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# A theoretical basis for the Harmonic Balance Method

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ABSTRACT

The Harmonic Balance Method provides a heuristic approach for finding truncated Fourier series as an approximation to the periodic solutions of ordinary differential equations. Another natural way for obtaining these types of approximations consists in applying numerical methods. In this paper we recover the pioneering results of Stokes and Urabe that provide a theoretical basis for proving that near these truncated series, whatever is the way they have been obtained, there are actual periodic solutions of the equation. We will restrict our attention to one-dimensional non-autonomous ordinary differential equations, and we apply the obtained results to a concrete example coming from a rigid cubic system.



## 1. Introduction and main results

Consider the real non-autonomous differential equation

$$x' = X(x, t), \tag{1}$$

where the prime denotes the derivative with respect to $t$, $X : \Omega \times [0, 2\pi] \to \mathbb{R}$ is a $C^2$-function, $2\pi$-periodic in $t$, and $\Omega \subset \mathbb{R}$ is a given open interval.

There are several methods for finding approximations to the periodic solutions of (1). For instance, the Harmonic Balance Method (HBM), recalled in Section 2.1, or simply the numerical approximations of the solutions of the differential equations. In any case, from all the methods we can get a truncated

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Fourier series, namely a trigonometric polynomial, that "approximates" an actual periodic solution of the equation. The aim of this work is to recover some old results of Stokes and Urabe that allow the use of these approximations to prove that near them there are actual periodic solutions and also provide explicit bounds, in the infinity norm, of the distance between both functions. To the best of our knowledge these results are rarely used in the papers dealing with HBM.

When the methods are applied to concrete examples one has to manage the coefficients of the truncated Fourier series that are rational numbers which renders the subsequent computations more difficult. See the example of Section 4. At this point we introduce in this setting a classical tool that as far as we know has never been used in this type of problems: we approximate all the coefficients of the truncated Fourier series by suitable convergents of their respective expansions in continuous fractions. This is done in such a way that by using these new coefficients we obtain a new approximate solution that is essentially at the same distance to the actual solution as the starting approximation. With this method we obtain trigonometric polynomials with nice rational coefficients that approximate the periodic solutions.

Before stating our main result, and following [4,5], we introduce some concepts. Let $\bar{x}(t)$ be a $2\pi$-periodic $\mathcal{C}^1$-function; we will say that $\bar{x}(t)$ is *noncritical* with respect to (1) if

$$\int_0^{2\pi} \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big)\, dt \neq 0. \tag{2}$$

Notice that if $\bar{x}(t)$ is a periodic solution of (1) then the concept of noncritical is equivalent to the one of being *hyperbolic*; see [2].

As we will see in Lemma 2.1, if $\bar{x}(t)$ is noncritical w.r.t. Eq. (1), the linear periodic system

$$y' = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) y + b(t)$$

has a unique periodic solution $y_b(t)$ for each smooth $2\pi$-periodic function $b(t)$. Moreover, once $X$ and $\bar{x}$ are fixed, there exists a constant $M$ such that

$$\|y_b\|_\infty \leqslant M\|b\|_2, \tag{3}$$

where as usual, for a continuous $2\pi$-periodic function $f$,

$$\|f\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} f^2(t)\, dt}, \qquad \|f\|_\infty = \max_{x\in\mathbb{R}} \big|f(x)\big| \quad \text{and} \quad \|f\|_2 \leqslant \|f\|_\infty.$$

Any constant satisfying (3) will be called a *deformation constant associated to $\bar{x}$ and $X$*. Finally, consider

$$s(t) := \bar{x}'(t) - X\big(\bar{x}(t), t\big). \tag{4}$$

We will say that $\bar{x}(t)$ is *an approximate solution of* (1) *with accuracy* $S = \|s\|_2$. For simplicity, if $\tilde{S} > S$, we also will say that $\bar{x}(t)$ has accuracy $\tilde{S}$. Notice that actual periodic solutions of (1) have accuracy 0; in this sense, the function $s(t)$ measures how far is $\bar{x}(t)$ from being an actual periodic solution of (1).

The next theorem improves some of the results of Stokes [4] and Urabe [5] in the one-dimensional setting. More concretely, in those papers they prove the existence and uniqueness of the periodic orbit when $4M^2KS < 1$. We present a similar proof with the small improvement $2M^2KS < 1$. Moreover our result gives, under an additional condition, the hyperbolicity of the periodic orbit.

**Theorem 1.1.** *Let $\bar{x}(t)$ be a $2\pi$-periodic $\mathcal{C}^1$-function such that*

- *it is noncritical w.r.t. Eq. (1) and has $M$ as a deformation constant,*
- *it has accuracy $S$ w.r.t. Eq. (1).*

*Given $I := [\min_{\{t\in\mathbb{R}\}} \bar{x}(t) - 2MS, \max_{\{t\in\mathbb{R}\}} \bar{x}(t) + 2MS] \subset \Omega$, let $K < \infty$ be a constant such that*

$$\max_{(x,t)\in I\times[0,2\pi]} \left| \frac{\partial^2}{\partial x^2} X(x,t) \right| \leqslant K.$$

*Therefore, if*

$$2M^2KS < 1,$$

*there exists a $2\pi$-periodic solution $x^*(t)$ of (1) satisfying*

$$\|x^* - \bar{x}\|_\infty \leqslant 2MS,$$

*and it is the unique periodic solution of the equation entirely contained in this strip. If in addition*

$$\left| \int_0^{2\pi} \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big)\, dt \right| > \frac{2\pi}{M},$$

*then the periodic orbit $x^*(t)$ is hyperbolic, and its stability is given by the sign of this integral.*

Once some approximate solution is guessed, for applying Theorem 1.1 we need to compute the three constants appearing in its statement. In general, $K$ and $S$ can be easily obtained. Recall for instance that $\|s\|_2$, when $s$ is a trigonometric polynomial, can be computed from Parseval's theorem. On the other hand, $M$ is much more difficult to estimate. In Lemma 2.3 we give a result useful for computing it in concrete cases, that is different from the approach used in [4–6].

Assuming that a non-autonomous differential equation has a hyperbolic periodic orbit, the results of [5] also guarantee that, if given a suitable trigonometric polynomial $\bar{r}(t)$ of a sufficiently high degree, we can apply the first part of Theorem 1.1. Intuitively, while the value of the accuracy $S$ goes to zero when we increase the degree of the trigonometric polynomial, the values $M$ and $K$ remain bounded. Thus, at some moment, it holds that $2M^2KS < 1$.

In Section 4 we apply Theorem 1.1 to study and localize the limit cycles of the rigid cubic system

$$\dot{x} = -y + \frac{x}{10}\big(1 - x - 10x^2\big),$$
$$\dot{y} = x + \frac{y}{10}\big(1 - x - 10x^2\big).$$

In polar coordinates it is written as $\dot{r} = r/10 - \cos(\theta)r^2/10 - \cos^2(\theta)r^3$, $\dot{\theta} = 1$, or equivalently,

$$r' = \frac{dr}{dt} = \frac{1}{10}r - \frac{1}{10}\cos(t)r^2 - \cos^2(t)r^3, \tag{5}$$

and it has a unique positive periodic orbit; see also [1]. Notice that we have renamed $\theta$ as $t$. We prove the following:

**Proposition 1.2.** *Consider the periodic function*

$$\bar{r}(t) = \frac{4}{9} - \frac{1}{693}\cos(t) - \frac{1}{51}\sin(t) - \frac{1}{653}\cos(2t) - \frac{1}{45}\sin(2t) - \frac{1}{780}\cos(3t).$$

*The differential equation* (5) *then has a periodic solution* $r^*(t)$ *such that*

$$\|\bar{r} - r^*\|_\infty \leqslant 0.042,$$

*which is hyperbolic and stable, and it is the only periodic solution of* (5) *contained in this strip.*

As we will see, in this example we will find computational difficulties to obtain the third approximation given by the HBM. Therefore we will get it first by numerically approaching the periodic solution, then by numerically computing the first terms of its Fourier series and finally by using the continuous fractions approach to simplify the values appearing in our computations. We also will see that the same approach works for other concrete rigid systems.

Similar examples for second-order differential equations have also been studied in [6].

## 2. Preliminary results

This section contains some technical lemmas that are useful for proving Theorem 1.1 and for obtaining in concrete examples the constants appearing in its statement. We also include a very short overview of the HBM adapted to our interests. See [3] for a more general point of view on the HBM.

As usual, given $A \subset \mathbb{R}$, $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$ denotes the *characteristic function of* $A$: the function takes the value 1 when $x \in A$, and the value is 0 otherwise.

**Lemma 2.1.** *Let $a(t)$ and $b(t)$ be continuous real $2\pi$-periodic functions. Consider the non-autonomous linear ordinary differential equation*

$$x' = a(t)x + b(t). \tag{6}$$

*If $A(2\pi) \neq 0$, where $A(t) := \int_0^t a(s)\,ds$, then for each $b(t)$ Eq.* (6) *has a unique $2\pi$-periodic solution $x_b(t) := \int_0^{2\pi} H(t,s)b(s)\,ds$, where the kernel $H(t,s)$ is given by the piecewise function*

$$H(t,s) = \frac{e^{A(t)}}{1 - e^{A(2\pi)}}\left[e^{-A(s)}\mathbf{1}_{[0,t]}(s) + e^{A(2\pi)-A(s)}\mathbf{1}_{[t,2\pi]}(s)\right]. \tag{7}$$

*Moreover* $\|x_b\|_\infty \leqslant 2\pi \max_{t\in[0,2\pi]} \|H(t,\cdot)\|_2 \|b\|_2$.

**Proof.** Since (6) is linear, its general solution is

$$x(t) = e^{A(t)}\left(x_0 + \int_0^t b(s)e^{-A(s)}\,ds\right). \tag{8}$$

If we impose that the solution is $2\pi$-periodic, i.e. $x(0) = x(2\pi)$, we get

$$x_0 = \frac{e^{A(2\pi)}}{1 - e^{A(2\pi)}}\int_0^{2\pi} b(s)e^{-A(s)}\,ds, \tag{9}$$

then (8) becomes

$$\begin{aligned} x_b(t) &= \frac{e^{A(t)}}{1-e^{A(2\pi)}}\left[e^{A(2\pi)}\int_0^{2\pi} b(s)e^{-A(s)}\,ds + \left(1-e^{A(2\pi)}\right)\int_0^t b(s)e^{-A(s)}\,ds\right] \\ &= \frac{e^{A(t)}}{1-e^{A(2\pi)}}\left[e^{A(2\pi)}\int_t^{2\pi} b(s)e^{-A(s)}\,ds + \int_0^t b(s)e^{-A(s)}\,ds\right] \\ &= \int_0^{2\pi} H(t,s)b(s)\,ds. \end{aligned}$$

Therefore, the first assertion follows. On the other hand, by the Cauchy–Schwarz inequality,

$$\left|x_b(t)\right| \leqslant \sqrt{\int_0^{2\pi} H^2(t,s)\,ds}\sqrt{\int_0^{2\pi} b^2(s)\,ds}.$$

Hence,

$$\|x_b\|_\infty \leqslant 2\pi \max_{t\in[0,2\pi]}\left\|H(t,\cdot)\right\|_2\|b\|_2.$$

This completes the proof. □

**Corollary 2.2.** *A deformation constant $M$ associated with $\bar{x}$ and $X$ is*

$$M := 2\pi \max_{t\in[0,2\pi]}\left\|H(t,\cdot)\right\|_2,$$

*where $H$ is given in (7) with $A(t)=\int_0^t \frac{\partial}{\partial x}X(\bar{x}(t),t)\,dt$.*

Now we prove a technical result that will allow us to compute in practice deformation constants. In fact we will find an upper bound of $M$ that will avoid the integration step needed in the computation of the norm $\|\cdot\|_2$. First, we introduce some notations.

Given a function $A:[0,2\pi]\to\mathbb{R}$, a positive number $\ell$, and a partition $t_i = ih$ with $i=0,1,\dots,N$, of the interval $[0,2\pi]$, where $h=2\pi/N$, we consider the function $L:[0,2\pi]\to\mathbb{R}$ given by the continuous linear piecewise function joining the points $(t_i, A(t_i)-\ell)$. Notice that $L(t)=\sum_{i=0}^{N-1} L_i(t)\mathbf{1}_{I_i}$, where $I_i=[t_i,t_{i+1}]$ and

$$L_i(t) = \frac{A(t_{i+1})-A(t_i)}{h}(t-t_i) + f(t_i) := -\frac{1}{2}(\alpha_i t + \beta_i).$$

We will say that $L$ is *an adequate lower bound of $A$* if it holds that $L(t) < A(t)$ for all $t \in [0,2\pi]$. Clearly, smooth functions always have adequate functions that approach them.

For each $m=0,1,\dots,N$ we define the function

$$\Psi_m(t) := \sum_{i=0}^{m-1} J_i + \lambda^2 \sum_{i=m-1}^{N-1} J_i + \left(1-\lambda^2\right)\frac{e^{\beta_m}}{\alpha_m}\left(e^{\alpha_m t} - e^{\alpha_m t_m}\right), \tag{10}$$

where $\lambda = e^{A(2\pi)}$, and

$$J_i := \int_{t_i}^{t_{i+1}} e^{-2L(s)}\,ds = \int_{t_i}^{t_{i+1}} e^{-2L_i(s)}\,ds = \frac{e^{\beta_i}}{\alpha_i}\left(e^{\alpha_i t_{i+1}} - e^{\alpha_i t_i}\right).$$

**Lemma 2.3.** *Let $L$ be an adequate lower bound of $A$, where $A$ is the function given in Lemma 2.1. Consider the functions $\Psi_m(t)$, with $m = 0, 1, \ldots, N-1$. Therefore, also following the notation introduced in that lemma, it holds that $\|x_b\|_\infty \leqslant N\|b\|_2$, where*

$$N = \frac{\sqrt{2\pi}}{|1-\lambda|} \max_{t\in[0,2\pi]} e^{A(t)} \sqrt{\sum_{m=0}^{N-1} \Psi_m(t)\mathbf{1}_{I_m}(t)}.$$

**Proof.** Recall that from Lemma 2.1, $\|x_b\|_\infty \leqslant M\|b\|_2$, where

$$M := 2\pi \max_{t\in[0,2\pi]} \left\|H(t,\cdot)\right\|_2.$$

Thus, we will find an upper bound of $M$. Since

$$H(t,s) = \frac{e^{A(t)}}{1-e^{A(2\pi)}}\left[e^{-A(s)}\mathbf{1}_{[0,t]}(s) + e^{A(2\pi)-A(s)}\mathbf{1}_{[t,2\pi]}(s)\right],$$

it holds that

$$\left\|H(t,\cdot)\right\|_2 = \frac{1}{\sqrt{2\pi}} \frac{e^{A(t)}}{|1-\lambda|}\sqrt{G(t)}$$

where

$$G(t) := \int_0^t e^{-2A(s)}\,ds + \lambda^2 \int_t^{2\pi} e^{-2A(s)}\,ds < \int_0^t e^{-2L(s)}\,ds + \lambda^2 \int_t^{2\pi} e^{-2L(s)}\,ds,$$

because $L(t) < A(t)$ for all $t \in [0, 2\pi]$.

Assume that $t \in I_m$. Thus,

$$\int_0^t e^{-2L(s)}\,ds = \sum_{i=0}^{m-1} J_i + \int_{t_m}^t e^{-2L_m(s)}\,ds,$$

$$\int_t^{2\pi} e^{-2L(s)}\,ds = \sum_{i=m}^{N-1} J_i + \int_t^{t_{m+1}} e^{-2L_m(s)}\,ds = \sum_{i=m-1}^{N-1} J_i - \int_{t_m}^t e^{-2L_m(s)}\,ds.$$

Therefore, for $t \in I_m$,

$$G(t) < \sum_{i=0}^{m-1} J_i + \lambda^2 \sum_{i=m-1}^{N-1} J_i + \left(1-\lambda^2\right)\int_{t_m}^t e^{\alpha_m s+\beta_m}\,ds = \Psi_m(t).$$

As a consequence, for $t \in [0, 2\pi]$,

$$G(t) < \sum_{m=0}^{N-1} \Psi_m(t) \mathbf{1}_{I_m}(t),$$

and the result follows. □

**Remark 2.4.** Notice that the above lemma provides a way for computing a deformation constant where there is no need of computing integrals. This will be very useful in concrete application, where the primitive of $e^{-2A(t)}$ is not computable, and so Corollary 2.2 is difficult to apply for obtaining $M$.

In the next result, which introduces the constant $K$ appearing in Theorem 1.1, $D^\circ$ denotes the topological interior of $D$.

**Lemma 2.5.** *Consider $X$ as in (1). Let $D$ be a closed interval, and let $\bar{x}(t)$ be a $2\pi$-periodic $\mathcal{C}^1$-function such that $\{\bar{x}(t)\colon t \in \mathbb{R}\} \subset D^\circ$. Define*

$$R(z,t) := X\bigl(\bar{x}(t) + z, t\bigr) - X\bigl(\bar{x}(t), t\bigr) - \frac{\partial}{\partial x} X\bigl(\bar{x}(t), t\bigr) z \tag{11}$$

*for all $z$ such that $\{\bar{x}(t) + z\colon t \in \mathbb{R}\} \subset D$. Then*

(i) $|R(z,t)| \leqslant \frac{K}{2} |z|^2$,
(ii) $|R(z,t) - R(\bar{z},t)| \leqslant K \max(|z|, |\bar{z}|) |z - \bar{z}|$,

*where*

$$K := \max_{(x,t) \in D \times [0,2\pi]} \left| \frac{\partial^2}{\partial x^2} X(x,t) \right|.$$

**Proof.** (i) By using Taylor's formula, for each $t$ it holds that

$$X\bigl(\bar{x}(t) + z, t\bigr) = X\bigl(\bar{x}(t), t\bigr) + \frac{\partial}{\partial x} X\bigl(\bar{x}(t), t\bigr) z + \frac{1}{2} \frac{\partial^2}{\partial x^2} X\bigl(\xi(t), t\bigr) z^2$$

for some $\xi(t) \in \langle \bar{x}(t), \bar{x}(t) + z \rangle$. Therefore

$$\bigl|R(z,t)\bigr| = \left| \frac{1}{2} \frac{\partial^2}{\partial x^2} X\bigl(\xi(t), t\bigr) \right| |z|^2 \leqslant \frac{K}{2} |z|^2,$$

as we wanted to prove.

(ii) From Rolle's theorem, for each fixed $t$ it follows that there exists $\eta(t) \in \langle z, \bar{z} \rangle$ such that

$$\bigl|R(z,t) - R(\bar{z},t)\bigr| \leqslant \left| \frac{\partial}{\partial z} R\bigl(\eta(t), t\bigr) \right| |z - \bar{z}|.$$

Applying again this theorem, but now to $\frac{\partial}{\partial z} R$, and by noticing that $\frac{\partial}{\partial z} R(z,t)|_{z=0} = 0$, we obtain

$$\left| \frac{\partial}{\partial z} R\bigl(\eta(t), t\bigr) \right| \leqslant \left| \frac{\partial^2}{\partial z^2} R\bigl(\omega(t), t\bigr) \right| \bigl|\eta(t)\bigr| = \left| \frac{\partial^2}{\partial x^2} X\bigl(\omega(t), t\bigr) \right| \bigl|\eta(t)\bigr| \leqslant K \bigl|\eta(t)\bigr|,$$

where $\omega(t) \in \langle 0, \eta(t)\rangle$. Note also that

$$\left|\eta(t)\right| \leqslant \max\left(|z|, |\bar{z}|\right).$$

Hence, the result follows combining the three inequalities. □

### 2.1. The Harmonic Balance Method

In this subsection we recall the HBM adapted to the setting of one-dimensional $2\pi$-periodic non-autonomous differential equations.

We are interested in finding periodic solutions of the $2\pi$-periodic differential equation (1), or equivalently, periodic functions which satisfy the following functional equation

$$\mathcal{F}\left(x(t)\right) := x'(t) - X\left(x(t), t\right) = 0. \tag{12}$$

Recall that any smooth $2\pi$-periodic function $x(t)$ can be written as its Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos(mt) + b_m \sin(mt)\right),$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(mt)\,dt, \quad \text{and} \quad b_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(mt)\,dt,$$

for all $m \geqslant 0$. Hence, it is natural to try to approximate the periodic solutions of the functional equation (12) by using truncated Fourier series, *i.e.* trigonometric polynomials.

Let us describe the HBM of order $N$. Consider a trigonometric polynomial

$$y_N(t) = \frac{r_0}{2} + \sum_{m=1}^{N}\left(r_m \cos(mt) + s_m \sin(mt)\right)$$

with unknowns $r_m = r_m(N)$, $s_m = s_m(N)$ for all $m \leqslant N$. Compute then the $2\pi$-periodic function $\mathcal{F}(y_N(t))$. It has also an associated Fourier series

$$\mathcal{F}\left(y_N(t)\right) = \frac{\mathcal{A}_0}{2} + \sum_{m=1}^{\infty}\left(\mathcal{A}_m \cos(mt) + \mathcal{B}_m \sin(mt)\right),$$

where $\mathcal{A}_m = \mathcal{A}_m(\mathbf{r}, \mathbf{s})$ and $\mathcal{B}_m = \mathcal{B}_m(\mathbf{r}, \mathbf{s})$, $m \geqslant 0$, with $\mathbf{r} = (r_0, r_1, \ldots, r_N)$ and $\mathbf{s} = (s_1, \ldots, s_N)$. The HBM consists of finding values $\mathbf{r}$ and $\mathbf{s}$ such that

$$\mathcal{A}_m(\mathbf{r}, \mathbf{s}) = 0 \quad \text{and} \quad \mathcal{B}_m(\mathbf{r}, \mathbf{s}) = 0 \quad \text{for } 0 \leqslant m \leqslant N. \tag{13}$$

The above set of equations is usually a very difficult nonlinear system of equations, and for this reason in various works including [3] and the references therein, only small values of $N$ are considered. We also remark that in general the coefficients of $y_N(t)$ and $y_{N+1}(t)$ do not coincide at all.

Notice that Eq. (13) is equivalent to

$$\int_0^{2\pi} \mathcal{F}\big(y_N(t)\big)\cos(mt)\,dt = 0 \quad \text{and} \quad \int_0^{2\pi} \mathcal{F}\big(y_N(t)\big)\sin(mt)\,dt = 0,$$

for $0 \leqslant m \leqslant N$.

The hope of the method is that the trigonometric polynomials found using this approach are "near" actual periodic solutions of the differential equation (1). In any case, as far as we know, the HBM for small $N$ is only a heuristic method that sometimes works quite well.

To end this subsection, we want to comment on a main difference between the non-autonomous case treated here and the autonomous one. In this second situation the periods of the searched periodic orbits, or equivalently their frequencies, are also treated as unknowns. The method then works similarly; see again [3].

## 3. Proof of the main result

**Proof of Theorem 1.1.** As a first step we prove the following result: consider the nonlinear differential equation

$$z' = X\big(z + \bar{x}(t), t\big) - X\big(\bar{x}(t), t\big) - s(t), \tag{14}$$

where $s(t)$ is given in (4). A $2\pi$-periodic function $z(t)$ is then a solution of (14) if and only if $z(t)+\bar{x}(t)$ is a $2\pi$-periodic solution of (1).

This is a consequence of the following equalities:

$$\begin{aligned}\big(z(t) + \bar{x}(t)\big)' &= \big[X\big(z(t) + \bar{x}(t), t\big) - X\big(\bar{x}(t), t\big) - s(t)\big] + \big[X\big(\bar{x}(t), t\big) + s(t)\big] \\ &= X\big(z(t) + \bar{x}(t), t\big).\end{aligned}$$

By using the function

$$R(z, t) = X\big(z + \bar{x}(t), t\big) - X\big(\bar{x}(t), t\big) - \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) z,$$

introduced in Lemma 2.5, Eq. (14) can be written as

$$z' = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) z + R(z, t) - s(t). \tag{15}$$

Let $\mathcal{P}$ be the space of $2\pi$-periodic $\mathcal{C}^0$-functions. To prove the first part of the theorem it suffices to see that Eq. (15) has a unique $\mathcal{C}^1$, $2\pi$-periodic solution $z^*(t)$, which belongs to the set

$$\mathcal{N} = \big\{z \in \mathcal{P}\colon \|z\|_\infty \leqslant 2MS\big\}.$$

To prove this last assertion, we will construct a contractive map $T : \mathcal{N} \to \mathcal{N}$. Because $\mathcal{N}$ is a complete space with the $\|\cdot\|_\infty$ norm, its fixed point will be a continuous function in $\mathcal{N}$ that will satisfy an integral equation, equivalent to (15). Finally we will see that this fixed point is in fact a $\mathcal{C}^1$ function, and it satisfies Eq. (15).

Let us define $T$. If $z \in \mathcal{N}$, then $T(z)$ is defined as the unique $2\pi$-periodic solution of the linear differential equation

$$y' = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) y + R\big(z(t), t\big) - s(t).$$

Notice that this map is well-defined, by Lemma 2.1, because $\bar{x}(t)$ is noncritical w.r.t. Eq. (1). Thus, $z_1$ satisfies

$$z_1' = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) z_1 + R\big(z(t), t\big) - s(t).$$

Let us prove that $T$ maps $\mathcal{N}$ into $\mathcal{N}$ and that it is a contraction. By Lemmas 2.1 and 2.5 and the hypotheses of the theorem

$$\begin{aligned}
\big\| T(z) \big\|_\infty = \|z_1\|_\infty &\leqslant M \big\| R\big(z(\cdot), \cdot\big) - s(\cdot) \big\|_2 \leqslant M\big( \big\| R\big(z(\cdot), \cdot\big) \big\|_2 + S\big) \\
&\leqslant M\big( \big\| R\big(z(\cdot), \cdot\big) \big\|_\infty + S\big) \leqslant M\left( \frac{K}{2} \|z\|_\infty^2 + S \right) \\
&\leqslant M\big(2KM^2S^2 + S\big) < 2MS,
\end{aligned}$$

where we have used in the last inequality that $2M^2KS < 1$.

To show that $T$ is a contraction on $\mathcal{N}$, take $z, \bar{z} \in \mathcal{N}$ and denote by $z_1 = T(z)$, $\bar{z}_1 = T(\bar{z})$. Then

$$\begin{aligned}
z_1' &= \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) z_1 + R\big(z(t), t\big) - s(t), \\
\bar{z}_1' &= \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) \bar{z}_1 + R\big(\bar{z}(t), t\big) - s(t).
\end{aligned}$$

Therefore,

$$(z_1 - \bar{z}_1)' = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big)(z_1 - \bar{z}_1) + R\big(z(t), t\big) - R\big(\bar{z}(t), t\big).$$

Again by Lemmas 2.1 and 2.5 and the hypotheses of the theorem,

$$\begin{aligned}
\big\| T(z) - T(\bar{z}) \big\|_\infty = \|z_1 - \bar{z}_1\|_\infty &\leqslant M \big\| R\big(z(\cdot), \cdot\big) - R\big(\bar{z}(\cdot), \cdot\big) \big\|_\infty \\
&\leqslant MK \max\big(\|z\|_\infty, \|\bar{z}\|_\infty\big) \|z - \bar{z}\|_\infty \leqslant 2M^2KS \|z - \bar{z}\|_\infty,
\end{aligned}$$

as we wanted to prove, because recall that $2M^2KS < 1$.

Thus, the sequence of functions $\{z_n(t)\}$ defined as

$$z_{n+1}'(t) = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) z_{n+1}(t) + R\big(z_n(t), t\big) - s(t),$$

with any $z_0(t) \in \mathcal{N}$, and $z_{n+1}(t)$ chosen to be periodic, converges uniformly to some function $x^*(t) \in \mathcal{N}$. In fact we also have that

$$z_{n+1}(t) = z_{n+1}(0) + \int_0^t \left( \frac{\partial}{\partial x} X\big(\bar{x}(w), w\big) z_{n+1}(w) + R\big(z_n(w), w\big) - s(w) \right) dw.$$

Therefore,

$$x^*(t) = x^*(0) + \int_0^t \left( \frac{\partial}{\partial x} X\big(\bar{x}(w), w\big) x^*(w) + R\big(x^*(w), w\big) - s(w) \right) dw.$$

We know that $x^*(t)$ is a continuous function, but from the above expression we obtain that it is indeed of class $\mathcal{C}^1$. Therefore $x^*(t)$ is a periodic solution of (15) and is the only one in $\mathcal{N}$, as we wanted to see.

To prove the hyperbolicity of $x^*(t)$, it suffices to show that

$$\int_0^{2\pi} \frac{\partial}{\partial x} X\big(x^*(t), t\big)\, dt \neq 0,$$

and study its sign; see [2]. We have that, fixed $t$,

$$\frac{\partial}{\partial x} X\big(x^*(t), t\big) = \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) + \frac{\partial^2}{\partial x^2} X\big(\xi(t), t\big)\big(x^*(t) - \bar{x}(t)\big),$$

for some $\xi(t) \in \langle x^*(t), \bar{x}(t)\rangle$. Therefore, since we have already proved that $|x^*(t) - \bar{x}(t)| < 2MS$,

$$\left| \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big) - \frac{\partial}{\partial x} X\big(x^*(t), t\big) \right| \leqslant 2KMS.$$

Then

$$\left| \int_0^{2\pi} \frac{\partial}{\partial x} X\big(\bar{x}(t), t\big)\, dt - \int_0^{2\pi} \frac{\partial}{\partial x} X\big(x^*(t), t\big)\, dt \right| \leqslant 4\pi KMS < \frac{2\pi}{M},$$

and the result follows because by hypothesis the first integral is, in absolute value, bigger than $2\pi/M$. $\square$

## 4. Applications

In this section we apply Theorem 1.1 for proving the existence and uniqueness of hyperbolic limit cycles, in a suitable region, of some planar rigid systems, which after some transformations can be converted into differential equation of the form (1). More precisely, we study some concrete cases of the family of rigid cubic systems

$$\dot{x} = -y - x\big(a + bx + x^2\big), \qquad \dot{y} = x - y\big(a + bx + x^2\big), \tag{16}$$

already considered in [1]. In that paper it is proved that (16) has at most one limit cycle, and when it exists, it is hyperbolic.

With our point of view we will find an explicit approximation of the limit cycle; see Proposition 1.2. In order to simplify the computations we first consider the case $a = -b = 1/10$, which in polar coordinates is written as (5):

$$r' = \frac{dr}{dt} = \frac{1}{10} r - \frac{1}{10}\cos(t) r^2 - \cos^2(t) r^3.$$

We want to find an approximation of the periodic solution of (5), which we will use for applying Theorem 1.1.

**First attempt: the HBM.** Following it, and according to Section 2.1, consider the equation

$$\mathcal{F}\big(r(t)\big) = r'(t) - \frac{1}{10}r + \frac{1}{10}\cos(t)r^2 + \cos^2(t)r^3 = 0, \tag{17}$$

which is clearly equivalent to (5).

Searching for a solution of the form $r(t) = r_0$ and imposing that the first harmonic of

$$\frac{1}{2}r_0^3 - \frac{1}{10}r_0 + \frac{1}{10}\cos(t)r_0^2 + \frac{1}{2}\cos(2t)r_0^3$$

vanishes, we obtain

$$\frac{1}{2}r_0\left(r_0^2 - \frac{1}{5}\right) = 0.$$

Hence $r_0 = \sqrt{5}/5 \approx 0.4472135954$ is the first-order solution given by the HBM. We obtain that the positive approximate solution is $r = \sqrt{5}/5$. For applying the second-order HBM we search for an approximation of the form

$$r(t) = r_0 + r_1\cos(t) + s_1\sin(t).$$

The vanishing of the coefficients of 1, $\cos(t)$ and $\sin(t)$ in $\mathcal{F}(r(t))$, provides the nonlinear system

$$\begin{aligned}
f(r_0, r_1, s_1) &:= \frac{1}{2}r_0^2 + \frac{9}{8}r_1^2 - \frac{1}{10} + \frac{3}{8}s_1^2 + \frac{1}{10}r_1 = 0,\\
g(r_0, r_1, s_1) &:= \frac{9}{4}r_0^2 r_1 - \frac{5}{8}r_1^3 + \frac{3}{8}r_1 s_1^2 + \frac{1}{10}r_0^2 + \frac{3}{40}r_1^2 + \frac{1}{40}s_1^2 - \frac{1}{10}r_1 + s_1 = 0,\\
h(r_0, r_1, s_1) &:= \frac{3}{4}r_0^2 s_1 + \frac{3}{8}r_1^2 s_1 + \frac{1}{8}s_1^3 + \frac{1}{20}r_1 s_1 - \frac{1}{10}s_1 - r_1 = 0.
\end{aligned}$$

Doing the resultants $\mathrm{Res}(f, g, r_0)$ and $\mathrm{Res}(f, h, r_0)$ we obtain respectively

$$\begin{gathered}
1775{r_1}^3 + 525r_1{s_1}^2 + 240{r_1}^2 + 20{s_1}^2 - 132r_1 - 400s_1 - 8 = 0,\\
105{r_1}^2 s_1 + 35{s_1}^3 + 8r_1 s_1 + 80r_1 - 4s_1 = 0.
\end{gathered}$$

Repeating the resultant between these last two equations with respect to $r_1$ we have

$$\begin{gathered}
17\,150\,000{s_1}^9 + 36\,970\,500{s_1}^7 - 35\,280\,000{s_1}^6 - 454\,252\,160{s_1}^5 - 9\,881\,600{s_1}^4\\
- 558\,027\,056{s_1}^3 + 264\,179\,200{s_1}^2 + 3\,704\,089\,600s_1 + 72\,704\,000 = 0.
\end{gathered}$$

The approximate real solution of this equation is $\tilde{s}_1 = -0.0196567414$, and then we have the respective approximate solutions $\tilde{r}_0 = 0.4471066159$, $\tilde{r}_1 = -0.0009814101$.

For our purposes we can consider simpler rational approximations of $\tilde{r}_0$, $\tilde{r}_1$ and $\tilde{s}_1$, with maintaining a similar accuracy. For finding these rational approximations, we seek them by performing the continued fraction expansion of these values. For instance,

$$\tilde{r}_0 = [0, 2, 4, 4, 2, 2, 2, 4, 2, 1, 1],$$

giving the convergents 1/2, 4/9, 17/38, 38/85, .... Similarly $\tilde{r}_1$ gives $-1/1018$, $-1/1019$, ... and $\tilde{s}_1$ gives $-1/50$, $-1/51$, .... At this point we have the following new candidate to be an approximation of the periodic solution

$$\bar{r}(t) = \frac{1}{2} - \frac{1}{1018}\cos(t) - \frac{1}{50}\sin(t).$$

Its accuracy w.r.t. Eq. (5) is

$$S = \left\| \bar{r}'(t) - \frac{1}{10}\bar{r}(t) + \frac{1}{10}\cos(t)\bar{r}(t)^2 + \cos^2(t)\bar{r}(t)^3 \right\|_2 \approx 0.046.$$

Doing all the computations needed to apply Theorem 1.1 we get that we are not under its hypotheses. Therefore we need to continue with the third-order HBM.

Performing the third-order approach we obtain five algebraic polynomial equations that we omit for the sake of simplicity. Unfortunately, neither using the resultant method as in the previous case, nor using the more sophisticated tool of Gröbner basis, our computers are able to obtain an approximate solution to start our theoretical analysis.

**A numerical approach.** First, we search for a numerical solution of (5) by using the Taylor series method. From this approximation we compute, again numerically, its first Fourier terms obtaining

$$\tilde{r}(t) = \sum_{k=0}^{3} r_k \cos(kt) + s_k \sin(kt),$$

where

$$\begin{aligned} &r_0 = 0.4483561517, \quad r_1 = -0.0024133439, \quad s_1 = -0.0193837572,\\ &r_2 = -0.0037463296, \quad s_2 = -0.0220176517,\\ &r_3 = -0.0012390886, \quad s_3 = 0.0003784656. \end{aligned}$$

The accuracy of $\tilde{r}(t)$ is 0.00289. If we take a new simpler approximation, using again some convergents of $r_k$ and $s_k$, we obtain

$$\bar{r}(t) = \frac{4}{9} - \frac{1}{693}\cos(t) - \frac{1}{51}\sin(t) - \frac{1}{653}\cos(2t) - \frac{1}{45}\sin(2t) - \frac{1}{780}\cos(3t), \tag{18}$$

with accuracy 0.00298, quite similar to the one of $\tilde{r}(t)$. Note that (18) is precisely the approximation of the periodic solution of (5) stated in Proposition 1.2.

**Proof of Proposition 1.2.** We already know that the accuracy of $\bar{r}(t)$ is $S := 0.003$. To apply Theorem 1.1 we will compute $M$ and $K$.

First we calculate $A(t) = \int_0^t \frac{\partial}{\partial r} X(\bar{r}(t), t)$.

$$\begin{aligned} A(t) = {} & \frac{2\,891\,685\,439}{72\,733\,752\,000} - \frac{347\,888\,350\,813\,299\,559}{1\,778\,094\,556\,332\,494\,400} t - \frac{561\,179}{36\,756\,720}\cos(t) - \frac{685\,338\,551}{8\,000\,712\,720}\sin(t) \\ & - \frac{757\,058\,717}{48\,004\,276\,320}\cos(2t) - \frac{40\,221\,206\,418\,131}{273\,447\,836\,421\,760}\sin(2t) - \frac{2\,923\,231}{576\,974\,475}\cos(3t) \end{aligned}$$

$$+\frac{37\,724\,429}{36\,003\,207\,240}\sin(3t)-\frac{353\,400\,139}{96\,008\,552\,640}\cos(4t)+\frac{17\,671\,001\,708\,653\,999}{42\,674\,269\,351\,979\,865\,600}\sin(4t)$$
$$+\frac{5\,358\,811}{300\,026\,727\,000}\cos(5t)+\frac{4\,708\,003}{20\,001\,781\,800}\sin(5t)+\frac{1537}{207\,810\,720}\cos(6t)$$
$$+\frac{43\,551\,971\,479}{1\,438\,264\,594\,166\,400}\sin(6t)+\frac{1}{327\,600}\cos(7t)-\frac{1}{4\,753\,840}\sin(7t)$$
$$-\frac{1}{12\,979\,200}\sin(8t).$$

Now, by using Lemma 2.3, we find a deformation constant $M$. In this case we use as a lower bound for $A$ the piecewise function $L$ formed by 7 straight lines and $\ell = 1/18$. We obtain that we can take $M = 7$. Therefore $2MS \approx 0.042$. Since it can be seen that $0.4 \leqslant \bar{r}(t) \leqslant 0.47$ we can consider the interval $I = [0.358, 0.512]$ in Theorem 1.1. In addition,

$$\max_{I\times[0,2\pi]}\left|\frac{\partial^2}{\partial r^2}X(r,t)\right| \leqslant \frac{1}{5}+6\|\bar{r}\|_\infty = \frac{1}{5}+6(0.512) = 3.272 =: K.$$

Finally, $2M^2KS \approx 0.962 < 1$, and the first part of Theorem 1.1 applies. Hence Eq. (5) has a periodic solution $r^*(t)$ satisfying

$$\left\|\bar{r}-r^*\right\|_\infty \leqslant 0.042, \tag{19}$$

which is the only one in this strip.

Moreover,

$$\left|\int_0^{2\pi}\frac{\partial}{\partial r}X\big(\bar{r}(t),t\big)\,dt\right| > 1.2.$$

Since $2\pi/M \approx 0.9$, the hyperbolicity of $r^*(t)$ follows by applying the second part of the theorem. □

Notice that the example of the system (16) that we have studied is $a = \lambda$ and $b = -\lambda$ with $\lambda = 1/10$. With the same techniques we see that the same function $\bar{r}(t)$ given in the statement of Proposition 1.2 is an approximation of the unique periodic orbit of the system when $|\lambda - 1/10| < 1/500$, which also satisfies (19).

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