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Control problems for weakly coupled systems with memory

Paola Loreti, Daniela Sforza *

Dipartimento di Scienze di Base e Applicate per l'Ingegneria Sezione di Matematica, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Roma, Italy

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Abstract

We investigate control problems for *wave–Petrovsky* coupled systems in the presence of memory terms. By writing the solutions as Fourier series, we are able to prove Ingham type estimates, and hence reachability results. Our findings have applications in viscoelasticity theory and linear acoustic theory.

MSC: 42A38; 45K05; 93C20

Keywords: Coupled systems; Convolution kernels; Fourier series; Ingham estimates; Reachability

1. Introduction

We will analyze control problems for *wave–Petrovsky* weakly coupled systems in the presence of memory terms. In particular, we will solve the reachability to a given target in a finite time, by using a harmonic approach based on Ingham type estimates.

In the papers [24,25] we studied reachability problems for a class of integro-differential equations

* Corresponding author.
E-mail addresses: paola.loreti@sbai.uniroma1.it (P. Loreti), daniela.sforza@sbai.uniroma1.it (D. Sforza).

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$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\,ds = 0, \quad t \in (0,T),\ x \in (0,\pi),$$

then generalized to spherical domains in [26] and to more general kernels in [29].

The interest for researching this type of control problems comes from the theory of viscoelasticity. Exponential kernels naturally arise in linear viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting–Thomson solids, see e.g. [32,34]. Despite of our paper that can be fitted in the classical theory of controllability, it is worth to mention that another type of control problems regards the possibility that the stresses, in addition to the motion, can be controlled. This different statement is motivated by many applications and for this reason considered very significative, see [34].

For other references in viscoelasticity theory see the seminal papers of Dafermos [2,3] and [33,18]. For other type of kernels, see [30].

As it is well known, viscoelastic relaxation kernels have to be completely monotone functions, that is, continuously differentiable to every order functions $K(t)$ satisfying

$$(-1)^n K^{(n)}(t) \ge 0 \quad \forall n \in \mathbb{N},\ \forall t \ge 0.$$

This class of relaxation kernels includes, as a significant case, the Prony sum

$$K(t) = \sum_{i=1}^{N} \beta_i e^{-\eta_i t}$$

with $\beta_i > 0$ and $\eta_i \ge 0$, $i = 1, \dots, N$. Prony-sum kernels have many implications for the dispersion and the attenuation phenomena in acoustic theory [7,8,36]. Moreover, the analysis of the 1-d wave equation of a vibrating string has analogies with seismic wave propagation [37]. It could be interesting to consider in the model the effect of viscosity as an attenuation phenomenon for seismic events.

Continuing along the lines traced by the research papers [24–26], we have done further investigations, which split into the following three directions a), b) and c).

a) The study of a more general relaxation kernel of Prony type in a single wave equation. This problem presents some difficulties with respect to the case of kernels consisting in a single exponential function, because we have to handle a more complicated spectral analysis, to compare the coefficients of the materials and to find conditions under which the reachability control problem may have a positive solution ([27], in preparation).
b) The analysis of weakly coupled systems of *wave–wave* type, with a memory term having a single-exponential kernel as in [25]. To find the eigenvalues, one has to study a fifth-degree equation: it turns out that the two couples of complex conjugate roots have the same asymptotic behavior ([28], preprint). See [12] for one of the first papers on wave–wave coupled PDE's without memory.
c) The study of weakly coupled systems of *wave–Petrovsky* type, again with memory terms consisting in a single-exponential kernel. The analysis of weakly coupled PDE's of wave–Petrovsky type without memory began in [13], where the harmonic analysis approach was successfully applied to get observability results.

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All these research lines need a deep analysis and extensive computations, with significant differences. In this paper we consider the third research problem c). We add to a wave equation an integral relaxation term and couple it with a Petrovsky type equation in the following way:

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \beta \displaystyle\int_0^t e^{-\eta(t-s)} u_{1xx}(s,x)\, ds + Au_2(t,x) = 0, \\ \quad t \in (0,T), \ x \in (0,\pi), \\ u_{2tt}(t,x) + u_{2xxxx}(t,x) + Bu_1(t,x) = 0, \end{cases} \tag{1.1}$$

$0 < \beta < \eta$, $A, B \in \mathbb{R}$, with null initial conditions

$$u_1(0,x) = u_{1t}(0,x) = u_2(0,x) = u_{2t}(0,x) = 0 \quad x \in (0,\pi), \tag{1.2}$$

and boundary conditions

$$u_1(t,0) = 0, \qquad u_1(t,\pi) = g_1(t) \quad t \in (0,T), \tag{1.3}$$

$$u_2(t,0) = u_{2xx}(t,0) = u_2(t,\pi) = 0, \qquad u_{2xx}(t,\pi) = g_2(t) \quad t \in (0,T). \tag{1.4}$$

We can consider g_i, $i = 1,2$, as control functions. The reachability problem consists in proving the existence of $g_i \in L^2(0,T)$ that steer a weak solution of system (1.1), subject to boundary conditions (1.3)–(1.4), from the null state to a given one in finite time. To better explain, we embrace the definition of reachability problem for systems with memory given by several authors in the literature, see for example [23,10,11,16,19,20,30,31].

Indeed, we mean the following: given $T > 0$ and

$$(u_{10}, u_{11}, u_{20}, u_{21}) \in L^2(0,\pi) \times H^{-1}(0,\pi) \times H_0^1(0,\pi) \times H^{-1}(0,\pi),$$

to find $g_i \in L^2(0,T)$ such that the weak solution u of problem (1.1)–(1.4) verifies the final conditions

$$u_1(T,x) = u_{10}(x), \qquad u_{1t}(T,x) = u_{11}(x), \quad x \in (0,\pi), \tag{1.5}$$

$$u_2(T,x) = u_{20}(x), \qquad u_{2t}(T,x) = u_{21}(x), \quad x \in (0,\pi). \tag{1.6}$$

We are able to bring about reachability results without any smallness assumption on the convolution kernels, as suggested by J.-L. Lions in [23, p. 258]. A common way to study exact controllability problems is the so-called Hilbert Uniqueness Method, introduced by Lagnese–Lions, see [15,21–23]. We will apply this method to system (1.1). The HUM method is based on a "uniqueness theorem" for the adjoint problem. To prove such uniqueness theorem we will employ some typical techniques of harmonic analysis, see [35,14]. This approach relies on Fourier series development for the solution (u_1, u_2) of the adjoint problem, which can be written as follows:

$$u_1(t,x) = \sum_{n=1}^{\infty}\left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right)\sin(nx)$$
$$+ \sum_{n=1}^{\infty}\left(D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right)\sin(nx), \tag{1.7}$$
$$u_2(t,x) = \sum_{n=1}^{\infty}\left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t}\right)\sin(nx)$$
$$+ e^{-\eta t}\sum_{n=1}^{\infty}\Re p_n\left(\frac{D_n}{\eta + ip_n} + \frac{\overline{D_n}}{\eta - ip_n}\right)\sin(nx), \tag{1.8}$$

where

$$r_n = \beta - \eta - \frac{\beta(\beta-\eta)^2}{\lambda_n} + O\left(\frac{1}{\lambda_n^{3/2}}\right), \tag{1.9}$$
$$\omega_n = \sqrt{\lambda_n} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_n}} + i\left(\frac{\beta}{2} - \frac{\beta(\beta-\eta)^2}{2\lambda_n}\right) + O\left(\frac{1}{\lambda_n^{3/2}}\right), \tag{1.10}$$
$$p_n = \lambda_n + \frac{AB}{2\lambda_n^3} + O\left(\frac{1}{\lambda_n^4}\right). \tag{1.11}$$

In this framework Ingham type estimates [9] play an important role. We need to establish inverse and direct inequalities for functions (1.7)–(1.8) evaluated at $x = \pi$, see (5.81) and (5.89) later on, obtaining them in the same sharp time of the nonintegral case.

In this approach the main difficulties are the following:

1. The study of the distribution of the eigenvalues on the complex plane. Indeed, the spectral analysis of the coupled system leads to a full fifth-degree equation governing the eigenvalues behavior. A method due to Haraux [6], subsequent to the seminal work of Ingham [9], enables us to consider only the asymptotic behavior of the eigenvalues related to the spatial operator. In order to get the asymptotic behavior of the eigenvalues, see (1.9)–(1.11), we need to develop an accurate asymptotic computation (see Section 4).
2. The generalization of Ingham's approach and the proof of the inverse inequality. This means that we are able to generalize the results contained in [9] and [6], see Theorems 5.19, 5.21 and Proposition 5.18 later on. In particular, a difficulty is the presence in u_2 of a series constant in time, but depending on the coefficients D_n, see (1.8). Due to its form, this series is difficult to handle. To overcome this impasse, as a first step we can neglect the dependence on D_n and treat the whole series simply as a constant. Following this approach, we have to use Haraux's method: we introduce the usual operator which annihilates the constant, so that we can apply an inverse estimate holding in the case the constant is null, and then recover the constant itself, see Theorem 5.10 later on.
3. Due to the finite speed of propagation, we expect the controllability time T to be sufficiently large. Indeed, we will find that $T > 2\pi/\gamma$, where γ is the gap of a branch of eigenvalues related to the integro-differential operator, see Theorem 6.1. The achievement of the time

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estimate $T > 2\pi/\gamma$ will require an accurate compensation in the analysis of the terms appearing in formulas (1.7) and (1.8), see Theorem 5.11 later on.

The plan of our paper is as follows. In Section 2 we give some preliminary results. In Section 3 we describe the Hilbert Uniqueness Method in an abstract setting. In Section 4 we give a detailed spectral analysis for a coupled system with memory. In Section 5 we prove our main results: Theorems 5.19, 5.21 and Proposition 5.18. Finally, in Section 6 we give a reachability result for a coupled system with memory.

2. Preliminaries

Let X be a real Hilbert space with scalar product $\langle\cdot\,,\,\cdot\rangle$ and norm $\|\cdot\|$. For any $T \in (0,\infty]$ we denote by $L^1(0,T;X)$ the usual spaces of measurable functions $v:(0,T)\to X$ such that one has

$$\|v\|_{1,T} := \int_0^T \big\|v(t)\big\|\,dt < \infty.$$

We shall use the shorter notation $\|v\|_1$ for $\|v\|_{1,\infty}$. We denote by $L^1_{loc}(0,\infty;X)$ the space of functions belonging to $L^1(0,T;X)$ for any $T\in(0,\infty)$. In the case of $X=\mathbb{R}$, we will use the abbreviations $L^1(0,T)$ and $L^1_{loc}(0,\infty)$ to denote the spaces $L^1(0,T;\mathbb{R})$ and $L^1_{loc}(0,\infty;\mathbb{R})$, respectively.

Classical results for integral equations (see, e.g., [5, Theorem 2.3.5]) ensure that, for any kernel $k\in L^1_{loc}(0,\infty)$ and $\psi\in L^1_{loc}(0,\infty;X)$, the problem

$$\varphi(t) - k*\varphi(t) = \psi(t), \qquad t\ge 0, \tag{2.1}$$

admits a unique solution $\varphi\in L^1_{loc}(0,\infty;X)$. In particular, if we take $\psi=k$ in (2.1), we can consider the unique solution $\varrho_k\in L^1_{loc}(0,\infty)$ of

$$\varrho_k(t) - k*\varrho_k(t) = k(t), \qquad t\ge 0.$$

Such a solution is called the *resolvent kernel* of k. Furthermore, for any ψ the solution φ of (2.1) is given by the variation of constants formula

$$\varphi(t) = \psi(t) + \varrho_k*\psi(t), \qquad t\ge 0,$$

where ϱ_k is the resolvent kernel of k.

We recall some results concerning integral equations in the case of decreasing exponential kernels, see for example [25, Corollary 2.2].

Proposition 2.1. *For $0<\beta<\eta$ and $T>0$ the following properties hold true.*

(i) *The resolvent kernel of $k(t)=\beta e^{-\eta t}$ is $\varrho_k(t)=\beta e^{(\beta-\eta)t}$.*

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(ii) *Given* $\psi \in L^1_{loc}(-\infty, T; X)$*, a function* $\varphi \in L^1_{loc}(-\infty, T; X)$ *is a solution of*

$$\varphi(t) - \beta \int_t^T e^{-\eta(s-t)}\varphi(s)\,ds = \psi(t) \quad t \le T,$$

if and only if

$$\varphi(t) = \psi(t) + \beta \int_t^T e^{(\beta-\eta)(s-t)}\psi(s)\,ds \quad t \le T.$$

Moreover, there exist two positive constants c_1, c_2 *depending on* β, η, T *such that*

$$c_1 \int_0^T \left|\varphi(t)\right|^2 dt \le \int_0^T \left|\psi(t)\right|^2 dt \le c_2 \int_0^T \left|\varphi(t)\right|^2 dt. \tag{2.2}$$

We state without proof an auxiliary result, useful in the sequel.

Lemma 2.2. *Given* $\beta, \eta \in \mathbb{R}$ *and* $\lambda, A, B \in \mathbb{R} \setminus \{0\}$*, a couple* (f, g) *of functions belonging to* $C^2([0,\infty))$ *is a solution of the system*

$$\begin{cases} f''(t) + \lambda f(t) - \lambda\beta \displaystyle\int_0^t e^{-\eta(t-s)} f(s)\,ds + Ag(t) = 0, \quad t \ge 0, \\ g''(t) + \lambda^2 g(t) + Bf(t) = 0, \quad t \ge 0, \end{cases} \tag{2.3}$$

if and only if $f \in C^5([0,\infty))$ *is a solution of the problem*

$$\begin{cases} f^{(5)}(t) + \eta f^{(4)}(t) + \big(\lambda + \lambda^2\big) f'''(t) + \big(\lambda(\eta-\beta) + \lambda^2\eta\big) f''(t) \\ \quad + \big(\lambda^3 - AB\big) f'(t) + \big(\lambda^3(\eta-\beta) - \eta AB\big) f(t) = 0 \quad t \ge 0, \\ f^{(4)}(0) = -\big(\lambda+\lambda^2\big) f''(0) + \lambda\beta f'(0) - \big(\eta\lambda\beta + \lambda^3 - AB\big) f(0), \end{cases} \tag{2.4}$$

and $g \in C^2([0,\infty))$ *is given by*

$$g(t) = -\frac{1}{A}\left[f''(t) + \lambda f(t) - \lambda\beta \int_0^t e^{-\eta(t-s)} f(s)\,ds \right]. \tag{2.5}$$

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3. The Hilbert Uniqueness Method

For reader's convenience, in this section we will describe the Hilbert Uniqueness Method for coupled systems. For another approach based on the ontoness of the solution operator, see e.g. [17,38].

Given $k \in L^1_{loc}(0, \infty)$ and $A, B \in \mathbb{R}$, we consider the following coupled system:

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \int_0^t k(t-s)u_{1xx}(s,x)\,ds + Au_2(t,x) = 0, \\ \quad t \in (0,T),\ x \in (0,\pi), \\ u_{2tt}(t,x) + u_{2xxxx}(t,x) + Bu_1(t,x) = 0, \end{cases} \tag{3.1}$$

with null initial conditions

$$u_1(0,x) = u_{1t}(0,x) = u_2(0,x) = u_{2t}(0,x) = 0 \quad x \in (0,\pi), \tag{3.2}$$

and boundary conditions, for $t \in (0,T)$,

$$u_1(t,0) = 0, \qquad u_1(t,\pi) = g_1(t), \tag{3.3}$$

$$u_2(t,0) = u_{2xx}(t,0) = 0, \qquad u_2(t,\pi) = g_2(t), \qquad u_{2xx}(t,\pi) = g_3(t). \tag{3.4}$$

For a reachability problem we mean the following: given $T > 0$ and taking $(u_{10}, u_{11}, u_{20}, u_{21})$ in a suitable space (to define later), find $g_i \in L^2(0,T)$, $i = 1,2,3$, such that the weak solution u of problem (3.1)–(3.4) verifies the final conditions, for $x \in (0,\pi)$,

$$\begin{cases} u_1(T,x) = u_{10}(x), \quad u_{1t}(T,x) = u_{11}(x), \\ u_2(T,x) = u_{20}(x), \quad u_{2t}(T,x) = u_{21}(x). \end{cases} \tag{3.5}$$

One can solve such reachability problems by the HUM method. To see that, we proceed as follows.

Given $(z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(0,\pi))^4$, we introduce the *adjoint* system of (3.1), that is

$$\begin{cases} z_{1tt}(t,x) - z_{1xx}(t,x) + \int_t^T k(s-t)z_{1xx}(s,x)\,ds + Bz_2(t,x) = 0, \\ \quad t \in (0,T),\ x \in (0,\pi), \\ z_{2tt}(t,x) + z_{2xxxx}(t,x) + Az_1(t,x) = 0, \\ z_1(t,0) = z_1(t,\pi) = z_2(t,0) = z_2(t,\pi) = z_{2xx}(t,0) = z_{2xx}(t,\pi) = 0, \end{cases} \tag{3.6}$$

with final data

$$z_1(T,\cdot) = z_{10}, \qquad z_{1t}(T,\cdot) = z_{11}, \qquad z_2(T,\cdot) = z_{20}, \qquad z_{2t}(T,\cdot) = z_{21}. \tag{3.7}$$

The above problem is well-posed, see e.g. [32]. Thanks to the regularity of the final data, the solution (z_1, z_2) of (3.6)–(3.7) is regular enough to consider the nonhomogeneous problem in $(0, T) \times (0, \pi)$

$$\begin{cases} \varphi_{1tt}(t,x) - \varphi_{1xx}(t,x) + \displaystyle\int_0^t k(t-s)\varphi_{1xx}(s,x)\,ds + A\varphi_2(t,x) = 0, \\ \varphi_{2tt}(t,x) + \varphi_{2xxxx}(t,x) + B\varphi_1(t,x) = 0, \\ \varphi_1(0,x) = \varphi_{1t}(0,x) = \varphi_2(0,x) = \varphi_{2t}(0,x) = 0 \quad x \in (0,\pi), \\ \varphi_1(t,0) = 0, \quad \varphi_1(t,\pi) = z_{1x}(t,\pi) - \displaystyle\int_t^T k(s-t)z_{1x}(s,\pi)\,ds \quad t \in [0,T], \\ \varphi_2(t,0) = \varphi_{2xx}(t,0) = 0, \quad \varphi_2(t,\pi) = -z_{2xxx}(t,\pi), \quad \varphi_{2xx}(t,\pi) = -z_{2x}(t,\pi). \end{cases} \tag{3.8}$$

As in the non-integral case, it can be proved that problem (3.8) admits a unique solution φ. So, we can introduce the following linear operator: for any $(z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(0,\pi))^4$ we define

$$\Psi(z_{10}, z_{11}, z_{20}, z_{21}) = \big(-\varphi_{1t}(T,\cdot), \varphi_1(T,\cdot), -\varphi_{2t}(T,\cdot), \varphi_2(T,\cdot)\big). \tag{3.9}$$

For any $(\xi_{10}, \xi_{11}, \xi_{20}, \xi_{21}) \in (C_c^\infty(0,\pi))^4$, let (ξ_1, ξ_2) be the solution of the following system

$$\begin{cases} \xi_{1tt}(t,x) - \xi_{1xx}(t,x) + \displaystyle\int_t^T k(s-t)\xi_{1xx}(s,x)\,ds + B\xi_2(t,x) = 0, \\ \xi_{2tt}(t,x) + \xi_{2xxxx}(t,x) + A\xi_1(t,x) = 0, \\ \xi_1(t,0) = \xi_1(t,\pi) = \xi_2(t,0) = \xi_2(t,\pi) = \xi_{2xx}(t,0) = \xi_{2xx}(t,\pi) = 0, \\ \xi_1(T,\cdot) = \xi_{10}, \quad \xi_{1t}(T,\cdot) = \xi_{11}, \quad \xi_2(T,\cdot) = \xi_{20}, \quad \xi_{2t}(T,\cdot) = \xi_{21}. \end{cases} \tag{3.10}$$

We will prove that

$$\begin{aligned} &\big\langle \Psi(z_{10}, z_{11}, z_{20}, z_{21}), (\xi_{10}, \xi_{11}, \xi_{20}, \xi_{21}) \big\rangle_{L^2(0,\pi)} \\ &= \int_0^T \varphi_1(t,\pi)\left(\xi_{1x}(t,\pi) - \int_t^T k(s-t)\xi_{1x}(s,\pi)\,ds\right) dt \\ &\quad - \int_0^T \varphi_{2xx}(t,\pi)\xi_{2x}(t,\pi)\,dt - \int_0^T \varphi_2(t,\pi)\xi_{2xxx}(t,\pi)\,dt. \end{aligned} \tag{3.11}$$

To this end, we multiply the first equation in (3.8) by ξ_1 and integrate on $(0, T) \times (0, \pi)$, so we have

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$$\int_0^\pi\int_0^T \varphi_{1tt}(t,x)\xi_1(t,x)\,dt\,dx - \int_0^T\int_0^\pi \varphi_{1xx}(t,x)\xi_1(t,x)\,dx\,dt$$
$$+\int_0^\pi\int_0^T\int_0^t k(t-s)\varphi_{1xx}(s,x)\,ds\,\xi_1(t,x)\,dt\,dx + A\int_0^T\int_0^\pi \varphi_2(t,x)\xi_1(t,x)\,dx\,dt = 0.$$

If we take into account that

$$\int_0^T\int_0^t k(t-s)\varphi_{1xx}(s,x)\,ds\,\xi_1(t,x)\,dt = \int_0^T \varphi_{1xx}(s,x)\int_s^T k(t-s)\xi_1(t,x)\,dt\,ds$$

and integrate by parts, then we have

$$\int_0^\pi \big(\varphi_{1t}(T,x)\xi_{10}(x) - \varphi_1(T,x)\xi_{11}(x)\big)\,dx + \int_0^\pi\int_0^T \varphi_1(t,x)\xi_{1tt}(t,x)\,dt\,dx$$
$$+\int_0^T \varphi_1(t,\pi)\xi_{1x}(t,\pi)\,dt - \int_0^T\int_0^\pi \varphi_1(t,x)\xi_{1xx}(t,x)\,dx\,dt$$
$$-\int_0^T \varphi_1(s,\pi)\int_s^T k(t-s)\xi_{1x}(t,\pi)\,dt\,ds + \int_0^\pi\int_0^T \varphi_1(s,x)\int_s^T k(t-s)\xi_{1xx}(t,x)\,dt\,ds\,dx$$
$$+A\int_0^T\int_0^\pi \varphi_2(t,x)\xi_1(t,x)\,dx\,dt = 0.$$

By recalling $\xi_{1tt} - \xi_{1xx} + \int_t^T k(s-t)\xi_{1xx}(s,\cdot)\,ds = -B\xi_2$, as a consequence of the above equation we obtain

$$\int_0^\pi \big(\varphi_{1t}(T,x)\xi_{10}(x) - \varphi_1(T,x)\xi_{11}(x)\big)\,dx$$
$$+\int_0^T \varphi_1(t,\pi)\left(\xi_{1x}(t,\pi) - \int_t^T k(s-t)\xi_{1x}(s,\pi)\,ds\right)dt$$
$$+\int_0^T\int_0^\pi \big(A\varphi_2(t,x)\xi_1(t,x) - B\varphi_1(t,x)\xi_2(t,x)\big)\,dx\,dt = 0. \tag{3.12}$$

In a similar way, we multiply the second equation in (3.8) by ξ_2 and integrate by parts on $(0,T)\times(0,\pi)$ to get

$$\int_0^\pi \big(\varphi_{2t}(T,x)\xi_{20}(x) - \varphi_2(T,x)\xi_{21}(x)\big)\,dx + \int_0^\pi \int_0^T \varphi_2(t,x)\xi_{2tt}(t,x)\,dt\,dx$$
$$- \int_0^T \big(\varphi_{2xx}(t,\pi)\xi_{2x}(t,\pi) + \varphi_2(t,\pi)\xi_{2xxx}(t,\pi)\big)\,dt$$
$$+ \int_0^T \int_0^\pi \varphi_2(t,x)\xi_{2xxxx}(t,x)\,dx\,dt + B\int_0^T \int_0^\pi \varphi_1(t,x)\xi_2(t,x)\,dx\,dt = 0,$$

whence, in virtue of

$$\xi_{2tt} + \xi_{2xxxx} = -A\xi_1,$$

we get

$$\int_0^\pi \big(\varphi_{2t}(T,x)\xi_{20}(x) - \varphi_2(T,x)\xi_{21}(x)\big)\,dx$$
$$- \int_0^T \big(\varphi_{2xx}(t,\pi)\xi_{2x}(t,\pi) + \varphi_2(t,\pi)\xi_{2xxx}(t,\pi)\big)\,dt$$
$$+ \int_0^T \int_0^\pi \big(B\varphi_1(t,x)\xi_2(t,x) - A\varphi_2(t,x)\xi_1(t,x)\big)\,dx\,dt = 0. \tag{3.13}$$

By summing Eqs. (3.12) and (3.13) and taking into account

$$\big\langle \Psi(z_{10},z_{11},z_{20},z_{21}), (\xi_{10},\xi_{11},\xi_{20},\xi_{21})\big\rangle_{L^2(0,\pi)}$$
$$= \int_0^\pi \big(\varphi_1(T,x)\xi_{11}(x) - \varphi_{1t}(T,x)\xi_{10}(x) + \varphi_2(T,x)\xi_{21}(x) - \varphi_{2t}(T,x)\xi_{20}(x)\big)\,dx,$$

we have that (3.11) holds true.

Now, taking $(\xi_{10},\xi_{11},\xi_{20},\xi_{21}) = (z_{10},z_{11},z_{20},z_{21})$ in (3.11), we have

$$\big\langle \Psi(z_{10},z_{11},z_{20},z_{21}), (z_{10},z_{11},z_{20},z_{21})\big\rangle_{L^2(0,\pi)}$$
$$= \int_0^T \Big|z_{1x}(t,\pi) - \int_t^T k(s-t)z_{1x}(s,\pi)\,ds\Big|^2\,dt$$
$$+ \int_0^T \big|z_{2x}(t,\pi)\big|^2\,dt + \int_0^T \big|z_{2xxx}(t,\pi)\big|^2\,dt. \tag{3.14}$$

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As a consequence, we can introduce a semi-norm on the space $(C_c^\infty(\Omega))^4$. Precisely, we define, for $(z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(\Omega))^4$,

$$\|(z_{10}, z_{11}, z_{20}, z_{21})\|_F^2 := \int_0^T \left| z_{1x}(t,\pi) - \int_t^T k(s-t) z_{1x}(s,\pi)\,ds \right|^2 dt + \int_0^T |z_{2x}(t,\pi)|^2\,dt + \int_0^T |z_{2xxx}(t,\pi)|^2\,dt. \tag{3.15}$$

If $k(t) = \beta e^{-\eta t}$, thanks to (2.2), $\|\cdot\|_F$ is a norm if and only if the following uniqueness theorem holds.

Theorem 3.1. *If (z_1, z_2) is the solution of problem (3.6)–(3.7) such that*

$$z_{1x}(t,\pi) = z_{2x}(t,\pi) = z_{2xxx}(t,\pi) = 0, \quad \forall t \in [0,T],$$

then

$$z_1(t,x) = z_2(t,x) = 0 \quad \forall (t,x) \in [0,T] \times [0,\pi].$$

If Theorem 3.1 holds true, then we can define the Hilbert space F as the completion of $(C_c^\infty(\Omega))^4$ for the norm (3.15). Moreover, the operator Ψ extends uniquely to a continuous operator, denoted again by Ψ, from F to the dual space F' in such a way that $\Psi : F \to F'$ is an isomorphism.

In conclusion, if we prove the uniqueness result given by Theorem 3.1 and

$$F = H_0^1(0,\pi) \times L^2(0,\pi) \times H_0^3(0,\pi) \times H^1(0,\pi)$$

with the equivalence of the respective norms, then we can solve the reachability problem (3.1)–(3.5) taking $(u_{10}, u_{11}, u_{20}, u_{21}) \in L^2(0,\pi) \times H^{-1}(0,\pi) \times H^{-1}(0,\pi) \times H^{-3}(0,\pi)$.

In addition, if $g_2(t) \equiv 0$ in the reachability problem (3.1)–(3.5), we must take $\varphi_2(t,\pi) \equiv 0$ in problem (3.8). So, in view of (3.11) formula (3.15) becomes

$$\|(z_{10}, z_{11}, z_{20}, z_{21})\|_F^2 := \int_0^T \left| z_{1x}(t,\pi) - \int_t^T k(s-t) z_{1x}(s,\pi)\,ds \right|^2 dt + \int_0^T |z_{2x}(t,\pi)|^2\,dt, \tag{3.16}$$

and the uniqueness result can be written as follows:

Theorem 3.2. *If (z_1, z_2) is the solution of problem (3.6)–(3.7) such that*

$$z_{1x}(t,\pi) = z_{2x}(t,\pi) = 0, \quad \forall t \in [0,T],$$

then

$$z_1(t,x)=z_2(t,x)=0 \quad \forall (t,x)\in[0,T]\times[0,\pi].$$

Finally, by proving the uniqueness result given by Theorem 3.2 and

$$F=H_0^1(0,\pi)\times L^2(0,\pi)\times H_0^1(0,\pi)\times H^{-1}(0,\pi)$$

with the equivalence of the respective norms, we can solve the reachability problem (3.1)–(3.5) for

$$(u_{10},u_{11},u_{20},u_{21})\in L^2(0,\pi)\times H^{-1}(0,\pi)\times H_0^1(0,\pi)\times H^{-1}(0,\pi).$$

4. Spectral analysis

In this section we will elaborate a detailed spectral analysis for the adjoint problem.

Let $L : D(L)\subset X\to X$ be a self-adjoint positive linear operator on X with dense domain $D(L)$ and let $\{\lambda_j\}_{j\ge 1}$ be a strictly increasing sequence of eigenvalues for the operator L with $\lambda_j>0$ and $\lambda_j\to\infty$ such that the sequence of the corresponding eigenvectors $\{w_j\}_{j\ge1}$ constitutes a Hilbert basis for X.

Fix two real numbers A, B and consider the following weakly coupled system:

$$\begin{cases} u_1''(t)+Lu_1(t)-\beta\displaystyle\int_0^t e^{-\eta(t-s)}Lu_1(s)\,ds+Au_2(t)=0, & t\ge 0,\\ u_2''(t)+L^2u_2(t)+Bu_1(t)=0, & t\ge0,\\ u_1(0)=u_{10}, \quad u_1'(0)=u_{11}, \\ u_2(0)=u_{20}, \quad u_2'(0)=u_{21}. \end{cases} \tag{4.1}$$

We have

$$u_{10}=\sum_{j=1}^{\infty}\alpha_{1j}w_j, \quad \alpha_{1j}=\langle u_{10},w_j\rangle, \qquad \sum_{j=1}^{\infty}\alpha_{1j}^2\lambda_j<\infty,$$

$$u_{11}=\sum_{j=1}^{\infty}\rho_{1j}w_j, \quad \rho_{1j}=\langle u_{11},w_j\rangle, \qquad \sum_{j=1}^{\infty}\rho_{1j}^2<\infty,$$

$$u_{20}=\sum_{j=1}^{\infty}\alpha_{2j}w_j, \quad \alpha_{2j}=\langle u_{20},w_j\rangle, \qquad \sum_{j=1}^{\infty}\alpha_{2j}^2\lambda_j<\infty,$$

$$u_{21}=\sum_{j=1}^{\infty}\rho_{2j}w_j, \quad \rho_{2j}=\langle u_{21},w_j\rangle, \qquad \sum_{j=1}^{\infty}\frac{\rho_{2j}^2}{\lambda_j}<\infty.$$

We will seek the solution $(u_1(t),u_2(t))$ of system (4.1) with components written as sums of series, that is

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$$u_1(t)=\sum_{j=1}^{\infty} f_{1j}(t)w_j, \qquad u_2(t)=\sum_{j=1}^{\infty} f_{2j}(t)w_j, \qquad f_{ij}(t)=\langle u_i(t), w_j\rangle, \quad i=1,2.$$

If we put the above expressions for $u_1(t)$ and $u_2(t)$ into (4.1) and multiply by w_j, $j\in\mathbb{N}$, then we have that $(f_{1j}(t), f_{2j}(t))$ is the solution of system

$$\begin{cases} f_{1j}''(t)+\lambda_j f_{1j}(t)-\lambda_j\beta\displaystyle\int_0^t e^{-\eta(t-s)}f_{1j}(s)\,ds+Af_{2j}(t)=0,\\ f_{2j}''(t)+\lambda_j^2 f_{2j}(t)+Bf_{1j}(t)=0,\\ f_{1j}(0)=\alpha_{1j}, \quad f_{1j}'(0)=\rho_{1j},\\ f_{2j}(0)=\alpha_{2j}, \quad f_{2j}'(0)=\rho_{2j}. \end{cases} \tag{4.2}$$

Thanks to Lemma 2.2, $(f_{1j}(t), f_{2j}(t))$ is the solution of problem (4.2) if and only if $f_{1j}(t)$ is the solution of the Cauchy problem

$$\begin{cases} f_{1j}^{(5)}(t)+\eta f_{1j}^{(4)}(t)+\big(\lambda_j^2+\lambda_j\big)f_{1j}'''(t)+\big(\eta\lambda_j^2+\lambda_j(\eta-\beta)\big)f_{1j}''(t)\\ \quad +\big(\lambda_j^3-AB\big)f_{1j}'(t)+\big(\lambda_j^3(\eta-\beta)-\eta AB\big)f_{1j}(t)=0, \quad t\ge 0,\\ f_{1j}(0)=\alpha_{1j}, \quad f_{1j}'(0)=\rho_{1j},\\ f_{1j}''(0)=-\lambda_j\alpha_{1j}-A\alpha_{2j}, \quad f_{1j}'''(0)=-A\rho_{2j}-\lambda_j\rho_{1j}+\lambda_j\beta\alpha_{1j},\\ f_{1j}^{(4)}(0)=\big(\lambda_j^2+\lambda_j\big)(\lambda_j\alpha_{1j}+A\alpha_{2j})+\lambda_j\beta\rho_{1j}-\lambda_j\eta\beta\alpha_{1j}-\big(\lambda_j^3-AB\big)\alpha_{1j}, \end{cases} \tag{4.3}$$

and $f_{2j}(t)$ is given by

$$f_{2j}(t)=-\frac{1}{A}\left[f_{1j}''(t)+\lambda_j f_{1j}(t)-\lambda_j\beta\int_0^t e^{-\eta(t-s)}f_{1j}(s)\,ds\right]. \tag{4.4}$$

We proceed to solve (4.3). To this end, we have to evaluate the solutions of the characteristic equation of the fifth degree

$$\begin{aligned} &\Lambda^5+\eta\Lambda^4+\big(\lambda_j^2+\lambda_j\big)\Lambda^3+\big(\eta\lambda_j^2+\lambda_j(\eta-\beta)\big)\Lambda^2\\ &\quad+\big(\lambda_j^3-AB\big)\Lambda+\lambda_j^3(\eta-\beta)-\eta AB=0. \end{aligned} \tag{4.5}$$

The asymptotic behavior of the solutions of Eq. (4.5) as $j\to\infty$ is as follows:

$$\Lambda_{1j}=\beta-\eta-\frac{\beta(\beta-\eta)^2}{\lambda_j}+O\left(\frac{1}{\lambda_j^2}\right)=\beta-\eta+O\left(\frac{1}{\lambda_j}\right), \tag{4.6}$$

$$\begin{aligned}
\Lambda_{2j} &= -\frac{\beta}{2} + \frac{\beta(\beta-\eta)^2}{2}\frac{1}{\lambda_j} + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}}\right] + O\left(\frac{1}{\lambda_j^{3/2}}\right) \\
&= -\frac{\beta}{2} + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}}\right] + O\left(\frac{1}{\lambda_j}\right), \qquad (4.7)
\end{aligned}$$

$$\Lambda_{3j} = \overline{\Lambda_{2j}} = -\frac{\beta}{2} - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}}\right] + O\left(\frac{1}{\lambda_j}\right), \qquad (4.8)$$

$$\Lambda_{4j} = -\frac{\beta AB}{2\lambda_j^5} + i\left(\lambda_j + \frac{AB}{2\lambda_j^3} + \frac{AB}{2\lambda_j^4} + \frac{AB}{2\lambda_j^5}\right) + O\left(\frac{1}{\lambda_j^6}\right) = i\lambda_j + O\left(\frac{1}{\lambda_j^3}\right), \qquad (4.9)$$

$$\Lambda_{5j} = \overline{\Lambda_{4j}} = -i\lambda_j + O\left(\frac{1}{\lambda_j^3}\right). \qquad (4.10)$$

Therefore, we can write the solution of (4.3) in the form

$$\begin{aligned}
f_{1j}(t) &= C_{1j}e^{t\Lambda_{1j}} + C_{2j}e^{t\Lambda_{2j}} + C_{3j}e^{t\Lambda_{3j}} + C_{4j}e^{t\Lambda_{4j}} + C_{5j}e^{t\Lambda_{5j}} \\
&= \sum_{k=1}^{5} C_{kj}e^{t\Lambda_{kj}}, \qquad (4.11)
\end{aligned}$$

where C_{kj} are complex numbers. To determine the coefficients C_{kj}, we have to impose the initial conditions in (4.3), that is we must solve the system

$$\begin{cases}
C_{1j} + C_{2j} + C_{3j} + C_{4j} + C_{5j} = f_{1j}(0), \\
\Lambda_{1j}C_{1j} + \Lambda_{2j}C_{2j} + \Lambda_{3j}C_{3j} + \Lambda_{4j}C_{4j} + \Lambda_{5j}C_{5j} = f'_{1j}(0), \\
\Lambda_{1j}^2C_{1j} + \Lambda_{2j}^2C_{2j} + \Lambda_{3j}^2C_{3j} + \Lambda_{4j}^2C_{4j} + \Lambda_{5j}^2C_{5j} = f''_{1j}(0), \\
\Lambda_{1j}^3C_{1j} + \Lambda_{2j}^3C_{2j} + \Lambda_{3j}^3C_{3j} + \Lambda_{4j}^3C_{4j} + \Lambda_{5j}^3C_{5j} = f'''_{1j}(0), \\
\Lambda_{1j}^4C_{1j} + \Lambda_{2j}^4C_{2j} + \Lambda_{3j}^4C_{3j} + \Lambda_{4j}^4C_{4j} + \Lambda_{5j}^4C_{5j} = f^{(4)}_{1j}(0).
\end{cases} \qquad (4.12)$$

Therefore, we have the following asymptotic behavior as $j \to \infty$ of the coefficients C_{kj}:

$$\begin{cases}
C_{1j} = \dfrac{\beta}{\lambda_j}\big(\rho_{1j} + \alpha_{1j}(\beta-\eta)\big) + (\alpha_{1j}+\rho_{1j})O\left(\dfrac{1}{\lambda_j^2}\right), \\
C_{2j} = \dfrac{\alpha_{1j}}{2} - \dfrac{i}{4\lambda_j^{1/2}}(\beta\alpha_{1j} + 2\rho_{1j}) + (\alpha_{1j}+\rho_{1j})O\left(\dfrac{1}{\lambda_j}\right), \\
C_{3j} = \overline{C_{2j}}, \\
C_{4j} = \dfrac{A\alpha_{2j}}{2\lambda_j^2} + (\alpha_{2j} - i\rho_{2j})\dfrac{A}{2\lambda_j^3} + (\alpha_{2j}+\rho_{2j})O\left(\dfrac{1}{\lambda_j^{7/2}}\right), \\
C_{5j} = \overline{C_{4j}}.
\end{cases} \qquad (4.13)$$

Thanks to the expressions of C_{kj} and Λ_{kj}, $k = 1, 2, 3$, we note that the function

$$f_{1j}^*(t) = C_{1j} e^{t\Lambda_{1j}} + C_{2j} e^{t\Lambda_{2j}} + \overline{C_{2j}} e^{t\overline{\Lambda_{2j}}}, \tag{4.14}$$

verifies the problem

$$\begin{cases} \left(f_{1j}^*\right)''(t) + \lambda_j f_{1j}^*(t) - \lambda_j \beta \displaystyle\int_0^t e^{-\eta(t-s)} f_{1j}^*(s)\, ds = 0, \\ f_{1j}^*(0) = \alpha_{1j}, \quad \left(f_{1j}^*\right)'(0) = \rho_{1j}, \end{cases}$$

see [25, Section 6]. Therefore, in view of (4.4) the coefficients f_{2j} are given by

$$\begin{aligned} f_{2j}(t) = &-\frac{1}{A}\left(C_{4j}\left(\Lambda_{4j}^2 + \lambda_j - \frac{\beta\lambda_j}{\eta + \Lambda_{4j}}\right)e^{t\Lambda_{4j}} + \beta e^{-\eta t}\lambda_j \frac{C_{4j}}{\eta + \Lambda_{4j}}\right) \\ &- \frac{1}{A}\left(\overline{C_{4j}}\left(\overline{\Lambda_{4j}}^2 + \lambda_j - \frac{\beta\lambda_j}{\eta + \overline{\Lambda_{4j}}}\right)e^{t\overline{\Lambda_{4j}}} + \beta e^{-\eta t}\lambda_j \frac{\overline{C_{4j}}}{\eta + \overline{\Lambda_{4j}}}\right) \quad t \geq 0. \end{aligned} \tag{4.15}$$

The proof of the following lemma is based on considerations similar to those used for analogous results in [24], but, for the sake of completeness, we prefer to give it.

Lemma 4.1. *The following estimates hold true:*

(i) *there exist some constants $c_1, c_2 > 0$ such that we have, for any $j \in \mathbb{N}$,*

$$\frac{c_1}{\lambda_j}\left(\alpha_{1j}^2 \lambda_j + \rho_{1j}^2\right) \leq |C_{2j}|^2 \leq \frac{c_2}{\lambda_j}\left(\alpha_{1j}^2 \lambda_j + \rho_{1j}^2\right); \tag{4.16}$$

(ii) *there exists a constant $c > 0$ such that we have, for any $j \in \mathbb{N}$,*

$$\frac{|C_{1j}|}{|C_{2j}|} \leq \frac{c}{\lambda_j^{1/2}}; \tag{4.17}$$

(iii) *there exist some constants $c_1, c_2 > 0$ such that we have, for any $j \in \mathbb{N}$,*

$$\frac{c_1}{\lambda_j^5}\left(\alpha_{2j}^2 \lambda_j + \frac{\rho_{2j}^2}{\lambda_j}\right) \leq |C_{4j}|^2 \leq \frac{c_2}{\lambda_j^5}\left(\alpha_{2j}^2 \lambda_j + \frac{\rho_{2j}^2}{\lambda_j}\right). \tag{4.18}$$

Proof. (i) First, we observe that

$$|C_{2j}|^2 = \frac{1}{4}\left(\alpha_{1j}^2 + \frac{\rho_{1j}^2}{\lambda_j}\right) + \alpha_{1j}^2 O\left(\frac{1}{\lambda_j}\right) + \alpha_{1j}\rho_{1j} O\left(\frac{1}{\lambda_j}\right) + \rho_{1j}^2 O\left(\frac{1}{\lambda_j^2}\right).$$

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We can assume that for any $j \in \mathbb{N}$ $\alpha_{1j} \neq 0$ or $\rho_{1j} \neq 0$, and hence by the previous formula we obtain

$$\frac{|C_{2j}|^2}{\alpha_{1j}^2 + \frac{\rho_{1j}^2}{\lambda_j}} = \frac{1}{4} + \frac{(\alpha_{1j}^2 + \frac{\rho_{1j}^2}{\lambda_j})O(\frac{1}{\lambda_j}) + \alpha_{1j}\frac{\rho_{1j}}{\lambda_j^{1/2}}O(\frac{1}{\lambda_j^{1/2}})}{\alpha_{1j}^2 + \frac{\rho_{1j}^2}{\lambda_j}} \to \frac{1}{4} \quad \text{as } j \to \infty,$$

so, (4.16) follows.

(ii) Since

$$|C_{1j}| \le \frac{|\alpha_{1j}|(\eta - \beta) + |\rho_{1j}|}{\lambda_j}\left(\beta + \frac{|\alpha_{1j}|O(\frac{1}{\lambda_j}) + |\rho_{1j}|O(\frac{1}{\lambda_j})}{|\alpha_{1j}|(\eta - \beta) + |\rho_{1j}|}\right),$$

we have, for any $j \in \mathbb{N}$,

$$|C_{1j}| \le c^* \frac{|\alpha_{1j}|(\eta - \beta) + |\rho_{1j}|}{\lambda_j}, \tag{4.19}$$

for some $c^* > 0$. Therefore, by using also (4.16) we get, for any $j \in \mathbb{N}$,

$$\frac{|C_{1j}|}{|C_{2j}|} \le \frac{c^*}{\sqrt{c_1\lambda_j}} \frac{|\alpha_{1j}|(\eta - \beta) + |\rho_{1j}|}{\sqrt{\alpha_{1j}^2\lambda_j + \rho_{1j}^2}} \le \frac{c}{\lambda_j^{1/2}},$$

so, we obtain (4.17).

(iii) Notice that

$$|C_{4j}|^2 = \frac{A^2}{4}\left(\frac{\alpha_{2j}^2}{\lambda_j^4} + \frac{\rho_{2j}^2}{\lambda_j^6}\right) + \alpha_{2j}^2 o\left(\frac{1}{\lambda_j^4}\right) + \rho_{2j}^2 o\left(\frac{1}{\lambda_j^6}\right),$$

and hence it follows

$$\frac{|C_{4j}|^2}{\frac{\alpha_{2j}^2}{\lambda_j^4} + \frac{\rho_{2j}^2}{\lambda_j^6}} = \frac{A^2}{4} + \frac{\alpha_{2j}^2 o(\frac{1}{\lambda_j^4}) + \rho_{2j}^2 o(\frac{1}{\lambda_j^6})}{\frac{\alpha_{2j}^2}{\lambda_j^4} + \frac{\rho_{2j}^2}{\lambda_j^6}} \to \frac{A^2}{4}, \quad \text{as } j \to \infty,$$

that is, (4.18) holds true. □

In conclusion, keeping in mind (4.11) and (4.15), the components $u_1(t)$ and $u_2(t)$ of the solution for the Cauchy problem (4.1) are given by

$$u_1(t) = \sum_{j=1}^{\infty}\left(C_{1j}e^{t\Lambda_{1j}} + C_{2j}e^{t\Lambda_{2j}} + \overline{C_{2j}}e^{t\overline{\Lambda_{2j}}} + C_{4j}e^{t\Lambda_{4j}} + \overline{C_{4j}}e^{t\overline{\Lambda_{4j}}}\right)w_j,$$

$$u_2(t) = -\frac{1}{A}\sum_{j=1}^{\infty} C_{4j}\left(\Lambda_{4j}^2 + \lambda_j - \frac{\beta\lambda_j}{\eta + \Lambda_{4j}}\right)e^{t\Lambda_{4j}}w_j$$
$$- \frac{1}{A}\sum_{j=1}^{\infty} \overline{C_{4j}}\left(\overline{\Lambda_{4j}}^2 + \lambda_j - \frac{\beta\lambda_j}{\eta + \overline{\Lambda_{4j}}}\right)e^{t\overline{\Lambda_{4j}}}w_j$$
$$- \frac{\beta}{A}e^{-\eta t}\sum_{j=1}^{\infty}\lambda_j\left(\frac{C_{4j}}{\eta + \Lambda_{4j}} + \frac{\overline{C_{4j}}}{\eta + \overline{\Lambda_{4j}}}\right)w_j$$

for any $t \geq 0$, where Λ_{kj} and C_{kj} are defined by formulas (4.6)–(4.10) and (4.13) respectively. We introduce, for any $n \geq 1$, the following numbers $r_n, R_n \in \mathbb{R}$ and $\omega_n, C_n, p_n, D_n \in \mathbb{C}$:

$$r_n = \Lambda_{1n} = \beta - \eta + O\left(\frac{1}{\lambda_n}\right),$$
$$\Re\omega_n = \Im\Lambda_{2n} = \sqrt{\lambda_n} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_n}} + O\left(\frac{1}{\lambda_n}\right),$$
$$\Im\omega_n = -\Re\Lambda_{2n} = \frac{\beta}{2} + O\left(\frac{1}{\lambda_n}\right),$$
$$\Re p_n = \Im\Lambda_{4n} = \lambda_n + O\left(\frac{1}{\lambda_n^3}\right), \tag{4.20}$$
$$\Im p_n = -\Re\Lambda_{4n} = O\left(\frac{1}{\lambda_n^5}\right), \tag{4.21}$$
$$R_n = C_{1n}, \qquad C_n = C_{2n}, \qquad D_n = C_{4n}. \tag{4.22}$$

Thanks to these notations, the functions u_1 and u_2 can be written in the form

$$u_1(t) = \sum_{n=1}^{\infty}\left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t} + D_n e^{ip_n t} + \overline{D_n}e^{-i\overline{p_n}t}\right)w_n, \tag{4.23}$$

$$u_2(t) = \sum_{n=1}^{\infty}\left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\right)w_n - \frac{\beta}{A}e^{-\eta t}\sum_{n=1}^{\infty}\Re p_n\left(\frac{D_n}{\eta + ip_n} + \frac{\overline{D_n}}{\eta - ip_n}\right)w_n, \tag{4.24}$$

for any $t \geq 0$, where

$$d_n = \frac{1}{A}\left(p_n^2 - \Re p_n + \frac{\beta\Re p_n}{\eta + ip_n}\right). \tag{4.25}$$

If it is not otherwise specified, in the following we will use the notation

$$\mathcal{D} = -\frac{\beta}{A}\sum_{n=1}^{\infty}\Re p_n\left(\frac{D_n}{\eta + ip_n} + \frac{\overline{D_n}}{\eta - ip_n}\right). \tag{4.26}$$

Lemma 4.2. *There exist constants $m_1, m_2 > 0$ such that*

$$\frac{m_1}{A}|p_n|^2 \le |d_n| \le \frac{m_2}{A}|p_n|^2 \quad \forall n \in \mathbb{N}. \tag{4.27}$$

Proof. We note that for n_0 sufficiently large we have, for any $n \ge n_0$,

$$A^2|d_n|^2 = \left|p_n^2 - \Re p_n + \frac{\beta \Re p_n}{\eta + ip_n}\right|^2 = |p_n|^4 \left|1 - \frac{\Re p_n}{p_n^2} + \frac{\beta \Re p_n}{p_n^2(\eta + ip_n)}\right|^2 \le \frac{3}{2}|p_n|^4,$$

and

$$A^2|d_n|^2 = \left|p_n^2 - \Re p_n + \frac{\beta \Re p_n}{\eta + ip_n}\right|^2 = |p_n|^4 \left|1 - \frac{\Re p_n}{p_n^2} + \frac{\beta \Re p_n}{p_n^2(\eta + ip_n)}\right|^2 \ge \frac{|p_n|^4}{2}.$$

Since ip_n is not a solution of the cubic equation

$$\Lambda^3 + \eta\Lambda^2 + \Re p_n \Lambda + \Re p_n(\eta - \beta) = 0,$$

we have for any $n \in \mathbb{N}$

$$1 - \frac{\Re p_n}{p_n^2} + \frac{\beta \Re p_n}{p_n^2(\eta + ip_n)} \ne 0,$$

whence

$$\min_{n \le n_0}\left|1 - \frac{\Re p_n}{p_n^2} + \frac{\beta \Re p_n}{p_n^2(\eta + ip_n)}\right| > 0, \qquad \max_{n \le n_0}\left|1 - \frac{\Re p_n}{p_n^2} + \frac{\beta \Re p_n}{p_n^2(\eta + ip_n)}\right| > 0.$$

Therefore, there exist constants $m_1, m_2 > 0$ such that (4.27) holds true. □

Remark 4.3. In the following section, we will skip the dependence on w_n in (4.23) and (4.24), because that is not restricting, as we will see in Theorem 6.1.

5. Ingham type inequalities

In this section we will establish the inverse and direct inequalities for (u_1, u_2), where

$$u_1(t) = \sum_{n=1}^{\infty}\left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right) \quad t \in \mathbb{R}, \tag{5.1}$$

$$u_2(t) = \sum_{n=1}^{\infty}\left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t}\right) + \mathcal{D} e^{-\eta t} \quad t \in \mathbb{R}, \tag{5.2}$$

$r_n, R_n, \mathcal{D} \in \mathbb{R}$ and $\omega_n, C_n, p_n, D_n \in \mathbb{C}$, $p_n \ne 0$, by assuming that

$$\lim_{n\to\infty}(\Re p_{n+1} - \Re p_n) = +\infty, \tag{5.3}$$

$$\lim_{n\to\infty}\Im p_n = 0, \tag{5.4}$$

and for some $\gamma > 0$, $\alpha \in \mathbb{R}$, $n' \in \mathbb{N}$, $\mu > 0$, $\nu > 1/2$, $m_1, m_2 > 0$

$$\liminf_{n\to\infty}(\Re\omega_{n+1} - \Re\omega_n) = \gamma, \tag{5.5}$$

$$\lim_{n\to\infty}\Im\omega_n = \alpha, \qquad r_n \le -\Im\omega_n \quad \forall n \ge n', \tag{5.6}$$

$$|R_n| \le \frac{\mu}{n^\nu}|C_n| \quad \forall n \ge n', \qquad |R_n| \le \mu|C_n| \quad \forall n \le n', \tag{5.7}$$

$$m_1|p_n|^2 \le |d_n| \le m_2|p_n|^2 \qquad \forall n \in \mathbb{N}. \tag{5.8}$$

We note that from (5.3) it follows

$$\lim_{n\to\infty}\frac{\Re p_n}{n} = +\infty, \tag{5.9}$$

see [1, p. 54] and from $\lim_{n\to\infty}|p_n| = +\infty$ and $p_n \ne 0$ it follows that there exists $a_0 > 0$ such that

$$|p_n| \ge a_0 \quad \forall n \in \mathbb{N}. \tag{5.10}$$

5.1. Preliminary results

First, to prove inverse type estimates we need to introduce an auxiliary function, see [4]. Indeed, we define

$$k(t) := \begin{cases} \sin\frac{\pi t}{T} & \text{if } t \in [0,T], \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

For the reader's convenience, we list some easy to check properties of k in the following lemma.

Lemma 5.1. *Set*

$$K(u) := \frac{\pi T}{\pi^2 - T^2u^2}, \qquad u \in \mathbb{C}; \tag{5.12}$$

the following properties hold for any $u \in \mathbb{C}$

$$\int_0^\infty k(t)e^{iut}\,dt = \left(1 + e^{iuT}\right)K(u), \tag{5.13}$$

$$\overline{K(u)} = K(\bar{u}), \tag{5.14}$$

$$\left|K(u)\right| = \left|K(\bar{u})\right|, \tag{5.15}$$

$$\left|K(u)\right| \le \frac{\pi T}{|T^2(\Re u)^2 - T^2(\Im u)^2 - \pi^2|}. \tag{5.16}$$

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The following result is a crucial tool in the proof of Ingham type inverse estimate.

Proposition 5.2. *Under assumptions (5.3)–(5.4), for $T > 0$, $\varepsilon \in (0, 1)$, $M > \frac{2\pi}{T(1-\varepsilon)}$ and for any complex number sequence $\{E_n\}$ such that $\sum_{n=1}^{\infty} |E_n|^2 < +\infty$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ independent of coefficients E_n such that if $E_n = 0$ for $n < n_0$, then we have*

$$\left| \int_0^\infty k(t) \left| \sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt - 2\pi T \sum_{n=n_0}^{\infty} \frac{1 + e^{-2\Im p_n T}}{\pi^2 + 4T^2(\Im p_n)^2} |E_n|^2 \right|$$
$$\le \frac{4\pi}{TM^2} \sum_{n=n_0}^{\infty} \left(1 + e^{-2\Im p_n T}\right) |E_n|^2. \tag{5.17}$$

Proof. First of all, we note that by using (5.13) we have

$$\begin{aligned}
&\int_0^\infty k(t) \left| \sum_{n=1}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \\
&= \int_0^\infty k(t) \sum_{n=1}^{\infty} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t}\right) \sum_{m=1}^{\infty} \left(\overline{E_m} e^{-i\overline{p_m} t} + E_m e^{ip_m t}\right) dt \\
&= \sum_{n,m=1}^{\infty} E_n \overline{E_m} \left(1 + e^{i(p_n - \overline{p_m})T}\right) K(p_n - \overline{p_m}) + \sum_{n,m=1}^{\infty} E_n E_m \left(1 + e^{i(p_n + p_m)T}\right) K(p_n + p_m) \\
&\quad + \sum_{n,m=1}^{\infty} \overline{E_n E_m} \left(1 + e^{-i(\overline{p_n + p_m})T}\right) K(\overline{p_n + p_m}) \\
&\quad + \sum_{n,m=1}^{\infty} \overline{E_n} E_m \left(1 + e^{-i(\overline{p_n} - p_m)T}\right) K(\overline{p_n} - p_m).
\end{aligned}$$

In view of (5.14) we have

$$\begin{aligned}
&\int_0^\infty k(t) \left| \sum_{n=1}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \\
&= 2 \sum_{n,m=1}^{\infty} \Re\left[E_n \overline{E_m} \left(1 + e^{i(p_n - \overline{p_m})T}\right) K(p_n - \overline{p_m})\right] \\
&\quad + 2 \sum_{n,m=1}^{\infty} \Re\left[E_n E_m \left(1 + e^{i(p_n + p_m)T}\right) K(p_n + p_m)\right].
\end{aligned}$$

For $m = n$ we have $K(p_n - \overline{p_n}) = K(2i\Im p_n) = \frac{\pi T}{\pi^2 + 4T^2(\Im p_n)^2}$. Therefore, we deduce

$$\int_0^\infty k(t)\left|\sum_{n=1}^\infty E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n}t}\right|^2 dt - 2\pi T\sum_{n=1}^\infty \frac{1+e^{-2\Im p_n T}}{\pi^2+4T^2(\Im p_n)^2}|E_n|^2$$
$$= 2\sum_{n,m=1,n\neq m}^\infty \Re\left[E_n\overline{E_m}\left(1+e^{i(p_n-\overline{p_m})T}\right)K(p_n-\overline{p_m})\right]$$
$$+ 2\sum_{n,m=1}^\infty \Re\left[E_n E_m\left(1+e^{i(p_n+p_m)T}\right)K(p_n+p_m)\right],$$

whence

$$\left|\int_0^\infty k(t)\left|\sum_{n=1}^\infty E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n}t}\right|^2 dt - 2\pi T\sum_{n=1}^\infty \frac{1+e^{-2\Im p_n T}}{\pi^2+4T^2(\Im p_n)^2}|E_n|^2\right|$$
$$\le 2\sum_{n,m=1,n\neq m}^\infty |E_n||E_m|\left(1+e^{-(\Im p_n+\Im p_m)T}\right)\left|K(p_n-\overline{p_m})\right|$$
$$+ 2\sum_{n,m=1}^\infty |E_n||E_m|\left(1+e^{-(\Im p_n+\Im p_m)T}\right)\left|K(p_n+p_m)\right|. \tag{5.18}$$

We observe that, in virtue of (5.15), we have

$$\left|K(p_n-\overline{p_m})\right| = \left|K(p_m-\overline{p_n})\right|,$$

whence

$$\sum_{n,m=1,n\neq m}^\infty |E_n||E_m|\left|K(p_n-\overline{p_m})\right|$$
$$\le \frac{1}{2}\sum_{n,m=1,n\neq m}^\infty \left(|E_n|^2+|E_m|^2\right)\left|K(p_n-\overline{p_m})\right|$$
$$= \frac{1}{2}\sum_{n=1}^\infty |E_n|^2\sum_{m=1,m\neq n}^\infty \left|K(p_n-\overline{p_m})\right| + \frac{1}{2}\sum_{m=1}^\infty |E_m|^2\sum_{n=1,n\neq m}^\infty \left|K(p_m-\overline{p_n})\right|$$
$$= \sum_{n=1}^\infty |E_n|^2\sum_{m=1,m\neq n}^\infty \left|K(p_n-\overline{p_m})\right|.$$

Similarly, we have

$$\sum_{n,m=1}^\infty |E_n||E_m|e^{-(\Im p_n+\Im p_m)T}\left|K(p_n-\overline{p_m})\right|$$
$$\le \sum_{n=1}^\infty e^{-2\Im p_n T}|E_n|^2\sum_{m=1}^\infty\left|K(p_n-\overline{p_m})\right|,$$

$$\sum_{n,m=1}^{\infty} |E_n||E_m|\left(1+e^{-(\Im p_n+\Im p_m)T}\right)\left|K(p_n+p_m)\right|$$
$$\leq \sum_{n=1}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2 \sum_{m=1}^{\infty}\left|K(p_n+p_m)\right|,$$

so plugging the above inequalities into formula (5.18), we obtain

$$\left|\int_0^\infty k(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt - 2\pi T\sum_{n=1}^{\infty}\frac{1+e^{-2\Im p_n T}}{\pi^2+4T^2(\Im p_n)^2}|E_n|^2\right|$$
$$\leq 2\sum_{n=1}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2 \sum_{m=1,m\neq n}^{\infty}\left|K(p_n-\overline{p_m})\right|$$
$$+2\sum_{n=1}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2 \sum_{m=1}^{\infty}\left|K(p_n+p_m)\right|. \tag{5.19}$$

In the following lemma we single out the estimates concerning the sums depending on K in the right-hand side of the above formula, because we will also use them in the proof of the direct estimate.

Lemma 5.3. *For any $\varepsilon \in (0,1)$ and $M > \frac{2\pi}{T(1-\varepsilon)}$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for any $n \geq n_0$ we have*

$$\sum_{m=n_0,m\neq n}^{\infty}\left|K(p_n-\overline{p_m})\right| \leq \frac{2\pi}{TM^2}, \tag{5.20}$$
$$\sum_{m=n_0}^{\infty}\left|K(p_n+p_m)\right| \leq \frac{4\pi}{TM^2}\sum_{m=n_0}^{\infty}\frac{1}{4m^2-1}. \tag{5.21}$$

Proof. To prove the first inequality, we observe that, thanks to (5.16), we get

$$\sum_{m=1,m\neq n}^{\infty}\left|K(p_n-\overline{p_m})\right| \leq \pi T\sum_{m=1,m\neq n}^{\infty}\frac{1}{|T^2(\Re p_n-\Re p_m)^2-T^2(\Im p_n+\Im p_m)^2-\pi^2|}. \tag{5.22}$$

From assumption (5.3) it follows that for any $M>0$ there exists $n_0 \in \mathbb{N}$ such that

$$\Re p_{n+1}-\Re p_n \geq M \quad \forall n \geq n_0,$$

whence

$$|\Re p_n-\Re p_m| \geq M|n-m|, \quad \forall n,m \geq n_0.$$

Thanks to the previous estimate, we have

$$T^2(\Re p_n - \Re p_m)^2 - T^2(\Im p_n + \Im p_m)^2 - \pi^2 \geq T^2M^2(n-m)^2 - T^2(\Im p_n + \Im p_m)^2 - \pi^2.$$

Moreover, since $\lim_{n\to\infty} \Im p_n = 0$, fix $0 < \varepsilon < 1$; for $n_0 \in \mathbb{N}$ sufficiently large we have

$$|\Im p_n| < \frac{M}{4}\varepsilon \quad \forall n \geq n_0,$$

so, for any $n, m \in \mathbb{N}$, $n, m \geq n_0$, we have

$$T^2(\Im p_n + \Im p_m)^2 + \pi^2 < \frac{1}{4}\big(T^2M^2\varepsilon^2 + 4\pi^2\big).$$

Now, as $M > \frac{2\pi}{T(1-\varepsilon)}$ we have $T^2M^2\varepsilon^2 + 4\pi^2 < T^2M^2$, so from the above inequality it follows

$$T^2(\Im p_n + \Im p_m)^2 + \pi^2 < \frac{1}{4}T^2M^2, \tag{5.23}$$

and hence for $m \neq n$,

$$T^2(\Re p_n - \Re p_m)^2 - T^2(\Im p_n + \Im p_m)^2 - \pi^2 \geq T^2M^2(n-m)^2 - \frac{1}{4}T^2M^2 > 0.$$

Putting the previous formula into (5.22), for any $n \geq n_0$ we obtain

$$\begin{aligned}
&\sum_{m=n_0, m\neq n}^{\infty} \big|K(p_n - \overline{p_m})\big| \\
&\leq 4\pi T \sum_{m=n_0, m\neq n}^{\infty} \frac{1}{4T^2M^2(m-n)^2 - T^2M^2} = \frac{4\pi}{TM^2} \sum_{m=n_0, m\neq n}^{\infty} \frac{1}{4(m-n)^2 - 1} \\
&\leq \frac{4\pi}{TM^2} \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} = \frac{2\pi}{TM^2} \sum_{j=1}^{\infty} \left(\frac{1}{2j-1} - \frac{1}{2j+1}\right) = \frac{2\pi}{TM^2},
\end{aligned}$$

that is (5.20).

As regards the second estimate, again by (5.16) we have

$$\sum_{m=n_0}^{\infty} \big|K(p_n + p_m)\big| \leq \pi T \sum_{m=n_0}^{\infty} \frac{1}{|T^2(\Re p_n + \Re p_m)^2 - T^2(\Im p_n + \Im p_m)^2 - \pi^2|}. \tag{5.24}$$

From (5.9), we have for any $M > 0$

$$\Re p_n \geq Mn, \quad \forall n \geq n_0.$$

By using the previous inequality and (5.23), we get for $M > \frac{2\pi}{T(1-\varepsilon)}$

$$T^2(\Re p_n + \Re p_m)^2 - T^2(\Im p_n + \Im p_m)^2 - \pi^2 \geq T^2M^2m^2 - \frac{1}{4}T^2M^2 = \frac{T^2M^2}{4}\big(4m^2 - 1\big).$$

Therefore from (5.24), by using the above estimate, we get

$$\sum_{m=n_0}^{\infty} \left|K(p_n + p_m)\right| \le \frac{4\pi}{TM^2} \sum_{m=n_0}^{\infty} \frac{1}{4m^2 - 1},$$

that is (5.21).

Proof of Proposition 5.2 (continued). If we assume $E_n = 0$ for any $n < n_0$, then from (5.19) and (5.20) we have

$$\left| \int_0^{\infty} k(t) \left| \sum_{n=n_0}^{\infty} E_n e^{i p_n t} + \overline{E_n} e^{-i \overline{p_n} t} \right|^2 dt - 2\pi T \sum_{n=n_0}^{\infty} \frac{1 + e^{-2\Im p_n T}}{\pi^2 + 4T^2 (\Im p_n)^2} |E_n|^2 \right|$$
$$\le \frac{4\pi}{TM^2} \sum_{n=n_0}^{\infty} \left(1 + e^{-2\Im p_n T}\right) |E_n|^2 + 2 \sum_{n=n_0}^{\infty} \left(1 + e^{-2\Im p_n T}\right) |E_n|^2 \sum_{m=n_0}^{\infty} \left|K(p_n + p_m)\right|. \tag{5.25}$$

Now, we observe that for $m \ge n_0$ we have

$$4m^2 - 1 \ge 4m^{3/2} n_0^{1/2} - 1 \ge n_0^{1/2} \left(4m^{3/2} - 1\right),$$

whence

$$\sum_{m=n_0}^{\infty} \frac{1}{4m^2 - 1} \le \frac{1}{n_0^{1/2}} \sum_{m=1}^{\infty} \frac{1}{4m^{3/2} - 1}.$$

Therefore from (5.21), by using the above inequality, we get

$$\sum_{m=n_0}^{\infty} \left|K(p_n + p_m)\right| \le \frac{4\pi}{TM^2 n_0^{1/2}} \sum_{n=1}^{\infty} \frac{1}{4n^{3/2} - 1},$$

whence

$$\sum_{n=n_0}^{\infty} \left(1 + e^{-2\Im p_n T}\right) |E_n|^2 \sum_{m=n_0}^{\infty} \left|K(p_n + p_m)\right|$$
$$\le \frac{4\pi}{TM^2 n_0^{1/2}} \sum_{n=1}^{\infty} \frac{1}{4n^{3/2} - 1} \sum_{n=n_0}^{\infty} \left(1 + e^{-2\Im p_n T}\right) |E_n|^2.$$

If we choose $n_0 \in \mathbb{N}$ large enough to satisfy the condition

$$\frac{2}{n_0^{1/2}} \sum_{n=1}^{\infty} \frac{1}{4n^{3/2} - 1} < \varepsilon,$$

we have

$$\sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2 \sum_{m=n_0}^{\infty}\left|K(p_n+p_m)\right| \le \frac{2\pi}{TM^2}\varepsilon \sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2.$$

Plugging the above inequality into (5.25) we get

$$\begin{aligned}&\left|\int_0^{\infty} k(t)\left|\sum_{n=n_0}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im p_n T}}{\pi^2+4T^2(\Im p_n)^2}|E_n|^2\right|\\ &\quad\le \frac{4\pi}{TM^2}(1+\varepsilon)\sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|E_n|^2.\end{aligned}$$

Finally, by substituting M with $M\sqrt{1+\varepsilon}$ we obtain (5.17). □

As for the inverse inequality, to prove direct estimates we need to introduce an auxiliary function. Let $T>0$ and define

$$k^*(t) := \begin{cases} \cos\frac{\pi t}{2T} & \text{if } |t|\le T,\\ 0 & \text{if } |t|>T.\end{cases} \tag{5.26}$$

For the sake of completeness, we list some easy to check properties of k^* in the following lemma.

Lemma 5.4. *Set*

$$K^*(u) := \frac{4T\pi}{\pi^2-4T^2u^2}, \qquad u\in\mathbb{C};$$

the following properties hold for any $u\in\mathbb{C}$

$$\int_{-\infty}^{\infty} k^*(t)e^{iut}\,dt = \cos(uT)K^*(u), \tag{5.27}$$

$$\overline{K^*(u)} = K^*(\bar{u}), \tag{5.28}$$

$$\left|K^*(u)\right| = \left|K^*(\bar{u})\right|. \tag{5.29}$$

If we set $K_T(u)=\frac{T\pi}{\pi^2-T^2u^2}$, then we have

$$K^*(u) = 2K_{2T}(u). \tag{5.30}$$

From now on $c(T)$ will denote a positive constant depending on T.

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Proposition 5.5. *Assume (5.3)–(5.4). Let $T>0$, $\varepsilon\in(0,1)$, $M>\frac{\pi}{T\sqrt{1-\varepsilon}}$ and $\{E_n\}$ be a complex number sequence such that $\sum_{n=1}^{\infty}|E_n|^2<+\infty$. There exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that if $E_n=0$ for $n<n_0$, then we have*

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt$$

$$\le 4c(T)\left(\frac{2T}{\pi}+\frac{\pi}{TM^2}\left(1+\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\right)\right)\sum_{n=n_0}^{\infty}|E_n|^2. \tag{5.31}$$

Proof. Let $k^*(t)$ be the function defined by (5.26). If we use (5.27) and (5.28), then we have

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt$$

$$=2\sum_{n,m=1}^{\infty}\Re\big(E_n\overline{E_m}\cos\big((p_n-\overline{p_m})T\big)K^*(p_n-\overline{p_m})\big)$$

$$+2\sum_{n,m=1}^{\infty}\Re\big(E_nE_m\cos\big((p_n+p_m)T\big)K^*(p_n+p_m)\big).$$

Applying the elementary estimates $\Re z\le|z|$ and $|\cos z|\le\cosh(\Im z)$, $z\in\mathbb{C}$, we obtain

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt$$

$$\le 2\sum_{n,m=1}^{\infty}|E_n||E_m|\cosh\big((\Im p_n+\Im p_m)T\big)\big[\big|K^*(p_n-\overline{p_m})\big|+\big|K^*(p_n+p_m)\big|\big].$$

Since the sequence $\{\Im p_n\}$ is bounded, for any $n,m\in\mathbb{N}$, we have

$$\cosh\big((\Im p_n+\Im p_m)T\big)\le e^{2T\sup|\Im p_n|},$$

and hence

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t}+\overline{E_n}e^{-i\overline{p_n}t}\right|^2 dt$$

$$\le 2e^{2T\sup|\Im p_n|}\sum_{n,m=1}^{\infty}|E_n||E_m|\big[\big|K^*(p_n-\overline{p_m})\big|+\big|K^*(p_n+p_m)\big|\big].$$

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In virtue of (5.29) we get $|K^*(p_n - \overline{p_m})| = |K^*(p_m - \overline{p_n})|$, so we have

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n}t}\right|^2 dt$$
$$\leq 2e^{2T\sup|\Im p_n|}\sum_{n=1}^{\infty}|E_n|^2\sum_{m=1}^{\infty}\left[\left|K^*(p_n - \overline{p_m})\right| + \left|K^*(p_n + p_m)\right|\right].$$

Taking into account the definition of K^* we have

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=1}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n}t}\right|^2 dt$$
$$\leq 8T\pi e^{2T\sup|\Im p_n|}\sum_{n=1}^{\infty}\frac{1}{\pi^2 + 8T^2(\Im p_n)^2}|E_n|^2$$
$$+ 2e^{2T\sup|\Im p_n|}\sum_{n=1}^{\infty}|E_n|^2\sum_{m=1,m\neq n}^{\infty}\left|K^*(p_n - \overline{p_m})\right|$$
$$+ 2e^{2T\sup|\Im p_n|}\sum_{n=1}^{\infty}|E_n|^2\sum_{m=1}^{\infty}\left|K^*(p_n + p_m)\right|. \tag{5.32}$$

Now, we note that in virtue of (5.30) we can apply Lemma 5.3: for any $\varepsilon \in (0, 1)$ and $M > \frac{\pi}{T\sqrt{1-\varepsilon}} = \frac{2\pi}{2T\sqrt{1-\varepsilon}}$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$\sum_{m=n_0,m\neq n}^{\infty}\left|K^*(p_n - \overline{p_m})\right| \leq \frac{4\pi}{2TM^2} = \frac{2\pi}{TM^2},$$
$$\sum_{m=n_0}^{\infty}\left|K^*(p_n + p_m)\right| \leq \frac{4\pi}{2TM^2}\sum_{m=n_0}^{\infty}\frac{1}{4m^2 - 1} \leq \frac{2\pi}{TM^2}\sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}.$$

In conclusion, assuming $E_n = 0$ for $n < n_0$ and putting the above formulas into (5.32), we get

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n}t}\right|^2 dt$$
$$\leq e^{2T\sup|\Im p_n|}\left(\frac{8T}{\pi} + \frac{4\pi}{TM^2}\left(1 + \sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}\right)\right)\sum_{n=n_0}^{\infty}|E_n|^2,$$

that is (5.31). □

5.2. Inverse and direct inequalities excluding a finite number of terms

Due to the asymptotic assumptions on data, some properties hold true for sufficiently large integers. For that reason, first we will show some inverse and direct inequalities in the special case when our series have a finite number of terms vanishing.

Before proceeding, we state the next result, that can be proved in the same way as in [25, Theorem 5.3], taking into account that the function $k(t)$ is non-negative.

From now on we denote with $c(T,\varepsilon)$ a positive constant depending on T and ε.

Theorem 5.6. *Under assumptions (5.5)–(5.7), for any $\varepsilon \in (0,1)$ and $T > \frac{2\pi}{\gamma(1-\varepsilon)}$ there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $c(T,\varepsilon) > 0$ such that if $C_n = 0$ for any $n < n_0$, then we have*

$$\int_0^\infty k(t)\left|\sum_{n=n_0}^\infty R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right|^2 dt$$
$$\geq c(T,\varepsilon)\sum_{n=n_0}^\infty \big(1 + e^{-2(\Im\omega_n - \alpha)T}\big)|C_n|^2. \tag{5.33}$$

In the following finding we give a lower bound for the first component of the solution of coupled system.

Theorem 5.7. *Under assumptions (5.3)–(5.7), for any $\varepsilon \in (0,1)$ and $T > \frac{2\pi}{\gamma(1-\varepsilon)}$, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $c(T,\varepsilon) > 0$ such that if $C_n = D_n = 0$ for any $n < n_0$, then we have*

$$\int_0^T \left|\sum_{n=n_0}^\infty R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt$$
$$\geq c(T,\varepsilon)\sum_{n=n_0}^\infty \big(1 + e^{-2(\Im\omega_n - \alpha)T}\big)|C_n|^2$$
$$- 2\pi T\sum_{n=n_0}^\infty \left(\frac{1}{\pi^2 + 4T^2(\Im p_n)^2} + \frac{2}{T^2\gamma^2}\right)\big(1 + e^{-2\Im p_n T}\big)|D_n|^2. \tag{5.34}$$

Proof. First of all, we set for any $t \geq 0$

$$F_1(t) = \sum_{n=1}^\infty \big(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\big) \in \mathbb{R},$$

$$F_2(t) = \sum_{n=1}^\infty \big(D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\big) \in \mathbb{R},$$

and observe that if $k(t)$ is the function defined by (5.11), we have to estimate the term

$$\int_0^\infty k(t)\big|F_1(t)+F_2(t)\big|^2\,dt.$$

Because of the elementary inequality $2|ab| \le \frac{1}{2}a^2 + 2b^2$, we observe that

$$\begin{aligned}\big|F_1(t)+F_2(t)\big|^2 &= \big|F_1(t)\big|^2 + 2F_1(t)F_2(t) + \big|F_2(t)\big|^2\\ &\ge \big|F_1(t)\big|^2 - \frac{1}{2}\big|F_1(t)\big|^2 - 2\big|F_2(t)\big|^2 + \big|F_2(t)\big|^2 = \frac{1}{2}\big|F_1(t)\big|^2 - \big|F_2(t)\big|^2.\end{aligned}$$

Since $k(t)$ is positive, from the above inequality we have

$$\int_0^\infty k(t)\big|F_1(t)+F_2(t)\big|^2\,dt \ge \frac{1}{2}\int_0^\infty k(t)\big|F_1(t)\big|^2\,dt - \int_0^\infty k(t)\big|F_2(t)\big|^2\,dt.$$

Therefore, in view of Theorem 5.6 we can apply (5.33) to get

$$\begin{aligned}&\int_0^\infty k(t)\big|F_1(t)+F_2(t)\big|^2\,dt\\ &\quad\ge \frac{1}{2}c(T,\varepsilon)\sum_{n=n_0}^\infty \big(1+e^{-2(\Im\omega_n-\alpha)T}\big)|C_n|^2 - \int_0^\infty k(t)\big|F_2(t)\big|^2\,dt, \qquad (5.35)\end{aligned}$$

for n_0 sufficiently large. To complete our proof, we must give an upper bound for the term $\int_0^\infty k(t)|F_2(t)|^2\,dt$. Indeed, if we take $E_n = D_n$ and $M = \gamma > \frac{2\pi}{T(1-\varepsilon)}$ in Proposition 5.2, then by formula (5.17) we have

$$\begin{aligned}\int_0^\infty k(t)\big|F_2(t)\big|^2\,dt &= \int_0^\infty k(t)\left|\sum_{n=n_0}^\infty D_n e^{ip_n t} + \overline{D_n}e^{-i\overline{p_n}t}\right|^2 dt\\ &\le 2\pi T\sum_{n=n_0}^\infty \frac{1+e^{-2\Im p_n T}}{\pi^2+4T^2(\Im p_n)^2}|D_n|^2 + \frac{4\pi}{T\gamma^2}\sum_{n=n_0}^\infty \big(1+e^{-2\Im p_n T}\big)|D_n|^2\\ &= 2\pi T\sum_{n=n_0}^\infty \left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} + \frac{2}{T^2\gamma^2}\right)\big(1+e^{-2\Im p_n T}\big)|D_n|^2.\end{aligned}$$

Therefore, putting the above estimate in (5.35) we have

$$\int_0^\infty k(t)\big|F_1(t)+F_2(t)\big|^2\,dt \ge \frac{1}{2}c(T,\varepsilon)\sum_{n=n_0}^\infty \big(1+e^{-2(\Im\omega_n-\alpha)T}\big)|C_n|^2$$
$$-2\pi T\sum_{n=n_0}^\infty\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}\right)\big(1+e^{-2\Im p_n T}\big)|D_n|^2,$$

whence, in virtue of the definition of $k(t)$, (5.34) follows. □

Proposition 5.8. *Assume (5.3), (5.4) and (5.8). Let $T>0$, $\varepsilon\in(0,1)$ and $M>\frac{2\pi}{T(1-\varepsilon)}$. There exist $n_0=n_0(\varepsilon)\in\mathbb{N}$ and $c(T,M,\varepsilon)>0$ such that if $D_n=0$ for any $n<n_0$, then we have*

$$\int_0^T\left|e^{\eta t}\sum_{n=n_0}^\infty\big(d_nD_ne^{ip_nt}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\big)\right|^2dt$$
$$\ge 2\pi Tm_1^2\sum_{n=n_0}^\infty\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\big(1+e^{-2\Im p_nT}\big)|D_n|^2|p_n|^4, \tag{5.36}$$

and

$$\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}>c(T,M,\varepsilon),\qquad \forall n\ge n_0. \tag{5.37}$$

Proof. We will use Proposition 5.2 again. Indeed, if we set

$$G(t)=\sum_{n=1}^\infty\big(d_nD_ne^{ip_nt}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\big)$$

and take $E_n=d_nD_n$, we can apply formula (5.17) for n_0 large enough:

$$\int_0^\infty k(t)\big|G(t)\big|^2dt\ge 2\pi T\sum_{n=n_0}^\infty\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\big(1+e^{-2\Im p_nT}\big)|D_n|^2|d_n|^2, \tag{5.38}$$

where $k(t)$ is the function defined by (5.11). Since $\lim_{n\to\infty}\Im p_n=0$, by taking n_0 large enough we have

$$|\Im p_n|<\frac{M\sqrt{\varepsilon}}{2\sqrt{2}}\qquad \forall n\ge n_0,$$

and hence, since $M>\frac{2\pi}{T(1-\varepsilon)}$, we get

$$\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}>\frac{1}{\pi^2+T^2M^2\varepsilon/2}-\frac{2}{T^2M^2}$$
$$=2\frac{T^2M^2(1-\varepsilon)-2\pi^2}{T^2M^2(2\pi^2+T^2M^2\varepsilon)},$$

that is, (5.37) holds true with $c(T,M,\varepsilon)=2\frac{T^2M^2(1-\varepsilon)-2\pi^2}{T^2M^2(2\pi^2+T^2M^2\varepsilon)}>0$. Thanks to (5.8), we have

$$\int_0^\infty k(t)\big|G(t)\big|^2\,dt \ge 2\pi T m_1^2\sum_{n=n_0}^\infty\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\left(1+e^{-2\Im p_nT}\right)|D_n|^2|p_n|^4,$$

whence, in virtue of the definition of $k(t)$,

$$\int_0^T \big|G(t)\big|^2\,dt \ge 2\pi T m_1^2\sum_{n=n_0}^\infty\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\left(1+e^{-2\Im p_nT}\right)|D_n|^2|p_n|^4. \tag{5.39}$$

Finally, in view of

$$\int_0^T \big|e^{\eta t}G(t)\big|^2\,dt=\int_0^T e^{2\eta t}\big|G(t)\big|^2\,dt\ge\int_0^T\big|G(t)\big|^2\,dt,$$

by (5.39) it follows (5.36). □

Now, we anticipate a result concerning direct estimates, because we will use it in the next theorem.

Proposition 5.9. *Assume (5.3), (5.4) and (5.8). Let $T>0$, $\varepsilon\in(0,1)$ and $M>\frac{\pi}{T\sqrt{1-\varepsilon}}$. There exist $c(T)>0$ and $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that if $D_n=0$ for any $n<n_0$, then we have*

$$\int_{-\infty}^\infty k^*(t)\left|\sum_{n=n_0}^\infty d_nD_ne^{ip_nt}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\right|^2dt\le c(T)\sum_{n=n_0}^\infty|D_n|^2|p_n|^4, \tag{5.40}$$

$$\int_{-T}^T \left|\sum_{n=n_0}^\infty d_nD_ne^{ip_nt}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\right|^2dt\le c(T)\sum_{n=n_0}^\infty|D_n|^2|p_n|^4. \tag{5.41}$$

Proof. We evaluate the integral by using Proposition 5.5: indeed, if we take

$$E_n=d_nD_n,$$

then from (5.31) it follows

$$\int_{-\infty}^\infty k^*(t)\left|\sum_{n=n_0}^\infty d_nD_ne^{ip_nt}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}t}\right|^2dt\le c(T)\sum_{n=n_0}^\infty|D_n|^2|d_n|^2. \tag{5.42}$$

Moreover, from (5.8) we get

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right|^2 dt \le c(T)\sum_{n=n_0}^{\infty}|D_n|^2|p_n|^4,$$

that is (5.40).

Now, if we consider the last inequality with the function k^* replaced by the analogous one relative to $2T$ instead of T, see (5.26), then we get

$$\int_{-2T}^{2T} \cos\frac{\pi t}{4T}\left|\sum_{n=n_0}^{\infty} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right|^2 dt \le c(2T)\sum_{n=n_0}^{\infty}|D_n|^2|p_n|^4,$$

whence, thanks to $\cos\frac{\pi t}{4T} \ge \frac{1}{\sqrt{2}}$ for $|t| \le T$, it follows

$$\int_{-T}^{T}\left|\sum_{n=n_0}^{\infty} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right|^2 dt \le \sqrt{2}c(2T)\sum_{n=n_0}^{\infty}|D_n|^2|p_n|^4,$$

that is (5.41). □

Theorem 5.10. *Assume (5.3), (5.4) and (5.8). Let $T > T_0 > 0$, $\varepsilon \in (0,1)$ and $M > \frac{2\pi}{T(1-\varepsilon)}$. If $\mathcal{D}$ is any real constant, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$, $C_0 > 0$ independent of T, $c(T) > 0$ and $c(T, M, \varepsilon) > 0$ such that if $D_n = 0$ for any $n < n_0$, then we have*

$$\int_0^T\left|e^{\eta t}\sum_{n=n_0}^{\infty}\left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right) + \mathcal{D}\right|^2 dt$$

$$\ge c(T)|\mathcal{D}|^2 + \pi C_0 T\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2 + 4T^2(\Im p_n)^2} - \frac{2}{T^2M^2}\right)\left(1 + e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4, \tag{5.43}$$

$$\frac{1}{\pi^2 + 4T^2(\Im p_n)^2} - \frac{2}{T^2M^2} > c(T, M, \varepsilon), \quad \forall n \ge n_0. \tag{5.44}$$

Proof. We introduce the function

$$G_1(t) = e^{\eta t}\sum_{n=1}^{\infty}\left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right) + \mathcal{D}.$$

To evaluate the integral of $G_1(t)$ on the left-hand side of (5.43), we will use the operator introduced by Haraux which annihilates the constant $\mathcal{D}$, see [6]. Indeed, if we take $\delta \in (T_0/4, T_0/2)$, then we have for any $t \in [0, T-\delta]$

$$\int_0^\delta \big(G_1(t)-G_1(t+s)\big)\,ds = e^{\eta t}\sum_{n=1}^\infty d_n\Big(\delta-\frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}\Big)D_n e^{ip_n t}$$
$$+\overline{d_n}\Big(\delta-\frac{e^{(\eta-i\overline{p_n})\delta}-1}{\eta-i\overline{p_n}}\Big)\overline{D_n}e^{-i\overline{p_n}t}. \tag{5.45}$$

We can apply Proposition 5.8 to the function $t\to\int_0^\delta(G_1(t)-G_1(t+s))\,ds$ in the interval $[0,T-\delta]$. If $M>\frac{2\pi}{T(1-\varepsilon)}$, we note that $\overline{M}=\frac{T}{T-\delta}M$ verifies $\overline{M}>\frac{2\pi}{(T-\delta)(1-\varepsilon)}$, so by (5.36) and (5.37) we have

$$\int_0^{T-\delta}\Big|\int_0^\delta \big(G_1(t)-G_1(t+s)\big)\,ds\Big|^2\,dt$$
$$\ge 2\pi(T-\delta)m_1^2\sum_{n=n_0}^\infty\Big(\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2}\Big)$$
$$\cdot\big(1+e^{-2\Im p_n(T-\delta)}\big)\Big|\delta-\frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}\Big|^2|D_n|^2|p_n|^4 \tag{5.46}$$

and

$$\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2}>0,\qquad \forall n\ge n_0.$$

We have to estimate $|\delta-\frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}|$. First, we observe that

$$\Big|\delta-\frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}\Big|\ge\delta-\frac{|e^{(\eta+ip_n)\delta}-1|}{|\eta+ip_n|}\ge\delta-\frac{e^{(\eta-\Im p_n)\delta}+1}{|\Re p_n|}.$$

Since the sequence $\{\Im p_n\}$ is bounded and $\delta<T_0$, we have

$$e^{(\eta-\Im p_n)\delta}\le e^{(\eta+\sup|\Im p_n|)\delta}\le e^{(\eta+\sup|\Im p_n|)T_0},$$

and hence

$$\Big|\delta-\frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}\Big|\ge\delta-\frac{2e^{(\eta+\sup|\Im p_n|)T_0}}{|\Re p_n|}.$$

Taking into account that $\lim_{n\to\infty}\Re p_n=+\infty$, for n_0 sufficiently large we have for any $n\ge n_0$

$$\frac{2e^{(\eta+\sup|\Im p_n|)T_0}}{\Re p_n}\le\frac{T_0}{4},$$

whence

$$\left|\delta - \frac{e^{(\eta+ip_n)\delta}-1}{\eta+ip_n}\right| \geq \delta - \frac{T_0}{4} > 0.$$

Plugging the above estimate into (5.46), we obtain

$$\begin{aligned}
&\int_0^{T-\delta}\left|\int_0^{\delta}\big(G_1(t)-G_1(t+s)\big)\,ds\right|^2 dt \\
&\quad \geq 2\pi(\delta-T_0/4)^2(T-\delta)m_1^2 \\
&\qquad \cdot \sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2}\right)\big(1+e^{-2\Im p_n(T-\delta)}\big)|D_n|^2|p_n|^4. \qquad (5.47)
\end{aligned}$$

Moreover, since $2\delta < T_0$ and the sequence $\{\Im p_n\}$ is bounded we get

$$e^{2\Im p_n\delta} \geq e^{-2\delta|\Im p_n|} \geq e^{-T_0|\Im p_n|} \geq e^{-T_0 \sup|\Im p_n|},$$

whence

$$1+e^{-2\Im p_n(T-\delta)} = 1+e^{-2\Im p_n T}e^{2\Im p_n\delta} \geq e^{-T_0\sup|\Im p_n|}\big(1+e^{-2\Im p_n T}\big).$$

In view of the above inequality, from (5.47) it follows

$$\begin{aligned}
&\int_0^{T-\delta}\left|\int_0^{\delta}\big(G_1(t)-G_1(t+s)\big)\,ds\right|^2 dt \\
&\quad \geq 2\pi(\delta-T_0/4)^2 e^{-T_0\sup|\Im p_n|}(T-\delta)m_1^2 \\
&\qquad \cdot \sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2}\right)\big(1+e^{-2\Im p_n T}\big)|D_n|^2|p_n|^4. \qquad (5.48)
\end{aligned}$$

By $(T-\delta)\overline{M} = TM$ and (5.37), for n_0 large enough we get for all $n \geq n_0$

$$\begin{aligned}
\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2} &= \frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{T^2M^2} \\
&> \frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2} > c(T,M,\varepsilon) > 0.
\end{aligned}$$

In addition, because of $\delta < T/2$, we have $\frac{T-\delta}{T} > \frac{1}{2}$, so

$$\begin{aligned}
(T-\delta)\left(\frac{1}{\pi^2+4(T-\delta)^2(\Im p_n)^2}-\frac{2}{(T-\delta)^2\overline{M}^2}\right) &> T\frac{T-\delta}{T}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right) \\
&> \frac{T}{2}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right).
\end{aligned}$$

Putting the above estimate into (5.48), we have

$$\begin{aligned}&\int_0^{T-\delta}\left|\int_0^{\delta}\big(G_1(t)-G_1(t+s)\big)\,ds\right|^2 dt\\&\quad\geq \pi(\delta-T_0/4)^2 e^{-T_0\sup|\Im p_n|}Tm_1^2\\&\qquad\cdot\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\big(1+e^{-2\Im p_nT}\big)|D_n|^2|p_n|^4,\end{aligned}\tag{5.49}$$

and, in addition, (5.44) holds true.

On the other hand

$$\begin{aligned}\int_0^{T-\delta}\left|\int_0^{\delta}\big(G_1(t)-G_1(t+s)\big)\,ds\right|^2 dt &\leq \delta\int_0^{T-\delta}\int_0^{\delta}\big|G_1(t)-G_1(t+s)\big|^2\,ds\,dt\\&\leq 2\delta\int_0^{T-\delta}\int_0^{\delta}\big(\big|G_1(t)\big|^2+\big|G_1(t+s)\big|^2\big)\,ds\,dt\\&\leq 2\delta^2\int_0^{T}\big|G_1(t)\big|^2\,dt+2\delta\int_0^{\delta}\int_0^{T-\delta}\big|G_1(t+s)\big|^2\,dt\,ds\\&= 2\delta^2\int_0^{T}\big|G_1(t)\big|^2\,dt+2\delta\int_0^{\delta}\int_s^{T-\delta+s}\big|G_1(y)\big|^2\,dy\,ds\\&\leq 2\delta^2\int_0^{T}\big|G_1(t)\big|^2\,dt+2\delta\int_0^{\delta}\int_0^{T}\big|G_1(y)\big|^2\,dy\,ds\\&= 4\delta^2\int_0^{T}\big|G_1(t)\big|^2\,dt,\end{aligned}$$

whence

$$\int_0^{T}\big|G_1(t)\big|^2\,dt\geq\frac{1}{4\delta^2}\int_0^{T-\delta}\left|\int_0^{\delta}\big(G_1(t)-G_1(t+s)\big)\,ds\right|^2 dt.$$

From the above estimate and (5.49), it follows

$$\int_0^{T}\big|G_1(t)\big|^2\,dt\geq 2\pi C_0T\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2M^2}\right)\big(1+e^{-2\Im p_nT}\big)|D_n|^2|p_n|^4,\tag{5.50}$$

where the constant $C_0 = \frac{(\delta - T_0/4)^2}{8\delta^2} e^{-T_0 \sup |\Im p_n|} m_1^2$ depends on T_0, but not on T. Moreover,

$$
\begin{aligned}
|\mathcal{D}|^2 &= \frac{1}{T} \int_0^T |\mathcal{D}|^2 \, dt \\
&= \frac{1}{T} \int_0^T \left| G_1(t) - e^{\eta t} \sum_{n=n_0}^{\infty} \left(d_n D_n e^{i p_n t} + \overline{d_n} \, \overline{D_n} e^{-i \overline{p_n} t} \right) \right|^2 dt \\
&\le \frac{2}{T} \left(\int_0^T \left| G_1(t) \right|^2 dt + \int_0^T \left| e^{\eta t} \sum_{n=n_0}^{\infty} \left(d_n D_n e^{i p_n t} + \overline{d_n} \, \overline{D_n} e^{-i \overline{p_n} t} \right) \right|^2 dt \right). \qquad (5.51)
\end{aligned}
$$

By (5.41), (5.50) and (5.44) we have

$$
\begin{aligned}
&\int_0^T \left| e^{\eta t} \sum_{n=n_0}^{\infty} \left(d_n D_n e^{i p_n t} + \overline{d_n} \, \overline{D_n} e^{-i \overline{p_n} t} \right) \right|^2 dt \\
&\quad \le e^{2\eta T} \int_0^T \left| \sum_{n=n_0}^{\infty} \left(d_n D_n e^{i p_n t} + \overline{d_n} \, \overline{D_n} e^{-i \overline{p_n} t} \right) \right|^2 dt \\
&\quad \le e^{2\eta T} c(T) \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 \le c(T) \int_0^T \left| G_1(t) \right|^2 dt.
\end{aligned}
$$

Plugging the above estimate into (5.51), we obtain

$$
\int_0^T \left| G_1(t) \right|^2 dt \ge c(T) |\mathcal{D}|^2.
$$

Finally, because of (5.50) we get

$$
\begin{aligned}
&\int_0^T \left| G_1(t) \right|^2 dt \\
&\quad \ge \pi C_0 T \sum_{n=n_0}^{\infty} \left(\frac{1}{\pi^2 + 4T^2 (\Im p_n)^2} - \frac{2}{T^2 M^2} \right) \left(1 + e^{-2\Im p_n T} \right) |D_n|^2 |p_n|^4 + c(T) |\mathcal{D}|^2,
\end{aligned}
$$

that is (5.43). □

Now, we are able to prove an inverse inequality in the special case when our series have a finite number of terms vanishing.

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Theorem 5.11. *Under assumptions (5.3)–(5.8), for any real constant $\mathcal{D}$, $\varepsilon \in (0,1)$ and $T > \frac{2\pi}{\gamma(1-\varepsilon)}$ there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $c(T,\varepsilon) > 0$ such that if $C_n = D_n = 0$ for any $n < n_0$, then we have*

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt$$
$$\geq c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty}\left(1+e^{-2(\Im\omega_n-\alpha)T}\right)|C_n|^2 + \sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4 + |\mathcal{D}|^2\right). \tag{5.52}$$

Proof. As a consequence of (5.34), (5.43) and (5.44) with $T_0 = \frac{2\pi}{\gamma}$ and $M = \gamma > \frac{2\pi}{T(1-\varepsilon)}$ we have, for n_0 large enough,

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|e^{\eta t}u_2(t)\right|^2\right) dt$$
$$\geq c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty}\left(1+e^{-2(\Im\omega_n-\alpha)T}\right)|C_n|^2 + |\mathcal{D}|^2\right)$$
$$+ C_0\pi T\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} - \frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4$$
$$- 2\pi T\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} + \frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2, \tag{5.53}$$

with

$$\frac{1}{\pi^2+4T^2(\Im p_n)^2} - \frac{2}{T^2\gamma^2} > c(T,\gamma,\varepsilon) > 0, \quad \forall n \geq n_0. \tag{5.54}$$

Now, we evaluate the sum

$$C_0\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} - \frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4$$
$$- 2\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} + \frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2$$
$$= \sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2} - \frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)$$
$$\cdot |D_n|^2\left[C_0|p_n|^4 - 2\frac{\frac{1}{\pi^2+4T^2(\Im p_n)^2} + \frac{2}{T^2\gamma^2}}{\frac{1}{\pi^2+4T^2(\Im p_n)^2} - \frac{2}{T^2\gamma^2}}\right]. \tag{5.55}$$

We note that

$$\frac{\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}}{\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}}=\frac{1+\frac{2(\pi^2+4T^2(\Im p_n)^2)}{T^2\gamma^2}}{1-\frac{2(\pi^2+4T^2(\Im p_n)^2)}{T^2\gamma^2}}=\frac{1+\frac{2\pi^2}{T^2\gamma^2}+\frac{8(\Im p_n)^2}{\gamma^2}}{1-(\frac{2\pi^2}{T^2\gamma^2}+\frac{8(\Im p_n)^2}{\gamma^2})}. \tag{5.56}$$

Since

$$\frac{2\pi^2}{T^2\gamma^2}<\frac{(1-\varepsilon)^2}{2},$$

and for n_0 sufficiently large

$$|\Im p_n|<\frac{1-\varepsilon}{4}\gamma, \quad \forall n\ge n_0,$$

we have

$$\frac{2\pi^2}{T^2\gamma^2}+\frac{8(\Im p_n)^2}{\gamma^2}<\frac{(1-\varepsilon)^2}{2}+\frac{(1-\varepsilon)^2}{2}=(1-\varepsilon)^2<1-\varepsilon.$$

Therefore, taking into account that the function $x\to\frac{1+x}{1-x}$ is strictly increasing on $(-\infty,1)$, from (5.56) we get

$$\frac{\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}}{\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}}<\frac{2-\varepsilon}{\varepsilon},$$

whence

$$C_0|p_n|^4-2\frac{\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}}{\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}}>C_0|p_n|^4-2\frac{2-\varepsilon}{\varepsilon}.$$

Recalling that the constant C_0 is independent of T, we can take $n_0\in\mathbb{N}$ (independent of T), large enough, so that it holds

$$\frac{C_0}{2}|p_n|^4>2\frac{2-\varepsilon}{\varepsilon} \quad \forall n\ge n_0,$$

and hence

$$C_0|p_n|^4-2\frac{\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}}{\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}}>\frac{C_0}{2}|p_n|^4.$$

Plugging the above formula into (5.55), we obtain

$$C_0\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4$$
$$-2\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}+\frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2$$
$$>\frac{C_0}{2}\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4.$$

Finally, from (5.53) it follows

$$\int_0^T\left(\left|u_1(t)\right|^2+\left|e^{\eta t}u_2(t)\right|^2\right)dt$$
$$\ge c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty}\left(1+e^{-2(\Im\omega_n-\alpha)T}\right)|C_n|^2+|\mathcal{D}|^2\right)$$
$$+T\pi\frac{C_0}{2}\sum_{n=n_0}^{\infty}\left(\frac{1}{\pi^2+4T^2(\Im p_n)^2}-\frac{2}{T^2\gamma^2}\right)\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4,$$

and hence, in view of (5.54), we obtain

$$\int_0^T\left(\left|u_1(t)\right|^2+\left|e^{\eta t}u_2(t)\right|^2\right)dt$$
$$\ge c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty}\left(1+e^{-2(\Im\omega_n-\alpha)T}\right)|C_n|^2+\sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4+|\mathcal{D}|^2\right).$$

In conclusion,

$$\int_0^T\left(\left|u_1(t)\right|^2+\left|u_2(t)\right|^2\right)dt\ge e^{-2\eta T}\int_0^T\left(\left|u_1(t)\right|^2+\left|e^{\eta t}u_2(t)\right|^2\right)dt$$
$$\ge e^{-2\eta T}c(T,\varepsilon)\sum_{n=n_0}^{\infty}\left(1+e^{-2(\Im\omega_n-\alpha)T}\right)|C_n|^2$$
$$+e^{-2\eta T}c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty}\left(1+e^{-2\Im p_n T}\right)|D_n|^2|p_n|^4+|\mathcal{D}|^2\right),$$

that is (5.52). □

As regards the direct inequality, first we recall the following result, see [25, Theorem 4.2].

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Theorem 5.12. *Under assumptions (5.5)–(5.7), for any $\varepsilon \in (0,1)$ and $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exist $c(T) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $C_n = 0$ for $n < n_0$, then we have*

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \le c(T) \sum_{n=n_0}^{\infty} |C_n|^2. \tag{5.57}$$

Theorem 5.13. *Assume (5.3), (5.4) and (5.8). Let $\mathcal{D}$ be any real constant, $T > 0$, $\varepsilon \in (0,1)$ and $M > \frac{\pi}{T\sqrt{1-\varepsilon}}$. There exist $c(T) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $D_n = 0$ for any $n < n_0$, then we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} + \mathcal{D} e^{-\eta t} \right|^2 dt \\ &\quad \le c(T) \left(\sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \right). \end{aligned} \tag{5.58}$$

Proof. Since the function $k^*(t)$ is positive, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} + \mathcal{D} e^{-\eta t} \right|^2 dt \\ &\quad \le 2 \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt + 2 \int_{-\infty}^{\infty} k^*(t) e^{-2\eta t}\, dt |\mathcal{D}|^2. \end{aligned} \tag{5.59}$$

We evaluate the first integral by using Proposition 5.9: indeed, from (5.40) we get

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \le c(T) \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4.$$

Putting the previous estimate in (5.59), we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} + \mathcal{D} e^{-\eta t} \right|^2 dt \\ &\quad \le 2c(T) \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + 2 \int_{-\infty}^{\infty} k^*(t) e^{-2\eta t}\, dt\, |\mathcal{D}|^2. \end{aligned} \tag{5.60}$$

In addition, formula (5.27) yields

$$\int_{-\infty}^{\infty} k^*(t) e^{-2\eta t}\, dt = \cosh(2\eta T) \frac{4T\pi}{\pi^2 + 16T^2\eta^2}.$$

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Because of the above formula from (5.60) it follows

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} + \mathcal{D}e^{-\eta t}\right|^2 dt$$

$$\le 2c(T)\sum_{n=n_0}^{\infty} |D_n|^2|p_n|^4 + \cosh(2\eta T)\frac{8T\pi}{\pi^2+16T^2\eta^2}|\mathcal{D}|^2,$$

that is (5.58). □

Finally, thanks to Theorems 5.12 and 5.13 we are able to prove an Ingham type direct estimate for the solution (u_1, u_2) of coupled systems in the special case when our series have a finite number of terms vanishing.

Theorem 5.14. *Under assumptions (5.3)–(5.8), for any real constant $\mathcal{D}$, $\varepsilon \in (0,1)$ and $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exist $c(T) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $C_n = D_n = 0$ for any $n < n_0$, then we have*

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c(T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2|p_n|^4 + |\mathcal{D}|^2\right). \tag{5.61}$$

Proof. First of all, since the function $k^*(t)$ is positive, we can write

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt$$

$$\le 2\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right|^2 dt$$

$$+ 2\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt.$$

So, we can apply Theorem 5.12: plugging into the above formula the inequality (5.57), we obtain

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt$$

$$\le c(T)\sum_{n=n_0}^{\infty} |C_n|^2 + 2\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt.$$

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In Proposition 5.5 we can take $E_n = D_n$ and $M = \gamma$, so by the previous inequality and (5.31) we get

$$\int_{-\infty}^{\infty} k^*(t)\left|\sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i p_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right|^2 dt$$
$$\le c(T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2\right).$$

Moreover, by the above estimate and (5.58) we obtain

$$\int_{-\infty}^{\infty} k^*(t)\left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c(T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right).$$

Now, if we consider the last inequality with the function k^* replaced by the analogous one relative to $2T$ instead of T, see (5.26), then we get

$$\int_{-2T}^{2T} \cos\frac{\pi t}{4T}\left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c(2T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right),$$

whence, thanks to $\cos\frac{\pi t}{4T} \ge \frac{1}{\sqrt{2}}$ for $|t| \le T$, it follows

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le \sqrt{2}c(2T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right).$$

So, the proof of (5.61) is complete. □

5.3. Haraux type estimates

To prove our results, we need to introduce a suitable family of operators which annihilate a finite number of terms in the Fourier series. For the reader's convenience, we proceed to recall the definition of operators, which was given in [25] and is slightly different from those introduced in [6] and [13].

Given $\delta > 0$ and $z \in \mathbb{C}$ arbitrarily, we define the linear operator $I_{\delta,z}$ as follows: for every continuous function $u : \mathbb{R} \to \mathbb{C}$ the function $I_{\delta,z}u : \mathbb{R} \to \mathbb{C}$ is given by the formula

$$I_{\delta,z}u(t) := u(t) - \frac{1}{\delta}\int_0^{\delta} e^{-izs} u(t+s)\, ds, \qquad t \in \mathbb{R}. \tag{5.62}$$

A list of properties connected with operators $I_{\delta,z}$ is now in order.

Lemma 5.15. *For any $\delta > 0$ and $z \in \mathbb{C}$ the following statements hold true.*

(i) $I_{\delta,z}(e^{izt}) = 0$.
(ii) *For any $z' \in \mathbb{C}$, $z' \neq z$, we have*

$$I_{\delta,z}\big(e^{iz't}\big) = \left(1 - \frac{e^{i(z'-z)\delta} - 1}{i(z'-z)\delta}\right)e^{iz't}.$$

(iii) *The linear operators $I_{\delta,z}$ commute, that is, for any $\delta' > 0$, $z' \in \mathbb{C}$ and continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$I_{\delta,z} I_{\delta',z'} u = I_{\delta',z'} I_{\delta,z} u.$$

(iv) *For any $T > 0$ and continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$\int_0^T \big|I_{\delta,z} u(t)\big|^2 \, dt \le 2\big(1 + e^{2|\Im z|\delta}\big) \int_0^{T+\delta} \big|u(t)\big|^2 \, dt. \tag{5.63}$$

We now define another operator

$$I_{\delta,r,\omega,p} := I_{\delta,-ir} \circ I_{\delta,\omega} \circ I_{\delta,-\overline{\omega}} \circ I_{\delta,p} \circ I_{\delta,-\overline{p}} \quad \delta > 0,\ r \in \mathbb{R},\ \omega, p \in \mathbb{C}, \tag{5.64}$$

where the symbol $\circ$ denotes the usual composition among operators.

By using Lemma 5.15 one can easily prove the following properties concerning operators $I_{\delta,r,\omega,p}$.

Lemma 5.16. *For any $\delta > 0$ and $r \in \mathbb{R}$, ω, $p \in \mathbb{C}$ the following statements hold true.*

(i) *If $z \in \{-ir, \omega, -\overline{\omega}, p, -\overline{p}\}$ we have $I_{\delta,r,\omega,p}(e^{izt}) = 0$.*
(ii) *For any $r' \in \mathbb{R}$, $r' \notin \{r, i\omega, -i\overline{\omega}, ip, -i\overline{p}\}$, we have*

$$I_{\delta,r,\omega,p}\big(e^{r't}\big) = \prod_{z \in \{r, i\omega, -i\overline{\omega}, ip, -i\overline{p}\}} \left(1 - \frac{e^{(r'-z)\delta} - 1}{(r'-z)\delta}\right)e^{r't}.$$

(iii) *For any $z' \in \mathbb{C}$, $z' \notin \{-ir, \omega, -\overline{\omega}, p, -\overline{p}\}$, we have*

$$I_{\delta,r,\omega,p}\big(e^{iz't}\big) = \prod_{z \in \{-ir, \omega, -\overline{\omega}, p, -\overline{p}\}} \left(1 - \frac{e^{i(z'-z)\delta} - 1}{i(z'-z)\delta}\right)e^{iz't}.$$

(iv) *The linear operators $I_{\delta,r,\omega,p}$ commute: for any $\delta' > 0$, $r' \in \mathbb{R}$, ω', $p' \in \mathbb{C}$ and continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$I_{\delta,r,\omega,p} I_{\delta',r',\omega',p'} u = I_{\delta',r',\omega',p'} I_{\delta,r,\omega,p} u.$$

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Corollary 5.17. *For any $T > 0$, $\delta > 0$, $r \in \mathbb{R}$, ω, $p \in \mathbb{C}$ and continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$\int_0^T \left|I_{\delta,r,\omega,p}u(t)\right|^2 dt \le 2^5\left(1+e^{2|r|\delta}\right)\left(1+e^{2|\Im\omega|\delta}\right)^2\left(1+e^{2|\Im p|\delta}\right)^2 \int_0^{T+5\delta} \left|u(t)\right|^2 dt. \tag{5.65}$$

Proof. By applying (5.63) repeatedly, we obtain

$$\begin{aligned}
\int_0^T \left|I_{\delta,r,\omega,p}u(t)\right|^2 dt &= \int_0^T \left|I_{\delta,-ir}I_{\delta,\omega}I_{\delta,-\overline{\omega}}I_{\delta,p}I_{\delta,-\overline{p}}u(t)\right|^2 dt \\
&\le 2^2\left(1+e^{2|\Im p|\delta}\right)^2 \int_0^{T+2\delta} \left|I_{\delta,-ir}I_{\delta,\omega}I_{\delta,-\overline{\omega}}u(t)\right|^2 dt \\
&\le 2^4\left(1+e^{2|\Im p|\delta}\right)^2\left(1+e^{2|\Im\omega|\delta}\right)^2 \int_0^{T+4\delta} \left|I_{\delta,-ir}u(t)\right|^2 dt \\
&\le 2^5\left(1+e^{2|\Im p|\delta}\right)^2\left(1+e^{2|\Im\omega|\delta}\right)^2\left(1+e^{2|r|\delta}\right) \int_0^{T+5\delta} \left|u(t)\right|^2 dt,
\end{aligned}$$

that is (5.65). $\square$

Proposition 5.18. *Let $\{\omega_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ be sequences of pairwise distinct numbers such that $\omega_n \ne p_m$, $\omega_n \ne \overline{p_m}$, $r_n \ne i\omega_m$, $r_n \ne ip_m$, $r_n \ne -\eta$, $p_n \ne 0$, for any $n, m \in \mathbb{N}$,*

$$\lim_{n\to\infty} |\omega_n| = \lim_{n\to\infty} |p_n| = +\infty, \tag{5.66}$$

and

$$|d_n| \ge m_1|p_n|^2 \quad \forall n \in \mathbb{N}\ (m_1 > 0). \tag{5.67}$$

Assume that $\mathcal{D}$ is any real constant and there exists $n_0 \in \mathbb{N}$ such that for any sequences $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$ verifying

$$R_n = C_n = D_n = 0 \quad \textit{for any } n < n_0,$$

the estimates

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \ge c_1\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2|p_n|^4 + |\mathcal{D}|^2\right), \tag{5.68}$$

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c_2\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2|p_n|^4 + |\mathcal{D}|^2\right), \tag{5.69}$$

are satisfied for some constants $c_1, c_2 > 0$.

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Then, there exists $C_1 > 0$ such that for any sequences $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$ the estimate

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \geq C_1 \left(\sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right) \tag{5.70}$$

holds.

Proof. To begin with, we will transform the functions

$$u_1(t) = \sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i p_n t} + \overline{D_n} e^{-i\overline{p_n} t}\right),$$

$$u_2(t) = \sum_{n=1}^{\infty} \left(d_n D_n e^{i p_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t}\right) + \mathcal{D} e^{-\eta t}$$

in such a way that the series have null terms corresponding to indices $n = 1, \dots, n_0 - 1$, because so we can apply our assumptions (5.68) and (5.69).

To this end, we fix $\varepsilon > 0$ and choose $\delta \in (0, \frac{\varepsilon}{5n_0})$. Let us denote by $\mathbb{I}$ the composition of all linear operators $I_{\delta, r_j, \omega_j, p_j}$, $j = 1, \dots, n_0 - 1$. We note that by Lemma 5.16(iv) the definition of $\mathbb{I}$ does not depend on the order of the operators $I_{\delta, r_j, \omega_j, p_j}$.

By using Lemma 5.16, we get

$$\mathbb{I}u_1(t) = \sum_{n=n_0}^{\infty} \left(R'_n e^{r_n t} + C'_n e^{i\omega_n t} + \overline{C'_n} e^{-i\overline{\omega_n} t} + D'_n e^{i p_n t} + \overline{D'_n} e^{-i\overline{p_n} t}\right),$$

$$\mathbb{I}u_2(t) = \sum_{n=n_0}^{\infty} \left(d_n D'_n e^{i p_n t} + \overline{d_n}\,\overline{D'_n} e^{-i\overline{p_n} t}\right) + \mathcal{D}' e^{-\eta t}$$

where

$$R'_n := R_n \prod_{j=1}^{n_0 - 1} \prod_{z \in \{r_j, i\omega_j, -i\overline{\omega_j}, i p_j, -i\overline{p_j}\}} \left(1 - \frac{e^{(r_n - z)\delta} - 1}{(r_n - z)\delta}\right),$$

$$C'_n := C_n \prod_{j=1}^{n_0 - 1} \prod_{z \in \{-i r_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(\omega_n - z)\delta} - 1}{i(\omega_n - z)\delta}\right),$$

$$D'_n := D_n \prod_{j=1}^{n_0 - 1} \prod_{z \in \{-i r_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta}\right),$$

$$\mathcal{D}' := \mathcal{D} \prod_{j=1}^{n_0 - 1} \prod_{z \in \{r_j, i\omega_j, -i\overline{\omega_j}, i p_j, -i\overline{p_j}\}} \left(1 + \frac{e^{-(\eta + z)\delta} - 1}{(\eta + z)\delta}\right).$$

Therefore, we are in condition to apply estimate (5.68) to functions $\mathbb{I}u_1(t)$ and $\mathbb{I}u_2(t)$:

$$\int_0^T \left(\left|\mathbb{I}u_1(t)\right|^2 + \left|\mathbb{I}u_2(t)\right|^2\right) dt \geq c_1 \left(\sum_{n=n_0}^{\infty} \left|C_n'\right|^2 + \sum_{n=n_0}^{\infty} \left|D_n'\right|^2 |p_n|^4 + \left|\mathcal{D}'\right|^2 \right). \tag{5.71}$$

Next, we choose $\delta \in (0, \frac{\varepsilon}{5n_0})$ such that for any $n \geq n_0$ none of the products

$$\prod_{j=1}^{n_0-1} \prod_{z \in \{-ir_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(\omega_n - z)\delta} - 1}{i(\omega_n - z)\delta}\right) \tag{5.72}$$

vanishes. This is possible because the analytic function

$$w \mapsto 1 - \frac{e^w - 1}{w}$$

does not vanish identically, and hence, since every number $\omega_n - z$ with $z \in \{-ir_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}$ is different from zero, we have to exclude only a countable set of values of δ.

Then, we note that there exists a constant $c' > 0$ such that for any $n \geq n_0$

$$\left| \prod_{j=1}^{n_0-1} \prod_{z \in \{-ir_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(\omega_n - z)\delta} - 1}{i(\omega_n - z)\delta}\right) \right|^2 \geq c'. \tag{5.73}$$

Indeed, it is sufficient to observe that for any fixed $j = 1, \ldots, n_0 - 1$ and $z \in \{-ir_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}$ we have

$$\left| \frac{e^{i(\omega_n - z)\delta} - 1}{i(\omega_n - z)\delta} \right| \leq \frac{e^{-\Im(\omega_n - z)\delta} + 1}{|\omega_n - z|\delta} \to 0 \quad \text{as } n \to \infty,$$

thanks to (5.66). As a result, the product in (5.72) tends to 1 as $n \to \infty$, so that it is minorized, e.g., by $1/2$ for n large enough. By repeating the same argumentations used to get (5.73), we also have

$$\left| \prod_{j=1}^{n_0-1} \prod_{z \in \{-ir_j, \omega_j, -\overline{\omega_j}, p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta}\right) \right|^2 \geq c'. \tag{5.74}$$

In addition, we can assume that

$$\left| \prod_{j=1}^{n_0-1} \prod_{z \in \{r_j, i\omega_j, -i\overline{\omega_j}, ip_j, -i\overline{p_j}\}} \left(1 + \frac{e^{-(\eta + z)\delta} - 1}{(\eta + z)\delta}\right) \right|^2 \geq c',$$

and hence

$$\left|\mathcal{D}'\right|^2 \geq c' |\mathcal{D}|^2.$$

Therefore, the above estimate and (5.71)–(5.74) yield

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$$\int_0^T \left(\left|\mathbb{I}u_1(t)\right|^2 + \left|\mathbb{I}u_2(t)\right|^2\right) dt \geq c' c_1 \left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \right). \tag{5.75}$$

On the other hand, applying (5.65) repeatedly with $r = r_j$, $\omega = \omega_j$ and $p = p_j$, $j = 1, \dots, n_0 - 1$, we have

$$\begin{aligned} &\int_0^T \left(\left|\mathbb{I}u_1(t)\right|^2 + \left|\mathbb{I}u_2(t)\right|^2\right) dt \\ &\leq 2^{5(n_0-1)} \prod_{j=1}^{n_0-1} \left(1 + e^{2|r_j|\delta}\right)\left(1 + e^{2|\Im\omega_j|\delta}\right)^2 \left(1 + e^{2|\Im p_j|\delta}\right)^2 \int_0^{T+5(n_0-1)\delta} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt. \end{aligned}$$

From the above inequality, by using (5.75) and $5n_0\delta < \varepsilon$, it follows

$$\begin{aligned} &\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \\ &\leq \frac{2^{5(n_0-1)}}{c' c_1} \prod_{j=1}^{n_0-1} \left(1 + e^{|r_j|\varepsilon/n_0}\right)\left(1 + e^{|\Im\omega_j|\varepsilon/n_0}\right)^2 \left(1 + e^{|\Im p_j|\varepsilon/n_0}\right)^2 \int_0^{T+\varepsilon} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt, \end{aligned}$$

whence, passing to the limit as $\varepsilon \to 0^+$, we have

$$\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \leq \frac{2^{10(n_0-1)}}{c' c_1} \int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt. \tag{5.76}$$

Moreover, thanks to the triangle inequality, we get

$$\begin{aligned} &\int_0^T \left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i p_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \\ &= \int_0^T \left| u_1(t) - \sum_{n=n_0}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i p_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right) \right|^2 dt \\ &\leq 2 \int_0^T \left|u_1(t)\right|^2 dt \\ &\quad + 2 \int_0^T \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i p_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \end{aligned} \tag{5.77}$$

and

$$\begin{aligned}
&\int_0^T \left| \sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{p_n}t} \right|^2 dt \\
&= \int_0^T \left| u_2(t) - \sum_{n=n_0}^{\infty} \left(d_n D_n e^{ip_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{p_n}t} \right) - \mathcal{D} e^{-\eta t} \right|^2 dt \\
&\le 2 \int_0^T |u_2(t)|^2 \, dt + 2 \int_0^T \left| \sum_{n=n_0}^{\infty} \left(d_n D_n e^{ip_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{p_n}t} \right) + \mathcal{D} e^{-\eta t} \right|^2 dt. \qquad (5.78)
\end{aligned}$$

Putting together (5.77) and (5.78) and using (5.69) and (5.76) we have

$$\begin{aligned}
&\int_0^T \left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n}t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n}t} \right|^2 dt \\
&\quad + \int_0^T \left| \sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{p_n}t} \right|^2 dt \\
&\le 2 \int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2 \right) dt + 2c_2 \left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \right) \\
&\le 2 \left(1 + c_2 \frac{2^{10(n_0-1)}}{c' c_1} \right) \int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2 \right) dt. \qquad (5.79)
\end{aligned}$$

Let us note that the expression

$$\begin{aligned}
&\int_0^T \left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n}t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n}t} \right|^2 dt \\
&\quad + \int_0^T \left| \sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{p_n}t} \right|^2 dt
\end{aligned}$$

is a positive semidefinite quadratic form of the variable

$$\left(\{R_n\}_{n<n_0}, \{C_n\}_{n<n_0}, \{d_n D_n\}_{n<n_0} \right) \in \mathbb{R}^{n_0-1} \times \mathbb{C}^{n_0-1} \times \mathbb{C}^{n_0-1}.$$

Moreover, it is positive *definite*, because the functions $e^{r_n t}$, $e^{i\omega_n t}$, $e^{ip_n t}$, $n < n_0$, are linearly independent. Hence, there exists a constant $c'' > 0$ such that

$$\int_0^T \left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt$$
$$+ \int_0^T \left| \sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \geq c'' \sum_{n=1}^{n_0-1} \left(|R_n|^2 + |C_n|^2 + |D_n|^2 |p_n|^4 \right),$$

taking into account (5.67). So, from (5.79) and the above inequality we deduce that

$$\sum_{n=1}^{n_0-1} \left(|C_n|^2 + |D_n|^2 |p_n|^4 \right) \leq \frac{2}{c''} \left(1 + c_2 \frac{2^{10(n_0-1)}}{c' c_1} \right) \int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2 \right) dt.$$

Finally, the above estimate and (5.76) yield the required inequality (5.70). □

5.4. *Inverse and direct inequalities*

We recall that

$$u_1(t) = \sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right),$$
$$u_2(t) = \sum_{n=1}^{\infty} \left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t},$$

with

$$\mathcal{D} = -\frac{\beta}{A} \sum_{n=1}^{\infty} \Re p_n \left(\frac{D_n}{\eta + ip_n} + \frac{\overline{D_n}}{\eta - i\overline{p_n}} \right). \tag{5.80}$$

Theorem 5.19. *Let $\{\omega_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ be sequences of pairwise distinct numbers such that $\omega_n \neq p_m$, $\omega_n \neq \overline{p_m}$, $r_n \neq i\omega_m$, $r_n \neq ip_m$, $r_n \neq -\eta$, $p_n \neq 0$, for any $n, m \in \mathbb{N}$. Assume*

$$\lim_{n\to\infty} (\Re p_{n+1} - \Re p_n) = +\infty,$$
$$\lim_{n\to\infty} \Im p_n = 0,$$

and for some $\gamma > 0$, $\alpha \in \mathbb{R}$, $n' \in \mathbb{N}$, $\mu > 0$, $\nu > 1/2$, $m_1, m_2 > 0$

$$\liminf_{n\to\infty} (\Re\omega_{n+1} - \Re\omega_n) = \gamma,$$
$$\lim_{n\to\infty} \Im\omega_n = \alpha, \qquad r_n \leq -\Im\omega_n \quad \forall n \geq n',$$
$$|R_n| \leq \frac{\mu}{n^\nu} |C_n| \quad \forall n \geq n', \qquad |R_n| \leq \mu |C_n| \quad \forall n \leq n',$$
$$m_1 |p_n|^2 \leq |d_n| \leq m_2 |p_n|^2 \quad \forall n \in \mathbb{N}.$$

Then, for any $T > 2\pi/\gamma$ we have

$$\int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \geq c_1(T)\left(\sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right), \tag{5.81}$$

where $c_1(T)$ is a positive constant.

Proof. Since $T > 2\pi/\gamma$, there exists $0 < \varepsilon < 1$ such that $T > \frac{2\pi}{\gamma(1-\varepsilon)}$. By applying Theorems 5.11 and 5.14, there exist $n_0 \in \mathbb{N}$, $c(T,\varepsilon) > 0$ and $c(T) > 0$ such that if $R_n = C_n = D_n = 0$ for $n < n_0$, then we have

$$\begin{aligned}
&c(T,\varepsilon)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right) \\
&\quad \leq \int_0^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \\
&\quad \leq c(T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right).
\end{aligned}$$

Finally, thanks to Proposition 5.18 we can conclude. □

Before to prove the direct inequality, we need a result, which allows us to recover a finite number of terms in the Fourier series.

Proposition 5.20. *Assume here that $\mathcal{D}$ is any real constant,*

$$|d_n| \leq m_2 |p_n|^2 \quad \forall n \in \mathbb{N}\ (m_2 > 0), \tag{5.82}$$

$$|p_n| \geq a_0 \quad \forall n \in \mathbb{N}\ (a_0 > 0), \tag{5.83}$$

and that there exists $n_0 \in \mathbb{N}$ such that for any sequences $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$ verifying

$$R_n = C_n = D_n = 0 \quad \textit{for any } n < n_0,$$

the estimate

$$\int_{-T}^T \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \leq c_2\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right) \tag{5.84}$$

is satisfied for some $c_2 > 0$. Then, for any sequences $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$ such that

$$|R_n| \leq \mu |C_n| \quad \textit{for any } n < n_0\ (\mu > 0), \tag{5.85}$$

the estimate

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le C_2 \left(\sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \right) \tag{5.86}$$

holds for some $C_2 > 0$.

Proof. Let $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$ be arbitrary sequences and assume that (5.85) holds. If we use (5.84), then we have

$$\begin{aligned}
&\int_{-T}^{T} \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \\
&\quad + \int_{-T}^{T} \left| \sum_{n=n_0}^{\infty} \left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t} \right|^2 dt \\
&\le c_2 \left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2 \right).
\end{aligned} \tag{5.87}$$

Now, we will prove that

$$\begin{aligned}
&\int_{-T}^{T} \left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \\
&\quad + \int_{-T}^{T} \left| \sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} \right|^2 dt \\
&\le c_2' \sum_{n=1}^{n_0-1} \left(|C_n|^2 + |D_n|^2 |p_n|^4 \right),
\end{aligned} \tag{5.88}$$

for some constant $c_2' > 0$. Indeed, by applying the Cauchy–Schwarz inequality we get

$$\begin{aligned}
&\left| \sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right|^2 \\
&\le \left(\sum_{n=1}^{n_0-1} |R_n| e^{r_n t} + 2|C_n| e^{-\Im\omega_n t} + 2|D_n| e^{-\Im p_n t} \right)^2 \\
&\le 12(n_0-1) \sum_{n=1}^{n_0-1} \left(|R_n|^2 e^{2r_n t} + |C_n|^2 e^{-2\Im\omega_n t} + |D_n|^2 e^{-2\Im p_n t} \right)
\end{aligned}$$

and in view also of (5.82)

$$\begin{aligned}\left|\sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right|^2 &\le 4\left(\sum_{n=1}^{n_0-1} |d_n D_n| e^{-\Im p_n t}\right)^2 \\ &\le 4(n_0-1)\sum_{n=1}^{n_0-1} |d_n|^2 |D_n|^2 e^{-2\Im p_n t} \\ &\le 4(n_0-1)m_2^2 \sum_{n=1}^{n_0-1} |p_n|^4 |D_n|^2 e^{-2\Im p_n t}.\end{aligned}$$

If we use the previous inequalities, (5.85) and (5.83), then we get

$$\begin{aligned}&\int_{-T}^{T}\left|\sum_{n=1}^{n_0-1} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n}t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n}t}\right|^2 dt \\ &\quad + \int_{-T}^{T}\left|\sum_{n=1}^{n_0-1} d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n}t}\right|^2 dt \\ &\le 12(n_0-1)\sum_{n=1}^{n_0-1} |C_n|^2 \int_{-T}^{T}\left(\mu^2 e^{2r_n t} + e^{-2\Im\omega_n t}\right) dt \\ &\quad + 4(n_0-1)\left(3a_0^{-4} + m_2^2\right)\sum_{n=1}^{n_0-1} |p_n|^4 |D_n|^2 \int_{-T}^{T} e^{-2\Im p_n t}\, dt,\end{aligned}$$

whence (5.88) follows with

$$c_2' = 12(n_0-1)\max_{n<n_0}\left\{\int_{-T}^{T}\left(\mu^2 e^{2r_n t} + e^{-2\Im\omega_n t} + \left(a_0^{-4} + m_2^2\right)e^{-2\Im p_n t}\right) dt\right\}.$$

Finally, from (5.87) and (5.88) we deduce that

$$\begin{aligned}&\int_{-T}^{T}\left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \\ &\quad \le 2c_2\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right) + 2c_2'\left(\sum_{n=1}^{n_0-1} |C_n|^2 + \sum_{n=1}^{n_0-1} |D_n|^2 |p_n|^4\right),\end{aligned}$$

so (5.86) holds with $C_2 = 2\max\{c_2, c_2'\}$. □

In conclusion we can prove the direct inequality.

Theorem 5.21. *Assume* $p_n \neq 0$,

$$\lim_{n\to\infty} (\Re p_{n+1} - \Re p_n) = +\infty,$$

$$\lim_{n\to\infty} \Im p_n = 0,$$

and for some $\gamma > 0$, $\alpha \in \mathbb{R}$, $n' \in \mathbb{N}$, $\mu > 0$, $\nu > 1/2$, $m_1, m_2 > 0$

$$\liminf_{n\to\infty} (\Re\omega_{n+1} - \Re\omega_n) = \gamma,$$

$$\lim_{n\to\infty} \Im\omega_n = \alpha, \qquad r_n \le -\Im\omega_n \quad \forall n \ge n',$$

$$|R_n| \le \frac{\mu}{n^\nu}|C_n| \quad \forall n \ge n', \qquad |R_n| \le \mu|C_n| \quad \forall n \le n',$$

$$m_1|p_n|^2 \le |d_n| \le m_2|p_n|^2 \quad \forall n \in \mathbb{N}.$$

Then, for any $T > \pi/\gamma$ *we have*

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c_2(T)\left(\sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4\right), \tag{5.89}$$

where $c_2(T)$ *is a positive constant.*

Proof. Since $T > \pi/\gamma$, there exists $0 < \varepsilon < 1$ such that $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$. By applying Theorem 5.14, there exist $c(T) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $C_n = D_n = 0$ for any $n < n_0$, then we have

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le c(T)\left(\sum_{n=n_0}^{\infty} |C_n|^2 + \sum_{n=n_0}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right).$$

We can use Proposition 5.20 to obtain, for any arbitrary sequences $\{R_n\}$, $\{C_n\}$ and $\{D_n\}$,

$$\int_{-T}^{T} \left(\left|u_1(t)\right|^2 + \left|u_2(t)\right|^2\right) dt \le C_2\left(\sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4 + |\mathcal{D}|^2\right) \tag{5.90}$$

for some $C_2 > 0$. Moreover, if we take

$$\mathcal{D} = -\frac{\beta}{A} \sum_{n=1}^{\infty} \Re p_n \left(\frac{D_n}{\eta + i p_n} + \frac{\overline{D_n}}{\eta - i p_n}\right),$$

then, for some $C > 0$, we have

$$|\mathcal{D}|^2 \le C \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4. \tag{5.91}$$

Indeed, we observe that

$$\left| \sum_{n=1}^{\infty} \Re p_n \left(\frac{D_n}{\eta + i p_n} + \frac{\overline{D_n}}{\eta - i \overline{p_n}} \right) \right| \le 2 \sum_{n=1}^{\infty} |p_n| \left| \Re \frac{D_n}{\eta + i p_n} \right| \le 2 \sum_{n=1}^{\infty} |p_n| \frac{|D_n|}{|\eta + i p_n|},$$

whence, thanks to $\lim_{n\to\infty} \frac{\Re p_n}{n} = +\infty$, we get

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \Re p_n \left(\frac{D_n}{\eta + i p_n} + \frac{\overline{D_n}}{\eta - i \overline{p_n}} \right) \right|^2 \\
&\quad \le 4 \sum_{n,m=1}^{\infty} \frac{|D_n||p_n|}{|\eta + i p_m|} \frac{|D_m||p_m|}{|\eta + i p_n|} \\
&\quad \le 2 \sum_{n=1}^{\infty} |D_n|^2 |p_n|^2 \sum_{m=1}^{\infty} \frac{1}{(\eta - \Im p_m)^2 + \Re p_m^2} + 2 \sum_{m=1}^{\infty} |D_m|^2 |p_m|^2 \sum_{n=1}^{\infty} \frac{1}{(\eta - \Im p_n)^2 + \Re p_n^2} \\
&\quad = 4 \sum_{n=1}^{\infty} \frac{1}{(\eta - \Im p_n)^2 + \Re p_n^2} \sum_{n=1}^{\infty} |D_n|^2 |p_n|^2.
\end{aligned}$$

Therefore, taking into account (5.10), we have that (5.91) holds true with

$$C = 4 a_0^{-2} \frac{\beta}{A} \sum_{n=1}^{\infty} \frac{1}{(\eta - \Im p_n)^2 + \Re p_n^2}.$$

In conclusion, from (5.90) and (5.91) it follows (5.89). □

6. A reachability result

Finally, by applying our abstract results of Sections 4 and 5 we are able to show our reachability result for wave–Petrovsky coupled systems with a memory term.

Theorem 6.1. *Let $\eta > 3\beta/2$. For any $T > 2\pi$ and*

$$(u_{10}, u_{11}, u_{20}, u_{21}) \in L^2(0,\pi) \times H^{-1}(0,\pi) \times H_0^1(0,\pi) \times H^{-1}(0,\pi),$$

there exist $g_i \in L^2(0,T)$, $i = 1, 2$, such that the weak solution (u_1, u_2) of system

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \beta \displaystyle\int_0^t e^{-\eta(t-s)} u_{1xx}(s,x)\,ds + A u_2(t,x) = 0, \\ \quad t \in (0,T), \ x \in (0,\pi), \\ u_{2tt}(t,x) + u_{2xxxx}(t,x) + B u_1(t,x) = 0, \end{cases} \tag{6.1}$$

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with null initial conditions

$$u_1(0,x) = u_{1t}(0,x) = u_2(0,x) = u_{2t}(0,x) = 0 \quad x \in (0,\pi), \tag{6.2}$$

and boundary conditions

$$u_1(t,0) = 0, \qquad u_1(t,\pi) = g_1(t) \quad t \in (0,T), \tag{6.3}$$

$$u_2(t,0) = u_{2xx}(t,0) = u_2(t,\pi) = 0, \qquad u_{2xx}(t,\pi) = g_2(t) \quad t \in (0,T) \tag{6.4}$$

verifies the final conditions

$$u_1(T,x) = u_{10}(x), \qquad u_{1t}(T,x) = u_{11}(x), \quad x \in (0,\pi), \tag{6.5}$$

$$u_2(T,x) = u_{20}(x), \qquad u_{2t}(T,x) = u_{21}(x), \quad x \in (0,\pi). \tag{6.6}$$

Proof. To prove our claim, we will apply the Hilbert Uniqueness Method described in Section 3.

Let $X = L^2(0,\pi)$ be endowed with the usual scalar product and norm

$$\|v\| := \left(\int_0^\pi |v(x)|^2\,dx\right)^{1/2} \qquad v \in L^2(0,\pi).$$

We consider the operator $L : D(L) \subset X \to X$ defined by

$$D(L) = H^2(0,\pi) \cap H_0^1(0,\pi),$$
$$Lv = -v_{xx} \quad v \in D(L).$$

It is well known that L is a self-adjoint positive operator on X with dense domain $D(L)$, $\{n^2\}_{n\ge1}$ is the sequence of eigenvalues for L and $\{\sin(nx)\}_{n\ge1}$ is the sequence of the corresponding eigenvectors. We can apply our spectral analysis (see Section 4) to the adjoint system of (6.1). Indeed, the adjoint system is given by

$$\begin{cases} z_{1tt}(t,x) - z_{1xx}(t,x) + \beta \displaystyle\int_t^T e^{-\eta(s-t)} z_{1xx}(s,x)\,ds + Bz_2(t,x) = 0, \\ \quad t \in (0,T),\ x \in (0,\pi), \\ z_{2tt}(t,x) + z_{2xxxx}(t,x) + Az_1(t,x) = 0, \\ z_1(t,0) = z_1(t,\pi) = z_2(t,0) = z_2(t,\pi) = z_{2xx}(t,0) = z_{2xx}(t,\pi) = 0, \end{cases} \tag{6.7}$$

with final data

$$z_1(T,\cdot) = z_{10}, \qquad z_{1t}(T,\cdot) = z_{11}, \qquad z_2(T,\cdot) = z_{20}, \qquad z_{2t}(T,\cdot) = z_{21}, \tag{6.8}$$

where

$$z_{10}(x)=\sum_{n=1}^{\infty}\alpha_{1n}\sin(nx),\qquad z_{11}(x)=\sum_{n=1}^{\infty}\rho_{1n}\sin(nx),\tag{6.9}$$

$$z_{20}(x)=\sum_{n=1}^{\infty}\alpha_{2n}\sin(nx),\qquad z_{21}(x)=\sum_{n=1}^{\infty}\rho_{2n}\sin(nx).\tag{6.10}$$

The solution (z_1, z_2) of system (6.7)–(6.8) can be written in the following way (see formulas (4.23)–(4.26)): for any $(t, x) \in [0, T] \times [0, \pi]$

$$\begin{aligned}
z_1(t,x)&=\sum_{n=1}^{\infty}\big(R_n e^{r_n(T-t)}+C_n e^{i\omega_n(T-t)}+\overline{C_n}e^{-i\overline{\omega_n}(T-t)}\big)\sin(nx)\\
&\quad+\sum_{n=1}^{\infty}\big(D_n e^{ip_n(T-t)}+\overline{D_n}e^{-i\overline{p_n}(T-t)}\big)\sin(nx),\\
z_2(t,x)&=\sum_{n=1}^{\infty}\big(d_n D_n e^{ip_n(T-t)}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}(T-t)}\big)\sin(nx)\\
&\quad-\frac{\beta}{A}e^{-\eta(T-t)}\sum_{n=1}^{\infty}\Re p_n\bigg(\frac{D_n}{\eta+ip_n}+\frac{\overline{D_n}}{\eta-ip_n}\bigg)\sin(nx).
\end{aligned}$$

In particular, by (4.22) and estimates (4.16) and (4.18) we have the following relationships between the coefficients C_n, D_n and the Fourier coefficients (see (6.9)–(6.10)) of the final data:

$$n^2|C_n|^2\asymp\alpha_{1n}^2n^2+\rho_{1n}^2,\qquad n^{10}|D_n|^2\asymp\alpha_{2n}^2n^2+\frac{\rho_{2n}^2}{n^2}.\tag{6.11}$$

Moreover, for any $t \in [0, T]$

$$\begin{aligned}
z_{1x}(t,\pi)&=\sum_{n=1}^{\infty}(-1)^n n\big(R_n e^{r_n(T-t)}+C_n e^{i\omega_n(T-t)}+\overline{C_n}e^{-i\overline{\omega_n}(T-t)}\big)\\
&\quad+\sum_{n=1}^{\infty}(-1)^n n\big(D_n e^{ip_n(T-t)}+\overline{D_n}e^{-i\overline{p_n}(T-t)}\big),\\
z_{2x}(t,\pi)&=\sum_{n=1}^{\infty}(-1)^n n\big(d_n D_n e^{ip_n(T-t)}+\overline{d_n}\,\overline{D_n}e^{-i\overline{p_n}(T-t)}\big)\\
&\quad-\frac{\beta}{A}e^{-\eta(T-t)}\sum_{n=1}^{\infty}(-1)^n n\Re p_n\bigg(\frac{D_n}{\eta+ip_n}+\frac{\overline{D_n}}{\eta-ip_n}\bigg).
\end{aligned}$$

We can apply Theorems 5.19 and 5.21 to $(z_{1x}(t, \pi), z_{2x}(t, \pi))$. Indeed, thanks to inequalities (5.81), (5.89) and (4.25)–(4.27) we have

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$$\int_0^T \left(\left|z_{1x}(t,\pi)\right|^2 + \left|z_{2x}(t,\pi)\right|^2\right) dt \asymp \sum_{n=1}^{\infty} n^2 \left(|C_n|^2 + \frac{1}{A^2}|D_n|^2|p_n|^4\right). \tag{6.12}$$

Therefore, if Ψ is the linear operator defined by (3.9) and

$$\Psi(z_{10}, z_{11}, z_{20}, z_{21}) = (-u_{11}, u_{10}, -u_{21}, u_{20}),$$

where

$$(z_{10}, z_{11}, z_{20}, z_{21}) \in H_0^1(0,\pi) \times L^2(0,\pi) \times H_0^1(0,\pi) \times H^{-1}(0,\pi),$$

taking into account (4.20), (4.21), (6.11) and the expressions of the norms

$$\|z_{10}\|^2_{H_0^1} = \sum_{n=1}^{\infty} \alpha_{1n}^2 n^2, \qquad \|z_{11}\|^2 = \sum_{n=1}^{\infty} \rho_{1n}^2,$$
$$\|z_{20}\|^2_{H_0^1} = \sum_{n=1}^{\infty} \alpha_{2n}^2 n^2, \qquad \|z_{21}\|^2_{H^{-1}} = \sum_{n=1}^{\infty} \frac{\rho_{2n}^2}{n^2},$$

we get

$$\int_0^T \left(\left|z_{1x}(t,\pi)\right|^2 + \left|z_{2x}(t,\pi)\right|^2\right) dt \asymp \|z_{10}\|^2_{H_0^1} + \|z_{11}\|^2 + \|z_{20}\|^2_{H_0^1} + \|z_{21}\|^2_{H^{-1}}.$$

Finally, Theorem 3.2 holds true, the space F introduced at the end of Section 3 is just

$$H_0^1(0,\pi) \times L^2(0,\pi) \times H_0^1(0,\pi) \times H^{-1}(0,\pi),$$

and the control functions are given by

$$g_1(t) = z_{1x}(t,\pi) - \beta \int_t^T e^{-\eta(s-t)} z_{1x}(s,\pi)\, ds, \qquad g_2(t) = -z_{2x}(t,\pi),$$

that is, our proof is complete. □

Remark 6.2. If the coupling term in the first equation is changed to u_{2xx} rather than u_2, we obtain the same reachability result.

First, we can repeat every argumentations done in Section 3, obtaining the same findings. For example, in this case the adjoint system is given by

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$$\begin{cases} z_{1tt}(t,x) - z_{1xx}(t,x) + \int_t^T k(s-t) z_{1xx}(s,x)\,ds + Bz_2(t,x) = 0, \\ \quad t \in (0,T),\ x \in (0,\pi), \\ z_{2tt}(t,x) + z_{2xxxx}(t,x) + Az_{1xx}(t,x) = 0, \\ z_1(t,0) = z_1(t,\pi) = z_2(t,0) = z_2(t,\pi) = z_{2xx}(t,0) = z_{2xx}(t,\pi) = 0, \end{cases}$$

with final data

$$z_1(T,\cdot) = z_{10}, \qquad z_{1t}(T,\cdot) = z_{11}, \qquad z_2(T,\cdot) = z_{20}, \qquad z_{2t}(T,\cdot) = z_{21}. \tag{6.13}$$

Moreover, we can reproduce the same calculations given in Section 4. In particular, we get that $\lambda_j A$ has to substitute A in every formula containing the coupling constant A. For example, in formula (4.13) the expression of coefficients C_{4j} becomes

$$C_{4j} = \frac{A\alpha_{2j}}{2\lambda_j} + (\alpha_{2j} - i\rho_{2j})\frac{A}{2\lambda_j^2} + (\alpha_{2j} + \rho_{2j})O\left(\frac{1}{\lambda_j^{5/2}}\right),$$

and hence estimate (4.18) must be written as

$$\frac{c_1}{\lambda_j^3}\left(\alpha_{2j}^2\lambda_j + \frac{\rho_{2j}^2}{\lambda_j}\right) \le |C_{4j}|^2 \le \frac{c_2}{\lambda_j^3}\left(\alpha_{2j}^2\lambda_j + \frac{\rho_{2j}^2}{\lambda_j}\right). \tag{6.14}$$

Therefore, by (4.22) and estimates (4.16) and (6.14) we have the following relationships between the coefficients C_n, D_n and the Fourier coefficients of the final data (6.13):

$$n^2|C_n|^2 \asymp \alpha_{1n}^2 n^2 + \rho_{1n}^2, \qquad n^6|D_n|^2 \asymp \alpha_{2n}^2 n^2 + \frac{\rho_{2n}^2}{n^2}. \tag{6.15}$$

In addition, taking into account (6.12) and recalling that we have to write n^2A in place of A, we have

$$\int_0^T \left(\left|z_{1x}(t,\pi)\right|^2 + \left|z_{2x}(t,\pi)\right|^2\right) dt \asymp \sum_{n=1}^\infty n^2\left(|C_n|^2 + \frac{1}{n^4A^2}|D_n|^2|p_n|^4\right),$$

whence we obtain again

$$\int_0^T \left(\left|z_{1x}(t,\pi)\right|^2 + \left|z_{2x}(t,\pi)\right|^2\right) dt \asymp \|z_{10}\|_{H_0^1}^2 + \|z_{11}\|^2 + \|z_{20}\|_{H_0^1}^2 + \|z_{21}\|_{H^{-1}}^2.$$

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