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Wave equations with strong damping in Hilbert spaces

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ABSTRACT

We study the Cauchy problem for abstract dissipative equations in Hilbert spaces generalizing wave equations with strong damping terms in $\mathbf{R}^N$ or exterior domains. Our main result is a generalized diffusion phenomenon: the long time asymptotics of strongly damped wave equations is a combination of solutions of diffusion and wave equations. In particular, we obtain sharp decay estimates. The proofs rely on the energy method in the Fourier space and its generalization based on the spectral theorem for self-adjoint operators.

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1. Introduction

We consider the abstract Cauchy problem in a real Hilbert space H for the following equation:

$$u''(t) + Au(t) + Au'(t) = 0, \quad t \in (0, \infty), \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{1.2}$$

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where $A : \mathcal{D}(A) \subset H \to H$ is a nonnegative self-adjoint operator in $(H, \|\cdot\|)$ with a dense domain $\mathcal{D}(A)$. In this case, the square root $A^{1/2} : V \to H$ of A is well defined on its domain $V := \mathcal{D}(A^{1/2})$ and also becomes a nonnegative self-adjoint operator. Furthermore, V is a Hilbert space with respect to the graph norm $\|u\|_V := (\|u\|^2 + \|A^{1/2}u\|^2)^{1/2}$. The range of the operator $A^{1/2}$ is defined by $\mathcal{R}(A^{1/2}) := \{A^{1/2}w:\ w \in V\}$ and the total energy $E_u(t)$ of the solution u is defined by

$$E_u(t) := \frac{1}{2}\big(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2\big).$$

It is easy to see that problem (1.1)–(1.2) has a unique mild solution $u \in C([0,+\infty); V) \cap C^1([0,+\infty); H)$ provided that $[u_0, u_1] \in V \times H$; see Proposition 2.1. The goal of this paper is to study the long time behavior of such solutions and to give several applications to strongly damped wave equations with variable coefficients in $\mathbf{R}^N$ or exterior domains. We use the energy method in the Fourier space to derive L^2–L^2 estimates for (1.1)–(1.2) in terms of the abstract diffusion semigroup e^{-tA}. Combined with L^1–L^2 decay estimates for the latter, our results imply sharp decay estimates for (1.1)–(1.2). The diffusion phenomenon is observed in the sense that the asymptotic behavior of solutions is determined by the spectrum of A near 0 and naturally described in terms of e^{-tA} up to exponentially small remainders. This point of view is shared with Radu, Todorova and Yordanov [26] studying the diffusion phenomenon of damped wave equations.

Let us briefly describe some previous results on the strongly damped wave equations

$$u_{tt} - \Delta u - \Delta u_t = 0, \tag{1.3}$$

where the spatial variables belong to an unbounded domain $\Omega \subset \mathbf{R}^N$ (in the case of $\Omega \neq \mathbf{R}^N$ the Dirichlet null boundary conditions are imposed). In a pioneering work, Ponce [22] derived L^q–L^p decay estimates for the Cauchy problem (1.3) in the case when the smooth initial data $(u_0, u_1) \in W^{N^*,1}(\mathbf{R}^N)$ with $N^* \geqslant \frac{N}{2} + 2$ have a special form $u_1(x) = \partial_{x_i} f(x)$ with some function $f(x)$ (more generally $u_1 \in \mathcal{R}(\sqrt{-\Delta})$). The explicit expression of the solutions through the Fourier transform plays an important role there. Applications to nonlinear problems are also discussed. Georgiev and Todorova [6] studied the global well-posedness of wave equations with strong damping and nonlinearity. These authors considered the intermediate case of strong damping with a (t,x)-dependent coefficient decaying at infinity. Furthermore, Shibata [28] reexamined the earlier paper [22] and not only obtained more general and detailed L^p–L^q estimates but also investigated the diffusion phenomenon for (1.3). Both works rely heavily on the explicit expressions of solutions through the Fourier transform, so their methods do not apply to strongly damped abstract wave equation, wave equations with variable coefficients, or wave equations in exterior domains. Notable exceptions are the equations whose coefficients stabilize sufficiently fast at infinity to guarantee the existence of generalized Fourier transform; see Racke [24]. A more general approach to (1.3) was developed by Ikehata [10]. The initial step, based on the multiplier method, provides an inequality relating the energy of u with the L^2 norms of u and ∇u. The crucial next step yields a uniform bound on these L^2-norms through the Ikehata–Matsuyama device: given a solution u of (1.3),

$$\tilde{u}(t,x) = \int_0^t u(s,x)\,ds, \quad t > 0, \tag{1.4}$$

defines a solution of the non-homogeneous (1.3): $\tilde{u}_{tt} - \Delta\tilde{u} - \Delta\tilde{u}_t = u_1 - \Delta u_0$. The multiplier method for this equation combined with the Hardy inequality yields the energy bound of $\tilde{u}$. Since $\tilde{u}_t = u$, the energy of $\tilde{u}$ is larger than the L^2 norm of u. The method of [10] applies to boundary value problems in exterior domain and leads to strong estimates. In particular, the decay estimates in two dimensions of [10] seem to be sharp.

The above survey of results on the strongly damped wave equation is, of course, far from complete. Omitted are also the results concerning bounded domains beginning with Greenberg, MacCamy and Mizel [5], Mizohata and Ukai [15], and the references therein.

The decay estimates and diffusion phenomenon of the damped wave equation in $\mathbf{R}^N$

$$u_{tt} - \Delta u + u_t = 0 \tag{1.5}$$

are well understood now. There are many similarities with the strongly damped equation (1.3). However, a significant difference is that the solutions of (1.3) remain oscillatory while the solutions of (1.5) approach Gaussian functions (solutions of $-\Delta u + u_t = 0$) at large times; see [26] and the references therein. The classical decay estimates for the Laplace operator in $\mathbf{R}^N$ were established by Matsumura [14]. Nishihara [17,18], Han and Milani [7] and Todorova and Yordanov [29] showed the diffusion phenomenon and global well-posedness for nonlinear perturbations. Recently, more precise estimates and generalizations were obtained by Hayashi, Kaikina and Naumkin [8], Hosono and Ogawa [9], Karch [13], Narazaki [16], Ono [19] (exterior domain), Todorova and Yordanov [30], Radu, Todorova and Yordanov [25] (variable coefficients) and Wirth [32] (time dependent coefficients). The case of elliptic operators with periodic coefficients was the next step towards abstract treatment. Orive, Zuazua and Pazoto [20] used Bloch wave decomposition to show the diffusion phenomenon and derive an asymptotic series for the solutions of (1.5). The study of abstract diffusion phenomenon was initiated by Ikehata [11] although his result was stated only for the Laplace operator in exterior domains. Soon afterwards Ikehata and Nishihara [12] gave a completely abstract form that was applicable to any nonnegative self-adjoint operator A replacing $-\Delta$ in (1.5). The final version of diffusion phenomenon, without logarithmic losses of t, was established by Chill and Haraux [1] and its extension to time-dependent coefficients was found by Yamazaki [33]. Unfortunately, none of the aforementioned results on abstract diffusion was suitable for decay estimates of the damped wave equation (1.5) in three or higher dimensions; the best estimate [1] has the form

$$\big\|u(t) - e^{-tA}\big\{u(0) + u_t(0)\big\}\big\|_V \leqslant Ct^{-1}\big(\big\|u(0)\big\|_V + \big\|u_t(0)\big\|\big), \quad t \geqslant 1,$$

hence no decay rate faster than t^{-1} can be seen. This drawback was addressed in a recent paper by Radu, Todorova and Yordanov [26] where the diffusion phenomenon was shown in the form

$$\begin{aligned}\big\|u(t) - e^{-tA}\big\{u(0) + u_t(0)\big\}\big\|_V \leqslant{}& Ce^{-t/16}\big(\big\|u(0)\big\|_V + \big\|u_t(0)\big\|\big)\\ &+ Ct^{-1}\big(\big\|e^{-tA/2}u(0)\big\| + \big\|e^{-tA/2}u_t(0)\big\|\big), \quad t \geqslant 1;\end{aligned}$$

hence the remainder decays faster than both the solution of $u_{tt} + Au + u_t = 0$ and solution of $Au + u_t = 0$, whenever these decay slower than exponentially. Approximations involving e^{-tA} are useful since the abstract diffusion semigroup in H is simpler than the semigroup generated by Eq. (1.1) in $V \times H$. Under additional assumptions on A, such as the Markov property, e^{-tA} is well understood and various decay estimates are available; see the survey by Saloff-Coste [27].

Here we will establish a more complex diffusion phenomenon for the strongly damped wave equation (Theorem 1.3), where the leading term will be

$$e^{-tA/2}\big(\cos tA^{1/2}u(0) + A^{-1/2}\sin tA^{1/2}u_t(0)\big).$$

In order to derive our main results, we rely on the energy method in the abstract Fourier space, which has its origin in Umeda, Kawashima and Shizuta [31]; see also Dharmawardane, Rivera and Kawashima [3], and Chill and Haraux [1]. Another important tool is the Ikehata–Matsuyama substitution (1.4).

The main results concerning the abstract problem read as follows, where constants $C > 0$ appearing in all results below are independent of the initial data.

Theorem 1.1. *For any initial data $[u_0, u_1] \in V \times H$, the corresponding mild solution $u(t)$ to problem (1.1)–(1.2) satisfies*

$$E_u(t) \leqslant C\big(\big\|e^{-t\eta A}u_0\big\|^2 + \big\|e^{-t\eta A}u_1\big\|^2\big) + Ce^{-\eta t}\big(\big\|A^{1/2}u_0\big\|^2 + \|u_1\|^2\big), \quad t > 0,$$

where $C > 0$ and $\eta > 0$ are constants.

Theorem 1.2. *If $u_0 \in V$ and $u_1 \in \mathcal{R}(A^{1/2})$, then the corresponding mild solution $u(t)$ to problem (1.1)–(1.2) satisfies*

$$\big\|u(t)\big\|^2 \leqslant C\big(\big\|e^{-t\eta A}u_0\big\|^2 + \big\|e^{-t\eta A}w\big\|^2\big) + Ce^{-\eta t}\big(\|u_0\|_V^2 + \|u_1\|^2\big), \quad t > 0. \tag{1.6}$$

Here $w \in V$ is a vector, such that $A^{1/2}w = u_1$, while $C > 0$ and $\eta > 0$ are constants.

Theorem 1.3. *Let $k \geqslant 0$ and assume that $[u_0, u_1] \in \mathcal{D}(A^{k+1/2}) \times \mathcal{D}(A^k)$.*

(I) *The mild solution $u(t)$ to problem (1.1)–(1.2) satisfies*

$$E_{A^k u}(t) \leqslant C(t+1)^{-2k}\big(\big\|e^{-t\zeta A}u_0\big\|^2 + \big\|e^{-t\zeta A}u_1\big\|^2\big) + Ce^{-\eta t}\big(\big\|A^{k+1/2}u_0\big\|^2 + \big\|A^k u_1\big\|^2\big),$$

where $C > 0$, $\eta > 0$ and $\zeta > 0$ are constants.

(II) *The mild solution $u(t)$ to problem (1.1)–(1.2) also satisfies*

$$\begin{aligned}\big\|A^k\big(u(t) - u_{dif}(t)\big)\big\|^2 \leqslant{}& C(t+1)^{-2k}\big(\big\|e^{-t\zeta A}u_0\big\|^2 + \big\|e^{-t\zeta A}u_1\big\|^2\big)\\ &+ Ce^{-\eta t}\big(\big\|A^k u_0\big\|_V^2 + \big\|A^k u_1\big\|^2\big),\end{aligned}$$

where $C > 0$, $\eta > 0$ and $\zeta > 0$ are constants; *the diffusion approximation is defined by*

$$u_{dif}(t) = e^{-tA/2}\big(\cos tA^{1/2}u_0 + A^{-1/2}\sin tA^{1/2}u_1\big). \tag{1.7}$$

In particular, the following estimate holds for any data $[u_0, u_1] \in V \times H$:

$$\big\|u(t)\big\|^2 \leqslant C(t+1)\big(\big\|e^{-t\zeta A}u_0\big\|^2 + \big\|e^{-t\zeta A}u_1\big\|^2\big) + Ce^{-\eta t}\big(\|u_0\|_V^2 + \|u_1\|^2\big).$$

Remark 1.1. Theorem 1.3 part (II) shows that in the case of the strongly damped wave equation there is no diffusion phenomenon in the usual sense, but there is a generalized diffusion phenomenon – an asymptotic formula for solutions that combines wave and heat flows (semiflows). It is important to mention that the wave flow in (1.7) is highly simplified since it is restricted to the low frequencies of the operator A by the heat semiflow. If we compare this result with the diffusion phenomenon for the damped wave equation (1.5) we observe how complex is the former compared with the latter [26].

Remark 1.2. The estimates in Theorems 1.1, 1.2 and 1.3 imply the decay of the energy and the L^2-norms under additional assumptions on the operator A such as the Markov property together with the Nash inequality (see [26] and the survey [27]). In this case the sharp decay of the $\|e^{-t\eta A}u_i\|^2$, $i = 0, 1$ is a classical result (see [2], for example) and the theorems above allow us to transfer this decay to the strongly damped wave equations. These results together with various applications of the asymptotic formula (1.7) will be presented elsewhere.

Remark 1.3. Notice that the rapid decay for the L^2-norm of solutions for the strongly damped wave equation in Theorem 1.2 applies only for very special initial data as in the assumption of Theorem 1.2. For general data, a weaker decay for the L^2-norm as in Theorem 1.3(II) holds.

Notation. Throughout this paper, $\|\cdot\|_q$ stands for the usual $L^q(\Omega)$-norm, where $\Omega \subset \mathbf{R}^N$ is an open set. In particular, we use $\|\cdot\|$ instead of $\|\cdot\|_2$ where no confusion can occur.

We will consider one concrete model of the abstract problem (1.1)–(1.2).

1.1. Application

Let $\Omega \subset \mathbf{R}^N$ be an exterior domain with smooth boundary $\partial\Omega$. Consider the following problem for the strongly damped wave equation with variable coefficients in exterior domain

$$u_{tt}(t,x) - \nabla\cdot\big(b(x)\nabla u\big) - \nabla\cdot\big(b(x)\nabla u_t\big) = 0, \quad (t,x)\in(0,\infty)\times\Omega, \tag{1.8}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x\in\Omega, \tag{1.9}$$

$$u|_{\partial\Omega} = 0, \quad t>0, \tag{1.10}$$

where $b(x) = (b_{ij}(x))$ $(1\leqslant i,j\leqslant N)$ be an $N\times N$ symmetric matrix with entries $b_{ij}\in C(\Omega)$.

Classical arguments can be used if $b(x) = const$ and $\Omega = \mathbf{R}^N$ or if the variable coefficients operator stabilized at the Laplacian sufficiently fast at infinity to guarantee the existence of generalized Fourier transform. We derive the decay of solutions of the problem (1.8)–(1.10) under quite general assumptions on b in the exterior domain.

We assume that

$$b_0\big(1+|x|\big)^{\beta}|\xi|^2 \leqslant b(x)\xi\cdot\xi \leqslant b_1\big(1+|x|\big)^{\beta}|\xi|^2, \quad \xi\in\mathbf{R}^N_\xi,\ x\in\Omega, \tag{1.11}$$

where $b_0>0$, $b_1>0$ and $\beta\in[0,2)$. For high-order estimates, we also assume that

$$\big\|\nabla^{2l-1}b(x)\big\|_{N\times N} \leqslant b_2\big(1+|x|\big)^{\beta_0}, \quad l=1,\dots,s_0, \tag{1.12}$$

where $\beta_0\geqslant\beta$ and $s_0\geqslant 1$. The latter conditions insure that $\mathcal{D}(A^{s_0})$ contains all smooth functions decaying like large powers of $|x|$ as $|x|\to\infty$, i.e., the high-order estimates hold for a natural set of initial data (u_0,u_1) (see Corollary 1.1 below).

In order to apply Theorems 1.1, 1.2 and 1.3 to problem (1.8)–(1.10), we need estimates for the corresponding parabolic one:

$$-\nabla\cdot\big(b(x)\nabla v\big) + v_t = 0, \quad (t,x)\in(0,\infty)\times\Omega, \tag{1.13}$$

$$v(0,x) = v_0(x), \quad x\in\Omega, \tag{1.14}$$

$$v|_{\partial\Omega} = 0, \quad t>0. \tag{1.15}$$

If $\beta=0$, the asymptotic behavior of $v(t,x)$ follows from precise two-sided bounds on the associated heat kernel; see Zhang [34]. If $\beta\in(0,2)$, however, the asymptotic behavior is less understood. We quote a recent result from Radu, Todorova and Yordanov [26, Corollary 4.4].

Theorem 1.4. *Let $N\geqslant 3$ and $v_0\in L^1(\Omega)$. Then the solution v to problem (1.13)–(1.15) satisfies*

$$\big\|v(t,\cdot)\big\|^2 \leqslant Ct^{-\frac{N}{2-\beta}}\|v_0\|_1^2, \quad t>0.$$

It remains to give rigorous definitions of the concrete Hilbert space and self-adjoint operator. Let $H := L^2(\Omega)$ and V be the completion of $C_0^\infty(\Omega)$ in the norm

$$u \mapsto \left(\int_\Omega |u(x)|^2 \, dx + \int_\Omega b(x) \nabla u(x) \cdot \nabla u(x) \, dx \right)^{1/2}.$$

Consider $A_0 u := -\nabla \cdot (b(x) \nabla u)$ for $u \in C_0^\infty(\Omega)$. It is easy to check that $(A_0 u, u) \geqslant 0$ for all $u \in C_0^\infty(\Omega)$, i.e., A_0 is a nonnegative symmetric operator in H. We define A: $\mathcal{D}(A) \subset H$ as the Friedrichs extension of A_0 in H. Then A becomes nonnegative and self-adjoint with $\mathcal{D}(A) = V \cap \{u \in H:\ \nabla \cdot (b(x) \nabla u) \in H\}$. In this case, the square root $A^{1/2}$ is well defined with $\mathcal{D}(A^{1/2}) = V$ and satisfies

$$\left\| A^{1/2} u \right\| = \left(\int_\Omega b(x) \nabla u(x) \cdot \nabla u(x) \, dx \right)^{1/2} \quad \text{for } u \in \mathcal{D}\big(A^{1/2}\big). \tag{1.16}$$

Remark 1.4. Proposition 2.1 below implies that the solution $u(t,x)$ to problem (1.8) with initial data $(u_0, u_1) \in \mathcal{D}(A^{1/2}) \times L^2(\Omega)$, where A is defined by (1.16) and (1.11) holds, is in the space $C([0,+\infty); V) \cap C^1([0,+\infty); L^2(\Omega))$.

Under these preparations, Theorem 1.4 can be restated in terms of A as follows:

$$\left\| e^{-tA} v_0 \right\|^2 \leqslant C t^{-\frac{N}{2-\beta}} \|v_0\|_1^2, \quad t > 0. \tag{1.17}$$

Combining (1.17) with Theorems 1.1 and 1.2, one obtains new decay estimates.

Theorem 1.5. *Let $N \geqslant 3$ and the initial data satisfy*

$$(u_0, u_1) \in \big(\mathcal{D}\big(A^{1/2}\big) \cap L^1(\Omega)\big) \times \big(L^2(\Omega) \cap L^1(\Omega)\big), \tag{1.18}$$

where $\mathcal{A}$ is defined by (1.16). If (1.11) holds, the solution u to problem (1.8) satisfies

$$E_u(t) \leqslant C t^{-\frac{N}{2-\beta}} \big(\|u_0\|_1^2 + \left\| A^{1/2} u_0 \right\|^2 + \|u_1\|_1^2 + \|u_1\|^2 \big),$$
$$\left\| u(t,\cdot) \right\|^2 \leqslant C t^{-\frac{N}{2-\beta}} \big(\|u_0\|_1^2 + \left\| A^{1/2} u_0 \right\|^2 + \|u_0\|^2 + \|w\|_1^2 + \|u_1\|^2 \big),$$

where $u_1 = A^{1/2} w$.

The next result shows a type of diffusion phenomenon for strongly damped wave equations in exterior domains.

Theorem 1.6. *Let $N \geqslant 3$ and $k \geqslant 0$ is fixed. Assume that (1.11) and (1.12) hold for all $s_0 \geqslant k$ and the initial data satisfy the condition (1.18). Then the solution $u(t,x)$ to problem (1.8) satisfies*

$$\left\| A^k \big(u(t,\cdot) - u_{dif}(t,\cdot) \big) \right\|^2 \leqslant C t^{-\frac{N}{2-\beta} - 2k} \big(\|u_0\|_1^2 + + \|u_1\|_1^2 + \left\| A^{k+\frac{1}{2}} u_0 \right\|^2 + \left\| A^k u_0 \right\|^2 + \left\| A^k u_1 \right\|^2 \big)$$

for $t \geqslant 1$, whenever the norms on the right hand side are finite. The diffusion approximation u_{dif} is given by

$$u_{dif}(t,x) = e^{-tA/2} \big(\cos t A^{1/2} u_0 + A^{-1/2} \sin t A^{1/2} u_1 \big).$$

Corollary 1.1. *Let $N \geqslant 3$ and fix $k \geqslant 1$. Assume that (1.11) and (1.12) hold for all $s_0 \geqslant k$, so that $\mathcal{D}(A^k)$ includes all Schwartz functions $S(\Omega)$ supported inside Ω. Then the estimates in Theorem 1.6 apply to any initial data, such that*

$$(u_0, u_1) \subset S(\Omega) \times S(\Omega), \qquad supp(u_0, u_1) \subset \Omega.$$

Remark 1.5. Our approach yields interesting results only when the spectrum of operator A accumulates at zero, which does not hold for uniformly elliptic operators of the type $-\nabla \cdot (b(x)\nabla u)$ in bounded domains with the Dirichlet boundary condition. This is the reason to exclude applications to problems in bounded domains where all norms of $u(t)$ decay exponentially and can be included in the remainder $O(e^{-\eta t})$.

This paper is organized as follows. In Section 2 we shall prove Theorems 1.1, 1.2 and 1.3 by dividing the proof into several lemmas.

2. Proofs of the main abstract theorems

We begin with the existence and uniqueness of solutions to (1.1)–(1.2). The proof is a standard application of semigroup theory and Lumer–Phillips theorem [21] (see also [4]). Let us rewrite the problem:

$$\frac{d}{dt}\mathcal{U}(t) = -\mathcal{A}\mathcal{U}(t), \qquad \mathcal{U}(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{for } \mathcal{U}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \text{ and } \mathcal{A} = \begin{pmatrix} 0 & -I \\ A & A \end{pmatrix}.$$

The new Hilbert space is $\mathcal{H} = V \times H$ equipped with the inner product

$$\big\langle [v_1, v_2], [w_1, w_2] \big\rangle_{\mathcal{H}} = \langle v_1, w_1 \rangle_H + \big\langle A^{1/2} v_1, A^{1/2} w_1 \big\rangle_H + \langle v_2, w_2 \rangle_H.$$

Thus, $\|[v_1, v_2]\|^2_{\mathcal{H}} = \|v_1\|^2_V + \|v_2\|^2_H$. Define $\mathcal{D}(\mathcal{A}) = \{[v_1, v_2] \in \mathcal{H}:\ v_1 + v_2 \in \mathcal{D}(A)\}$.

Proposition 2.1. *The operator $-\mathcal{A}$ generates a C^0 semigroup on $\mathcal{H}$; hence problem (1.1)–(1.2) has a unique mild solution $[u(t), u'(t)] = e^{-t\mathcal{A}}[u_0, u_1]$ and $[u, u'] \in C([0, \infty); V \times H)$.*

Proof. We will apply the Lumer–Phillips theorem to $\mathcal{B} = -\mathcal{A} - I$, since $-\mathcal{A}$ is not exactly a dissipative operator in $\mathcal{H}$. Then the result for $\mathcal{A}$ follows from $e^{-\mathcal{A}t} = e^t e^{\mathcal{B}t}$. Here are the Lumer–Phillips conditions for $\mathcal{B}$: (a) $\mathcal{B}$ is a closed operator; (b) $\mathcal{B}$ is a dissipative operator, i.e., $\lambda\|\mathcal{U}\|_{\mathcal{H}} \leqslant \|\mathcal{B}\mathcal{U} - \lambda\mathcal{U}\|_{\mathcal{H}}$ for any $\lambda > 0$ and $\mathcal{U} \in \mathcal{D}(\mathcal{B})$; (c) $\mathcal{D}(\mathcal{B})$ is dense in $\mathcal{H}$; (d) $\mathcal{R}(I - \mathcal{B}) = \mathcal{H}$.

(a) Assume that $\mathcal{D}(\mathcal{B}) \ni \mathcal{U}_n \to \mathcal{U}$ and $\mathcal{B}\mathcal{U}_n \to \mathcal{V}$ in $\mathcal{H}$, where

$$\mathcal{U}_n = \begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This means $u_1^n \to u_1$ in V, $u_2^n \to u_2$ in H and

$$\begin{aligned} -u_1^n + u_2^n &\to v_1 \quad \text{in } V, \\ -u_2^n - A\big(u_1^n + u_2^n\big) &\to v_2 \quad \text{in } H, \end{aligned}$$

so that we obtain $u_1 \in V$ and $u_2 \in H$. Thus $u_1^n + u_2^n \to u_1 + u_2$ and $A(u_1^n + u_2^n) \to -(v_2 + u_2)$. We recall that A is a closed operator in H, so $u_1 + u_2 \in D(A)$ and $A(u_1 + u_2) = -(v_2 + u_2)$. In other words, $[u_1, u_2] \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A})$ and $\mathcal{B}([u_1, u_2]) = [v_1, v_2]$, i.e., the operator $\mathcal{B}$ is closed.

(b) It is easy to calculate

$$\begin{aligned}\langle \mathcal{BU}, \mathcal{U}\rangle_{\mathcal{H}} &= -\langle u_1, u_1\rangle_H + \langle u_2, u_1\rangle_H \\ &\quad - \langle A^{1/2}u_1, A^{1/2}u_1\rangle_H + \langle A^{1/2}u_2, A^{1/2}u_1\rangle_H \\ &\quad - \langle Au_1, u_2\rangle_H - \langle Au_2, u_2\rangle_H - \langle u_2, u_2\rangle_H.\end{aligned}$$

From $|\langle u, v\rangle_H| \leqslant (\|u\|_H^2 + \|v\|_H^2)/2$, we see that $\langle \mathcal{BU}, \mathcal{U}\rangle_{\mathcal{H}} \leqslant 0$. Hence

$$\langle \mathcal{BU} - \lambda\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = \langle \mathcal{BU}, \mathcal{U}\rangle_{\mathcal{H}} - \lambda\|\mathcal{U}\|_{\mathcal{H}}^2 \leqslant -\lambda\|\mathcal{U}\|_{\mathcal{H}}^2, \quad \mathcal{U} \in \mathcal{D}(\mathcal{B}).$$

We have $\lambda\|\mathcal{U}\|_{\mathcal{H}}^2 \leqslant |\langle \mathcal{BU} - \lambda\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}}| \leqslant \|\mathcal{BU} - \lambda\mathcal{U}\|_{\mathcal{H}}\|\mathcal{U}\|_{\mathcal{H}}$ or $\lambda\|\mathcal{U}\|_{\mathcal{H}} \leqslant \|\mathcal{BU} - \lambda\mathcal{U}\|_{\mathcal{H}}$.

(c) It is clear that $\mathcal{D}(\mathcal{B})$ is dense in $\mathcal{H}$, since $D(A)$ is dense in H.

(d) We will solve $(I - \mathcal{B})\mathcal{U} = \mathcal{V}$ for every $\mathcal{V} \in \mathcal{H}$:

$$\begin{aligned}2u_1 - u_2 &= v_1, \\ 2u_2 + A(u_1 + u_2) &= v_2.\end{aligned}$$

Using that A is a nonnegative self-adjoint operator in H, we find

$$\begin{aligned}u_1 &= (A + 2)(4I + 3A)^{-1}v_1 + (4I + 3A)^{-1}v_2, \\ u_2 &= -A(4I + 3A)^{-1}v_1 + 2(4I + 3A)^{-1}v_2.\end{aligned}$$

Thus, conditions (a)–(d) are verified. □

By the spectral theorem for self-adjoint operators (see [23, Theorem VIII.4, p. 260] and [1]), the operator A is unitary equivalent to a multiplication operator on some L^2-space. Therefore, it is sufficient to consider the case

$$\begin{aligned}H &= L^2(M; d\mu), \\ (Au)(\xi) &= a(\xi)u(\xi), \quad \xi \in M,\ u \in \mathcal{D}(A),\end{aligned}$$

where $(M, d\mu)$ is a measure space and $a(\xi)$ is a nonnegative μ-measurable real-valued function. In this case, we see that

$$\begin{aligned}\mathcal{D}(A) &= \{u \in H\colon au \in H\} = L^2\big(M; \big(1 + a^2\big)\,d\mu\big), \\ V &= \mathcal{D}\big(A^{1/2}\big) = L^2\big(M; (1 + a)\,d\mu\big).\end{aligned}$$

With this notation, if u is the mild solution to problem (1.1)–(1.2), then the function $u(t, \xi)$ satisfies the ordinary differential equation

$$u'' + a(\xi)u + a(\xi)u' = 0, \quad t > 0,\ \xi \in M, \tag{2.1}$$

together with the initial conditions

$$u(0, \xi) = u_0(\xi), \qquad u'(0, \xi) = u_1(\xi), \quad \xi \in M. \tag{2.2}$$

The total energy $E_u(t)$ is redefined as

$$E_u(t) = \frac{1}{2}\int_M \left|u'(t,\xi)\right|^2 d\mu + \frac{1}{2}\int_M a(\xi)\left|u(t,\xi)\right|^2 d\mu.$$

For the solution $u(t,\xi)$ to problem (2.1)–(2.2), we also set

$$\begin{aligned}
E_0(t,\xi) &:= \frac{1}{2}\left|u'\right|^2 + \frac{1}{2}a(\xi)|u|^2,\\
E(t,\xi) &:= \frac{1}{2}\left|u'\right|^2 + \frac{1}{2}a(\xi)|u|^2 + \beta\rho(\xi)u'u + \frac{1}{2}\beta\rho(\xi)a(\xi)|u|^2,\\
F(t,\xi) &:= a(\xi)\left|u'\right|^2 + \beta\rho(\xi)a(\xi)|u|^2,\\
R(t,\xi) &:= \beta\rho(\xi)\left|u'\right|^2.
\end{aligned}$$

The small parameter $\beta > 0$ is specified later and the function $\rho : M \to \mathbf{R}$ is defined by

$$\rho(\xi) = \rho_\varepsilon(\xi) = \begin{cases} \varepsilon a(\xi), & \xi \in M_1 \cup M_3,\\ \varepsilon, & \xi \in M_2, \end{cases}$$

where

$$\begin{aligned}
M_1 &:= \left\{\xi \in M\colon\ 0 < a(\xi) \leqslant 1\right\},\\
M_2 &:= \left\{\xi \in M\colon\ a(\xi) \geqslant 1\right\},\\
M_3 &:= \left\{\xi \in M\colon\ a(\xi) = 0\right\}.
\end{aligned}$$

Here $\varepsilon > 0$ is another small parameter specified later. Some of the sets M_i might be empty.

Let us consider problem (2.1)–(2.2) for a fixed ξ in the measure space M. By multiplying both sides of (2.1) with u_t, and further with $\beta\rho(\xi)u$, one has

$$\frac{d}{dt}E_0(t,\xi) + a(\xi)\left|u'\right|^2 = 0, \tag{2.3}$$

$$\frac{d}{dt}\left\{\beta\rho(\xi)u'u + \frac{1}{2}\beta a(\xi)\rho(\xi)|u|^2\right\} + \beta a(\xi)\rho(\xi)|u|^2 = \beta\rho(\xi)\left|u'\right|^2. \tag{2.4}$$

Adding (2.3) and (2.4), one has

$$\frac{d}{dt}E(t,\xi) + F(t,\xi) = R(t,\xi). \tag{2.5}$$

We first prove

Lemma 2.1. *For any $\varepsilon > 0$ and $\beta > 0$, it is true that*

$$R(t,\xi) \leqslant \beta\varepsilon F(t,\xi), \quad \xi \in M.$$

Proof. In the case when $\xi \in M_1$, it follows from the definition on $\rho(\xi)$ that

$$R(t,\xi) = \beta\frac{\rho(\xi)}{a(\xi)}a(\xi)\left|u'\right|^2 = \varepsilon\beta a(\xi)\left|u'\right|^2 \leqslant \varepsilon\beta F(t,\xi),$$

and in the case when $\xi \in M_2$ one has

$$R(t,\xi) = \beta\rho(\xi)\left|u'\right|^2 = \varepsilon\beta a(\xi)\left|u'\right|^2 \leqslant \varepsilon\beta F(t,\xi).$$

Hence, in both cases, it follows that

$$R(t,\xi) \leqslant \varepsilon\beta F(t,\xi), \tag{2.6}$$

for any $\xi \in M$ with $\xi \notin M_3$. However, (2.6) also holds for $\xi \in M_3$ by the definition of R and F. □

It is a consequence of Lemma 2.1 that

$$\frac{d}{dt}E(t,\xi) + (1-\varepsilon\beta)F(t,\xi) \leqslant 0, \tag{2.7}$$

provided that the parameters $\varepsilon > 0$ and $\beta > 0$.

Lemma 2.2. *There is a positive constant K_1, depending on ε, and β, and independent of ξ such that the following inequalities hold for all $\xi \in M$ with $\xi \notin M_3$:*

$$\frac{\rho(\xi) + \beta\rho(\xi)^2 a(\xi)^{-1}}{2a(\xi)} \leqslant K_1,$$
$$\frac{1}{2\beta} + \rho(\xi) \leqslant K_1.$$

Using Lemma 2.2 one can derive the following

Lemma 2.3. *For all $\xi \in M$ and all $t > 0$ it is true that*

$$\rho(\xi)E(t,\xi) \leqslant K_1 F(t,\xi).$$

Proof. It suffices to verify the case $\xi \notin M_3$. Indeed, since

$$u'u \leqslant \frac{1}{2}\left(\frac{1}{a(\xi)}\left|u'\right|^2 + a(\xi)|u|^2\right),$$

it follows from the definition of $E(t,\xi)$ that

$$\begin{aligned}\rho(\xi)E(t,\xi) &\leqslant \frac{1}{2}\rho(\xi)\left|u'\right|^2 + \frac{1}{2}\rho(\xi)a(\xi)|u|^2 + \frac{1}{2}\beta\rho(\xi)^2\frac{1}{a(\xi)}\left|u'\right|^2 \\ &\quad + \frac{1}{2}\beta\rho(\xi)^2 a(\xi)|u|^2 + \frac{1}{2}\beta\rho(\xi)^2 a(\xi)|u|^2 \\ &\leqslant \frac{\rho(\xi) + \beta\rho(\xi)^2 a(\xi)^{-1}}{2a(\xi)}a(\xi)\left|u'\right|^2 + \beta\rho(\xi)a(\xi)|u|^2\frac{1+2\beta\rho(\xi)}{2\beta}.\end{aligned}$$

Thus, Lemma 2.2 implies

$$\rho(\xi)E(t,\xi) \leqslant K_1 a(\xi)\left|u'\right|^2 + K_1\beta\rho(\xi)a(\xi)|u|^2,$$

which is the desired estimate. □

Combining Lemma 2.3 and inequality (2.7), one has

$$\frac{d}{dt}E(t,\xi) + (1-\varepsilon\beta)K_1^{-1}\rho(\xi)E(t,\xi) \leqslant 0, \tag{2.8}$$

for any $\xi \in M$. From (2.8) it follows that $E(t,\xi)$ is a Lyapunov functional and

$$E(t,\xi) \leqslant e^{-\alpha\rho_\varepsilon(\xi)t}E(0,\xi), \tag{2.9}$$

where $\alpha := (1-\varepsilon\beta)K_1^{-1}$ with sufficiently small $\varepsilon > 0$ and $\beta > 0$. It remains to obtain an equivalent but more convenient estimate in terms of $E_0(t,\xi)$. If $\xi \notin M_3$, we observe that

$$\pm\beta\rho(\xi)u'u \leqslant \frac{\beta}{2}a(\xi)\rho(\xi)|u|^2 + \frac{\rho(\xi)}{2}\frac{|u'|^2}{a(\xi)}. \tag{2.10}$$

The definition of $E(t,\xi)$ and (2.10) with $-$ sign yield

$$E(t,\xi) \geqslant \frac{1}{2}\left(1 - \frac{\rho(\xi)}{a(\xi)}\right)\left|u'\right|^2 + \frac{1}{2}a(\xi)|u|^2.$$

If we choose $\varepsilon \in (0,1)$ small, the definition of $\rho(\xi)$ shows that

$$1 - \frac{\rho(\xi)}{a(\xi)} \geqslant (1-\varepsilon) > 0 \tag{2.11}$$

for $\xi \notin M_3$. Hence, the lower bound on $E(t,\xi)$ becomes

$$E(t,\xi) \geqslant (1-\varepsilon)E_0(t,\xi). \tag{2.12}$$

Inequality (2.12) also holds true for $\xi \in M_3$. Thus, from (2.9) one has

$$E_0(t,\xi) \leqslant (1-\varepsilon)^{-1}e^{-\alpha\rho_\varepsilon(\xi)t}E(0,\xi), \quad \xi \in M. \tag{2.13}$$

For the upper bound of $E(t,\xi)$ in terms of $E_0(t,\xi)$, we use (2.10) with $+$ sign. If $\xi \notin M_3$, then

$$E(t,\xi) \leqslant \frac{1}{2}\left|u'\right|^2 + \frac{1}{2}a(\xi)|u|^2 + \beta a(\xi)\rho(\xi)|u|^2 + \frac{1}{2}\frac{\rho(\xi)}{a(\xi)}\left|u'\right|^2.$$

Since

$$\frac{\rho(\xi)}{a(\xi)} \leqslant \varepsilon, \qquad \rho(\xi) \leqslant \varepsilon,$$

for $\xi \notin M_3$, it follows that

$$E(t,\xi) \leqslant \max\{1+\beta\varepsilon, 1+\varepsilon\} E_0(t,\xi), \quad \xi \notin M_3. \tag{2.14}$$

Notice that (2.14) also holds true with $\xi \in M_3$. By (2.13) and (2.14) with $t=0$ one arrives at the following important estimate.

Lemma 2.4. *There exist constants $C = C(\beta,\varepsilon) > 0$ and $\alpha = \alpha(\beta,\varepsilon) > 0$, such that*

$$E_0(t,\xi) \leqslant C e^{-\alpha\rho_\varepsilon(\xi)t} E_0(0,\xi), \quad t > 0,\ \xi \in M,$$

provided that the two parameters $\beta > 0$ and $\varepsilon > 0$ are small enough.

Let us now prove the main Theorem 1.1. This part seems to be rather standard.

Proof of Theorem 1.1. By Lemma 2.4 one has

$$\begin{aligned} E_u(t) &= \Big(\int\limits_{M_1\cup M_3} + \int\limits_{M_2} \Big) E_0(t,\xi)\, d\mu \\ &\leqslant C \Big(\int\limits_{M_1\cup M_3} + \int\limits_{M_2} \Big) e^{-\alpha\rho_\varepsilon(\xi)t} E_0(0,\xi)\, d\mu = I_{low} + I_{high}. \end{aligned} \tag{2.15}$$

It is easy to see that

$$\begin{aligned} I_{high} &\leqslant C \int\limits_{M_2} e^{-\alpha\varepsilon t} \big(|u_1(\xi)|^2 + a(\xi)|u_0(\xi)|^2 \big)\, d\mu \\ &\leqslant C e^{-\alpha\varepsilon t} \int\limits_{M} \big(|u_1(\xi)|^2 + a(\xi)|u_0(\xi)|^2 \big)\, d\mu \leqslant C e^{-2\eta t} E_u(0), \end{aligned} \tag{2.16}$$

where $\eta = \alpha\varepsilon/2$. On the other hand, the low frequency part satisfies

$$\begin{aligned} I_{low} &\leqslant C \int\limits_{M_1\cup M_3} \big(e^{-\alpha\varepsilon a(\xi)t} |u_1(\xi)|^2 + e^{-\alpha\varepsilon a(\xi)t} a(\xi) |u_0(\xi)|^2 \big)\, d\mu \\ &\leqslant C \int\limits_{M} \big(e^{-\alpha\varepsilon a(\xi)t} |u_1(\xi)|^2 + e^{-\alpha\varepsilon a(\xi)t} |u_0(\xi)|^2 \big)\, d\mu \\ &= C \big\| e^{-t\eta A} u_1 \big\|^2 + C \big\| e^{-t\eta A} u_0 \big\|^2, \end{aligned} \tag{2.17}$$

where $\eta = \alpha\varepsilon/2$. Adding (2.15)–(2.17) gives the desired estimate. □

Proof of Theorem 1.2. For the solution $u(t)$ to problem (1.1)–(1.2), we set

$$\chi(t) = \int\limits_0^t u(s)\, ds.$$

Then the new function $\chi(t)$ satisfies

$$\chi''(t) + A\chi(t) + A\chi'(t) = u_1 + Au_0, \quad t \in (0,\infty), \tag{2.18}$$

$$\chi(0) = 0, \qquad \chi'(0) = u_0. \tag{2.19}$$

Using spectral analysis, similar to (2.1)–(2.2), we reduce (2.18)–(2.19) to an equivalent problem:

$$\chi'' + a(\xi)\chi + a(\xi)\chi' = u_1 + a(\xi)u_0, \quad (t,\xi) \in (0,\infty) \times M, \tag{2.20}$$

$$\chi(0,\xi) = 0, \qquad \chi'(0,\xi) = u_0(\xi), \quad \xi \in M. \tag{2.21}$$

When $\xi \notin M_3$, we can define a new function

$$W(t,\xi) = \chi(t,\xi) - u_0(\xi) - \frac{u_1(\xi)}{a(\xi)}.$$

Then $W(t,\xi)$ solves the following problem:

$$W''(t,\xi) + a(\xi)W(t,\xi) + a(\xi)W'(t,\xi) = 0, \quad (t,\xi) \in (0,\infty) \times (M \setminus M_3), \tag{2.22}$$

$$W(0,\xi) = -u_0(\xi) - \frac{u_1(\xi)}{a(\xi)}, \qquad W'(0,\xi) = u_0(\xi), \quad \xi \notin M_3,\ t > 0. \tag{2.23}$$

Notice that (2.22) and (2.23) are satisfied only in the case $\xi \notin M_3$. However, through the same computations as in Lemma 2.4 toward (2.22)–(2.23), we can derive

$$\left|W'(t,\xi)\right|^2 + a(\xi)\left|W(t,\xi)\right|^2 \leqslant Ce^{-\alpha\rho(\xi)t}\left(\left|u_0(\xi)\right|^2 + a(\xi)\left|u_0(\xi) + \frac{u_1(\xi)}{a(\xi)}\right|^2\right) \tag{2.24}$$

for all $\xi \notin M_3$. Using $W'(t,\xi) = \chi'(t,\xi) = u(t,\xi)$ and integrating both sides of (2.24) on the set $\{\xi\colon\ \delta \leqslant a(\xi)\}$ with small $\delta > 0$, one has the estimate

$$\begin{aligned}\int\limits_{\delta\leqslant a(\xi)} \left|u(t,\xi)\right|^2 d\mu &\leqslant \int\limits_{\delta\leqslant a(\xi)} \left(\left|W'(t,\xi)\right|^2 + a(\xi)\left|W(t,\xi)\right|^2\right) d\mu \\ &\leqslant C\int\limits_{\delta\leqslant a(\xi)} e^{-\alpha\rho(\xi)t}\left(\left|u_0(\xi)\right|^2 + a(\xi)\left|u_0(\xi)\right|^2 + \frac{|u_1(\xi)|^2}{a(\xi)}\right) d\mu.\end{aligned} \tag{2.25}$$

We split the right hand side into four integrals depending on $a(\xi) \leqslant 1$ or $a(\xi) > 1$:

$$\begin{aligned}&\int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha\varepsilon a(\xi)t}\left(\left|u_0(\xi)\right|^2 + a(\xi)\left|u_0(\xi)\right|^2\right) d\mu + e^{-\alpha\varepsilon t}\|u_0\|_V^2 \\ &\quad + \int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha a(\xi)t}\left[a(\xi)\right]^{-1}\left|u_1(\xi)\right|^2 d\mu + e^{-\alpha\varepsilon t}\|u_1\|^2.\end{aligned}$$

Observe that

$$\int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha\varepsilon a(\xi)t}\big(\big|u_0(\xi)\big|^2+a(\xi)\big|u_0(\xi)\big|^2\big)\,d\mu\leqslant C\big\|e^{-t(\alpha\varepsilon)/2A}u_0\big\|^2.$$

Then (2.25) implies

$$\int\limits_{\delta\leqslant a(\xi)}\big|u(t,\xi)\big|^2\,d\mu\leqslant C\int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha\varepsilon a(\xi)t}\big[a(\xi)\big]^{-1}\big|u_1(\xi)\big|^2\,d\mu + C\big\|e^{-t\eta A}u_0\big\|^2+Ce^{-2\eta t}\big(\|u_0\|_V^2+\|u_1\|^2\big),\tag{2.26}$$

where $\eta=\alpha\varepsilon/2>0$. In the case $u_1\in\mathcal{R}(A^{1/2})$, we have $u_1(\xi)=a(\xi)^{1/2}w(\xi)$ for some $w\in V$. Then

$$\int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha\varepsilon a(\xi)t}\big[a(\xi)\big]^{-1}\big|u_1(\xi)\big|^2\,d\mu=\int\limits_{\delta\leqslant a(\xi)\leqslant 1} e^{-\alpha\varepsilon a(\xi)t}\big|w(\xi)\big|^2\,d\mu\leqslant\big\|e^{-t\eta A}w\big\|^2.\tag{2.27}$$

After substituting in (2.26) one has to take the limit $\delta\downarrow 0$ to get (1.6). □

Proof of Theorem 1.3. (I) Since A^k commutes with time derivatives, $\chi=A^ku$ satisfies the same equation with different initial data:

$$\chi''(t)+A\chi(t)+A\chi'(t)=0,\quad t\in(0,\infty),$$
$$\chi(0)=A^ku_0,\qquad \chi'(0)=A^ku_1.$$

If $t\leqslant 1$, the claim follows from the standard energy estimate $E_\chi(t)\leqslant E_\chi(0)$. If $t>1$, we apply Theorem 1.1 to $\chi=A^ku$:

$$E_{A^ku}(t)\leqslant C\big(\big\|e^{-t\eta A}A^ku_1\big\|^2+\big\|e^{-t\eta A}A^ku_0\big\|^2\big)+Ce^{-t\eta}\big(\big\|A^ku_1\big\|^2+\big\|A^{k+1/2}u_0\big\|^2\big).$$

The first two terms are further bounded using

$$\big\|e^{-\eta tA}A^ku_i\big\|\leqslant\big\|e^{-\eta tA/2}A^k\big\|_{\mathrm{op}}\big\|e^{-\eta tA/2}u_i\big\|,\quad i=0,1,$$

and $\|e^{-t\eta A/2}A^k\|_{\mathrm{op}}\leqslant\sup_{a\geqslant 0}e^{-t\eta a/2}a^k\leqslant Ct^{-k}$. It follows that

$$\big\|e^{-\eta tA}A^ku_i\big\|\leqslant Ct^{-k}\big\|e^{-t\zeta A}u_i\big\|,\quad i=0,1,$$

where $\zeta=\eta/2$. This completes the proof of claim (I).

(II) The claim is trivial for $t\leqslant 1$, as both $\|u(t)\|^2$ and $\|u_{dif}(t)\|^2$ are estimated by

$$C\big(\big\|A^ku_1\big\|^2+\big\|A^ku_0\big\|_V^2\big).$$

In fact, we have

$$\|A^k u(t)\| \leqslant \|A^k u_0\| + \int_0^t \|A^k u_s(s)\|\,ds \quad \text{and} \quad E_{A^k u}(t) \leqslant E_{A^k u}(0),$$
$$\|A^k u_{dif}(t)\| \leqslant \|A^k u_0\| + t\|A^k u_1\|.$$

Hence we assume that $t > 1$ and rewrite Eq. (2.1) as

$$\big(e^{ta(\xi)/2}u\big)'' + a(\xi)e^{ta(\xi)/2}u = \frac{1}{4}\big[a(\xi)\big]^2 e^{ta(\xi)/2}u.$$

It is sufficient to consider only $\xi \in M_1 \cup M_3$, as $\xi \in M_2$ will have exponentially small contributions to $u(t)$ and $u_{dif}(t)$; we will use the same notations for the projections of these functions onto $M_1 \cup M_3$. Solving the equation for $u(t,\xi)$,

$$u(t,\xi) = e^{-ta(\xi)/2}\big(\cos t\big[a(\xi)\big]^{1/2}u_0(\xi) + \big[a(\xi)\big]^{-1/2}\sin t\big[a(\xi)\big]^{1/2}\big(u_1(\xi) + a(\xi)u_0(\xi)/2\big)\big)$$
$$+ \frac{1}{4}\int_0^t e^{-(t-s)a(\xi)/2}\big[a(\xi)\big]^{3/2}\sin(t-s)\big[a(\xi)\big]^{1/2}u(s,\xi)\,ds.$$

Thus $u(t) = u_{dif}(t) + r(t)$, where $u_{dif}(t)$ is defined in Theorem 1.3 while $r(t)$ is defined by

$$r(t) = \frac{1}{4}\int_0^t e^{-(t-s)A/2}A^{3/2}\sin(t-s)A^{1/2}u(s)\,ds.$$

The question becomes how to estimate $\|A^k r(t)\|^2$. We begin with

$$\|A^k r(t)\| \leqslant C\int_0^t \|e^{-(t-s)A/2}A^{k+3/2}u(s)\|\,ds$$

and then split the integral on $[0,t]$ into $[0,t/2]\cup[t/2,t]$:

$$\|A^k r(t)\| \leqslant C\int_0^{t/2} \|e^{-tA/4}A^{k+3/2}u(s)\|\,ds + C\int_{t/2}^t \|A^{k+3/2}u(s)\|\,ds.$$

The two terms satisfy different estimates:

$$\int_0^{t/2} \|e^{-tA/4}A^{k+3/2}u(s)\|\,ds \leqslant Ct^{-k}\max_{s\in[0,t/2]}\|A^{1/2}e^{-tA/8}u(s)\|, \tag{2.28}$$

$$\int_{t/2}^t \|A^{k+3/2}u(s)\|\,ds \leqslant Ct\max_{s\in[t/2,t]}\|A^{1/2}A^{k+1}u(s)\|. \tag{2.29}$$

We can complete the upper bound on (2.29) by applying claim (I) to $u(s)$ and by choosing smaller constants ζ and η. The bound on (2.28) follows from the energy estimate when the initial data are chosen as $[e^{-tA/8}u_0,\ e^{-tA/8}u_1]$. This completes the estimate of $\|A^k(u(t)-u_{dif}(t))\|$.

Using the estimate of $\|r(t)\|$ and observing that $\|A^{-1/2}\sin tA^{1/2}e^{-tA/2}u_1\| \leqslant t\|e^{-tA/2}u_1\|$, we readily derive the estimate of $\|u(t)\|$. □

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