(v_2) \big\|_{Y_s} \leqslant \frac{1}{2} \|v_1 - v_2\|_{Y_s}.$$

The map Γ is a contraction whose fixed point $u \in Y_s$ is the desired solution satisfying

$$\big\| u(\cdot,t) \big\|_{H^s} \leqslant 2C_s e^{-\gamma t} \|u_0\|_{H^s}$$

for any $t \geqslant 0$. $\square$

Now we turn to the issue of the global stability of the system (6.21)–(6.22). First we show that the system (6.21)–(6.22) is globally exponentially stable in the space $H^1(\mathbb{T})$.

Theorem 6.7. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$, and let $\gamma > 0$ be as in Lemma 6.3. Then for any $R_0 > 0$, there exists a constant $C^* > 0$ such that for any $u_0 \in H^1(\mathbb{T})$ with $\|u_0\|_{H^1} \leqslant R_0$, the corresponding solution u of (6.21)–(6.22) satisfies*

$$\big\| u(\cdot,t) \big\|_{H^1} \leqslant C^* e^{-\gamma t} \|u_0\|_{H^1} \quad \textit{for all } t \geqslant 0. \tag{6.27}$$

Theorem 6.7 is a direct consequence of the following observability inequality.

Proposition 6.8. *Let $R_0 > 0$ be given. Then there exist two positive numbers T and β such that for any $u_0 \in H^1(\mathbb{T})$ satisfying*

$$\|u_0\|_{H^1} \leqslant R_0, \tag{6.28}$$

the corresponding solution u of (6.21)–(6.22) satisfies

$$\|u_0\|_{H^1}^2 \leqslant \beta \int_0^T \big\| au(t) \big\|_{H^1}^2 \, dt. \tag{6.29}$$

Indeed, if (6.29) holds, then it follows from the energy identity

$$\|u(t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 - 2\int_0^t \|au(\tau)\|_{H^1}^2\, d\tau \quad \forall t \geqslant 0 \tag{6.30}$$

that

$$\|u(T)\|_{H^1}^2 \leqslant \left(1 - 2\beta^{-1}\right)\|u_0\|_{H^1}^2.$$

Thus

$$\|u(mT)\|_{H^1}^2 \leqslant \left(1 - 2\beta^{-1}\right)^m \|u_0\|_{H^1}^2$$

which gives by the semigroup property

$$\|u(t)\|_{H^1} \leqslant Ce^{-\kappa t}\|u_0\|_{H^1} \quad \text{for all } t \geqslant 0, \tag{6.31}$$

for some positive constants $C = C(R_0)$, $\kappa = \kappa(R_0)$.

Finally, we can replace κ by the γ given in Lemma 6.3. Indeed, let $t' = \kappa^{-1}\log[1 + CR_0\delta^{-1}]$, where δ is as given in Proposition 6.6. Then for $\|u_0\|_{H^1} \leqslant R_0$, $\|u(t')\|_{H^1} < \delta$, hence for all $t \geqslant t'$

$$\|u(t)\|_{H^1} \leqslant C_1'\|u(t')\|_{H^1} e^{-\gamma(t-t')} \leqslant \left(C_1'\delta/R_0\right)\|u_0\|_{H^1} e^{-\gamma(t-t')} \leqslant C^* e^{-\gamma t}\|u_0\|_{H^1}$$

where $C^* = (C_1'\delta/R_0)e^{\gamma t'}$. $\square$

Now we present a proof of Proposition 6.8. Pick for the moment any $T > 2\pi/|c|$ (its value will be specified later on). We prove the estimate (6.29) by contradiction. If (6.29) is not true, then for any $n \geqslant 1$ (6.21)–(6.22) admits a solution $u_n \in C([0,T]; H^1(\mathbb{T}))$ satisfying

$$\|u_n(0)\|_{H^1} \leqslant R_0 \tag{6.32}$$

and

$$\int_0^T \|au_n(t)\|_{H^1}^2\, dt < \frac{1}{n}\|u_{0,n}\|_{H^1}^2 \tag{6.33}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n := \|u_{0,n}\|_{H^1} \leqslant R_0$, one can choose a subsequence of (α_n), still denoted by (α_n), such that $\lim_{n\to\infty}\alpha_n = \alpha$. Note that $\alpha_n > 0$ for all n, by (6.33). Set $v_n = u_n/\alpha_n$ for all $n \geqslant 1$. Then

$$v_{n,t} - v_{n,txx} - cv_{n,xxx} + (c+1)v_{n,x} + \alpha_n v_n v_{n,x} = -a\left(1 - \partial_x^2\right)[av_n] \tag{6.34}$$

and

$$\int_0^T \|av_n\|_{H^1}^2\, dt < \frac{1}{n}. \tag{6.35}$$

Because of

$$\|v_n(0)\|_{H^1} = 1, \tag{6.36}$$

the sequence (v_n) is bounded in $L^\infty(0,T;H^1(\mathbb{T}))$, while $(v_{n,t})$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}))$. From Aubin–Lions' lemma and a diagonal process, we infer that we can extract a subsequence of (v_n), still denoted (v_n), such that

$$v_n \to v \quad \text{in } C\big([0,T];H^s(\mathbb{T})\big) \quad \forall s<1, \tag{6.37}$$

$$v_n \to v \quad \text{in } L^\infty\big(0,T;H^1(\mathbb{T})\big) \quad \text{weak}^* \tag{6.38}$$

for some $v \in L^\infty(0,T;H^1(\mathbb{T})) \cap C([0,T];H^s(\mathbb{T}))$ for all $s<1$. Note that, by (6.37)–(6.38), we have that

$$\alpha_n v_n v_{n,x} \to \alpha v v_x \quad \text{in } L^\infty\big(0,T;L^2(\mathbb{T})\big) \quad \text{weak}^*. \tag{6.39}$$

Furthermore, by (6.35),

$$\int_0^T \|av\|_{H^1}^2\,dt \leqslant \liminf_{n\to\infty} \int_0^T \|av_n\|_{H^1}^2\,dt = 0. \tag{6.40}$$

Thus, v solves

$$v_t - v_{txx} - cv_{xxx} + (c+1)v_x + \alpha v v_x = 0 \quad \text{on } \mathbb{T}\times(0,T), \tag{6.41}$$

$$v = 0 \quad \text{on } \omega\times(0,T), \tag{6.42}$$

where ω is given in (6.3). According to Theorem 5.1, $v \equiv 0$ on $\mathbb{T}\times(0,T)$.

We claim that (v_n) is *linearizable* in the sense of [9]; that is, if (w_n) denotes the sequence of solutions to the linear KdV–BBM equation with the same initial data

$$w_{n,t} - w_{n,txx} - cw_{n,xxx} + (c+1)w_{n,x} = -a\big(1-\partial_x^2\big)[aw_n], \tag{6.43}$$

$$w_n(x,0) = v_n(x,0), \tag{6.44}$$

then

$$\sup_{0\leqslant t\leqslant T} \|v_n(t) - w_n(t)\|_{H^1} \to 0 \quad \text{as } n\to\infty. \tag{6.45}$$

Indeed, if $d_n = v_n - w_n$, then d_n solves

$$d_{n,t} - d_{n,txx} - cd_{n,xxx} + (c+1)d_{n,x} = -a\big(1-\partial_x^2\big)[ad_n] - \alpha_n v_n v_{n,x},$$

$$d_n(0) = 0.$$

Since $\|W_a(t)\|_{\mathcal{L}(H^1(\mathbb{T}))} \leqslant 1$, we have from Duhamel formula that for $t\in[0,T]$

$$\|d_n(t)\|_{H^1} \leqslant \int_0^T \big\|\big(1-\partial_x^2\big)^{-1}(\alpha_n v_n v_{n,x})(\tau)\big\|_{H^1}\,d\tau.$$

Combined with (6.37) and to the fact that $v \equiv 0$, this gives (6.45). By Lemma 6.3, we have that

$$\left\| w_n(t) \right\|_{H^1} \leqslant C_1 e^{-\gamma t} \left\| w_n(0) \right\|_{H^1} \quad \text{for all } t \geqslant 0. \tag{6.46}$$

From (6.46) and the energy identity for (6.43)–(6.44), namely

$$\left\| w_n(t) \right\|_{H^1}^2 - \left\| w_n(0) \right\|_{H^1}^2 = -2 \int_0^t \left\| a w_n(\tau) \right\|_{H^1}^2 \, d\tau, \tag{6.47}$$

we have for $Ce^{-\lambda T} < 1$

$$\left\| w_n(0) \right\|_{H^1}^2 \leqslant 2 \left(1 - C_1^2 e^{-2\gamma T}\right)^{-1} \int_0^T \left\| a w_n(\tau) \right\|_{H^1}^2 \, d\tau. \tag{6.48}$$

Combined with (6.35) and (6.45), this yields $\|v_n(0)\|_{H^1} = \|w_n(0)\|_{H^1} \to 0$, which contradicts (6.36). This completes the proof of Proposition 6.8 and of Theorem 6.7. $\square$

Next we show that the system (6.21)–(6.27) is exponentially stable in the space $H^s(\mathbb{T})$ for any $s \geqslant 1$.

Theorem 6.9. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$ and $\gamma > 0$ be as given in Lemma 6.3. For any given $s \geqslant 1$ and $R_0 > 0$, there exists a constant $C > 0$ depending only on s and R_0 such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R_0$, the corresponding solution u of (6.21)–(6.22) satisfies*

$$\left\| u(\cdot, t) \right\|_{H^s} \leqslant C e^{-\gamma t} \|u_0\|_{H^s} \quad \text{for all } t \geqslant 0. \tag{6.49}$$

Proof. As before, rewrite the system in its integral form

$$u(t) = W_a(t) u_0 - \frac{1}{2} \int_0^t W_a(t - \tau) \left(1 - \partial_x^2\right)^{-1} (u u_x)(\tau) \, d\tau.$$

For $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R_0$, applying Lemma 6.3, Lemma 6.5 and Theorem 6.7 yields that, for any $1 \leqslant s \leqslant 1 + \frac{1}{10}$,

$$\begin{aligned}
\left\| u(\cdot, t) \right\|_{H^s} &\leqslant C_s e^{-\gamma t} \|u_0\|_{H^s} + \frac{C_s c_{1,1,s}}{2} \int_0^t e^{-\gamma(t-\tau)} \left\| u(\cdot, \tau) \right\|_{H^1}^2 \, d\tau \\
&\leqslant C_s e^{-\gamma t} \|u_0\|_{H^s} + \frac{C_s c_{1,1,s} (C^*)^2}{2} \int_0^t e^{-\gamma(t-\tau)} e^{-2\gamma\tau} \|u_0\|_{H^1}^2 \, d\tau \\
&\leqslant \left(C_s + \frac{C_s c_{1,1,s} (C^*)^2}{2\gamma} \|u_0\|_{H^1} \right) e^{-\gamma t} \|u_0\|_{H^s}
\end{aligned}$$

for any $t \geqslant 0$. Thus the estimate (6.49) holds for $1 \leqslant s \leqslant m_1 := 1 + \frac{1}{10}$. Similarly, for $m_1 \leqslant s \leqslant m_2 := 1 + \frac{2}{10}$, we have for $\|u_0\|_{H^s} \leqslant R_0$

$$\begin{aligned}\left\|u(\cdot,t)\right\|_{H^s} &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{m_1,m_1,s}}{2}\int_0^t e^{-\gamma(t-\tau)}\left\|u(\cdot,\tau)\right\|_{H^{m_1}}^2\,d\tau \\ &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + C(s,m_1,R_0)\int_0^t e^{-\gamma(t-\tau)}e^{-2\gamma\tau}\|u_0\|_{H^{m_1}}^2\,d\tau \\ &\leqslant \left(C_s + C(s,m_1,R_0)\|u_0\|_{H^{m_1}}\gamma^{-1}\right)e^{-\gamma t}\|u_0\|_{H^s}.\end{aligned}$$

Thus the estimate (6.49) holds for $1 \leqslant s \leqslant m_2 := 1 + \frac{2}{10}$. Repeating this argument yields that the estimate (6.49) holds for $1 \leqslant s \leqslant m_k := 1 + \frac{k}{10}$ for $k = 1, 2, \ldots$. $\square$

Acknowledgments

The authors would like to thank E. Zuazua for having brought to their attention Ref. [1], and anonymous referees for their very valuable comments. L.R. was partially supported by the Agence Nationale de la Recherche, Project CISIFS, grant ANR-09-BLAN-0213-02. B.-Y.Z. was partially supported by a grant from the Simons Foundation (#201615 to Bing-Yu Zhang).

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