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Global weak solutions of 3D compressible MHD with discontinuous initial data and vacuum ☆

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ABSTRACT

In this paper, we study the global existence of weak solutions to the Cauchy problem of the three-dimensional equations for compressible isentropic magnetohydrodynamic flows subject to discontinuous initial data. It is assumed here that the initial energy is suitably small in L^2, and that the initial density and the gradients of initial velocity/magnetic field are bounded in L^∞ and L^2, respectively. This particularly implies that the initial data may contain vacuum states and the oscillations of solutions could be arbitrarily large. As a byproduct, we also prove the global existence of smooth solutions with strictly positive density and small initial energy.

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of a conducting fluid (plasma) in an electromagnetic field with a very wide range of applications. Because the dynamic motion of the fluid and the magnetic field interact on each other and both the hydrodynamic and electrodynamic effects in the motion are strongly coupled, the problems of MHD system are considerably complicated. The governing equations for the motion of three-dimensional compressible isentropic magnetohydrodynamic

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flows, derived from fluid mechanics with appropriate modifications to account for electrical forces, have the following form (see, e.g., [2,20]):

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = (\nabla \times H) \times H + \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \tag{1.2}$$

$$H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \qquad \operatorname{div} H = 0, \tag{1.3}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the spatial variable and $t \geqslant 0$ is the time. The unknown functions ρ, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$, $P = P(\rho)$ and $H = (H^1, H^2, H^3) \in \mathbb{R}^3$ are the density, velocity, pressure and magnetic field, respectively. The constants μ and λ are the shear and bulk viscosity coefficients of the flow and satisfy the physical restrictions:

$$\mu > 0 \quad \text{and} \quad \lambda + \frac{2}{3}\mu \geqslant 0. \tag{1.4}$$

The constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields. Here we only consider the isentropic flows in which the equation of state reads

$$P(\rho) = A\rho^\gamma \quad \text{with } A > 0 \text{ and } \gamma > 1 \tag{1.5}$$

where $A > 0$ and $\gamma > 1$ are some physical parameters.

In this paper, we are interested in an initial value problem of (1.1)–(1.5) subject to the following initial conditions:

$$(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x) \quad \text{for all } x \in \mathbb{R}^3, \tag{1.6}$$

and the far-field behavior:

$$\rho(x,t) \to \tilde{\rho} > 0, \qquad (u, H)(x,t) \to (0,0) \quad \text{as } |x| \to \infty, \tag{1.7}$$

where $\tilde{\rho} > 0$ is the fixed reference density.

There have been numerous studies on the MHD problem by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges, see, for example, [2,3,9,12,13,20] and the references therein. In particular, if there is no electromagnetic effect, i.e. $H \equiv 0$, then (1.1)–(1.5) reduce to the compressible Navier–Stokes equations for isentropic flows, which have also been studied by many people, see, for example, [4,6,7,11,16–18] among others. For multi-dimensional compressible MHD flows, Kawashima [13] first considered the global existence of smooth solutions when the initial data are close to a non-vacuum equilibrium in H^3-norm. In the Lions' framework [16] (see also Feireisl et al. [5]), Hu and Wang [9] studied the global existence of weak solutions, the so-called finite energy weak solution, to the compressible MHD equations with general initial data and suitably large adiabatic exponent γ when the initial energy is merely finite. The solution obtained in [9] may have large oscillations and contain vacuum, however, it possess rather little regularity and only satisfies the equations in a very weak sense. Recently, assume that the viscosity coefficients μ and λ fulfill the following additional/non-physical conditions:

$$0 < \mu < \xi \triangleq 2\mu + \lambda < \left(\frac{3}{2} + \frac{\sqrt{21}}{6}\right)\mu. \tag{1.8}$$

Suen and Hoff [19] proved the global existence of weak solutions of (1.1)–(1.3) when the initial data (ρ_0, u_0, H_0) satisfies

$$\begin{cases} \left\|(\rho_0 - \tilde{\rho}, u_0, H_0)\right\|_{L^2} \quad \text{is sufficiently small}, \\ 0 < \underline{\rho} \leqslant \inf \rho_0(x) \leqslant \sup \rho_0(x) \leqslant \bar{\rho} < \infty, \\ \|u_0\|_{L^p} + \|H_0\|_{L^p} < \infty \quad \text{for some } p > 6. \end{cases} \tag{1.9}$$

It is worth mentioning that the additional restriction (1.8) on viscosity coefficients seems unsatisfactory and non-physical, and moreover, the positive lower bound of initial density in $(1.9)_2$ indicates there is absent of vacuum initially.

It is well known that the discontinuous solutions (namely, weak solutions) are fundamental and important in both the physical and mathematical theory. Moreover, as emphasized in many papers (see, e.g. [8,15,16,21]), the possible presence of vacuum is one of the major difficulties in the study of mathematical theory of compressible fluids. So, the main purpose of this paper is to study the global existence and large-time behavior of weak solutions of (1.1)–(1.7) when the initial density may contain vacuum states. A great deal of information on the partial regularity of velocity, vorticity and magnetic field will be also obtained. Our study is mainly motivated by a recent paper due to Huang, Li and Xin [11], where the authors established the global well-posedness of classical solution with large oscillations and vacuum to the Cauchy problem of three-dimensional Navier–Stokes equations for compressible isentropic flows in a very technical and subtle way.

To state the main results in a precise way, we first introduce some notations and conventions which will be used throughout the paper. Let

$$\int f \, \mathrm{d}x = \int_{\mathbb{R}^3} f \, \mathrm{d}x \quad \text{and} \quad \partial_i f \triangleq \frac{\partial f}{\partial x_i}.$$

For $k \in \mathbb{Z}^+$ and $r > 1$, the standard homogeneous and inhomogeneous Sobolev spaces for scalar/vector functions are denoted by (see, e.g. [1]):

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad D^{k,r} = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \;\middle|\; \left\|\nabla^k u\right\|_{L^r} < \infty\right\}, \quad W^{k,r} = L^r \cap D^{k,r}, \\ H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \left\{u \in L^6 \;\middle|\; \|\nabla u\|_{L^2} < \infty\right\}, \quad \|u\|_{D^{k,r}} = \left\|\nabla^k u\right\|_{L^r}. \end{cases}$$

Weak solutions of (1.1)–(1.7) are defined in a usual way.

Definition 1.1. A pair of functions (ρ, u, H) is said to be a weak solution of (1.1)–(1.7) provided that $(\rho - \tilde{\rho}, \rho u, H) \in C([0,\infty); H^{-1}(\mathbb{R}^3))$, $u \in L^2_{\mathrm{loc}}(0,\infty; D^1)$, $H \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; D^1)$, and $\operatorname{div} H(\cdot,t) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$ for $t > 0$. Moreover, the following identities hold for any test function $\psi \in \mathcal{D}(\mathbb{R}^3 \times (t_1, t_2))$ with $t_2 \geqslant t_1 \geqslant 0$ and $j = 1, 2, 3$:

$$\int \rho \psi(x,t) \, \mathrm{d}x \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int (\rho \psi_t + \rho u \cdot \nabla \psi) \, \mathrm{d}x \, \mathrm{d}t,$$

$$\int \rho u^j \psi(x,t) \, \mathrm{d}x \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int \left(\mu \nabla u^j \cdot \nabla \psi + (\mu + \lambda)(\operatorname{div} u) \psi_{x_j}\right) \mathrm{d}x \, \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \int \left(\rho u^j \psi_t + \rho u^j u \cdot \nabla \psi + P(\rho) \psi_{x_j} + \frac{1}{2} |H|^2 \psi_{x_j} - H^j H \cdot \nabla \psi\right) \mathrm{d}x \, \mathrm{d}t,$$

and

$$\int H^j \psi(x,t)\,\mathrm{d}x\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int \left(H^j\psi_t + H^j u\cdot\nabla\psi - u^j H\cdot\nabla\psi - \nu\nabla H^j\cdot\nabla\psi\right)\mathrm{d}x\,\mathrm{d}t.$$

For any given initial data (ρ_0, u_0, H_0), we define the initial energy C_0 as follows:

$$C_0 \triangleq \int \left(\frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|H_0|^2 + G(\rho_0)\right)\mathrm{d}x, \tag{1.10}$$

where $G(\rho)$ is the potential energy density defined by

$$G(\rho) \triangleq \rho\int_{\tilde{\rho}}^{\rho}\frac{P(s)-P(\tilde{\rho})}{s^2}\,\mathrm{d}s. \tag{1.11}$$

It is clear that

$$c_1(\tilde{\rho},\bar{\rho})(\rho-\tilde{\rho})^2 \leqslant G(\rho) \leqslant c_2(\tilde{\rho},\bar{\rho})(\rho-\tilde{\rho})^2 \quad \text{for } \tilde{\rho}>0,\ 0\leqslant\rho\leqslant 2\bar{\rho},$$

where c_1, c_2 are positive constants depending only on $\tilde{\rho}$ and $\bar{\rho}$.

We are now ready to state our main results.

Theorem 1.1. *For given positive numbers M_1, M_2 (not necessarily small) and $\bar{\rho}\geqslant\tilde{\rho}+1$, assume that the initial data (ρ_0, u_0, H_0) satisfies*

$$\begin{cases} 0\leqslant \inf\rho_0 \leqslant \sup\rho_0 \leqslant \bar{\rho}, & \operatorname{div} H_0 = 0, \\ \|\nabla u_0\|_{L^2}^2 \leqslant M_1, & \|\nabla H_0\|_{L^2}^2 \leqslant M_2. \end{cases} \tag{1.12}$$

Then there exists a positive constant ε, depending on μ, λ, ν, γ, A, $\tilde{\rho}$, $\bar{\rho}$, M_1 and M_2, such that if

$$C_0 \leqslant \varepsilon, \tag{1.13}$$

then there exists a weak solution (ρ, u, H) of (1.1)–(1.7) in the sense of Definition 1.1 satisfying

$$0\leqslant\rho(x,t)\leqslant 2\bar{\rho} \quad \text{for all } x\in\mathbb{R}^3,\ t\geqslant 0, \tag{1.14}$$

and

$$\lim_{t\to\infty}\int\left(|\rho-\tilde{\rho}|^p + \rho^{1/2}|u|^4 + |H|^q\right)\mathrm{d}x = 0, \tag{1.15}$$

where $p\in(2,+\infty)$ and $q\in(2,6]$.

Remark 1.1. Compared with the results obtained in [19], the initial vacuum now is allowed. Moreover, it is worth mentioning that D^1 only embeds into L^6, but not into L^p with $p>6$.

Theorem 1.1 will be proved by constructing weak solutions as limits of smooth solutions. Roughly speaking, we first utilize Kawashima's theorem (see Lemma 2.4) to guarantee the local existence of smooth solutions with strictly positive density, then extend the smooth non-vacuum solutions globally in time just under the condition that the initial energy is suitably small (see Theorem 4.1), and finally let the lower bound of the initial density go to zero. So, for the proof of Theorem 1.1, it suffices to derive some global a priori estimates which are independent of the lower bound of density. However, due to the presence of vacuum states, the analysis (especially, the energy estimates of $A_1(T)$, $A_2(T)$) here is completely different than that in [19]. To overcome the difficulties induced by vacuum, we shall make use of some ideas developed in [11]. As that in [11], it turns out that the key step is to obtain the time-independent upper bound of the density. However, because of the influence of magnetic field and its interaction with the hydrodynamic motion, the problem of MHD considered becomes more complicated than that of Navier–Stokes equations. For example, since the material derivative $\dot{u} \triangleq u_t + u \cdot \nabla u$ is strongly associated with the density in the presence of vacuum, some additional difficulties will arise when we deal with the magnetic force $(\nabla \times H) \times H$ and the convection term $\nabla \times (u \times H)$. These difficulties will be circumvented by using Sobolev inequalities and the important relations among the velocity u, pressure P, magnetic field H, vorticity ω and effective viscous flux F (see Lemma 2.2) in a subtle way.

The remainder of this paper is organized as follows. We first collect some useful inequalities and basic results in Section 2. The global-in-time a priori estimates will be proved in Section 3. Finally, Theorem 1.1 will be proved in Section 4 by constructing weak solutions as limits of smooth solutions.

2. Auxiliary lemmas

In this section, we state some auxiliary lemmas, which will be frequently used in the sequel. We start with the well-known Gagliardo–Nirenberg inequality (see, for instance, [1,14]).

Lemma 2.1. *Assume that $f \in H^1$ and $g \in L^q \cap D^{1,r}$ with $q > 1$ and $r > 3$. Then for any $p \in [2,6]$, there exists a positive constant C, depending only on p, q and r, such that*

$$\|f\|_{L^p} \leqslant C\|f\|_{L^2}^{(6-p)/(2p)}\|\nabla f\|_{L^2}^{(3p-6)/2p}, \tag{2.1}$$

$$\|g\|_{L^\infty} \leqslant C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

As it was pointed out in [4–7,16], the effective viscous flux plays an important role in the mathematical theory of compressible fluid dynamics. However, due to the additional presence of magnetic field, we need to define the effective viscous flux in a slightly different manner. More precisely, let F and ω be the (modified) effective flux and vorticity defined by

$$F \triangleq (2\mu + \lambda)\operatorname{div} u - \big(P(\rho) - P(\tilde{\rho})\big) - \frac{1}{2}|H|^2 \quad \text{and} \quad \omega \triangleq \nabla \times u. \tag{2.3}$$

Due to $\operatorname{div} H = 0$, one has

$$(\nabla \times H) \times H = H \cdot \nabla H - \frac{1}{2}\nabla |H|^2.$$

So, it follows from (1.2) that

$$\Delta F = \operatorname{div}(\rho\dot{u} - H \cdot \nabla H), \qquad \mu\Delta\omega = \nabla \times (\rho\dot{u} - H \cdot \nabla H), \tag{2.4}$$

where " $\dot{}$ " denotes the material derivative, i.e.,

$$\dot{f} := \partial_t f + u \cdot \nabla f. \tag{2.5}$$

In view of Lemma 2.1 and the classical estimates of elliptic system, we have

Lemma 2.2. *There exists a generic positive constant C, depending only on μ and λ, such that for any $2 \leqslant p \leqslant 6$,*

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leqslant C\big(\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p}\big), \tag{2.6}$$

$$\|\nabla u\|_{L^p} \leqslant C\big(\|F\|_{L^p} + \|\omega\|_{L^p} + \big\|P(\rho) - P(\tilde{\rho})\big\|_{L^p} + \|H\|_{L^{2p}}^2\big). \tag{2.7}$$

Proof. An application of the L^p-estimate of elliptic systems to (2.4) gives (2.6). On the other hand, since $-\Delta u = -\nabla \operatorname{div} u + \nabla \times \omega$, it holds that

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla \operatorname{div} u + \nabla(-\Delta)^{-1}\nabla \times \omega,$$

which, combined with the standard L^p-estimate and (2.3), yields that $\forall p \in [2, 6]$,

$$\begin{aligned}\|\nabla u\|_{L^p} &\leqslant C\big(\|\operatorname{div} u\|_{L^p} + \|\omega\|_{L^p}\big)\\ &\leqslant C\big(\|F\|_{L^p} + \|\omega\|_{L^p} + \big\|P(\rho) - P(\tilde{\rho})\big\|_{L^p} + \|H\|_{L^{2p}}^2\big).\end{aligned}$$

This finishes the proof of Lemma 2.2. □

The next lemma is due to Zlotnik [22], which will be used to prove the uniform (in time) upper bound of density.

Lemma 2.3. *Assume that $y \in W^{1,1}(0, T)$ solves the ODE system:*

$$y' = g(y) + b'(t) \quad \text{on } [0, T], \qquad y(0) = y_0, \tag{2.8}$$

where $b \in W^{1,1}(0, T)$, $g \in C(\mathbb{R})$ and $g(\infty) = -\infty$. If there are two non-negative numbers $N_0, N_1 \geqslant 0$ satisfying

$$b(t_2) - b(t_1) \leqslant N_0 + N_1(t_2 - t_1) \quad \text{for all } 0 \leqslant t_1 < t_2 \leqslant T, \tag{2.9}$$

then it holds that

$$y(t) \leqslant \max\big\{y_0, \xi^*\big\} + N_0 < \infty \quad \text{on } [0, T], \tag{2.10}$$

where $\xi^* \in \mathbb{R}$ is a constant such that

$$g(\xi) \leqslant -N_1 \quad \text{for } \xi \geqslant \xi^*. \tag{2.11}$$

Finally, we need the local-in-time existence theorem of (1.1)–(1.7) in the case that the initial density is strictly away from vacuum (see [13]).

Lemma 2.4. *Assume that the initial data (ρ_0, u_0, H_0) satisfies*

$$(\rho_0 - \tilde{\rho}, u_0, H_0) \in H^3, \qquad \operatorname{div} H_0 = 0 \quad \text{and} \quad \inf \rho_0 > 0. \tag{2.12}$$

Then there exists a positive time T_0, which may depend on $\inf \rho_0$, such that the Cauchy problem (1.1)–(1.7) has a unique smooth solution (ρ, u, H) on $\mathbb{R}^3 \times [0, T_0]$ satisfying

$$\rho(x,t)>0 \quad \textit{for all } x\in\mathbb{R}^3,\ t\in[0,T_0], \tag{2.13}$$

$$\rho-\tilde{\rho}\in C\big([0,T_0];H^3\big)\cap C^1\big([0,T_0];H^2\big), \tag{2.14}$$

and

$$(u,H)\in C\big([0,T_0];H^3\big)\cap C^1\big([0,T_0];H^1\big)\cap L^2\big([0,T_0];H^4\big). \tag{2.15}$$

3. A priori estimates

This section is devoted to the global-in-time a priori estimates. To begin, let $T>0$ be fixed and assume that (ρ,u,H) is a smooth solution of (1.1)–(1.7) defined on $\mathbb{R}^3\times(0,T]$. For simplicity, we set $\sigma(t)\triangleq\min\{1,t\}$ and define

$$A_1(T)\triangleq\sup_{0\leqslant t\leqslant T}\sigma\big(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2\big)+\int_0^T\sigma\big(\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2+\|H_t\|_{L^2}^2+\big\|\nabla^2H\big\|_{L^2}^2\big)\,\mathrm{d}t, \tag{3.1}$$

$$A_2(T)\triangleq\sup_{0\leqslant t\leqslant T}\sigma^2\big(\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2+\big\|\nabla^2H\big\|_{L^2}^2+\|H_t\|_{L^2}^2\big)+\int_0^T\sigma^2\big(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2\big)\,\mathrm{d}t \tag{3.2}$$

and

$$A_3(T)\triangleq\sup_{0\leqslant t\leqslant T}\big(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2\big). \tag{3.3}$$

The proof of Theorem 1.1 is based on the following energy estimates of (u,H) and uniform upper bound of ρ.

Proposition 3.1. *Assume that (ρ_0,u_0,H_0) satisfies (1.12) and that (ρ,u,H) is a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3\times(0,T]$. There exist two positive constants ε and K, depending on μ, λ, ν, γ, A, $\bar{\rho}$, $\tilde{\rho}$, M_1 and M_2, such that if*

$$\begin{cases}0\leqslant\rho(x,t)\leqslant2\bar{\rho} \quad \textit{for } (x,t)\in\mathbb{R}^3\times[0,T],\\ A_1(T)+A_2(T)\leqslant2C_0^{1/2}, \quad A_3\big(\sigma(T)\big)\leqslant3K,\end{cases} \tag{3.4}$$

then one has

$$\begin{cases}0\leqslant\rho(x,t)\leqslant7\bar{\rho}/4 \quad \textit{for } (x,t)\in\mathbb{R}^3\times[0,T],\\ A_1(T)+A_2(T)\leqslant C_0^{1/2}, \quad A_3\big(\sigma(T)\big)\leqslant2K,\end{cases} \tag{3.5}$$

provided that

$$C_0\leqslant\varepsilon. \tag{3.6}$$

Proof. As a result of Lemmas 3.2, 3.5 and 3.7, one gets (3.5) provided K and ε are chosen as the ones in Lemmas 3.2 and 3.7, respectively. □

For simplicity, throughout this section we denote by C the various generic positive constants, which may depend on μ, λ, ν, γ, A, $\bar{\rho}$, $\tilde{\rho}$, M_1 and M_2, but are independent of T. We also sometimes write $C(\alpha)$ to emphasize the dependence on α.

We begin the proof of Proposition 3.1 with the standard energy estimate of (ρ,u,H).

Lemma 3.1. *Let (ρ, u, H) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$. Then,*

$$\sup_{0\leqslant t\leqslant T}\int\left(G(\rho)+\frac{1}{2}\rho|u|^2+\frac{1}{2}|H|^2\right)\mathrm{d}x + \int_0^T\left(\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2+\nu\|\nabla H\|_{L^2}^2\right)\mathrm{d}t\leqslant C_0. \tag{3.7}$$

Proof. Due to $\operatorname{div} H = 0$, it is easy to check that

$$\nabla\times(u\times H)=H\cdot\nabla u-u\cdot\nabla H-H\operatorname{div} u,$$
$$(\nabla\times H)\times H=H\cdot\nabla H-\frac{1}{2}\nabla|H|^2\quad\text{and}\quad -\nabla\times(\nabla\times H)=\Delta H.$$

Thus, multiplying (1.1), (1.2) and (1.3) by $G'(\rho)$, u and H, respectively, integrating the resulting equations by parts over $\mathbb{R}^3 \times (0, T)$, and adding them together, one easily obtains (3.7). □

The next lemma is concerned with the estimate of $A_3(T)$, which plays an important role in the proofs of $A_1(T)$, $A_2(T)$ and the uniform upper bound of density.

Lemma 3.2. *Suppose that the conditions of Proposition 3.1 hold. Then there exist positive constants K and ε_1, depending on μ, λ, ν, γ, A, $\bar{\rho}$, $\tilde{\rho}$, M_1 and M_2, such that*

$$A_3\big(\sigma(T)\big)+\int_0^{\sigma(T)}\left(\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2+\|H_t\|_{L^2}^2+\big\|\nabla^2H\big\|_{L^2}^2\right)\mathrm{d}t\leqslant 2K, \tag{3.8}$$

and moreover,

$$\sup_{0\leqslant t\leqslant T}\|\nabla H\|_{L^2}^2+\int_0^T\left(\|H_t\|_{L^2}^2+\big\|\nabla^2H\big\|_{L^2}^2\right)\mathrm{d}t\leqslant C\|\nabla H_0\|_{L^2}^2, \tag{3.9}$$

provided $A_3(\sigma(T)) \leqslant 3K$ and $C_0 \leqslant \varepsilon_1$.

Proof. Multiplying (1.2) by u_t and integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\nabla u\|_{L^2}^2+(\mu+\lambda)\|\operatorname{div} u\|_{L^2}^2\right)+\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}|H|^2(\operatorname{div} u)-H\cdot\nabla u\cdot H+\big(P(\rho)-P(\tilde{\rho})\big)(\operatorname{div} u)\right)\mathrm{d}x\\
&\qquad+\int\big(H_t\cdot\nabla u\cdot H+H\cdot\nabla u\cdot H_t-H\cdot H_t(\operatorname{div} u)\big)\,\mathrm{d}x\\
&\qquad-\int P_t(\operatorname{div} u)\,\mathrm{d}x+\int\rho u\cdot\nabla u\cdot\dot{u}\,\mathrm{d}x.
\end{aligned}\tag{3.10}$$

On the other hand, it follows from (1.3) that

$$
\begin{aligned}
&\nu \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla H\|_{L^2}^2 + \left(\|H_t\|_{L^2}^2 + \nu^2\left\|\nabla^2 H\right\|_{L^2}^2\right) \\
&\quad = \int |H_t - \nu \Delta H|^2 \,\mathrm{d}x = \int |H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u|^2 \,\mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

Thus, adding (3.10) and (3.11) together gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\|\nabla u\|_{L^2}^2 + \frac{1}{2}(\mu+\lambda)\|\operatorname{div} u\|_{L^2}^2 + \nu\|\nabla H\|_{L^2}^2\right) \\
&\quad + \left(\left\|\rho^{1/2}\dot{u}\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \nu^2\left\|\nabla^2 H\right\|_{L^2}^2\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}|H|^2(\operatorname{div} u) - H\cdot\nabla u\cdot H + \big(P(\rho)-P(\tilde{\rho})\big)(\operatorname{div} u)\right)\mathrm{d}x \\
&\quad + \int\big(H_t\cdot\nabla u\cdot H + H\cdot\nabla u\cdot H_t - H\cdot H_t(\operatorname{div} u)\big)\,\mathrm{d}x - \int P_t(\operatorname{div} u)\,\mathrm{d}x \\
&\quad + \int \rho u\cdot\nabla u\cdot\dot{u}\,\mathrm{d}x + \int |H\cdot\nabla u - u\cdot\nabla H - H\operatorname{div} u|^2\,\mathrm{d}x \\
&\triangleq \frac{\mathrm{d}}{\mathrm{d}t}I_0 + \sum_{i=1}^{4} I_i. \qquad (3.12)
\end{aligned}
$$

By Lemma 2.1 and Cauchy–Schwarz inequality, we have for any $\eta > 0$ that

$$
\begin{aligned}
I_1 &\leqslant C\|H\|_{L^\infty}\|H_t\|_{L^2}\|\nabla u\|_{L^2} \\
&\leqslant C\|\nabla H\|_{L^2}^{1/2}\left\|\nabla^2 H\right\|_{L^2}^{1/2}\|H_t\|_{L^2}\|\nabla u\|_{L^2} \\
&\leqslant \eta\left(\left\|\nabla^2 H\right\|_{L^2}^2 + \|H_t\|_{L^2}^2\right) + C(\eta)\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2. \qquad (3.13)
\end{aligned}
$$

To deal with I_2, we first observe from (1.1) and (2.3) that

$$
P_t = -u\cdot\nabla\big(P - P(\tilde{\rho})\big) - \gamma P\operatorname{div} u, \qquad P(\rho) = A\rho^\gamma \qquad (3.14)
$$

and

$$
\operatorname{div} u = \frac{1}{2\mu+\lambda}\left(F + \big(P - P(\tilde{\rho})\big) + \frac{1}{2}|H|^2\right).
$$

So, after integrating by parts we infer from (2.6), (3.4) and (3.7) that

$$
\begin{aligned}
I_2 &= \int\left(\gamma P(\operatorname{div} u)^2 + \frac{1}{2\mu+\lambda}\left(F + \big(P-P(\tilde{\rho})\big) + \frac{1}{2}|H|^2\right)u\cdot\nabla\big(P-P(\tilde{\rho})\big)\right)\mathrm{d}x \\
&\leqslant C\left(\|\nabla u\|_{L^2}^2 + \left\|P-P(\tilde{\rho})\right\|_{L^4}^4\right) + C\|\nabla u\|_{L^2}\big(\|\nabla F\|_{L^2} + \|H\nabla H\|_{L^2}\big)\left\|P-P(\tilde{\rho})\right\|_{L^3} \\
&\leqslant C\big(\|\nabla u\|_{L^2}^2 + C_0\big) + CC_0^{1/3}\|\nabla u\|_{L^2}\left(\left\|\rho^{1/2}\dot{u}\right\|_{L^2} + \|H\nabla H\|_{L^2}\right) \\
&\leqslant \eta\left(\left\|\rho^{1/2}\dot{u}\right\|_{L^2}^2 + \left\|\nabla^2 H\right\|_{L^2}^2\right) + C(\eta)\left(C_0 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^{4/3}\|\nabla H\|_{L^2}^2\right), \qquad (3.15)
\end{aligned}
$$

where we have also used the following inequality (due to (2.1)):

$$\|H\|_{L^{12}}^2 \leqslant C\|H\nabla H\|_{L^2} \leqslant C\|H\|_{L^6}\|\nabla H\|_{L^3} \leqslant C\|\nabla H\|_{L^2}^{3/2}\|\nabla^2 H\|_{L^2}^{1/2}. \tag{3.16}$$

Using (2.1), (2.6), (2.7), (3.4), (3.7) and (3.16), we deduce that

$$\begin{aligned}
I_3 &\leqslant \|\rho^{1/2}\dot{u}\|_{L^2}\|\nabla u\|_{L^3}\|u\|_{L^6} \leqslant \|\rho^{1/2}\dot{u}\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{L^6}^{1/2} \\
&\leqslant \|\rho^{1/2}\dot{u}\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\big(\|\rho^{1/2}\dot{u}\|_{L^2}^{1/2} + \|H\nabla H\|_{L^2}^{1/2} + \|P-P(\tilde{\rho})\|_{L^6}^{1/2}\big) \\
&\leqslant \|\rho^{1/2}\dot{u}\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\big(1 + \|\rho^{1/2}\dot{u}\|_{L^2}^{1/2} + \|\nabla H\|_{L^2}^{3/4}\|\nabla^2 H\|_{L^2}^{1/4}\big) \\
&\leqslant \eta\big(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2\big) + C(\eta)\big(\|\nabla u\|_{L^2}^3 + \|\nabla u\|_{L^2}^6 + \|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2\big),
\end{aligned} \tag{3.17}$$

and finally,

$$\begin{aligned}
I_4 &\leqslant C\big(\|u\|_{L^6}^2\|\nabla H\|_{L^3}^2 + \|\nabla u\|_{L^2}^2\|H\|_{L^\infty}^2\big) \\
&\leqslant C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2} \\
&\leqslant \eta\|\nabla^2 H\|_{L^2}^2 + C(\eta)\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2.
\end{aligned} \tag{3.18}$$

Thanks to (3.4), (3.7) and (2.1), we find

$$\begin{aligned}
I_0 &\leqslant C\|\nabla u\|_{L^2}\big(\|P-P(\tilde{\rho})\|_{L^2} + \|H\|_{L^4}^2\big) \leqslant C\|\nabla u\|_{L^2}\big(C_0^{1/2} + \|H\|_{L^2}^{1/2}\|\nabla H\|_{L^2}^{3/2}\big) \\
&\leqslant C\|\nabla u\|_{L^2}\big(C_0^{1/2} + C_0^{1/4}\|\nabla H\|_{L^2}^{3/2}\big) \leqslant \eta\|\nabla u\|_{L^2}^2 + C(\eta)\big(C_0 + C_0^{1/2}\|\nabla H\|_{L^2}^3\big).
\end{aligned} \tag{3.19}$$

Thus, putting (3.13), (3.15), (3.17) and (3.18) into (3.12), integrating it over $(0, \sigma(T))$, and using Cauchy–Schwarz inequality, by virtue of (3.4), (3.7) and (3.19) we conclude that (choosing $\eta > 0$ sufficiently small)

$$\begin{aligned}
&A_3\big(\sigma(T)\big) + \int_0^{\sigma(T)} \big(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2\big)\,\mathrm{d}t \\
&\leqslant C\big(1 + \|\nabla u_0\|_{L^2}^2 + \|\nabla H_0\|_{L^2}^2\big) + CC_0^{1/2}\sup_{0\leqslant t\leqslant\sigma(T)}\|\nabla H\|_{L^2}^3 \\
&\quad + C\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^4\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)\,\mathrm{d}t \\
&\leqslant C(M_1, M_2) + CC_0^{1/2}A_3^{3/2}\big(\sigma(T)\big) + C\sup_{0\leqslant t\leqslant\sigma(T)}\|\nabla u\|_{L^2}^4\int_0^{\sigma(T)}\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big)\,\mathrm{d}t \\
&\leqslant C(M_1, M_2) + CC_0^{1/2}A_3^{3/2}\big(\sigma(T)\big) + CC_0A_3^2\big(\sigma(T)\big) \\
&\leqslant K + CC_0^{1/2}A_3^2\big(\sigma(T)\big),
\end{aligned} \tag{3.20}$$

where $K \triangleq C(M_1, M_2)$. As an immediate result of (3.20), one obtains (3.8) provided it holds that $A_3(\sigma(T)) \leqslant 3K$ and $C_0 \leqslant \varepsilon_{1,1} \triangleq \min\{1, (9CK)^{-2}\}$.

To prove (3.9), we deduce from (3.11) and (3.18) that

$$\sup_{0\leqslant t\leqslant T}\|\nabla H\|_{L^2}^2+\int_0^T\left(\|H_t\|_{L^2}^2+\left\|\nabla^2 H\right\|_{L^2}^2\right)\mathrm{d}t$$

$$\leqslant C\|\nabla H_0\|_{L^2}^2+C_1\sup_{0\leqslant t\leqslant T}\|\nabla H\|_{L^2}^2\int_0^T\|\nabla u\|_{L^2}^4\,\mathrm{d}t. \tag{3.21}$$

On the other hand, it follows from (3.4), (3.7) and (3.8) that

$$\begin{aligned}\int_0^T\|\nabla u\|_{L^2}^4\,\mathrm{d}t&=\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^4\,\mathrm{d}t+\int_{\sigma(T)}^T\|\nabla u\|_{L^2}^4\,\mathrm{d}t\\&\leqslant\sup_{0\leqslant t\leqslant\sigma(T)}\|\nabla u\|_{L^2}^2\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2\,\mathrm{d}t+\sup_{\sigma(T)\leqslant t\leqslant\sigma(T)}\left(\sigma\|\nabla u\|_{L^2}^2\right)\int_{\sigma(T)}^T\|\nabla u\|_{L^2}^2\,\mathrm{d}t\\&\leqslant C(K)C_0.\end{aligned}\tag{3.22}$$

Thus, if C_0 is chosen to be such that

$$C_0\leqslant\varepsilon_1\triangleq\min\left\{\varepsilon_{1,1},\left(2C_1C(K)\right)^{-1}\right\},$$

then substituting (3.22) into (3.21) immediately leads to (3.9). The proof of Lemma 3.2 is therefore complete. □

Next we derive preliminary bounds for $A_1(T)$ and $A_2(T)$.

Lemma 3.3. *Assume that the conditions of Proposition 3.1 hold. Then,*

$$A_1(T)\leqslant CC_0+\int_0^T\sigma^2\left\|P-P(\tilde\rho)\right\|_{L^4}^4\,\mathrm{d}t \tag{3.23}$$

and

$$A_2(T)\leqslant CC_0+CA_1(T)+C\int_0^T\sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t, \tag{3.24}$$

provided $C_0\leqslant\varepsilon_1$.

Proof. Multiplying (3.12) by $\sigma(t)$, integrating the resulting equation over $(0,T)$, and using the similar arguments as those in the derivations of (3.13), (3.15) and (3.17)–(3.19), we infer from (3.4), (3.7), (3.9), (3.22) and Cauchy–Schwarz inequality that

$$
\begin{aligned}
A_1(T) \leqslant{}& CC_0 + C\int_0^T \big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\big)\,\mathrm{d}t \\
&+ CC_0^{1/2}\sup_{0\leqslant t\leqslant T}\|\nabla H\|_{L^2}\sup_{0\leqslant t\leqslant T}\big(\sigma\|\nabla H\|_{L^2}^2\big) \\
&+ C\int_0^T \big(\sigma\|\nabla u\|_{L^2}^6 + \sigma\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2 + \sigma^2\big\|P - P(\tilde\rho)\big\|_{L^4}^4\big)\,\mathrm{d}t \\
&+ C\int_0^T\!\!\int \sigma'\left|\frac{1}{2}|H|^2(\operatorname{div} u) - H\cdot\nabla u\cdot H + \big(P(\rho) - P(\tilde\rho)\big)(\operatorname{div} u)\right|\mathrm{d}x\,\mathrm{d}t \\
\leqslant{}& CC_0 + CC_0^{1/2}\sup_{0\leqslant t\leqslant T}\sigma\big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) + C\int_0^T \sigma^2\big\|P - P(\tilde\rho)\big\|_{L^4}^4\,\mathrm{d}t \\
&+ C\int_0^T \sigma'\big(\|\nabla u\|_{L^2}^2 + \|H\|_{L^4}^4 + \big\|P - P(\tilde\rho)\big\|_{L^2}^2\big)\,\mathrm{d}t \\
\leqslant{}& CC_0 + C\int_0^T \sigma^2\big\|P - P(\tilde\rho)\big\|_{L^4}^4\,\mathrm{d}t,
\end{aligned}
$$

which proves (3.23). Note here that $0 \leqslant \sigma' \leqslant 1$ for $t \geqslant 0$ and $\sigma' = 0$ for $t > 1$.

To estimate $A_2(T)$, applying the operator $\sigma^2\dot u^j[\partial_t + \operatorname{div}(u\cdot)]$ to both sides of j-th equation of (1.2) and integrating them by parts over $\mathbb{R}^3$, we obtain after summing up that

$$
\begin{aligned}
L \triangleq{}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \sigma^2\rho|\dot u|^2\,\mathrm{d}x - \mu\int \sigma^2\dot u^j\big[\Delta u_t^j + \operatorname{div}\big(u\Delta u^j\big)\big]\,\mathrm{d}x \\
&- (\lambda+\mu)\int \sigma^2\dot u^j\big[\partial_t\partial_j\operatorname{div} u + \operatorname{div}(u\partial_j\operatorname{div} u)\big]\,\mathrm{d}x \\
={}& \int \sigma\sigma'\rho|\dot u|^2\,\mathrm{d}x - \int \sigma^2\dot u^j\big[\partial_j P_t + \operatorname{div}(\partial_j P u)\big]\,\mathrm{d}x \\
&- \int \sigma^2\dot u^j\big[\partial_j\big(H^iH_t^i\big) + \operatorname{div}\big(H^i\partial_j H^i u\big)\big]\,\mathrm{d}x \\
&+ \int \sigma^2\dot u^j\big[\partial_t\big(H^i\partial_i H^j\big) + \operatorname{div}\big(H^i\partial_i H^j u\big)\big]\,\mathrm{d}x \triangleq \sum_{i=1}^4 R_i.
\end{aligned}
\tag{3.25}
$$

Here and in what follows, we use the Einstein convention that repeated indices denote the summation over the indices.

We are now in a position of estimating some terms in (3.25). First, the second term on the left-hand side can be estimated from below as follows:

$$
-\mu\int \sigma^2\dot u^j\big[\Delta u_t^j + \operatorname{div}\big(u\Delta u^j\big)\big]\,\mathrm{d}x
$$

$$
\begin{aligned}
&= \mu \int \sigma^2 \big(\partial_k \dot{u}^j \partial_k u_t^j - \partial_{ik}^2 \dot{u}^j u^i \partial_k u^j - \partial_i \dot{u}^j \partial_k u^i \partial_k u^j \big)\,\mathrm{d}x \\
&= \mu \int \sigma^2 \big(|\nabla \dot{u}|^2 + \partial_k \dot{u}^j \partial_i u^i \partial_k u^j - \partial_k \dot{u}^j \partial_k u^i \partial_i u^j - \partial_i \dot{u}^j \partial_k u^i \partial_k u^j \big)\,\mathrm{d}x \\
&\geqslant \frac{7\mu}{8} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 - C\sigma^2 \|\nabla u\|_{L^4}^4.
\end{aligned}
$$

In a similar manner,

$$
-(\lambda+\mu) \int \sigma^2 \dot{u}^j \big[\partial_t \partial_j \operatorname{div} u + \operatorname{div}(u \partial_j \operatorname{div} u) \big]\,\mathrm{d}x \geqslant \frac{\mu+\lambda}{2} \sigma^2 \|\operatorname{div} \dot{u}\|_{L^2}^2 - C\sigma^2 \|\nabla u\|_{L^4}^4,
$$

and consequently,

$$
L \geqslant \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \sigma^2 \rho |\dot{u}|^2\,\mathrm{d}x + \frac{7\mu}{8} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 - C\sigma^2 \|\nabla u\|_{L^4}^4. \tag{3.26}
$$

For the second term R_2 on the right-hand side, using (3.14) and integrating by parts, by (3.4) and Cauchy–Schwarz inequality we have

$$
\begin{aligned}
R_2 &= \int \sigma^2 \big(-\gamma P(\rho) \partial_i u^i \partial_j \dot{u}^j + P(\rho) \partial_i \big(u^i \partial_j \dot{u}^j \big) - P(\rho) \partial_j \big(u^i \partial_i \dot{u}^j \big) \big)\,\mathrm{d}x \\
&= \int \sigma^2 \big(-\gamma P(\rho) \partial_i u^i \partial_j \dot{u}^j + P(\rho) \partial_i u^i \partial_j \dot{u}^j - P(\rho) \partial_j u^i \partial_i \dot{u}^j \big)\,\mathrm{d}x \\
&\leqslant \frac{\mu}{8} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 + C\sigma^2 \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.27}
$$

By virtue of (2.1), (3.7) and (3.9), we obtain after integrating by parts that

$$
\begin{aligned}
R_3 &\leqslant C\sigma^2 \|\nabla \dot{u}\|_{L^2} \big(\|H\|_{L^3} \|H_t\|_{L^6} + \|u\|_{L^6} \|H\|_{L^6} \|\nabla H\|_{L^6} \big) \\
&\leqslant C\sigma^2 \|\nabla \dot{u}\|_{L^2} \big(\|H\|_{L^2}^{1/2} \|\nabla H\|_{L^2}^{1/2} \|\nabla H_t\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla H\|_{L^2} \big\|\nabla^2 H\big\|_{L^2} \big) \\
&\leqslant \frac{\mu}{8} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 + C\sigma^2 \big(C_0^{1/2} \|\nabla H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \big\|\nabla^2 H\big\|_{L^2}^2 \big),
\end{aligned} \tag{3.28}
$$

and similarly, using the fact that $\operatorname{div} H = 0$, we have

$$
\begin{aligned}
R_4 &= -\int \sigma^2 \big(\partial_i \dot{u}^j \big(H^j H_t^i + H^i H_t^j \big) + \partial_k \dot{u}^j H^i \partial_i H^j u^k \big)\,\mathrm{d}x \\
&\leqslant \frac{\mu}{8} \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 + C\sigma^2 \big(C_0^{1/2} \|\nabla H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \big\|\nabla^2 H\big\|_{L^2}^2 \big).
\end{aligned} \tag{3.29}
$$

Thus, putting (3.26)–(3.29) into (3.25) and integrating the resulting inequality over $(0,T)$, we infer from (3.4) and (3.7) that

$$
\sup_{0 \leqslant t \leqslant T} \big(\sigma^2 \big\|\rho^{1/2} \dot{u}\big\|_{L^2}^2 \big) + \int_0^T \sigma^2 \|\nabla \dot{u}\|_{L^2}^2\,\mathrm{d}t
$$

$$\leqslant C\int_0^T\left(\sigma\left\|\rho^{1/2}\dot{u}\right\|_{L^2}^2+\sigma^2\|\nabla u\|_{L^2}^2\right)\mathrm{d}t+C\int_0^T\sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t$$
$$+CC_0^{1/2}\int_0^T\sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t+C\sup_{0\leqslant t\leqslant T}\left(\sigma\|\nabla u\|_{L^2}^2\right)\int_0^T\sigma\left\|\nabla^2H\right\|_{L^2}^2\mathrm{d}t$$
$$\leqslant CC_0+CA_1(T)+C\int_0^T\sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t. \tag{3.30}$$

Differentiating (1.3) with respect to t and multiplying the resulting equations by σ^2H_t in L^2, we obtain after integrating by parts over $\mathbb{R}^3\times(0,T)$ that

$$\frac{1}{2}\sup_{0\leqslant t\leqslant T}\left(\sigma^2\|H_t\|_{L^2}^2\right)+\nu\int_0^T\sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t$$
$$=\int_0^T\sigma\sigma'\|H_t\|_{L^2}^2\,\mathrm{d}t+\int_0^T\!\!\int\sigma^2(H\cdot\nabla\dot{u}\cdot H_t+\dot{u}\cdot\nabla H_t\cdot H)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\!\!\int\sigma^2\left(H^i\partial_iH_t^j-H^k\partial_jH_t^k\right)\left(u\cdot\nabla u^j\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\!\!\int\sigma^2(H_t\cdot\nabla u-H_t\operatorname{div}u-u\cdot\nabla H_t)\cdot H_t\,\mathrm{d}x\,\mathrm{d}t\triangleq\sum_{i=1}^4R^i, \tag{3.31}$$

where we have used $\operatorname{div}H=0$, $u_t=\dot{u}-u\cdot\nabla u$ and

$$-\int\dot{u}\cdot\nabla H\cdot H_t\,\mathrm{d}x=\int\left(H\cdot H_t(\operatorname{div}\dot{u})+\dot{u}\cdot\nabla H_t\cdot H\right)\mathrm{d}x.$$

The second term on the right-hand side of (3.31) can be estimated as follows, using (2.1), (3.4), (3.7) and (3.9):

$$R^2\leqslant C\int_0^T\sigma^2\|H\|_{L^2}^{1/2}\|\nabla H\|_{L^2}^{1/2}\left(\|H_t\|_{L^6}\|\nabla\dot{u}\|_{L^2}+\|\dot{u}\|_{L^6}\|\nabla H_t\|_{L^2}\right)\mathrm{d}t$$
$$\leqslant\frac{\nu}{4}\int_0^T\sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t+CC_0^{1/2}\int_0^T\sigma^2\|\nabla\dot{u}\|_{L^2}^2\,\mathrm{d}t$$
$$\leqslant CC_0+\frac{\nu}{4}\int_0^T\sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t. \tag{3.32}$$

In view of (3.4), (3.9), (3.16) and (3.22), we have

$$
\begin{aligned}
R^3 &\leqslant C\int_0^T \sigma^2\|H\|_{L^{12}}\|\nabla H_t\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^4}\,\mathrm{d}t\\
&\leqslant C\int_0^T \sigma^2\big\|\nabla^2 H\big\|_{L^2}^{1/4}\|\nabla H_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\,\mathrm{d}t\\
&\leqslant \frac{\nu}{4}\int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t + C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t + C\sup_{0\leqslant t\leqslant T}\big(\sigma\big\|\nabla^2 H\big\|_{L^2}\big)\int_0^T\|\nabla u\|_{L^2}^4\,\mathrm{d}t\\
&\leqslant CC_0 + \frac{\nu}{4}\int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t + C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t,
\end{aligned}
\tag{3.33}
$$

and similarly,

$$
\begin{aligned}
R^4 &\leqslant C\int_0^T \sigma^2\big(\|H_t\|_{L^4}^2\|\nabla u\|_{L^2} + \|u\|_{L^6}\|\nabla H_t\|_{L^2}\|H_t\|_{L^3}\big)\,\mathrm{d}t\\
&\leqslant C\int_0^T \sigma^2\|H_t\|_{L^2}^{1/2}\|\nabla H_t\|_{L^2}^{3/2}\|\nabla u\|_{L^2}\,\mathrm{d}t\\
&\leqslant \frac{\nu}{4}\int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t + C\sup_{0\leqslant t\leqslant T}\big(\sigma\|\nabla u\|_{L^2}^2\big)^2\int_0^T\|H_t\|_{L^2}^2\,\mathrm{d}t\\
&\leqslant CC_0 + \frac{\nu}{4}\int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t.
\end{aligned}
\tag{3.34}
$$

Thus, combining (3.32)–(3.34) with (3.31) shows that

$$
\sup_{0\leqslant t\leqslant T}\big(\sigma^2\|H_t\|_{L^2}^2\big) + \int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2\,\mathrm{d}t \leqslant CC_0 + CA_1(T) + \int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t. \tag{3.35}
$$

Moreover, it follows from (1.3), the L^2-estimate of elliptic system and Lemma 2.1 that

$$
\begin{aligned}
\big\|\nabla^2 H\big\|_{L^2}^2 &\leqslant C\big(\|H_t\|_{L^2}^2 + \|u\nabla H\|_{L^2}^2 + \|H\nabla u\|_{L^2}^2\big)\\
&\leqslant C\big(\|H_t\|_{L^2}^2 + \|u\|_{L^6}^2\|\nabla H\|_{L^3}^2 + \|H\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\big)\\
&\leqslant C\big(\|H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}\big\|\nabla^2 H\big\|_{L^2}\big)\\
&\leqslant \frac{1}{2}\big\|\nabla^2 H\big\|_{L^2}^2 + C\big(\|H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2\big),
\end{aligned}
$$

which, together with (3.30) and (3.35), gives

$$
\begin{aligned}
A_2(T) &\leqslant CC_0 + CA_1(T) + C\sup_{0\leqslant t\leqslant T}\left(\left(\sigma\|\nabla u\|_{L^2}^2\right)^2\|\nabla H\|_{L^2}^2\right) + C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t \\
&\leqslant CC_0 + CA_1(T) + C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t,
\end{aligned}
$$

where we have also used (3.4) and (3.9). The proof of Lemma 3.3 is therefore complete. □

Clearly, we still need to deal with $\|\nabla u\|_{L^4}$ and $\|P - P(\tilde\rho)\|_{L^4}$.

Lemma 3.4. *Let F, ω be the ones defined in (2.3), and let the conditions of Proposition 3.1 be satisfied. Then,*

$$
\int_0^T \sigma^2\left(\|\nabla u\|_{L^4}^4 + \|F\|_{L^4}^4 + \|\omega\|_{L^4}^4 + \left\|P - P(\tilde\rho)\right\|_{L^4}^4\right)\mathrm{d}t \leqslant CC_0^{3/4}, \tag{3.36}
$$

provided $C_0 \leqslant \varepsilon_1$.

Proof. In view of the standard L^p-estimate, we have

$$
\begin{aligned}
\int_0^T \sigma^2\|\nabla u\|_{L^4}^4\,\mathrm{d}t &\leqslant C\int_0^T \sigma^2\left(\|\operatorname{div} u\|_{L^4}^4 + \|\omega\|_{L^4}^4\right)\mathrm{d}t \\
&\leqslant C\int_0^T \sigma^2\left(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4 + \left\|P - P(\tilde\rho)\right\|_{L^4}^4 + \|H\|_{L^8}^8\right).
\end{aligned} \tag{3.37}
$$

The right-hand side of (3.37) will be estimated term by term as follows. First, it follows from (2.3), (3.4), (3.7), (3.8) and (3.9) that

$$
\|F\|_{L^2} + \|\omega\|_{L^2} \leqslant \left(\|\nabla u\|_{L^2} + \left\|P - P(\tilde\rho)\right\|_{L^2} + \|H\|_{L^4}^2\right) \leqslant C, \tag{3.38}
$$

and hence, using Lemmas 2.1, 2.2, (3.4), (3.7), (3.9), (3.16) and (3.38), we see that

$$
\begin{aligned}
\int_0^T \sigma^2\left(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4\right)\mathrm{d}t &\leqslant C\int_0^T \sigma^2\left(\|F\|_{L^2} + \|\omega\|_{L^2}\right)\left(\|\nabla F\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3\right)\mathrm{d}t \\
&\leqslant C\int_0^T \sigma^2\left\|\rho^{1/2}\dot u\right\|_{L^2}^3\mathrm{d}t + C\int_0^T \sigma^2\|\nabla H\|_{L^2}^{9/2}\left\|\nabla^2 H\right\|_{L^2}^{3/2}\mathrm{d}t \\
&\leqslant C\sup_{0\leqslant t\leqslant T}\left(\sigma\left\|\rho^{1/2}\dot u\right\|_{L^2}\right)\int_0^T \sigma\|\rho^{1/2}\dot u\|_{L^2}^2\,\mathrm{d}t
\end{aligned}
$$

$$+C\sup_{0\leqslant t\leqslant T}\left(\sigma\left\|\nabla^2 H\right\|_{L^2}\right)^{3/2}\int_0^T\|\nabla H\|_{L^2}^2\,\mathrm{d}t$$

$$\leqslant CA_2^{1/2}(T)A_1(T)+CC_0\leqslant CC_0^{3/4}. \tag{3.39}$$

Next, multiplying (3.14) by $3\sigma^2(P-P(\tilde\rho))^2$, integrating the resulting equation by parts over $\mathbb{R}^3\times(0,T)$ and using the effective viscous flux F, we obtain

$$\begin{aligned}
&\frac{1}{2\mu+\lambda}\int_0^T\sigma^2\left\|P-P(\tilde\rho)\right\|_{L^4}^4\,\mathrm{d}t\\
&=\int\sigma^2\big(P-P(\tilde\rho)\big)^3(\cdot,T)\,\mathrm{d}x-2\int_0^T\sigma\sigma'\int\big(P-P(\tilde\rho)\big)^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{2\mu+\lambda}\int_0^T\sigma^2\int\big(P-P(\tilde\rho)\big)^3\left(F+\frac12|H|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad+3\int_0^T\sigma^2\int\gamma P(\operatorname{div}u)\big(P-P(\tilde\rho)\big)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant CC_0+C\int_0^T\sigma^2\left\|P-P(\tilde\rho)\right\|_{L^4}^3\big(\|F\|_{L^4}+\|H\|_{L^8}^2\big)\,\mathrm{d}t\\
&\quad+C\int_0^T\sigma^2\left\|P-P(\tilde\rho)\right\|_{L^4}^2\|\nabla u\|_{L^2}\,\mathrm{d}t\\
&\leqslant C(\delta)C_0^{3/4}+\delta\int_0^T\sigma^2\left\|P-P(\tilde\rho)\right\|_{L^4}^4\,\mathrm{d}t+C(\delta)\int_0^T\sigma^2\|H\|_{L^8}^8\,\mathrm{d}t,
\end{aligned}\tag{3.40}$$

where we have used (3.4), (3.7), (3.39) and Cauchy–Schwarz inequality. Noting that,

$$\begin{aligned}
\int_0^T\sigma^2\|H\|_{L^8}^8\,\mathrm{d}t&\leqslant C\int_0^T\sigma^2\|H\|_{L^\infty}^4\|H\|_{L^4}^4\,\mathrm{d}t\\
&\leqslant C\int_0^T\sigma^2\|H\|_{L^2}\|\nabla H\|_{L^2}^5\left\|\nabla^2 H\right\|_{L^2}^2\,\mathrm{d}t\\
&\leqslant C_0^{1/2}\int_0^T\sigma^2\left\|\nabla^2 H\right\|_{L^2}^2\,\mathrm{d}t\leqslant CC_0
\end{aligned}$$

due to Lemma 2.1, (3.4), (3.7) and (3.9). This, together with (3.40), gives (choosing $\delta > 0$ small enough)

$$\int_0^T \sigma^2 \left(\left\| P - P(\tilde{\rho}) \right\|_{L^4}^4 + \|H\|_{L^8}^8 \right) dt \leqslant C C_0^{3/4}. \tag{3.41}$$

Thus, combining (3.37) with (3.39) and (3.41) leads to the desired estimate of (3.36). □

Lemma 3.5. *Suppose that the conditions of Proposition 3.1 hold. Then there exists a positive constant ε_2, depending on μ, λ, ν, γ, A, $\bar{\rho}$, $\tilde{\rho}$, M_1 and M_2, such that*

$$A_1(T) + A_2(T) \leqslant C_0^{1/2}, \tag{3.42}$$

provided $C_0 \leqslant \varepsilon_2$.

Proof. Indeed, it follows from (3.23), (3.24) and (3.36) that

$$A_1(T) + A_2(T) \leqslant C C_0^{3/4}.$$

Thus, if C_0 is chosen to be such that

$$C_0 \leqslant \varepsilon_2 \triangleq \min\left\{\varepsilon_1, C^{-4}\right\},$$

then one immediately obtains (3.42). □

In order to complete the proof of Proposition 3.1, we still need to estimate the upper bound of density. To this end, we first prove

Lemma 3.6. *Suppose that the conditions of Proposition 3.1 hold. Then,*

$$\sup_{0 \leqslant t \leqslant T} \left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + \int_0^T \left(\left\| \rho^{1/2} \dot{u} \right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \left\| \nabla^2 H \right\|_{L^2}^2 \right) dt \leqslant C, \tag{3.43}$$

$$\sup_{0 \leqslant t \leqslant T} \sigma \left(\left\| \rho^{1/2} \dot{u} \right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \left\| \nabla^2 H \right\|_{L^2}^2 \right) + \int_0^T \left(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right) dt \leqslant C, \tag{3.44}$$

provided $C_0 \leqslant \varepsilon_2$.

Proof. As an immediate consequence of Lemmas 3.2 and 3.5, one gets (3.43). The proof of (3.44) is similar to the one used for (3.24). More precisely, applying the operator $\sigma \dot{u}[\partial_t + \operatorname{div}(u\cdot)]$ to both sides of (1.2) and integrating the resulting equation by parts over $\mathbb{R}^3 \times (0, T)$, we deduce from (3.7), (3.42), (3.43) as well as (3.26) and (3.27) (with σ^2 replaced by σ) that

$$\sup_{0 \leqslant t \leqslant T} \left(\sigma \left\| \rho^{1/2} \dot{u} \right\|_{L^2}^2 \right) + \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 \, dt$$

$$\leqslant C + \int_0^T \sigma \int \left(|\nabla u|^4 + |\nabla \dot{u}||H||H_t| + |\nabla \dot{u}||H||u||\nabla H| \right) \mathrm{d}t$$

$$\leqslant C + C \int_0^T \sigma \left(\|\nabla u\|_{L^4}^4 + \|\nabla \dot{u}\|_{L^2} \|H\|_{L^\infty} \|H_t\|_{L^2} + \|\nabla \dot{u}\|_{L^2} \|H\|_{L^6} \|u\|_{L^6} \|\nabla H\|_{L^6} \right) \mathrm{d}t$$

$$\leqslant C + \frac{1}{2} \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 \,\mathrm{d}t + C \int_0^T \sigma \left(\|\nabla u\|_{L^4}^4 + \left\|\nabla^2 H\right\|_{L^2}^2 + \left\|\nabla^2 H\right\|_{L^2} \|H_t\|_{L^2}^2 \right) \mathrm{d}t$$

$$\leqslant C + \frac{1}{2} \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 \,\mathrm{d}t + C \int_0^T \sigma \|\nabla u\|_{L^4}^4 \,\mathrm{d}t + C \sup_{0 \leqslant t \leqslant T} \left(\sigma \left\|\nabla^2 H\right\|_{L^2} \right) \int_0^T \|H_t\|_{L^2}^2 \,\mathrm{d}t$$

$$\leqslant C + C \int_0^T \sigma \|\nabla u\|_{L^4}^4 \,\mathrm{d}t + \frac{1}{2} \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 \,\mathrm{d}t. \tag{3.45}$$

It follows from Lemmas 2.1, 2.2, (3.4), (3.7), (3.16), (3.22), (3.36), (3.42) and (3.43) that

$$\begin{aligned}
\int_0^T \sigma \|\nabla u\|_{L^4}^4 \,\mathrm{d}t &\leqslant C \int_0^T \sigma \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 \,\mathrm{d}t \\
&\leqslant C \int_0^T \sigma \|\nabla u\|_{L^2} \left(\left\|\rho^{1/2} \dot{u}\right\|_{L^2}^3 + \left\|P - P(\tilde{\rho})\right\|_{L^6}^3 + \|H\|_{L^{12}}^6 + \|H \nabla H\|_{L^2}^3 \right) \\
&\leqslant C \int_0^T \sigma \|\nabla u\|_{L^2} \left(\left\|\rho^{1/2} \dot{u}\right\|_{L^2}^3 + \left\|P - P(\tilde{\rho})\right\|_{L^4}^2 + \|\nabla H\|_{L^2}^{9/2} \left\|\nabla^2 H\right\|_{L^2}^{3/2} \right) \\
&\leqslant C \sup_{0 \leqslant t \leqslant T} \left(\sigma \left\|\rho^{1/2} \dot{u}\right\|_{L^2}^2 \right)^{1/2} \int_0^T \left\|\rho^{1/2} \dot{u}\right\|_{L^2}^2 \,\mathrm{d}t \\
&\quad + C \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \left\|\nabla^2 H\right\|_{L^2}^2 + \sigma^2 \left\|P - P(\tilde{\rho})\right\|_{L^4}^4 \right) \\
&\leqslant C + C \sup_{0 \leqslant t \leqslant T} \left(\sigma \left\|\rho^{1/2} \dot{u}\right\|_{L^2}^2 \right)^{1/2},
\end{aligned}$$

which, together with (3.45) and Young inequality, immediately results in

$$\sup_{0 \leqslant t \leqslant T} \left(\sigma \left\|\rho^{1/2} \dot{u}\right\|_{L^2}^2 \right) + \int_0^T \sigma \left(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 \right) \mathrm{d}t \leqslant C. \tag{3.46}$$

With the help of (3.43) and (3.46), similar to the derivation of (3.35), one easily obtains the estimates of magnetic field H stated in (3.44). □

We are now ready to prove the uniform upper bound of density.

Lemma 3.7. *Suppose that the conditions of Proposition 3.1 hold. Then there exists a positive constant ε, depending on μ, λ, ν, γ, A, $\bar{\rho}$, $\tilde{\rho}$, M_1 and M_2, such that*

$$\sup_{x\in\mathbb{R}^3,\, t\in[0,T]} \rho(x,t) \leqslant \frac{7}{4}\bar{\rho}, \tag{3.47}$$

provided $C_0 \leqslant \varepsilon$.

Proof. Let $D_t\rho \triangleq \rho_t + u\cdot\nabla\rho$. Then it follows from (1.1) and (2.3) that

$$D_t\rho = g(\rho) + b'(t),$$

where

$$g(\rho) \triangleq -\frac{A\rho}{2\mu+\lambda}\left(\rho^\gamma - \tilde{\rho}^\gamma\right), \qquad b(t) = -\frac{1}{2\mu+\lambda}\int_0^t \rho\left(\frac{1}{2}|H|^2 + F\right)\mathrm{d}s.$$

In order to apply Zlotnik's inequality (i.e. Lemma 2.3), we need to deal with $b(t)$. To do so, we first observe from (3.7), (3.9) and (3.42) that

$$\begin{aligned}\|F\|_{L^2} &\leqslant C\left(\|\nabla u\|_{L^2} + \left\|P - P(\tilde{\rho})\right\|_{L^2} + \|H\|_{L^2}^{1/2}\|\nabla H\|_{L^2}^{3/2}\right)\\ &\leqslant CC_0^{1/4} + C\sigma^{-1/2}\left(\sigma\|\nabla u\|_{L^2}^2\right)^{1/2} \leqslant CC_0^{1/4}\left(1+\sigma^{-1/2}\right),\end{aligned} \tag{3.48}$$

and thus, using Lemmas 2.1, 2.2, (3.7), (3.43) and (3.44), we obtain

$$\begin{aligned}&\int_0^{\sigma(T)} \left(\left\|\rho^{1/2}H\right\|_{L^\infty}^2 + \|\rho F\|_{L^\infty}\right)\mathrm{d}t\\ &\leqslant C\int_0^{\sigma(T)} \left(\|\nabla H\|_{L^2}\left\|\nabla^2 H\right\|_{L^2} + \|F\|_{L^2}^{1/4}\|\nabla F\|_{L^6}^{3/4}\right)\mathrm{d}t\\ &\leqslant CC_0^{1/2} + CC_0^{1/16}\int_0^{\sigma(T)}\left(1+\sigma^{-1/8}\right)\left(\|\rho\dot{u}\|_{L^6}^{3/4} + \|H\cdot\nabla H\|_{L^6}^{3/4}\right)\mathrm{d}t\\ &\leqslant CC_0^{1/2} + CC_0^{1/16}\int_0^{\sigma(T)}\left(1+\sigma^{-1/8}\right)\left(\|\nabla\dot{u}\|_{L^2}^{3/4} + \left\|\nabla^2 H\right\|_{L^2}^{9/8}\right)\mathrm{d}t\\ &\leqslant CC_0^{1/2} + CC_0^{1/16}\left(\int_0^{\sigma(T)}\left(1+\sigma^{-4/5}\right)\mathrm{d}t\right)^{5/8}\left(\int_0^{\sigma(T)}\sigma\|\nabla\dot{u}\|_{L^2}^2\,\mathrm{d}t\right)^{3/8}\end{aligned}$$

$$
\begin{aligned}
&+ CC_0^{1/16}\left(\int_0^{\sigma(T)}\left(1+\sigma^{-2/7}\right)\mathrm{d}t\right)^{7/16}\left(\int_0^{\sigma(T)}\left\|\nabla^2 H\right\|_{L^2}^2\,\mathrm{d}t\right)^{9/16} \\
&\leqslant CC_0^{1/16}.
\end{aligned}
$$

This particularly implies that for any $0 \leqslant t_1 < t_2 \leqslant \sigma(T)$,

$$
\left|b(t_2)-b(t_1)\right| \leqslant C\int_0^{\sigma(T)}\left(\left\|\rho^{1/2}H\right\|_{L^\infty}^2+\|\rho F\|_{L^\infty}\right)\mathrm{d}t \leqslant CC_0^{1/16}.
$$

So, for $t \in [0,\sigma(T)]$ we can choose N_0, N_1 and ξ^* in Lemma 2.3 as follows:

$$
N_0 = CC_0^{1/16}, \qquad N_1 = 0 \quad \text{and} \quad \xi^* = \tilde{\rho}.
$$

Noting that

$$
g(\xi) = -\frac{A\xi}{2\mu+\lambda}\left(\xi^\gamma - \tilde{\rho}^\gamma\right) \leqslant -N_1 = 0 \quad \text{for all } \xi \geqslant \xi^* = \tilde{\rho},
$$

we thus deduce from (2.10) that

$$
\sup_{0\leqslant t\leqslant\sigma(T)}\left\|\rho(t)\right\|_{L^\infty} \leqslant \max\{\bar{\rho},\tilde{\rho}\} + N_0 \leqslant \bar{\rho} + CC_0^{1/16} \leqslant \frac{3}{2}\bar{\rho}, \tag{3.49}
$$

provided C_0 is chosen to be such that

$$
C_0 \leqslant \varepsilon_{3,1} \triangleq \min\left\{\varepsilon_2, \left(\frac{\bar{\rho}}{2C}\right)^{16}\right\}.
$$

It is clear that $\|F(t)\|_{L^2} \leqslant C$ for all $t \in [0,T]$ due to (3.7) and (3.43). Hence, we have by Cauchy–Schwarz inequality and (3.42) that for any $\sigma(T) \leqslant t_1 < t_2 \leqslant T$,

$$
\begin{aligned}
\left|b(t_2)-b(t_1)\right| &\leqslant \int_{t_1}^{t_2}\left(\left\|\rho^{1/2}H\right\|_{L^\infty}^2+\|\rho F\|_{L^\infty}\right)\mathrm{d}t \\
&\leqslant CC_0^{1/2} + C\int_{t_1}^{t_2}\left(\|\nabla\dot{u}\|_{L^2}^{3/4}+\left\|\nabla^2 H\right\|_{L^2}^{9/8}\right)\mathrm{d}t \\
&\leqslant \frac{A}{2\mu+\lambda}(t_2-t_1) + CC_0^{1/2} + C\int_{t_1}^{t_2}\left(\sigma^2\|\nabla\dot{u}\|_{L^2}^2+\sigma\left\|\nabla^2 H\right\|_{L^2}^2\right)\mathrm{d}t \\
&\leqslant \frac{A}{2\mu+\lambda}(t_2-t_1) + CC_0^{1/2}.
\end{aligned}
$$

Thus, for $t \in [\sigma(T), T]$ we can choose N_0, N_1 and ξ^* in Lemma 2.3 as follows:

$$N_0 = CC_0^{1/2}, \qquad N_1 = \frac{A}{2\mu+\lambda} \quad \text{and} \quad \xi^* = \tilde{\rho} + 1.$$

Since

$$g(\xi) = -\frac{A\xi}{2\mu+\lambda}\left(\xi^\gamma - \tilde{\rho}^\gamma\right) \leqslant -N_1 \quad \text{for } \xi \geqslant \xi^* = \tilde{\rho}+1,$$

we thus infer from (2.10) and (3.49) that

$$\sup_{\sigma(T)\leqslant t\leqslant T} \left\|\rho(t)\right\|_{L^\infty} \leqslant \max\left\{\frac{3}{2}\bar{\rho}, \tilde{\rho}+1\right\} + N_0 \leqslant \frac{3}{2}\bar{\rho} + CC_0^{1/2} \leqslant \frac{7}{4}\bar{\rho}, \tag{3.50}$$

provided

$$C_0 \leqslant \varepsilon \triangleq \min\left\{\varepsilon_{3,1}, \left(\frac{\bar{\rho}}{4C}\right)^2\right\}.$$

Combining (3.49) and (3.50) finishes the proof of (3.47). □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by constructing weak solutions as limits of smooth solutions. So, we first prove the global-in-time existence of smooth solutions with smooth initial data which is strictly away from vacuum and is only of small energy.

Theorem 4.1. *Assume that (ρ_0, u_0, H_0) satisfies (2.12). Then for any $0 < T < \infty$, there exists a unique smooth solution (ρ, u, H) of (1.1)–(1.7) on $\mathbb{R}^3 \times [0, T]$ satisfying (2.13)–(2.15) with T_0 being replaced by T, provided the initial energy C_0 satisfies the smallness condition (1.13) with $\varepsilon > 0$ being the same one as in (3.6) of Proposition 3.1.*

Proof. The standard local existence theorem (i.e. Lemma 2.4) shows that the Cauchy problem (1.1)–(1.7) admits a unique local smooth solution (ρ, u, H) on $\mathbb{R}^3 \times [0, T_0]$, where $T_0 > 0$ may depend on $\inf \rho_0$.

In view of (3.1)–(3.3), we have

$$A_1(0) = A_2(0) = 0, \qquad A_3(0) = M_1 + M_2 \leqslant 3K, \qquad 0 \leqslant \rho_0 \leqslant \bar{\rho}.$$

So, by continuity argument we see that there exists a positive time $T_1 \in (0, T_0]$ such that (3.4) holds for $T = T_1$. Set

$$T_* = \sup\left\{T \mid (3.4) \text{ holds}\right\}. \tag{4.1}$$

Then it is clear that $T_* \geqslant T_1 > 0$.

We claim that

$$T_* = \infty. \tag{4.2}$$

Otherwise, $T_* < \infty$. Then due to $C_0 \leqslant \varepsilon$, it follows from Proposition 3.1 that (3.5) holds for any $0 \leqslant T \leqslant T_*$. This, together with Proposition 4.1 (see below) and Lemma 2.4, implies there exists a

$T^* > T_*$ such that (3.4) holds for $T = T^*$. This contradicts (4.1), and thus, (4.2) holds. As a result, we deduce from Proposition 4.1 that (ρ, u, H) is in fact the unique smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times [0, T]$ for any $0 < T < \infty$. $\square$

Proposition 4.1. *Let (ρ, u, H) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times [0, T]$ with initial data (ρ_0, u_0, H_0) satisfying (2.12) and the small-energy condition (1.13). Then,*

$$\rho(x,t) > 0 \quad \textit{for all } x \in \mathbb{R}^3,\ t \in [0, T], \tag{4.3}$$

and

$$\sup_{0 \leqslant t \leqslant T} \big\|(\rho - \tilde{\rho}, u, H)\big\|_{H^3} + \int_0^T \big\|(u, H)\big\|_{H^4}^2 \,\mathrm{d}t \leqslant \tilde{C}. \tag{4.4}$$

Here and in what follows, for simplicity we denote by $\tilde{C}$ the various positive constants which depend on μ, λ, ν, γ, A, $\tilde{\rho}$, $\bar{\rho}$, $\|(\rho_0 - \tilde{\rho}, u_0, H_0)\|_{H^3}$, $\inf \rho_0(x)$ and T as well.

Proof. The positive lower bound of density in (4.3) is an immediate result of (4.4), which indeed only depends on the bound of $\|\operatorname{div} u\|_{L^1(0,T;L^\infty)}$. So we only need to prove (4.4). As that in [19], the key point here is to estimate $\|\nabla u\|_{L^1(0,T;L^\infty)}$ and $\|\nabla \rho\|_{L^\infty(0,T;L^p)}$ with $p \in [2, 6]$, which will be achieved by using the Beale–Kato–Majda's type inequality developed in [10,11].

Step I. To begin, we first notice that (due to $\inf \rho_0 > 0$ and (2.12))

$$\dot{u}(\cdot, 0) = \rho_0^{-1}\left(\mu \Delta u_0 + (\mu + \lambda)\nabla \operatorname{div} u_0 - \nabla P(\rho_0) - \frac{1}{2}\nabla|H_0| + H_0 \cdot \nabla H_0\right) \in H^1. \tag{4.5}$$

In view of Proposition 3.1, we have

$$\rho(x,t) \leqslant C < \infty \quad \text{for all } x \in \mathbb{R}^3,\ t \in [0, T], \tag{4.6}$$

$$\sup_{0 \leqslant t \leqslant T} \big(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\big) + \int_0^T \big(\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \big\|\nabla^2 H\big\|_{L^2}^2\big)\,\mathrm{d}t \leqslant C, \tag{4.7}$$

and moreover, similar to the derivation of (3.44), by using (3.7), (4.6) and (4.7) we also infer from (4.5) that

$$\sup_{0 \leqslant t \leqslant T} \big(\big\|\rho^{1/2}\dot{u}\big\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \big\|\nabla^2 H\big\|_{L^2}^2\big) + \int_0^T \big(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2\big)\,\mathrm{d}t \leqslant \tilde{C}(T). \tag{4.8}$$

Step II. This step is concerned with the estimate of the gradient of density. To do this, operating ∇ to both sides of (1.1) and multiplying the resulting equation by $|\nabla \rho|^{p-2}\nabla \rho$ with $p \geqslant 2$, we obtain after integrating by parts over $\mathbb{R}^3$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \rho\|_{L^p} \leqslant C\|\nabla u\|_{L^\infty}\|\nabla \rho\|_{L^p} + C\big\|\nabla^2 u\big\|_{L^p}. \tag{4.9}$$

By the standard L^p-estimate of elliptic system, we infer from (1.2) that

$$\left\|\nabla^2 u\right\|_{L^p} \leqslant C\big(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p} + \|H\nabla H\|_{L^p}\big). \tag{4.10}$$

In order to deal with $\|\nabla u\|_{L^\infty}$, we recall the following Beale–Kato–Majda's type inequality (see [10,11]):

$$\|\nabla u\|_{L^\infty} \leqslant C\big(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}\big)\ln\big(e + \left\|\nabla^2 u\right\|_{L^q}\big) + C\|\nabla u\|_{L^2} + C, \tag{4.11}$$

where $\omega = \nabla \times u$ and $q > 3$.

So, choosing $p = q = 6$ in (4.9)–(4.11), and using Lemma 2.1 and (4.6)–(4.8), we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^6} &\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^6} + C\big(\|\rho\dot{u}\|_{L^6} + \|\nabla P\|_{L^6} + \|H\nabla H\|_{L^6}\big)\\
&\leqslant C\|\nabla\rho\|_{L^6}\big(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}\big)\ln\big(e + \|\nabla\dot{u}\|_{L^2}\big)\\
&\quad + C\|\nabla\rho\|_{L^6}\big(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}\big)\ln\big(e + \|\nabla\rho\|_{L^6}\big)\\
&\quad + C\big(\|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^6} + 1\big).
\end{aligned} \tag{4.12}$$

Define

$$f(t) \triangleq e + \|\nabla\rho\|_{L^6}, \qquad g(t) \triangleq 1 + \|\nabla\dot{u}\|_{L^2} + \big(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}\big)\ln\big(e + \|\nabla\dot{u}\|_{L^2}\big).$$

Hence, it follows from (4.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \leqslant Cg(t)f(t) + Cg(t)f(t)\ln f(t),$$

which particularly implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln f(t) \leqslant Cg(t) + Cg(t)\ln f(t). \tag{4.13}$$

Next we estimate $g(t)$. Indeed, by Lemmas 2.1, 2.2 and (4.6)–(4.8), we have

$$\begin{aligned}
\int_0^T g(t)\,\mathrm{d}t &\leqslant \tilde{C} + \tilde{C}\int_0^T \big(\|\nabla\dot{u}\|_{L^2}^2 + \|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\big)\,\mathrm{d}t\\
&\leqslant \tilde{C} + \tilde{C}\int_0^T \big(\|F\|_{L^\infty}^2 + \left\|P - P(\tilde{\rho})\right\|_{L^\infty}^2 + \|H\|_{L^\infty}^4 + \|\omega\|_{L^\infty}^2\big)\,\mathrm{d}t\\
&\leqslant \tilde{C} + \tilde{C}\int_0^T \big(\|F\|_{L^2}^{1/2}\|\nabla F\|_{L^6}^{3/2} + \|\omega\|_{L^2}^{1/2}\|\nabla\omega\|_{L^6}^{3/2}\big)\,\mathrm{d}t\\
&\leqslant \tilde{C} + \tilde{C}\int_0^T \big(\|\rho\dot{u}\|_{L^6}^{3/2} + \|H\nabla H\|_{L^6}^{3/2}\big)\,\mathrm{d}t\\
&\leqslant \tilde{C} + \tilde{C}\int_0^T \|\nabla\dot{u}\|_{L^2}^2\,\mathrm{d}t \leqslant \tilde{C}.
\end{aligned} \tag{4.14}$$

This, together with (4.13) and Gronwall inequality, gives

$$\sup_{0\leqslant t\leqslant T}\|\nabla\rho\|_{L^6}\leqslant\tilde{C},\tag{4.15}$$

which, combined with (4.10), (4.11) and (4.14), also yields

$$\int_0^T\|\nabla u\|_{L^\infty}\,\mathrm{d}t\leqslant\tilde{C}.\tag{4.16}$$

As a result, one also easily deduces from (4.9) and (4.10) that

$$\sup_{0\leqslant t\leqslant T}\big(\|\nabla\rho\|_{L^2}+\big\|\nabla^2u\big\|_{L^2}\big)\leqslant\tilde{C}.\tag{4.17}$$

Step III. By virtue of (4.5)–(4.8) and (4.15)–(4.17), one can derive the estimates of the higher-order derivatives of (ρ,u,H) in a similar way as that in [19], basing on the elementary L^2-energy method. The details are omitted here for simplicity. The proof of Proposition 4.1 is therefore complete. □

With the help of Theorem 4.1, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $j_\delta(x)$ be a standard mollifier with width δ. Define the approximate initial data $(\rho_0^\delta,u_0^\delta,H_0^\delta)$ as follows:

$$\rho_0^\delta=j_\delta*\rho_0+\delta,\qquad u_0^\delta=j_\delta*u_0,\qquad H_0^\delta=j_\delta*H_0.$$

Then Theorem 4.1 can be applied to obtain a global smooth solution $(\rho^\delta,u^\delta,H^\delta)$ of (1.1)–(1.7) with the initial data $(\rho_0^\delta,u_0^\delta,H_0^\delta)$ satisfying (3.4) for all $t>0$ uniformly in δ.

In view of Lemma 2.2 and (3.4), we see from Sobolev embedding theorem that

$$\begin{aligned}\big\langle u^\delta(\cdot,t)\big\rangle^{1/2}&\leqslant C\big(1+\big\|\nabla u^\delta\big\|_{L^6}\big)\\&\leqslant C\big(1+\big\|F^\delta\big\|_{L^6}+\big\|\omega^\delta\big\|_{L^6}+\big\|P^\delta-P(\tilde{\rho})\big\|_{L^6}+\big\|H^\delta\big\|_{L^{12}}^2\big)\\&\leqslant C\big(1+\big\|\rho^\delta\dot{u}^\delta\big\|_{L^2}+\big\|H^\delta\nabla H^\delta\big\|_{L^2}\big)\\&\leqslant C(\tau),\quad t\geqslant\tau>0,\end{aligned}\tag{4.18}$$

where F^δ, ω^δ and P^δ are the functions F, ω and P with (ρ,u,H) being replaced by $(\rho^\delta,u^\delta,H^\delta)$. Here, we also used $\langle\cdot\rangle^\alpha$ to denote the Hölder norm with Hölder exponent $\alpha\in(0,1)$.

In addition to (4.18), one also has

$$\left|u^\delta(x,t)-\frac{1}{|B_R(x)|}\int_{B_R(x)}u^\delta(y,t)\,\mathrm{d}y\right|\leqslant C(\tau)R^{1/2},$$

and hence,

$$\begin{aligned}
&\left|u^\delta(x,t_2)-u^\delta(x,t_1)\right| \\
&\quad\leqslant \frac{1}{|B_R(x)|}\int_{t_1}^{t_2}\int_{B_R(x)}\left|u_t^\delta(y,t)\right|\mathrm{d}y\,\mathrm{d}t + C(\tau)R^{1/2} \\
&\quad\leqslant CR^{-3/2}|t_2-t_1|^{1/2}\left(\int_{t_1}^{t_2}\int_{B_R(x)}\left|u_t^\delta(y,t)\right|^2\mathrm{d}y\,\mathrm{d}t\right)^{1/2} + C(\tau)R^{1/2} \\
&\quad\leqslant CR^{-3/2}|t_2-t_1|^{1/2}\left(\int_{t_1}^{t_2}\int_{B_R(x)}\left(\left|\dot{u}^\delta\right|^2+\left|u^\delta\right|^2\left|\nabla u^\delta\right|^2\right)\mathrm{d}y\,\mathrm{d}t\right)^{1/2} + C(\tau)R^{1/2}. \qquad (4.19)
\end{aligned}$$

Since it holds for any $0<\tau\leqslant t_1<t_2<\infty$ that

$$\begin{aligned}
\int_{t_1}^{t_2}\int\left|\dot{u}^\delta\right|^2\mathrm{d}x\,\mathrm{d}t &\leqslant C(\bar{\rho},\tilde{\rho})\int_{t_1}^{t_2}\int\left(\rho^\delta\left|\dot{u}^\delta\right|^2+\left|\rho^\delta-\tilde{\rho}\right|^2\left|\dot{u}^\delta\right|^2\right)\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C(\tau,\bar{\rho},\tilde{\rho}) + C(\bar{\rho},\tilde{\rho})\int_{t_1}^{t_2}\left\|\nabla\dot{u}^\delta\right\|_{L^2}^2\left\|\rho^\delta-\tilde{\rho}\right\|_{L^3}^2\mathrm{d}t \\
&\leqslant C(\tau,\bar{\rho},\tilde{\rho})
\end{aligned}$$

and

$$\int_{t_1}^{t_2}\int\left|u^\delta\right|^2\left|\nabla u^\delta\right|^2\mathrm{d}x\,\mathrm{d}t \leqslant C(\bar{\rho},\tilde{\rho})\sup_{t\geqslant\tau}\left\|u^\delta\right\|_{L^\infty}^2\int_{t_1}^{t_2}\left\|\nabla u^\delta\right\|_{L^2}^2\mathrm{d}t \leqslant C(\tau,\bar{\rho},\tilde{\rho}),$$

we thus infer from (4.19) that

$$\left|u^\delta(x,t_2)-u^\delta(x,t_1)\right| \leqslant C(\tau)R^{-3/2}|t_2-t_1|^{1/2}+C(\tau)R^{1/2}$$

for any $0<\tau\leqslant t_1<t_2<\infty$. Thus, choosing $R=|t_2-t_1|^{1/4}$, we get

$$\left|u^\delta(x,t_2)-u^\delta(x,t_1)\right| \leqslant C(\tau)|t_2-t_1|^{1/8}, \quad 0<\tau\leqslant t_1<t_2<\infty. \qquad (4.20)$$

The same estimates in (4.18) and (4.20) also hold for the magnetic filed H^δ. Thus, we have proved that $\{u^\delta\}$ and $\{H^\delta\}$ are uniform Hölder continuity away from $t=0$. As a result, it follows from Ascoli–Arzela theorem that

$$u^\delta\to u, \qquad H^\delta\to H \quad \text{uniformly on compact sets in } \mathbb{R}^3\times(0,\infty). \qquad (4.21)$$

Moreover, by the argument in [16] (see also [5]), we know that

$$\rho^\delta\to\rho \quad \text{strongly in } L^p\big(\mathbb{R}^3\times(0,\infty)\big),\ \forall p\in[2,\infty). \qquad (4.22)$$

Therefore, passing to the limit as $\delta\to 0$, by (4.21), (4.22) we obtain the limited function (ρ,u,H) which is indeed a weak solution of (1.1)–(1.7) in the sense of Definition 1.1 and satisfies (3.4) for all

$T \geqslant 0$. The large-time behavior of (ρ, u, H) in (1.15) is an immediate result of the uniform bounds established in Section 3 and can be proved in a similar manner as that in [11]. The proof of Theorem 1.1 is thus complete. $\square$

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