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Asymptotic stability of a nonlinear Korteweg–de Vries equation with critical lengths

Jixun Chu [a,b,1], Jean-Michel Coron [c,2], Peipei Shang [d,*,3]

[a] *Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China*

[b] *Université Pierre et Marie Curie-Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France*

[c] *Institut universitaire de France and Université Pierre et Marie Curie-Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France*

[d] *Department of Mathematics, Tongji University, Shanghai 200092, China*

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Abstract

We study an initial–boundary-value problem of a nonlinear Korteweg–de Vries equation posed on the finite interval $(0, 2k\pi)$ where k is a positive integer. The whole system has Dirichlet boundary condition at the left end-point, and both of Dirichlet and Neumann homogeneous boundary conditions at the right end-point. It is known that the origin is not asymptotically stable for the linearized system around the origin. We prove that the origin is (locally) asymptotically stable for the nonlinear system if the integer k is such that the kernel of the linear Korteweg–de Vries stationary equation is of dimension 1. This is for example the case if $k = 1$.

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* Corresponding author.

E-mail addresses: chujixun@126.com (J. Chu), coron@ann.jussieu.fr (J.-M. Coron), peipeishang@hotmail.com (P. Shang).

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1. Introduction

This article is concerned with the following initial–boundary-value problem of the Korteweg–de Vries (KdV) equation posed on a finite interval

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = 0, \\ y(0,x) = y_0 \in L^2(0,L). \end{cases} \tag{1.1}$$

The KdV equation was first derived by Boussinesq in [4] (see, in particular, Eq. (283 bis), p. 360) and Korteweg and de Vries in [26] in order to describe the propagation of small amplitude long water waves in a uniform channel. This equation is now commonly used to model unidirectional propagation of small amplitude long waves in nonlinear dispersive systems.

Since in many physical applications the region is finite, people are also interested in properties of the KdV equations on a finite spatial domain. Moreover, Bona and Winther pointed out in [3] that the term y_x, which is already in [4], cannot be removed in the KdV equations to model the water waves when x denotes the spatial coordinate in a fixed frame. We refer to [1,2,12,18,20,22,27,35] for the well-posedness results of initial–boundary-value problems of the KdV equations posed on a finite interval. From control theory point of view, we refer to [7,38] for an overall review and recent progress on different kinds of KdV equations. In particular, when the spatial domain is a finite interval, we refer to [6,14,15,19,36,37,45] for the controllability and to [8,23,30,31,34] for some stabilization results. We refer to [10,24,25,28,39–41] for studies on the KdV equations with periodic boundary conditions.

Rosier introduced in [36] the following set of critical lengths

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}}; \, j, l \in \mathbb{N}^* \right\}$$

for the following KdV control system

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = u(t), \\ y(0,x) = y_0, \end{cases} \tag{1.2}$$

where $u(t) \in \mathbb{R}$ is the control. We refer to [9,14,36] for the well-posedness and controllability of system (1.2). Especially, Rosier proved in [36] that (1.2) is locally controllable around the origin by analyzing the corresponding linearized system and by means of Banach fixed point theorem, provided that the spatial domain is not critical, i.e. $L \notin \mathcal{N}$. However, this method does not work when $L \in \mathcal{N}$, since the corresponding linearized system of (1.2) around the origin is not any more controllable in this case. By using the “power series expansion” method, Coron and

Crépeau in [14] obtained the local exact controllability around the origin of the nonlinear KdV equation (1.2) with the critical length $L = 2k\pi$ (i.e. taking $j = l = k$ in $\mathcal{N}$), where k is a positive integer such that (see [13, Theorem 8.1 and Remark 8.2])

$$\left(j^2 + l^2 + jl = 3k^2 \text{ and } (j, l) \in (\mathbb{N} \setminus \{0\})^2\right) \Rightarrow (j = l = k)\,. \tag{1.3}$$

The cases with the other critical lengths have been studied by Cerpa in [6] and by Cerpa and Crépeau in [9] with the same method, where the authors have proved that the nonlinear term yy_x gives the local exact controllability around the origin.

If $L \notin \mathcal{N}$, it is proved by Perla Menzala, Vasconcellos and Zuazua in [34] that 0 is exponentially stable for the linearized equation (1.4)

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, x) = y_0 \in L^2(0, L)\,, \end{cases} \tag{1.4}$$

of (1.1) around 0. Furthermore, it is also proved in [34] that 0 is locally asymptotically stable for system (1.1). However, when $L \in \mathcal{N}$, it has been proved by Rosier in [36] that (1.4) admits a family of non-trivial solutions of the form $e^{\lambda t} v_\lambda(x)$ for some $\lambda \in i\mathbb{R}$, where $v_\lambda \in C^\infty([0, L]) \setminus \{0\}$ satisfies

$$\begin{cases} \lambda v_\lambda(x) + v_\lambda'(x) + v_\lambda'''(x) = 0, \\ v_\lambda(0) = v_\lambda(L) = v_\lambda'(0) = v_\lambda'(L) = 0. \end{cases}$$

For these critical lengths, it is therefore interesting to study the influence of the nonlinear term yy_x on the local asymptotic stability of 0 for the nonlinear KdV equation (1.1). This article is concerned with the stability property for system (1.1) with the special critical lengths $L = 2k\pi$, where k is a positive integer such that (1.3) holds.

Center manifolds play an important role in studying nonlinear systems. We refer to [5,11,21,29,42] and the references therein for center manifold theories on abstract Cauchy problems in Banach spaces. The authors in [5,21,29] investigated directly the evolution equations and gave some sufficient conditions for the existence and smoothness of center manifolds. While, the authors in [11] presented a general result on the invariant manifolds together with associated invariant foliations of the state space, which can be applied directly to C^1 semigroups in Banach space. But the method presented in [11] has no extension to the case of C^l-smoothness with $l > 1$. In [42], by using the method of graph transforms, some classical results about smoothness of invariant manifolds for maps and the technique of "lifting", the existence, smoothness and attractivity of invariant manifolds for evolutionary process on general Banach spaces are proved when the nonlinear perturbation has a small global Lipschitz constant and is locally C^l-smooth near the trivial solution. Because of the existence of the nonlinear term in (1.1), the results presented in [5,29] do not work for our system. Moreover, due to the fact that, whatever is $L > 0$, the linear operator in our system (1.1) does not satisfy the resolvent estimates provided by [21], we cannot apply directly the results given in [21]. Thanks to the center manifold results given in [42], in this article, we show the existence and smoothness of a center manifold of (1.1) with

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$L = 2k\pi$, where k is a positive integer such that (1.3) holds, and obtain that the asymptotic stability property can be determined by a reduced system of dimension one. Furthermore, by studying the stability on this reduced one dimensional system, we obtain the local asymptotic stability of 0 for the original system (1.1). The main result of this article is the following theorem.

Theorem 1.1. *Let us assume that the positive integer k is such that (1.3) holds. Then $0 \in L^2(0, L)$ is (locally) asymptotically stable for the nonlinear KdV equation (1.1). More precisely:*

(i) *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\|y_0\|_{L^2(0,L)} < \delta$, then*

$$\|y(t, \cdot)\|_{L^2(0,L)} < \varepsilon, \quad \forall t \geq 0.$$

(ii) *There exists $\delta_1 > 0$ such that, if $\|y_0\|_{L^2(0,L)} < \delta_1$, then*

$$\lim_{t \to +\infty} \|y(t, \cdot)\|_{L^2(0,L)} = 0.$$

Remark 1.1. (a) The existence of $\delta(\varepsilon)$ is trivial and well known. In fact, one can take $\delta(\varepsilon) = \varepsilon$ since $t \in [0, +\infty) \mapsto \|y(t, \cdot)\|_{L^2(0,L)}$ is nonincreasing (see also Lemma 3.1 below). The nontrivial part of Theorem 1.1 is property (ii). (b) It is proved in [16, Theorem 1 and Comments] that, for every $L > 0$, there are non-zero stationary solutions to (1.1). In particular δ_1 cannot be taken arbitrary large in (ii) and $0 \in L^2(0, L)$ is not globally asymptotically stable for the nonlinear KdV equation (1.1). (c) Let us emphasize that, for $k := 1$, one has (1.3) and that, as proved in [13, Proposition 8.3], there are infinitely many positive integers k such that (1.3) holds.

The organization of this paper is as follows: First, in Section 2, some basic properties of the linearized system (1.4) are given. Then, in Section 3, we prove some properties of a nonlocal modification of the KdV equation (1.1) and then deduce the existence and smoothness of the center manifold. Finally, in Section 4, we analyze the dynamic on the center manifold, which concludes the proof of the main result, i.e. Theorem 1.1.

2. Preliminary

Until the end of this manuscript, k is a positive integer such that (1.3) holds and, unless otherwise specified, $L = 2k\pi$. In this section, we give some properties for the linearized system (1.4).

Set $X := L^2(0, L)$. Let $A : D(A) \to X$ be the linear operator defined by

$$A\psi = -\psi_x - \psi_{xxx}$$

with

$$D(A) = \left\{ \psi \in H^3(0, L) : \psi(0) = \psi(L) = \psi_x(L) = 0 \right\}.$$

It is easily verified that both A and its adjoint A^* are dissipative. The following proposition follows from [33, Corollary 4.4, Chapter 1]. See also [36].

Proposition 2.1. *A generates a C_0-semigroup of contractions on $L^2(0, L)$.*

From now on, we denote by $\{S(t)\}_{t\geq 0}$ the C_0-semigroup associated with A. Then $S(t)y_0$ is the mild solution of the linearized system (1.4) for any given initial data $y_0 \in L^2(0,L)$. By Proposition 2.1, we obtain the following lemma directly.

Lemma 2.1. *For every $y_0 \in L^2(0,L)$, we have*

$$\|S(t)y_0\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Furthermore, the following Kato smoothing effect is given by Rosier [36, Proposition 3.2].

Lemma 2.2. *For every $y_0 \in L^2(0,L)$ and for every $T > 0$, we have $S(t)y_0 \in L^2\left(0,T;H^1(0,L)\right)$ and*

$$\|S(t)y_0\|_{L^2(0,T;H^1(0,L))} \leq \left(\frac{4T+L}{3}\right)^{\frac{1}{2}} \|y_0\|_{L^2(0,L)}.$$

Proceeding as in [32], we can prove the following two results.

Lemma 2.3. *There exists a constant $C > 0$ such that for any $y_0 \in H_0^1(0,L)$, the solution $S(t)y_0$ of (1.4) fulfills*

$$\|S(t)y_0\|_{H_0^1(0,L)} \leq C\|y_0\|_{H_0^1(0,L)}, \quad \forall t \geq 0.$$

Proof. For any $U_0 \in D(A)$, let us define $U(t) := S(t)U_0$. Let $V(t) = U_t(t) = AU(t)$. Then V is the mild solution of the system

$$\begin{cases} V_t = AV, \\ V(0) = AU_0 \in L^2(0,L). \end{cases}$$

Hence, it follows from Lemma 2.1 that

$$\|V(t)\|_{L^2(0,L)} \leq \|V(0)\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Since $V(t) = AU(t)$, $V(0) = AU_0$, and the norms $\|U\|_{L^2(0,L)} + \|AU\|_{L^2(0,L)}$ and $\|U\|_{D(A)}$ are equivalent on $D(A)$, we conclude that, for some constant $C_1 > 0$ independent of U_0 and $t \geq 0$, we have

$$\|U(t)\|_{D(A)} \leq C_1\|U_0\|_{D(A)}.$$

Then the result of Lemma 2.3 follows by a standard interpolation argument. □

Our next proposition shows that $\{S(t)\}_{t\geq 0}$ is a compact semigroup.

Proposition 2.2. *Let $T > 0$. There exists a constant $C > 0$ such that, for every $y_0 \in L^2(0, L)$, we have*

$$\|S(t)\, y_0\|_{H_0^1(0,L)} \leq \frac{C}{\sqrt{t}} \|y_0\|_{L^2(0,L)}, \quad \forall t \in (0, T]. \tag{2.1}$$

Consequently, the C_0-semigroup $\{S(t)\}_{t \geq 0}$ generated by A is compact.

Proof. Let $T > 0$ be fixed. For every $t \in (0, T]$ and for every $y_0 \in L^2(0, L)$, by Lemma 2.2, the estimate

$$\|S(\cdot) y_0\|_{L^2(0,\frac{t}{2};H_0^1(0,L))} \leq \left(\frac{2t + L}{3}\right)^{\frac{1}{2}} \|y_0\|_{L^2(0,L)} \tag{2.2}$$

holds. Then, arguing by contradiction, we get the existence of $\tau \in (0, t/2]$ such that

$$\|S(\tau)\, y_0\|_{H_0^1(0,L)} \leq \left(\frac{2t + L}{3}\right)^{\frac{1}{2}} \sqrt{\frac{2}{t}} \|y_0\|_{L^2(0,L)}, \quad \forall y_0 \in L^2(0, L). \tag{2.3}$$

Now it follows from Lemma 2.3 and (2.3) that there exists $C' = C'(T) > 0$ such that, for every $t \in (0, T]$ and every $y_0 \in L^2(0, L)$,

$$\begin{aligned}
\|S(t)\, y_0\|_{H_0^1(0,L)} &= \|S(t - \tau)\, S(\tau)\, y_0\|_{H_0^1(0,L)} \\
&\leq C\, \|S(\tau)\, y_0\|_{H_0^1(0,L)} \\
&\leq C \left(\frac{2t + L}{3}\right)^{\frac{1}{2}} \sqrt{\frac{2}{t}} \|y_0\|_{L^2(0,L)} \\
&\leq \frac{C'}{\sqrt{t}} \|y_0\|_{L^2(0,L)}.
\end{aligned}$$

Thus, for any given $T > 0$, (2.1) holds. Since $H^1(0, L)$ is compactly embedded in $L^2(0, L)$, we conclude that $S(t)$ is compact. □

Let us now consider the spectral properties of the operator A. Firstly, we give the definition of growth bound and essential growth bound of the infinitesimal generator of a linear C_0-semigroup. We refer to [43, Definition 4.15, p. 170] for this definition.

Definition 2.1. Let $K : D(K) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{S_K(t)\}_{t \geq 0}$ on a Banach space X. We define $\omega_0(K) \in [-\infty, +\infty)$ the *growth bound* of K by

$$\omega_0(K) := \lim_{t \to +\infty} \frac{\ln\left(\|S_K(t)\|_{\mathcal{L}(X)}\right)}{t}.$$

The *essential growth bound* $\omega_{0,ess}(K) \in [-\infty, +\infty)$ of K is defined by

$$\omega_{0,ess}(K) := \lim_{t \to +\infty} \frac{\ln(\|S_K(t)\|_{ess})}{t},$$

where $\|S_K(t)\|_{ess}$ is the essential norm of $S_K(t)$ defined by

$$\|S_K(t)\|_{ess} = \kappa\left(S_K(t) B_X(0,1)\right),$$

where $B_X(0,1) := \{x \in X : \|x\|_X \le 1\}$ and, for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \le \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

The following result is proved by Webb [43, Proposition 4.11, p. 166, Proposition 4.13, p. 170] and by Engel and Nagel [17, Corollary 2.11, p. 241].

Theorem 2.1. *Let $K : D(K) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{S_K(t)\}_{t\ge 0}$ on a Banach space X. Then*

$$\omega_0(K) = \max\left(\omega_{0,ess}(K), \max_{\lambda\in\sigma(K)\setminus\sigma_{ess}(K)} Re(\lambda)\right).$$

Assume in addition that $\omega_{0,ess}(K) < \omega_0(K)$. Then for each $\gamma \in \left(\omega_{0,ess}(K), \omega_0(K)\right]$, $\{\lambda \in \sigma(K) : Re(\lambda) \ge \gamma\} \subset \sigma_p(K)$ is nonempty, finite and contains only poles of the resolvent of K.

As a consequence of Proposition 2.2 and Theorem 2.1, one has the following lemma.

Lemma 2.4. *All the spectra of the linear operator A are point spectra, i.e., $\sigma(A) = \sigma_p(A)$ and $\omega_0(A) = \max_{\lambda\in\sigma(A)} Re(\lambda)$. Moreover, for each $\gamma \in (-\infty, \omega_0(A)]$, $\{\lambda \in \sigma(A) : Re(\lambda) \ge \gamma\}$ is nonempty, finite and contains only poles of the resolvent of A.*

From Lemma 2.1 and Lemma 2.4, one has

Lemma 2.5. *For every $\lambda \in \sigma(A)$,* $\mathrm{Re}(\lambda) \le 0$.

Let us now prove the following lemma.

Lemma 2.6. *One has $\sigma_p(A) \cap i\mathbb{R} = \{0\}$. Moreover, the kernel of A is $a(1-\cos x)$, $a \in \mathbb{R}$.*

Proof. We have $\lambda \in \sigma_p(A) \cap i\mathbb{R}$ if and only if there exists $\psi \in H^3(0,L)\setminus\{0\}$ such that

$$\begin{cases} \lambda\psi + \psi_x + \psi_{xxx} = 0, \\ \psi(0) = \psi(L) = \psi_x(L) = 0. \end{cases} \tag{2.4}$$

Multiplying Eq. (2.4) by $\overline{\psi}$, and then integrating over $[0, L]$, we obtain

$$\lambda \int_0^L \psi\overline{\psi} dx + \int_0^L \psi_x \overline{\psi} dx + \int_0^L \psi_{xxx}\overline{\psi} dx = 0. \tag{2.5}$$

Taking the real part of (2.5), we have

$$\int_0^L \frac{\psi_x \bar{\psi} + \bar{\psi}_x \psi}{2}\, dx + \int_0^L \frac{\psi_{xxx} \bar{\psi} + \bar{\psi}_{xxx} \psi}{2}\, dx = 0. \tag{2.6}$$

Integrating by parts in (2.6) and using (2.4), we get

$$\psi_x (0) = 0.$$

Hence, $\lambda \in \sigma_p (A) \cap i\mathbb{R}$ if and only if there exists $\psi \in H^3 (0, L) \setminus \{0\}$ such that

$$\begin{cases} \lambda\psi + \psi_x + \psi_{xxx} = 0, \\ \psi (0) = \psi (L) = \psi_x (0) = \psi_x (L) = 0, \end{cases}$$

and the result of this lemma follows directly from (1.3) and from the proof of Rosier [36, Lemma 3.5]. □

Combining Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain the following corollary.

Corollary 2.2. $0 \in \sigma (A) = \sigma_p (A)$ *and the other eigenvalues of* A *have negative real parts which are bounded away from* 0, *i.e., there exists* $\Lambda > 0$ *such that all the nonzero eigenvalues of* A *have a real part which is less than* $-\Lambda$.

3. Existence and smoothness of the center manifold

This section is devoted to showing the existence and smoothness of the center manifold for system (1.1) with $L = 2k\pi$ by applying the results given in [42]. We would like to mention that the linear operator A in our system (1.1) with $L = 2k\pi$ does not satisfy the resolvent estimates required in [21]. In particular, A does not generate an analytic semigroup, and, therefore, we cannot apply the results given in [21] to show the existence and smoothness of the center manifold.

In order to apply the results given in [42], we need to show that the nonlinear perturbation has a small global Lipschitz constant. To that end, we modify the nonlinear part of the original system (1.1) by using some smooth cut-off mapping, and consider the following equation

$$\begin{cases} y_t + y_x + y_{xxx} + \Phi_\varepsilon(\|y\|_{L^2(0,L)}) y y_x = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, x) = y_0(x) \in L^2(0, L). \end{cases} \tag{3.1}$$

Here $\varepsilon > 0$ is small enough, and $\Phi_\varepsilon : [0, +\infty) \to [0, 1]$ is defined by

$$\Phi_\varepsilon (x) = \Phi \left(\frac{x}{\varepsilon}\right), \quad \forall x \in [0, +\infty),$$

where $\Phi \in C^\infty ([0, +\infty) ; [0, 1])$ satisfies

$$\Phi(x) = \begin{cases} 1, & \text{when } x \in [0, \frac{1}{2}], \\ 0, & \text{when } x \in [1, +\infty), \end{cases}$$

and

$$\Phi' \leq 0.$$

It can be readily checked that

$$\begin{aligned} \Phi_\varepsilon(x) &= 1, \quad \text{when } x \in [0, \frac{\varepsilon}{2}], \\ \Phi_\varepsilon(x) &= 0, \quad \text{when } x \in [\varepsilon, +\infty). \end{aligned} \tag{3.2}$$

Moreover, there exists some constant $C > 0$ such that

$$0 \leq -\Phi'_\varepsilon(x) \leq \frac{C}{\varepsilon}, \quad \forall x \in [0, +\infty). \tag{3.3}$$

In (3.3) and in the following, C denotes various positive constants, which may vary from line to line, but do not depend on $\varepsilon \in (0, 1]$ and $y_0 \in L^2(0, L)$.

3.1. *Well-posedness of (3.1)*

In this section, we prove the following proposition on the global (in positive time) existence and uniqueness of the solution to system (3.1).

Proposition 3.1. *For every $y_0 \in L^2(0, L)$, there exists a unique mild solution*

$$y \in C([0, +\infty); L^2(0, L)) \cap L^2_{loc}\big([0, +\infty); H_0^1(0, L)\big)$$

of (3.1).

In order to prove this proposition, one first points out that

Lemma 3.1. *Let $T > 0$. If*

$$y \in C([0, T]; L^2(0, L)) \cap L^2\big(0, T; H_0^1(0, L)\big)$$

is a mild solution of (3.1), then

$$\frac{d}{dt}\left(\int_0^L y^2(t, x)\, dx\right) \leq 0.$$

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Proof. We multiply $y_t + y_x + y_{xxx} + \Phi_\varepsilon(\|y\|_{L^2(0,L)})yy_x = 0$ by y and integrate over $[0, L]$. Using the boundary conditions in (3.1) and integrations by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + \frac{1}{2}y_x^2(t,0) = 0.$$

The lemma follows. □

By Lemma 3.1, in order to prove Proposition 3.1, it is sufficient to prove local (in positive time) existence and uniqueness of the solution to system (3.1).

Proposition 3.2. *Let $\varepsilon > 0, \eta > 0$. There exists $T > 0$ such that for every $y_0 \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} \leq \eta$, there exists a unique solution $y \in C\left([0,T]; L^2(0,L)\right) \cap L^2\left(0,T; H_0^1(0,L)\right)$ of* (3.1).

Proof. The case where $\Phi_\varepsilon \equiv 1$ is proved in [34]. Adapting the proof given in [34], we get the existence of T together with the existence and uniqueness of mild solution y. We briefly give the proof since some estimates given in the proof will be used later on.

Using the variation of constants formula, system (3.1) can be written in the following integral form:

$$\begin{aligned} y(t,\cdot) &= S(t)y_0 - \int_0^t S(t-s)\Phi_\varepsilon\left(\|y(s,\cdot)\|_{L^2(0,L)}\right)y(s,\cdot)y_x(s,\cdot)\,ds \\ &:= \left[\phi(y)\right](t). \end{aligned} \tag{3.4}$$

We will show that the nonlinear map ϕ is a contraction from $Y_T := C\left([0,T]; L^2(0,L)\right) \cap L^2\left(0,T; H_0^1(0,L)\right)$ into itself when $T > 0$ is small enough.

Firstly, we prove that ϕ maps continuously Y_T into itself. Let us first show that if $y \in Y_T$, $\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right)yy_x \in L^1\left(0,T; L^2(0,L)\right)$ and the map $y \to \Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right)yy_x$ is continuous. Indeed, let $y, z \in Y_T$. Applying the triangular inequality, Hölder's inequality and Sobolev's embedding $H_0^1(0,L) \subset C^0([0,L])$ together with (3.3), we get

$$\begin{aligned} &\left\|\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right)yy_x - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right)zz_x\right\|_{L^1(0,T;L^2(0,L))} \\ &\quad\leq \|(yy_x - zz_x)\|_{L^1(0,T;L^2(0,L))} + \left\|\left[\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right)\right]zz_x\right\|_{L^1(0,T;L^2(0,L))} \\ &\quad\leq \|(y-z)y_x + (y_x - z_x)z\|_{L^1(0,T;L^2(0,L))} + \frac{C}{\varepsilon}\left\|\|y-z\|_{L^2(0,L)}zz_x\right\|_{L^1(0,T;L^2(0,L))} \\ &\quad\leq \int_0^T \|(y-z)y_x\|_{L^2(0,L)}dt + \int_0^T \|(y_x - z_x)z\|_{L^2(0,L)}dt \\ &\qquad + \frac{C}{\varepsilon}\int_0^T \|y-z\|_{L^2(0,L)}\|zz_x\|_{L^2(0,L)}dt \end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^T \|y-z\|_{L^\infty(0,L)}\|y_x\|_{L^2(0,L)}\,dt + C\int_0^T \|z\|_{L^\infty(0,L)}\|y_x-z_x\|_{L^2(0,L)}\,dt \\
&\quad + \frac{C}{\varepsilon}\|y-z\|_{L^\infty(0,T;L^2(0,L))}\int_0^T \|z\|_{L^\infty(0,L)}\|z_x\|_{L^2(0,L)}\,dt \\
&\leq C\|y-z\|_{L^2(0,T;L^\infty(0,L))}\|y_x\|_{L^2(0,T;L^2(0,L))} \\
&\quad + C\|z\|_{L^2(0,T;L^\infty(0,L))}\|y_x-z_x\|_{L^2(0,T;L^2(0,L))} \\
&\quad + \frac{C}{\varepsilon}\|y-z\|_{L^\infty(0,T;L^2(0,L))}\|z\|_{L^2(0,T;L^\infty(0,L))}\|z_x\|_{L^2(0,T;L^2(0,L))}\,.
\end{aligned} \tag{3.5}
$$

By the classical Gagliardo–Nirenberg inequality, we have

$$
\|u\|_{L^\infty(0,L)} \leq C\|u\|_{L^2(0,L)}^{\frac{1}{2}}\|u_x\|_{L^2(0,L)}^{\frac{1}{2}}, \quad \forall u\in H_0^1(0,L)\,. \tag{3.6}
$$

Hence,

$$
\begin{aligned}
\int_0^T \|u\|_{L^\infty(0,L)}^2\,dt &\leq C\int_0^T \|u\|_{L^2(0,L)}\|u_x\|_{L^2(0,L)}\,dt \\
&\leq C\|u\|_{L^\infty(0,T;L^2(0,L))}\int_0^T \|u_x\|_{L^2(0,L)}\,dt \\
&\leq C\|u\|_{L^\infty(0,T;L^2(0,L))}T^{\frac{1}{2}}\|u_x\|_{L^2(0,T;L^2(0,L))}\,.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\|u\|_{L^2(0,T;L^\infty(0,L))} &\leq C\|u\|_{L^\infty(0,T;L^2(0,L))}^{\frac{1}{2}}T^{\frac{1}{4}}\|u_x\|_{L^2(0,T;L^2(0,L))}^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{4}}\|u\|_{Y_T}, \quad \forall u\in Y_T\,.
\end{aligned}
$$

Thus, it follows from (3.5) that

$$
\begin{aligned}
&\left\|\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right)yy_x - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right)zz_x\right\|_{L^1(0,T;L^2(0,L))} \\
&\quad\leq CT^{\frac{1}{4}}\|y-z\|_{Y_T}\|y_x\|_{L^2(0,T;L^2(0,L))} + CT^{\frac{1}{4}}\|z\|_{Y_T}\|y_x-z_x\|_{L^2(0,T;L^2(0,L))} \\
&\qquad + \frac{C}{\varepsilon}\|y-z\|_{L^\infty(0,T;L^2(0,L))}T^{\frac{1}{4}}\|z\|_{Y_T}\|z_x\|_{L^2(0,T;L^2(0,L))} \\
&\quad\leq \|y-z\|_{Y_T}T^{\frac{1}{4}}C\left(\|y\|_{Y_T}+\|z\|_{Y_T}+\frac{1}{\varepsilon}\|z\|_{Y_T}^2\right),
\end{aligned} \tag{3.7}
$$

which implies that $\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right)yy_x \in L^1\left(0,T;L^2(0,L)\right)$ and that the map

$$y \to \Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x$$

is continuous from Y_T to $L^1\left(0, T; L^2(0, L)\right)$.

By Proposition 4.1 in [36], we obtain that

$$\int_0^t S(t-s)\,\Phi_\varepsilon\left(\|y(s,\cdot)\|_{L^2(0,L)}\right) y(s,\cdot)\, y_x(s,\cdot)\, ds$$

lies in Y_T, and the map

$$\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x \to \int_0^t S(t-s)\,\Phi_\varepsilon\left(\|y(s,\cdot)\|_{L^2(0,L)}\right) y(s,\cdot)\, y_x(s,\cdot)\, ds$$

is continuous. This fact, together with the continuity of the map $y \to \Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x$ from Y_T to $L^1\left(0, T; L^2(0, L)\right)$ and $S(t)\, y_0 \in Y_T$ (thanks to Lemma 2.1 and Lemma 2.2), leads to the conclusion that ϕ maps continuously Y_T into itself.

Let us now prove that ϕ is a contraction in a suitable ball B_R of Y_T when $T > 0$ is small enough. Obviously,

$$\phi(y) - \phi(z) = -\int_0^t S(t-s)\left[\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x(s) - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right) z z_x(s)\right] ds.$$

In view of the proof of Proposition 4.1 in [36] and (3.7), we deduce that

$$\begin{aligned}
&\|\phi(y) - \phi(z)\|_{Y_T} \\
&\le \left(1 + \left(\frac{T+2L}{3}\right)^{\frac{1}{2}}\right) \left\|\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right) z z_x\right\|_{L^1(0,T;L^2(0,L))} \\
&\le C\left(1+\sqrt{T}\right) \left\|\Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right) y y_x - \Phi_\varepsilon\left(\|z\|_{L^2(0,L)}\right) z z_x\right\|_{L^1(0,T;L^2(0,L))} \\
&\le C\left(1+\sqrt{T}\right) \|y - z\|_{Y_T}\, T^{\frac{1}{4}} \left(\|y\|_{Y_T} + \|z\|_{Y_T} + \frac{1}{\varepsilon}\|z\|_{Y_T}^2\right),
\end{aligned} \tag{3.8}$$

which shows that ϕ is a contraction in the ball B_R of Y_T if

$$C\left(1+\sqrt{T}\right) T^{\frac{1}{4}} \left(2R + \frac{1}{\varepsilon} R^2\right) < 1. \tag{3.9}$$

Therefore, the proof will be complete if we could show that for a suitable choice of R and T satisfying (3.9), the map ϕ sends B_R into itself.

It can be deduced from the definition of $\phi(y)$ given in (3.4), Lemma 2.1, Lemma 2.2 and (3.8) with $z = 0$ that there exists $\bar{C} > 0$ independent of $\varepsilon \in (0, 1]$, $y_0 \in L^2(0, L)$ and $T > 0$, such that

$$\begin{aligned}\|\phi(y)\|_{Y_T} &\le \left(1+\left(\frac{4T+L}{3}\right)^{\frac{1}{2}}\right)\|y_0\|_{L^2(0,L)} + \|y\|_{Y_T}^2 T^{\frac{1}{4}} C\left(1+\sqrt{T}\right)\\ &\le \left(1+\left(\frac{4T+L}{3}\right)^{\frac{1}{2}}\right)\|y_0\|_{L^2(0,L)} + R^2 T^{\frac{1}{4}} C\left(1+\sqrt{T}\right)\\ &\le \bar{C}\left(1+\sqrt{T}\right)\left(\|y_0\|_{L^2(0,L)} + R^2 T^{\frac{1}{4}}\right), \quad \forall y \in B_R.\end{aligned}$$

Now let $\|y_0\|_{L^2(0,L)} \le \eta$, and set $R := 2\eta\bar{C}$. Then

$$\|\phi(y)\|_{Y_T} \le \eta\bar{C}\left(1+\sqrt{T}\right)\left(1+4\bar{C}^2\eta T^{\frac{1}{4}}\right), \quad \forall y \in B_R. \tag{3.10}$$

It is clear that we can choose $T > 0$ sufficiently small such that

$$\left(1+\sqrt{T}\right)\left(1+4\eta\bar{C}^2 T^{\frac{1}{4}}\right) \le 2,$$

which, together with (3.10) implies that ϕ maps B_R into itself. Moreover, decreasing T if necessary allows us to guarantee (3.9) as well. The proof of Proposition 3.2 is complete. □

Proposition 3.3. *There exists $C > 0$ such that for every $\varepsilon > 0$, for every $y_0 \in L^2(0, L)$ and for every $T > 0$, the unique solution of (3.1) satisfies*

$$\|y\|^2_{L^2(0,T;H_0^1(0,L))} \le \frac{8T+2L}{3}\|y_0\|^2_{L^2(0,L)} + CT\|y_0\|^4_{L^2(0,L)}. \tag{3.11}$$

Proof. Proceeding as in [36], we multiply the first equation in (3.1) by xy and integrate over $(0, L) \times (0, T)$. Then, by Lemma 3.1, we obtain

$$\begin{aligned}&\int_0^T\int_0^L y_x^2\,dx\,dt + \frac{1}{3}\int_0^L xy^2(T,x)\,dx\\ &= \frac{1}{3}\int_0^T\int_0^L y^2\,dx\,dt + \frac{1}{3}\int_0^L xy_0^2\,dx - \frac{2}{3}\int_0^T \Phi_\varepsilon(\|y\|_{L^2(0,L)})\int_0^L xy^2y_x\,dx\,dt\\ &\le \frac{T+L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{2}{3}\int_0^T \Phi_\varepsilon(\|y\|_{L^2(0,L)})\left|\int_0^L xy^2y_x\,dx\right|dt.\end{aligned} \tag{3.12}$$

Since

$$\int_0^L xy^2y_x\,dx = -\frac{1}{3}\int_0^L y^3\,dx,$$

it follows from (3.12) that

$$\begin{aligned}
&\int_0^T\int_0^L y_x^2 dxdt + \frac{1}{3}\int_0^L xy^2(T,x)\,dx\\
&\quad\le \frac{T+L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{2}{9}\int_0^T \Phi_\varepsilon(\|y\|_{L^2(0,L)})\int_0^L |y|^3\,dxdt\\
&\quad\le \frac{T+L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{2}{9}\int_0^T\int_0^L |y|^3\,dxdt.
\end{aligned}$$

Hence, with $\|\psi\|^2_{H_0^1(0,L)} := \|\psi\|^2_{L^2(0,L)} + \|\psi_x\|^2_{L^2(0,L)}$,

$$\|y\|^2_{L^2(0,T;H_0^1(0,L))} \le \frac{4T+L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{2}{9}\int_0^T\int_0^L |y|^3\,dxdt. \tag{3.13}$$

Furthermore, by Lemma 3.1, the continuous Sobolev embedding $H_0^1(0,L)\subset C^0([0,L])$, the Poincaré inequality and Hölder's inequality, we have

$$\begin{aligned}
\int_0^T\int_0^L |y|^3\,dxdt &\le C\int_0^T \|y\|_{H_0^1(0,L)}\left(\int_0^L |y|^2\,dx\right)dt\\
&\le C\|y_0\|^2_{L^2(0,L)}\int_0^T \|y\|_{H_0^1(0,L)}\,dt\\
&\le C\|y_0\|^2_{L^2(0,L)}\sqrt{T}\left(\int_0^T \|y\|^2_{H_0^1(0,L)}\,dt\right)^{\frac{1}{2}}\\
&= C\sqrt{T}\|y_0\|^2_{L^2(0,L)}\|y\|_{L^2(0,T;H_0^1(0,L))}.
\end{aligned}$$

Now, using the above inequality in (3.13) we have

$$\begin{aligned}
&\|y\|^2_{L^2(0,T;H_0^1(0,L))}\\
&\quad\le \frac{4T+L}{3}\|y_0\|^2_{L^2(0,L)} + C\sqrt{T}\|y_0\|^2_{L^2(0,L)}\|y\|_{L^2(0,T;H_0^1(0,L))}\\
&\quad\le \frac{4T+L}{3}\|y_0\|^2_{L^2(0,L)} + CT\|y_0\|^4_{L^2(0,L)} + \frac{1}{2}\|y\|^2_{L^2(0,T;H_0^1(0,L))}.
\end{aligned}$$

Therefore, we get

$$\|y\|^2_{L^2(0,T;H_0^1(0,L))} \le \frac{8T+2L}{3}\|y_0\|^2_{L^2(0,L)} + CT\|y_0\|^4_{L^2(0,L)}.$$

This concludes the proof of Proposition 3.3. □

Remark 3.1. According to Proposition 3.3, we have, for every $\tau \in [0, T]$,

$$\|y\|^2_{L^2\left(\tau,T;H_0^1(0,L)\right)} \leq \frac{8(T-\tau)+2L}{3}\|y(\tau,\cdot)\|^2_{L^2(0,L)} + C(T-\tau)\|y(\tau,\cdot)\|^4_{L^2(0,L)}.$$

It follows that, if $\tau \in [0, T]$ is such that $\|y(\tau,\cdot)\|_{L^2(0,L)} = \varepsilon$, then

$$\begin{aligned}\|y\|^2_{L^2\left(\tau,T;H_0^1(0,L)\right)} &\leq \frac{8(T-\tau)+2L}{3}\varepsilon^2 + C(T-\tau)\varepsilon^4 \\ &\leq \frac{8T+2L}{3}\varepsilon^2 + CT\varepsilon^4.\end{aligned}$$

Lemma 3.2. *Let $T > 0$. There exist $\eta > 0$ and $C > 0$, such that, for every $\varepsilon \in (0, 1]$ and for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2(0,L)} \leq \eta$, there exists a unique mild solution $y : [0, T] \times [0, L] \to \mathbb{R}$ of (3.1) which satisfies*

$$\|y(t,\cdot)\|_{H_0^1(0,L)} \leq \frac{C}{\sqrt{t}}\|y_0\|_{L^2(0,L)}, \quad \forall t \in (0, T].$$

Proof. From Proposition 2.2 and (3.4), we deduce that

$$\begin{aligned}\|y(t,\cdot)\|_{H_0^1(0,L)} &\leq \|S(t)y_0\|_{H_0^1(0,L)} \\ &\quad + \int_0^t \left\|S(t-s)\Phi_\varepsilon\left(\|y(s,\cdot)\|_{L^2(0,L)}\right)y(s,\cdot)y_x(s,\cdot)\right\|_{H_0^1(0,L)} ds \\ &\leq \frac{C}{\sqrt{t}}\|y_0\|_{L^2(0,L)} + \int_0^t \frac{C}{\sqrt{t-s}}\|y(s,\cdot)y_x(s,\cdot)\|_{L^2(0,L)} ds. \qquad (3.14)\end{aligned}$$

As a consequence of Lemma 3.1 and (3.6), we have

$$\begin{aligned}\|y(s,\cdot)y_x(s,\cdot)\|_{L^2(0,L)} &\leq \|y(s,\cdot)\|_{L^\infty(0,L)}\|y_x(s,\cdot)\|_{L^2(0,L)} \\ &\leq C\|y(s,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\|y_x(s,\cdot)\|^{\frac{3}{2}}_{L^2(0,L)} \\ &\leq C\|y_0\|^{\frac{1}{2}}_{L^2(0,L)}\|y(s,\cdot)\|^{\frac{3}{2}}_{H_0^1(0,L)}. \qquad (3.15)\end{aligned}$$

Substituting (3.15) into (3.14), we obtain

$$\|y(t,\cdot)\|_{H_0^1(0,L)} \leq \frac{C}{\sqrt{t}}\|y_0\|_{L^2(0,L)} + \|y_0\|^{\frac{1}{2}}_{L^2(0,L)}\int_0^t \frac{C}{\sqrt{t-s}}\|y(s,\cdot)\|^{\frac{3}{2}}_{H_0^1(0,L)} ds,$$

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i.e.

$$\sqrt{t}\,\|y(t,\cdot)\|_{H_0^1(0,L)} \leq C\,\|y_0\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}^{\frac{1}{2}}\sqrt{t}\int_0^t \frac{C}{s^{\frac{3}{4}}\sqrt{t-s}}\left(\sqrt{s}\,\|y(s,\cdot)\|_{H_0^1(0,L)}\right)^{\frac{3}{2}} ds. \tag{3.16}$$

Let $\overline{C} > C$. We claim that there exists $\eta > 0$ (small enough) such that, for every $\varepsilon \in (0,1]$ and for every $y_0 \in L^2(0,L)$ such that $\|y_0\|_{L^2(0,L)} \leq \eta$, we have

$$\xi(t) \leq \overline{C}\,\|y_0\|_{L^2(0,L)}, \quad \forall t \in (0,T], \tag{3.17}$$

where $\xi(t) := \sqrt{t}\,\|y(t,\cdot)\|_{H_0^1(0,L)}$. By a simple density argument, we may assume that $y_0 \in H_0^1(0,L)$. Then, by [1, Theorem 1.3 with $s=1$]

$$y \in C^0([0,+\infty); H_0^1(0,L)). \tag{3.18}$$

(In fact [1, Theorem 1.3] is dealing with (1.1) and not with (3.1); however the proof given there can be adapted to (3.1).) Property (3.18) implies that the integral term in the right hand side of (3.16) tends to 0 as $t \to 0$. Hence, if $y_0 \neq 0$ (which can be assumed, since (3.17) holds for $y_0 = 0$) and if (3.17) is not valid, there exists $\tau \in (0,T]$ such that

$$\xi(\tau) = \overline{C}\,\|y_0\|_{L^2(0,L)} \quad \text{and} \quad \xi(t) < \overline{C}\,\|y_0\|_{L^2(0,L)}, \quad \forall t \in (0,\tau). \tag{3.19}$$

Thus by (3.16), we have

$$\begin{aligned}
\xi(\tau) &\leq C\,\|y_0\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}^{\frac{1}{2}}\sqrt{\tau}\int_0^\tau \frac{C}{s^{\frac{3}{4}}\sqrt{\tau-s}}\left(\overline{C}\,\|y_0\|_{L^2(0,L)}\right)^{\frac{3}{2}} ds \\
&= C\,\|y_0\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}^{2}\sqrt{\tau}\,C\left(\overline{C}\right)^{\frac{3}{2}}\int_0^\tau \frac{1}{s^{\frac{3}{4}}\sqrt{\tau-s}} ds \\
&= \|y_0\|_{L^2(0,L)}\left(C + \|y_0\|_{L^2(0,L)}\sqrt{\tau}\,C\left(\overline{C}\right)^{\frac{3}{2}}\int_0^\tau \frac{1}{s^{\frac{3}{4}}\sqrt{\tau-s}} ds\right).
\end{aligned}$$

Hence, if $\|y_0\|_{L^2(0,L)}$ is small enough but not 0, we get $\xi(\tau) < \overline{C}\,\|y_0\|_{L^2(0,L)}$, which leads to a contradiction with (3.19). This concludes the proof of Lemma 3.2. □

3.2. Properties of the semigroup generated by (3.1)

Let

$$\tilde{S}(t) : L^2(0, L) \to L^2(0, L), \quad t \geq 0$$

be the semigroup on $L^2(0, L)$ defined by

$$\tilde{S}(t)(y_0)(x) := y(t, x),$$

where $y(t, x)$ is the unique solution of (3.1) with respect to the initial value $y_0 \in L^2(0, L)$. Let $T > 0$. Then, for every $t \in [0, T]$, $\tilde{S}(t)$ can be decomposed as

$$\tilde{S}(t) = S(t) + R(t),$$

or equivalently,

$$y(t, x) = z(t, x) + \alpha(t, x),$$

where, as above, for every $y_0 \in L^2(0, L)$, $z(t, \cdot) := S(t) y_0$ is the unique solution of

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t, 0) = z(t, L) = 0, \\ z_x(t, L) = 0, \\ z(0, x) = y_0 \end{cases}$$

and $\alpha(t, \cdot) := R(t) y_0$ is the unique solution of

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} + \Phi_\varepsilon\left(\|z + \alpha\|_{L^2(0,L)}\right)(z_x \alpha + \alpha_x z + z_x z + \alpha_x \alpha) = 0, \\ \alpha(t, 0) = \alpha(t, L) = 0, \\ \alpha_x(t, L) = 0, \\ \alpha(0, x) = 0. \end{cases}$$

Let

$$M := \{\alpha\varphi : \alpha \in \mathbb{R}\},$$

where

$$\varphi(x) = \frac{1}{\sqrt{3k\pi}}(1 - \cos x). \tag{3.20}$$

Let us recall that, by Lemma 2.6, $\varphi(x)$ is an eigenfunction of the linear operator A for the linearized system (1.4) corresponding to the eigenvalue 0 and M is the eigenspace corresponding to this eigenvalue. Then we can do the following decomposition of $X = L^2(0, L)$:

$$X = M \oplus M^\perp.$$

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The projection $P : X \to M$ is given by

$$Py(t,x) = p(t)\varphi(x),$$

where

$$p(t) := \int_0^L y(t,x)\varphi(x)dx, \tag{3.21}$$

and the projection $Q : X \to M^\perp$ is given by $I - P$.

One easily checks that

$$S(t) \text{ leaves } M \text{ and } M^\perp \text{ invariant and } S(t) \text{ commutes with } P \text{ and } Q. \tag{3.22}$$

Denote by $S_1(t) : M \to M$ and $S_2(t) : M^\perp \to M^\perp$ the restriction of $S(t)$ on M and $M^\perp$ respectively. Then

$$S_1(t) = \mathrm{Id}, \quad \forall t \geq 0, \tag{3.23}$$

where Id is the identity on $L^2(0, L)$. Moreover, by Proposition 2.2 and Corollary 2.2, there exist $N \geq 1$ and $\omega > 0$ such that

$$\|S_2(t)\| \leq Ne^{-\omega t}, \quad \forall t \geq 0. \tag{3.24}$$

3.2.1. Global Lipschitzianity of the map $R(t) : L^2(0, L) \to L^2(0, L)$

The aim of this part is to prove and estimate the global Lipschitzianity of the map $R(t) : L^2(0, L) \to L^2(0, L)$. To that end, we consider

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} + \Phi_\varepsilon\left(\|\alpha + z\|_{L^2(0,L)}\right)(z_x\alpha + \alpha_x z + z_x z + \alpha_x\alpha) = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = 0, \\ \alpha(0,x) = 0, \end{cases}$$

and

$$\begin{cases} \bar\alpha_t + \bar\alpha_x + \bar\alpha_{xxx} + \Phi_\varepsilon\left(\|\bar z + \bar\alpha\|_{L^2(0,L)}\right)(\bar z_x\bar\alpha + \bar\alpha_x\bar z + \bar z_x\bar z + \bar\alpha_x\bar\alpha) = 0, \\ \bar\alpha(t,0) = \bar\alpha(t,L) = 0, \\ \bar\alpha_x(t,L) = 0, \\ \bar\alpha(0,x) = 0, \end{cases}$$

where z is the solution of

$$\begin{cases} z_t + z_x + z_{xxx} = 0, \\ z(t,0) = z(t,L) = 0, \\ z_x(t,L) = 0, \\ z(0,x) = y_0 \in L^2(0,L), \end{cases}$$

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and $\bar{z}$ is the solution of

$$\begin{cases} \bar{z}_t + \bar{z}_x + \bar{z}_{xxx} = 0, \\ \bar{z}(t,0) = \bar{z}(t,L) = 0, \\ \bar{z}_x(t,L) = 0, \\ z(0,x) = \bar{y}_0 \in L^2(0,L). \end{cases}$$

Set

$$\Delta := \alpha - \bar{\alpha}, \qquad y := \alpha + z, \qquad \bar{y} := \bar{z} + \bar{\alpha},$$
$$\Phi_1 := \Phi_\varepsilon\left(\|y\|_{L^2(0,L)}\right), \qquad \Phi_2 := \Phi_\varepsilon\left(\|\bar{y}\|_{L^2(0,L)}\right).$$

Then we obtain

$$\begin{cases} \Delta_t + \Delta_x + \Delta_{xxx} = -\Phi_1 y y_x + \Phi_2 \bar{y}\bar{y}_x \\ \qquad = \Phi_1\left[-(\alpha+z)\Delta_x - (\bar{\alpha}_x + z_x)\Delta - \bar{\alpha}(z-\bar{z})_x - \bar{\alpha}_x(z-\bar{z}) - z_x z + \bar{z}_x\bar{z}\right] \\ \qquad\quad - (\Phi_1 - \Phi_2)(\bar{z}_x\bar{\alpha} + \bar{\alpha}_x\bar{z} + \bar{z}_x\bar{z} + \bar{\alpha}_x\bar{\alpha}), \\ \Delta(t,0) = \Delta(t,L) = 0, \\ \Delta_x(t,L) = 0, \\ \Delta(0,x) = 0. \end{cases} \tag{3.25}$$

Moreover, by the definition of Φ_1, Φ_2 and (3.2), we get

$$\Phi_1 = \Phi_2 = 0, \quad \forall\, \|y\|_{L^2(0,L)} \geq \varepsilon,\ \forall\, \|\bar{y}\|_{L^2(0,L)} \geq \varepsilon. \tag{3.26}$$

We first give the following estimate of the L^2-norm of Δ.

Lemma 3.3. *Let $T > 0$. Then there exists $C > 0$ such that*

$$\|\Delta(t,\cdot)\|_{L^2(0,L)} \leq C, \quad \forall t \in [0,T],\ \forall \varepsilon \in (0,1],\ \forall y_0 \in L^2(0,L),\ \forall \bar{y}_0 \in L^2(0,L).$$

Proof. By integrating by parts in

$$\int_0^L \Delta\left(\Delta_t + \Delta_x + \Delta_{xxx} + \Phi_1 y y_x - \Phi_2 \bar{y}\bar{y}_x\right) dx = 0,$$

we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Delta^2 dx + \frac{1}{2}\Delta_x^2(t,0) = -\Phi_1\int_0^L \Delta y y_x dx + \Phi_2 \int_0^L \Delta \bar{y}\bar{y}_x dx. \tag{3.27}$$

Note that $\Delta(t,0) = \Delta(t,L) = 0$, by the continuous Sobolev embedding $H_0^1(0,L) \subset C^0([0,L])$ and the Poincaré inequality, we obtain

$$\left|\int_0^L \Delta \bar{y}\bar{y}_x dx\right| \le \|\bar{y}\|_{L^\infty(0,L)} \int_0^L |\Delta \bar{y}_x|\, dx$$

$$\le C\, \|\bar{y}\|_{H_0^1(0,L)} \int_0^L |\Delta \bar{y}_x|\, dx$$

$$\le C\, \|\bar{y}_x\|_{L^2(0,L)} \int_0^L |\Delta \bar{y}_x|\, dx.$$

In the above inequalities and in the following, C, unless otherwise specified, denotes various positive constants which may vary from line to line but are independent of $t \in [0, T]$, $\varepsilon \in (0, 1]$, $y_0 \in L^2(0, L)$ and $\bar{y}_0 \in L^2(0, L)$. Thus,

$$\left|\int_0^L \Delta \bar{y}\bar{y}_x dx\right| \le C\, \|\bar{y}_x\|^2_{L^2(0,L)} \|\Delta\|_{L^2(0,L)}.$$

Similarly, we have

$$\left|\int_0^L \Delta y y_x dx\right| \le C\, \|y_x\|^2_{L^2(0,L)} \|\Delta\|_{L^2(0,L)}.$$

Hence, it follows from (3.27) that

$$\frac{d}{dt}\int_0^L \Delta^2 dx + \Delta_x^2(t,0) \le C\left(\Phi_1 \|y_x\|^2_{L^2(0,L)} + \Phi_2 \|\bar{y}_x\|^2_{L^2(0,L)}\right) \|\Delta\|_{L^2(0,L)}.$$

In particular,

$$\frac{d}{dt}\int_0^L \Delta^2 dx \le C\left(\Phi_1 \|y_x\|^2_{L^2(0,L)} + \Phi_2 \|\bar{y}_x\|^2_{L^2(0,L)}\right) \|\Delta\|_{L^2(0,L)}.$$

By Lemma 17 in [14] and Remark 3.1, we get

$$\int_0^L \Delta^2 dx \le 3\left(\int_0^t C\left(\Phi_1 \|y_x\|^2_{L^2(0,L)} + \Phi_2 \|\bar{y}_x\|^2_{L^2(0,L)}\right) dt\right)^2$$

$$\le 3C^2\left(2\left(\frac{8T+2L}{3}\varepsilon^2 + CT\varepsilon^4\right)\right)^2, \quad \forall t \in [0, T].$$

The result follows. □

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For the sake of simplicity, we denote from now on by $L^2(L^2)$ the norm $L^2(0,T;L^2(0,L))$.

Lemma 3.4. *Let $T>0$. Then there exists $C>0$ such that*

$$\begin{aligned}
&\|\Delta(t,\cdot)\|_{L^2(0,L)} \\
&\quad \leqslant \int_0^T \Big[\Phi_1 \left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right) \left\|(z-\bar{z})_x\right\|_{L^2(0,L)} \\
&\qquad + |\Phi_1 - \Phi_2|\, \|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}^{\frac{1}{2}} \left\|(\bar{z}+\bar{\alpha})_x\right\|_{L^2(0,L)}^{\frac{3}{2}}\Big] dt \\
&\qquad \times \exp\left[C\left(1 + \left\|\sqrt{\Phi_1}\alpha_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1} z_x\right\|_{L^2(L^2)}^2\right)\right],
\end{aligned}$$

for every $t \in [0,T]$, for every $\varepsilon \in (0,1]$, for every $y_0 \in L^2(0,L)$ and for every $\bar{y}_0 \in L^2(0,L)$.

Proof. We multiply the first equation of (3.25) by $2x\Delta$ and then integrate over $[0,L]$. By integrating by parts and using the boundary conditions of (3.25), we get

$$\begin{aligned}
&\frac{d}{dt}\int_0^L x\Delta^2 dx + 3\int_0^L \Delta_x^2 dx \\
&\quad = \int_0^L \Delta^2 dx + \Phi_1 \times \Bigg(-2\int_0^L x\alpha\Delta\Delta_x dx + 4\int_0^L x\bar{\alpha}\Delta\Delta_x dx + 2\int_0^L xz\Delta\Delta_x dx \\
&\qquad + 2\int_0^L \bar{\alpha}\Delta^2 dx + 2\int_0^L z\Delta^2 dx - 2\int_0^L x\Delta\bar{\alpha}\,(z-\bar{z})_x\, dx \\
&\qquad - 2\int_0^L x\Delta\bar{\alpha}_x\,(z-\bar{z})\, dx - 2\int_0^L x\Delta z_x\,(z-\bar{z})\, dx - 2\int_0^L x\Delta\bar{z}\,(z-\bar{z})_x\, dx\Bigg) \\
&\qquad - (\Phi_1 - \Phi_2)\int_0^L 2x\Delta\left(\bar{z}_x\bar{\alpha} + \bar{\alpha}_x\bar{z} + \bar{z}_x\bar{z} + \bar{\alpha}_x\bar{\alpha}\right) dx.
\end{aligned} \tag{3.28}$$

Note that $\alpha(t,0) = \alpha(t,L) = 0$, by the continuous Sobolev embedding $H_0^1(0,L) \subset C^0([0,L])$ and the Poincaré inequality, there exists $C = C(L) > 0$ such that

$$2\left|\int_0^L x\alpha\Delta\Delta_x dx\right| \leq C\,\|\alpha_x\|_{L^2(0,L)} \int_0^L |x\Delta\Delta_x|\, dx.$$

Thus,

$$
\begin{aligned}
2\left|\int_0^L x\alpha\Delta\Delta_x dx\right| &\leq C\left\|\alpha_x\right\|_{L^2(0,L)}\left\|\Delta_x\right\|_{L^2(0,L)}\left\|x\Delta\right\|_{L^2(0,L)} \\
&\leq \frac{1}{2}\left\|\Delta_x\right\|_{L^2(0,L)}^2 + \frac{1}{2}\left(C\left\|\alpha_x\right\|_{L^2(0,L)}\left\|x\Delta\right\|_{L^2(0,L)}\right)^2 \\
&\leq \frac{1}{2}\int_0^L \Delta_x^2 dx + C\left\|\alpha_x\right\|_{L^2(0,L)}^2\int_0^L x\Delta^2 dx. \qquad (3.29)
\end{aligned}
$$

Similarly,

$$
4\left|\int_0^L x\bar{\alpha}\Delta\Delta_x dx\right| \leq \frac{1}{2}\int_0^L \Delta_x^2 dx + C\left\|\bar{\alpha}_x\right\|_{L^2(0,L)}^2\int_0^L x\Delta^2 dx, \qquad (3.30)
$$

$$
2\left|\int_0^L xz\Delta\Delta_x dx\right| \leq \frac{1}{2}\int_0^L \Delta_x^2 dx + C\left\|z_x\right\|_{L^2(0,L)}^2\int_0^L x\Delta^2 dx. \qquad (3.31)
$$

Note that $\bar{\alpha}(t,0) = \bar{\alpha}(t,L) = 0$, by the continuous Sobolev embedding $H_0^1(0,L) \subset C^0([0,L])$ and the Poincaré inequality, we have

$$
2\left|\int_0^L \bar{\alpha}\Delta^2 dx\right| \leq C\left\|\bar{\alpha}_x\right\|_{L^2(0,L)}\int_0^L \Delta^2 dx. \qquad (3.32)
$$

From (3.32) and Lemma 16 in [14] with $a := \min\left\{\frac{1}{\sqrt{2}}C^{-\frac{1}{2}}\left\|\bar{\alpha}_x\right\|_{L^2(0,L)}^{-\frac{1}{2}}, L\right\}$, there exists $C = C(L) > 0$ such that

$$
2\left|\int_0^L \bar{\alpha}\Delta^2 dx\right| \leq \frac{1}{4}\int_0^L \Delta_x^2 dx + C\left(\left\|\bar{\alpha}_x\right\|_{L^2(0,L)}^{\frac{3}{2}} + \left\|\bar{\alpha}_x\right\|_{L^2(0,L)}\right)\int_0^L x\Delta^2 dx. \qquad (3.33)
$$

Similarly, we have

$$
2\left|\int_0^L z\Delta^2 dx\right| \leq \frac{1}{4}\int_0^L \Delta_x^2 dx + C\left(\left\|z_x\right\|_{L^2(0,L)}^{\frac{3}{2}} + \left\|z_x\right\|_{L^2(0,L)}\right)\int_0^L x\Delta^2 dx. \qquad (3.34)
$$

By Lemma 16 in [14], there exists $C = C(L) > 0$ such that

$$
\int_0^L \Delta^2 dx \leq \frac{1}{2}\int_0^L \Delta_x^2 dx + C\int_0^L x\Delta^2 dx. \qquad (3.35)
$$

We have

$$2\left|\int_0^L x\Delta\bar{\alpha}\,(z-\bar{z})_x\,dx\right| \le C\,\|\bar{\alpha}_x\|_{L^2(0,L)}\left|\int_0^L x\Delta\,(z-\bar{z})_x\,dx\right|$$
$$\le C\,\|\bar{\alpha}_x\|_{L^2(0,L)}\left(\int_0^L x^2\Delta^2 dx\right)^{\frac{1}{2}}\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$
$$\le C\,\|\bar{\alpha}_x\|_{L^2(0,L)}\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}. \tag{3.36}$$

Similarly, we can obtain

$$2\left|\int_0^L x\Delta\bar{\alpha}_x\,(z-\bar{z})\,dx\right| \le C\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}\|\bar{\alpha}_x\|_{L^2(0,L)}, \tag{3.37}$$

$$2\left|\int_0^L x\Delta z_x\,(z-\bar{z})\,dx\right| \le C\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}\|z_x\|_{L^2(0,L)}, \tag{3.38}$$

$$2\left|\int_0^L x\Delta\bar{z}\,(z-\bar{z})_x\,dx\right| \le C\,\|\bar{z}_x\|_{L^2(0,L)}\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}. \tag{3.39}$$

Moreover, we have

$$\left|\int_0^L 2x\Delta\,(\bar{z}_x\bar{\alpha}+\bar{\alpha}_x\bar{z}+\bar{z}_x\bar{z}+\bar{\alpha}_x\bar{\alpha})\,dx\right|$$
$$=2\left|\int_0^L x\Delta\,(\bar{z}_x+\bar{\alpha}_x)\,(\bar{z}+\bar{\alpha})\,dx\right|$$
$$\le 2\,\|\bar{z}+\bar{\alpha}\|_{L^\infty(0,L)}\int_0^L\left|x\Delta\,(\bar{z}+\bar{\alpha})_x\right|dx$$
$$\le 2\sqrt{L}\,\|\bar{z}+\bar{\alpha}\|_{L^\infty(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|_{L^2(0,L)}\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}. \tag{3.40}$$

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Then, using the Gagliardo–Nirenberg inequality (3.6), it follows from (3.40) that

$$\left| \int_0^L 2x\Delta\left(\bar{z}_x\bar{\alpha} + \bar{\alpha}_x\bar{z} + \bar{z}_x\bar{z} + \bar{\alpha}_x\bar{\alpha}\right)dx \right|$$
$$\leq C\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)} \left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)} \left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}. \tag{3.41}$$

Thus, using (3.28) to (3.41), we get

$$\frac{d}{dt}\int_0^L x\Delta^2 dx + \frac{1}{2}\int_0^L \Delta_x^2 dx$$
$$\leq C\left(1 + \Phi_1\left(\|\alpha_x\|^2_{L^2(0,L)} + \|\bar{\alpha}_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)}\right)\right)\int_0^L x\Delta^2 dx$$
$$+ C\Big[\Phi_1\left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$
$$+ |\Phi_1 - \Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big]\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}. \tag{3.42}$$

In particular,

$$\frac{d}{dt}\int_0^L x\Delta^2 dx$$
$$\leq C\left(1 + \Phi_1\left(\|\alpha_x\|^2_{L^2(0,L)} + \|\bar{\alpha}_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)}\right)\right)\int_0^L x\Delta^2 dx$$
$$+ C\Big[\Phi_1\left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$
$$+ |\Phi_1 - \Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big]\left(\int_0^L x\Delta^2 dx\right)^{\frac{1}{2}}.$$

Then, by Lemma 17 in [14], we get

$$\int_0^L x\Delta^2 dx \leq W, \quad \forall t \in [0, T] \tag{3.43}$$

with

$$W := 3C^2\Big[\int_0^T \Big(\big(\|\Phi_1\bar{\alpha}_x\|_{L^2(0,L)} + \|\Phi_1 z_x\|_{L^2(0,L)} + \|\Phi_1\bar{z}_x\|_{L^2(0,L)}\big)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$

$$+ |\Phi_1 - \Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big)dt\Big]^2$$

$$\times \exp\left[C\left(T + \left\|\sqrt{\Phi_1}\alpha_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1}z_x\right\|^2_{L^2(L^2)}\right)\right]. \tag{3.44}$$

Now integrating (3.42) over $[0, T]$ and using (3.43), we have

$$\int_0^L x\Delta^2(T,x)\,dx + \frac{1}{2}\int_0^T\int_0^L \Delta_x^2 dxdt$$

$$\leq C\int_0^T \left(1 + \Phi_1\left(\|\alpha_x\|^2_{L^2(0,L)} + \|\bar{\alpha}_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)}\right)\right)dt\,W$$

$$+ C\int_0^T \Big[\Phi_1\left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$

$$+ |\Phi_1 - \Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big]dt\,W^{\frac{1}{2}}.$$

Then it follows that

$$\frac{1}{2}\int_0^T\int_0^L \Delta_x^2 dxdt \tag{3.45}$$

$$\leqslant C\int_0^T \left(1 + \Phi_1\left(\|\alpha_x\|^2_{L^2(0,L)} + \|\bar{\alpha}_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)}\right)\right)dt\,W + \frac{1}{2}W$$

$$+ \frac{1}{2}\Big[C\int_0^T \Big(\Phi_1\left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)}$$

$$+ |\Phi_1 - \Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big)dt\Big]^2. \tag{3.46}$$

Hence, combining (3.46) with (3.44), we obtain

$$\begin{aligned}
&\int_0^T\int_0^L \Delta_x^2 dxdt \\
&\leqslant C\Big[\int_0^T \Big(\big(\|\Phi_1\bar{\alpha}_x\|_{L^2(0,L)}+\|\Phi_1 z_x\|_{L^2(0,L)}+\|\Phi_1\bar{z}_x\|_{L^2(0,L)}\big)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)} \\
&\quad +|\Phi_1-\Phi_2|\,\|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}^{\frac{1}{2}}\left\|(\bar{z}+\bar{\alpha})_x\right\|_{L^2(0,L)}^{\frac{3}{2}}\Big)dt\Big]^2 \\
&\quad \times \exp\left[C\left(T+\left\|\sqrt{\Phi_1}\alpha_x\right\|_{L^2(L^2)}^2+\left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|_{L^2(L^2)}^2+\left\|\sqrt{\Phi_1}z_x\right\|_{L^2(L^2)}^2\right)\right]. \qquad (3.47)
\end{aligned}$$

We multiply the first equation of (3.25) by Δ and integrate over $[0, L]$. Using the boundary conditions of (3.25) and integrations by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \Delta^2 dx+\frac{1}{2}\Delta_x^2(t,0) \\
&=\Phi_1\times\Big(-\int_0^L \alpha\Delta_x\Delta dx+2\int_0^L \bar{\alpha}\Delta_x\Delta dx+\int_0^L z\Delta_x\Delta dx \\
&\quad -\int_0^L \bar{\alpha}(z-\bar{z})_x\,\Delta dx-\int_0^L \bar{\alpha}_x(z-\bar{z})\,\Delta dx \\
&\quad -\int_0^L z_x(z-\bar{z})\,\Delta dx-\int_0^L \bar{z}(z-\bar{z})_x\,\Delta dx\Big) \\
&\quad -(\Phi_1-\Phi_2)\int_0^L \Delta\left(\bar{z}_x\bar{\alpha}+\bar{\alpha}_x\bar{z}+\bar{z}_x\bar{z}+\bar{\alpha}_x\bar{\alpha}\right)dx. \qquad (3.48)
\end{aligned}$$

It can be readily checked that

$$\begin{aligned}
\left|\int_0^L \alpha\Delta_x\Delta dx\right| &\leq \frac{1}{2}\int_0^L \Delta_x^2 dx+\frac{1}{2}\int_0^L \Delta^2\alpha^2 dx \\
&\leq \frac{1}{2}\int_0^L \Delta_x^2 dx+C\,\|\alpha_x\|_{L^2(0,L)}^2\int_0^L \Delta^2 dx. \qquad (3.49)
\end{aligned}$$

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Similarly, we have

$$\left| 2\int_0^L \bar{\alpha} \Delta_x \Delta dx \right| \le \frac{1}{2} \int_0^L \Delta_x^2 dx + C \left\| \bar{\alpha}_x \right\|_{L^2(0,L)}^2 \int_0^L \Delta^2 dx, \tag{3.50}$$

and

$$\left| \int_0^L z \Delta_x \Delta dx \right| \le \frac{1}{2} \int_0^L \Delta_x^2 dx + C \left\| z_x \right\|_{L^2(0,L)}^2 \int_0^L \Delta^2 dx. \tag{3.51}$$

Similarly to (3.36), we get the following inequalities

$$\left| \int_0^L \bar{\alpha} \left(z - \bar{z}\right)_x \Delta dx \right| \le C \left\| \bar{\alpha}_x \right\|_{L^2(0,L)} \left\| (z - \bar{z})_x \right\|_{L^2(0,L)} \left(\int_0^L \Delta^2 dx \right)^{\frac{1}{2}}, \tag{3.52}$$

$$\left| \int_0^L \bar{\alpha}_x \left(z - \bar{z}\right) \Delta dx \right| \le C \left\| (z - \bar{z})_x \right\|_{L^2(0,L)} \left\| \bar{\alpha}_x \right\|_{L^2(0,L)} \left(\int_0^L \Delta^2 dx \right)^{\frac{1}{2}}, \tag{3.53}$$

$$\left| \int_0^L z_x \left(z - \bar{z}\right) \Delta dx \right| \le C \left\| (z - \bar{z})_x \right\|_{L^2(0,L)} \left\| z_x \right\|_{L^2(0,L)} \left(\int_0^L \Delta^2 dx \right)^{\frac{1}{2}}, \tag{3.54}$$

$$\left| \int_0^L \bar{z} \left(z - \bar{z}\right)_x \Delta dx \right| \le C \left\| \bar{z}_x \right\|_{L^2(0,L)} \left\| (z - \bar{z})_x \right\|_{L^2(0,L)} \left(\int_0^L \Delta^2 dx \right)^{\frac{1}{2}}. \tag{3.55}$$

Moreover, for the last term on the right-hand side of (3.48), using the same argument as for (3.41), we have

$$\begin{aligned} &\left| \int_0^L \Delta \left(\bar{z}_x \bar{\alpha} + \bar{\alpha}_x \bar{z} + \bar{z}_x \bar{z} + \bar{\alpha}_x \bar{\alpha} \right) dx \right| \\ &\quad \le C \left\| \bar{z} + \bar{\alpha} \right\|_{L^2(0,L)}^{\frac{1}{2}} \left\| (\bar{z} + \bar{\alpha})_x \right\|_{L^2(0,L)}^{\frac{3}{2}} \left(\int_0^L \Delta^2 dx \right)^{\frac{1}{2}}. \end{aligned} \tag{3.56}$$

Hence, by (3.48) to (3.56), we deduce that

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$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \Delta^2 dx \\
&\le \frac{3}{2}\int_0^L \Delta_x^2 dx + C\Phi_1\left(\|\alpha_x\|^2_{L^2(0,L)} + \|\bar{\alpha}_x\|^2_{L^2(0,L)} + \|z_x\|^2_{L^2(0,L)}\right)\int_0^L \Delta^2 dx \\
&\quad + C\Big[\Phi_1\left(2\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)} \\
&\quad + |\Phi_1-\Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big]\left(\int_0^L \Delta^2 dx\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, by (3.47) and Lemma 17 in [14], we get that, for every $t \in [0, T]$,

$$
\begin{aligned}
&\|\Delta(t,\cdot)\|_{L^2(0,L)} \\
&\leqslant \Big[\int_0^T \Big(\Phi_1\left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)}\right)\left\|(z-\bar{z})_x\right\|_{L^2(0,L)} \\
&\quad + |\Phi_1-\Phi_2|\,\|\bar{z}+\bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)}\left\|(\bar{z}+\bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)}\Big)dt\Big]^2 \\
&\quad \times \exp\left[C\left(1+\left\|\sqrt{\Phi_1}\alpha_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1}z_x\right\|^2_{L^2(L^2)}\right)\right].
\end{aligned}
$$

This completes the proof of Lemma 3.4. □

Now we are in a position to prove the following proposition on the global Lipschitzianity of the map $R(t)$. With our notation, we have

$$R(t)y_0 - R(t)\bar{y}_0 = \alpha(t,\cdot) - \bar{\alpha}(t,\cdot) = \Delta(t,\cdot).$$

Proposition 3.4. *Let $T > 0$. There exist $\varepsilon_0 \in (0, 1]$ and $\tilde{C} : (0, \varepsilon_0] \to (0, +\infty)$ such that*

$$\|\Delta(t,\cdot)\|_{L^2(0,L)} \le \tilde{C}(\varepsilon)\|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall \bar{y}_0, y_0 \in L^2(0,L),\ \forall t \in [0,T],\ \forall \varepsilon \in (0,\varepsilon_0], \tag{3.57}$$

$$\tilde{C}(\varepsilon) \to 0 \quad as\ \varepsilon \to 0^+. \tag{3.58}$$

Proof. Let

$$\Delta_{\max} := \Delta_{\max}(\varepsilon) := \sup_{t\in[0,T]} \|\Delta(t,\cdot)\|_{L^2(0,L)}.$$

Let us point out that, by Lemma 3.3, $\Delta_{\max} < +\infty$. We claim that

$$\Delta_{\max} \le \varepsilon C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max} \right), \quad \forall \bar{y}_0, y_0 \in L^2(0, L). \tag{3.59}$$

Then letting ε be small enough such that $1 - \varepsilon C > 0$, we obtain

$$\Delta_{\max} \le \frac{\varepsilon C}{1 - \varepsilon C} \|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall \bar{y}_0, y_0 \in L^2(0, L).$$

Consequently, we get

$$\|\Delta\|_{L^2(0,L)} \le \frac{\varepsilon C}{1 - \varepsilon C} \|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall t \in [0, T], \ \forall \bar{y}_0, y_0 \in L^2(0, L),$$

and the result follows. Hence, in order to prove Proposition 3.4, we only need to show that (3.59) holds in the following cases:

(i) $\|\bar{y}_0\|_{L^2(0,L)}, \|y_0\|_{L^2(0,L)} \ge \varepsilon$, and $\|\bar{y}\|_{L^2(0,L)}, \|y\|_{L^2(0,L)} \ge \varepsilon$, $\forall t \in [0, T]$;
(ii) $\|\bar{y}_0\|_{L^2(0,L)}, \|y_0\|_{L^2(0,L)} \ge \varepsilon$, there exists $\tau \in [0, T]$ such that $\|\bar{y}(\tau, \cdot)\|_{L^2(0,L)} = \varepsilon$, and $\|y(t, \cdot)\|_{L^2(0,L)} \ge \varepsilon$, $\forall t \in [0, T]$;
(iii) $\|\bar{y}_0\|_{L^2(0,L)}, \|y_0\|_{L^2(0,L)} \ge \varepsilon$, there exist $\tau, \varsigma \in [0, T]$, $\varsigma \ge \tau$, such that $\|\bar{y}(\tau, \cdot)\|_{L^2(0,L)} = \varepsilon$, and $\|y(\varsigma, \cdot)\|_{L^2(0,L)} = \varepsilon$;
(iv) $\|\bar{y}_0\|_{L^2(0,L)} \le \varepsilon$ and $\|y\|_{L^2(0,L)} \ge \varepsilon$, $\forall t \in [0, T]$;
(v) $\|\bar{y}_0\|_{L^2(0,L)} \le \varepsilon$, $\|y_0\|_{L^2(0,L)} \ge \varepsilon$, and there exists $\tau \in [0, T]$ such that $\|y(\tau, \cdot)\|_{L^2(0,L)} = \varepsilon$;
(vi) $\|\bar{y}_0\|_{L^2(0,L)} \le \varepsilon$, $\|y_0\|_{L^2(0,L)} \le \varepsilon$.

By Lemma 3.4, for every $t \in [0, T]$, we have

$$\begin{aligned}
&\|\Delta(t, \cdot)\|_{L^2(0,L)} \\
&\le \int_0^T \Big[\Phi_1 \left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)} \right) \left\|(z - \bar{z})_x\right\|_{L^2(0,L)} \\
&\quad + |\Phi_1 - \Phi_2| \, \|\bar{z} + \bar{\alpha}\|^{\frac{1}{2}}_{L^2(0,L)} \left\|(\bar{z} + \bar{\alpha})_x\right\|^{\frac{3}{2}}_{L^2(0,L)} \Big] dt \\
&\quad \times \exp\left[C \left(1 + \left\|\sqrt{\Phi_1}\alpha_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|^2_{L^2(L^2)} + \left\|\sqrt{\Phi_1} z_x\right\|^2_{L^2(L^2)} \right) \right].
\end{aligned} \tag{3.60}$$

Furthermore, by using Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}
&\int_0^T \Phi_1 \left(\|\bar{\alpha}_x\|_{L^2(0,L)} + \|z_x\|_{L^2(0,L)} + \|\bar{z}_x\|_{L^2(0,L)} \right) \left\|(z - \bar{z})_x\right\|_{L^2(0,L)} dt \\
&\quad \le \left\|(z - \bar{z})_x\right\|_{L^2(L^2)} \left(\|\Phi_1 \bar{\alpha}_x\|_{L^2(L^2)} + \|\Phi_1 z_x\|_{L^2(L^2)} + \|\Phi_1 \bar{z}_x\|_{L^2(L^2)} \right) \\
&\quad \le C \|y_0 - \bar{y}_0\|_{L^2(0,L)} \left(\|\Phi_1 \bar{\alpha}_x\|_{L^2(L^2)} + \|\Phi_1 z_x\|_{L^2(L^2)} + \|\Phi_1 \bar{z}_x\|_{L^2(L^2)} \right).
\end{aligned} \tag{3.61}$$

Applying the mean value theorem, noticing that $\|\alpha - \bar{\alpha}\|_{L^2(0,L)} \le \Delta_{\max}$, $\forall t \in [0, T]$, and by Lemma 2.1,

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$$\|z-\bar{z}\|_{L^2(0,L)} \le \|y_0-\bar{y}_0\|_{L^2(0,L)}, \quad \forall t\in[0,T],$$

we get

$$\begin{aligned}
|\Phi_1-\Phi_2| &\le \left|\Phi_\varepsilon'(\theta)\right| \left|\|z+\alpha\|_{L^2(0,L)} - \|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}\right| \\
&\le \left|\Phi_\varepsilon'(\theta)\right| \|(z+\alpha)-(\bar{z}+\bar{\alpha})\|_{L^2(0,L)} \\
&\le \left|\Phi_\varepsilon'(\theta)\right| \left(\|z-\bar{z}\|_{L^2(0,L)} + \|\alpha-\bar{\alpha}\|_{L^2(0,L)}\right) \\
&\le \left|\Phi_\varepsilon'(\theta)\right| \left(\|y_0-\bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right),
\end{aligned} \tag{3.62}$$

where

$$\theta=\theta(t)\in\left(\min\{\|z+\alpha\|_{L^2(0,L)}, \|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}\}, \max\{\|z+\alpha\|_{L^2(0,L)}, \|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}\}\right).$$

Thus, combining (3.60), (3.61) and (3.62), we arrive at

$$\begin{aligned}
&\|\Delta(t,\cdot)\|_{L^2(0,L)} \\
&\le \Bigg(C\,\|y_0-\bar{y}_0\|_{L^2(0,L)} \left(\|\Phi_1\bar{\alpha}_x\|_{L^2(L^2)} + \|\Phi_1 z_x\|_{L^2(L^2)} + \|\Phi_1\bar{z}_x\|_{L^2(L^2)}\right) \\
&\quad + \left(\|y_0-\bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right) \int_0^T \left|\Phi_\varepsilon'(\theta)\right| \|\bar{z}+\bar{\alpha}\|_{L^2(0,L)}^{\frac{1}{2}} \left\|(\bar{z}+\bar{\alpha})_x\right\|_{L^2(0,L)}^{\frac{3}{2}} dt \Bigg) \\
&\quad \times \exp\left(C\left(1+\left\|\sqrt{\Phi_1}\alpha_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1}z_x\right\|_{L^2(L^2)}^2\right)\right).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\Delta_{\max} &\le \Bigg[C\,\|y_0-\bar{y}_0\|_{L^2(0,L)} \left(\|\Phi_1\bar{\alpha}_x\|_{L^2(L^2)} + \|\Phi_1 z_x\|_{L^2(L^2)} + \|\Phi_1\bar{z}_x\|_{L^2(L^2)}\right) \\
&\quad + \left(\|y_0-\bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right) \int_0^T \left|\Phi_\varepsilon'(\theta)\right| \|\bar{y}\|_{L^2(0,L)}^{\frac{1}{2}} \|\bar{y}_x\|_{L^2(0,L)}^{\frac{3}{2}} dt \Bigg] \\
&\quad \times \exp\left(C\left(1+\left\|\sqrt{\Phi_1}\alpha_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|_{L^2(L^2)}^2 + \left\|\sqrt{\Phi_1}z_x\right\|_{L^2(L^2)}^2\right)\right).
\end{aligned} \tag{3.63}$$

For case (i), by (3.26), we have $\Phi_1=\Phi_2=0$, $\forall t\in[0,T]$, it follows directly from (3.60) that

$$\Delta_{\max}=0.$$

For case (ii), by (3.26), we have

$$\Phi_1\equiv 0, \quad \forall t\in[0,T]. \tag{3.64}$$

In view of Lemma 3.1, we have

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$$\|\bar{y}(t,\cdot)\|_{L^2(0,L)} \geq \varepsilon, \quad \forall t \in [0,\tau], \tag{3.65}$$

and

$$\|\bar{y}(t,\cdot)\|_{L^2(0,L)} \leq \|\bar{y}(\tau,\cdot)\|_{L^2(0,L)} = \varepsilon, \quad \forall t \in [\tau,T]. \tag{3.66}$$

Consequently, it follows from (3.26) and (3.65) that

$$\Phi_2 \equiv 0, \quad \forall t \in [0,\tau].$$

From (3.3), (3.63), (3.64), (3.65) and (3.66), we get that

$$\Delta_{\max} \leq \exp(C)\left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right) \int_\tau^T \frac{C}{\varepsilon}\varepsilon^{\frac{1}{2}} \|\bar{y}_x\|^{\frac{3}{2}}_{L^2(0,L)}\, dt. \tag{3.67}$$

From now on, we assume that $\varepsilon \in (0,\eta]$, where $\eta > 0$ is chosen as in Lemma 3.2. Thanks to Lemma 3.2 and (3.66), we have

$$\|\bar{y}_x(t,\cdot)\|_{L^2(0,L)} \leq \frac{C}{\sqrt{t-\tau}}\|\bar{y}(\tau,\cdot)\|_{L^2(0,L)} = \frac{C}{\sqrt{t-\tau}}\varepsilon, \quad \forall t \in [\tau,T]. \tag{3.68}$$

Replacing (3.68) into (3.67), we obtain

$$\begin{aligned}\Delta_{\max} &\leq \varepsilon C\left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right) \int_\tau^T \frac{1}{(t-\tau)^{\frac{3}{4}}} dt \\ &\leq \varepsilon C\left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right).\end{aligned}$$

For case (iii), by (3.26) and Lemma 3.1, we have

$$\Phi_1 = 0, \quad \forall t \in [0,\varsigma], \tag{3.69}$$

$$\Phi_2 = 0, \quad \forall t \in [0,\tau], \tag{3.70}$$

and (3.66) still holds. In particular,

$$\|\bar{y}(\varsigma,\cdot)\|_{L^2(0,L)} \leq \varepsilon. \tag{3.71}$$

It follows from Lemma 2.2 and (3.69) that

$$\begin{aligned}\|\Phi_1 \bar{z}_x\|_{L^2(0,T;L^2(0,L))} &= \|\Phi_1 \bar{z}_x\|_{L^2(\varsigma,T;L^2(0,L))} \leq \|\bar{z}_x\|_{L^2(\varsigma,T;L^2(0,L))} \leq C\|\bar{z}(\varsigma,\cdot)\|_{L^2(0,L))} \\ &\leq C\|\bar{z}(\tau,\cdot)\|_{L^2(0,L))} = C\|\bar{y}(\tau,\cdot)\|_{L^2(0,L)} = \varepsilon C, \end{aligned} \tag{3.72}$$

$$\begin{aligned}\|\Phi_1 z_x\|_{L^2(0,T;L^2(0,L))} &= \|\Phi_1 z_x\|_{L^2(\varsigma,T;L^2(0,L))} \leq \|z_x\|_{L^2(\varsigma,T;L^2(0,L))} \leq C\|z(\varsigma,\cdot)\|_{L^2(0,L))} \\ &= C\|y(\varsigma,\cdot)\|_{L^2(0,L)} = \varepsilon C,\end{aligned} \tag{3.73}$$

and

$$\left\| \sqrt{\Phi_1} z_x \right\|_{L^2(0,T;L^2(0,L))} \leq \varepsilon C. \tag{3.74}$$

By Remark 3.1, (3.69), (3.71) and (3.72), we have

$$\begin{aligned} \|\Phi_1 \bar{\alpha}_x\|_{L^2(0,T;L^2(0,L))} &\leq \|\Phi_1 \bar{y}_x\|_{L^2(0,T;L^2(0,L))} + \|\Phi_1 \bar{z}_x\|_{L^2(0,T;L^2(0,L))} \\ &\leq \|\bar{y}_x\|_{L^2(\varsigma,T;L^2(0,L))} + \|\Phi_1 \bar{z}_x\|_{L^2(0,T;L^2(0,L))} \\ &\leq \varepsilon C, \end{aligned} \tag{3.75}$$

and

$$\left\| \sqrt{\Phi_1} \bar{\alpha}_x \right\|_{L^2(0,T;L^2(0,L))} \leq \varepsilon C. \tag{3.76}$$

Similarly, we obtain

$$\left\| \sqrt{\Phi_1} \alpha_x \right\|_{L^2(0,T;L^2(0,L))} \leq \varepsilon C. \tag{3.77}$$

Moreover, for this case, (3.68) still holds. Now it follows from (3.3), (3.63), (3.66), (3.68), (3.69), (3.70), (3.72) to (3.77) that

$$\begin{aligned} \Delta_{\max} &\leq C \left(C\varepsilon \|y_0 - \bar{y}_0\|_{L^2(0,L)} + \varepsilon \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max} \right) \int_\tau^T \frac{1}{(t-\tau)^{\frac{3}{4}}} dt \right) \\ &\leq \varepsilon C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max} \right). \end{aligned}$$

For case (iv), by (3.26), we have $\Phi_1 \equiv 0$, $\forall t \in [0, T]$. It follows from (3.63) that

$$\Delta_{\max} \leq C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max} \right) \int_0^T \left| \Phi'(\theta) \right| \|\bar{y}\|_{L^2(0,L)}^{\frac{1}{2}} \|\bar{y}_x\|_{L^2(0,L)}^{\frac{3}{2}} dt. \tag{3.78}$$

By Lemma 3.1, we have

$$\|\bar{y}(t,\cdot)\|_{L^2(0,L)} \leq \|\bar{y}_0\|_{L^2(0,L)} \leq \varepsilon, \quad \forall t \in [0,T]. \tag{3.79}$$

Moreover, thanks to Lemma 3.2, we have

$$\|\bar{y}_x(t,\cdot)\|_{L^2(0,L)} \leq \frac{C}{\sqrt{t}} \|\bar{y}_0\|_{L^2(0,L)} \leq \frac{C}{\sqrt{t}} \varepsilon, \quad \forall t \in [0,T]. \tag{3.80}$$

Then it follows from (3.3), (3.78) to (3.80) that

$$\Delta_{\max} \le \varepsilon C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right) \int_0^T \frac{1}{t^{\frac{3}{4}}} dt$$
$$\le \varepsilon C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right).$$

For case (v), similarly to case (iii), we have

$$\|\Phi_1 \bar{\alpha}_x\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \qquad \|\Phi_1 z_x\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \tag{3.81}$$

$$\|\Phi_1 \bar{z}_x\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \qquad \left\|\sqrt{\Phi_1}\alpha_x\right\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \tag{3.82}$$

$$\left\|\sqrt{\Phi_1}\bar{\alpha}_x\right\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \qquad \left\|\sqrt{\Phi_1} z_x\right\|_{L^2(0,T;L^2(0,L))} \le \varepsilon C, \tag{3.83}$$

Moreover, (3.79) and (3.80) still hold. Thanks to (3.3), (3.79) and (3.80), we have

$$\int_0^T \left|\Phi'(\theta)\right| \|\bar{y}\|_{L^2(0,L)}^{\frac{1}{2}} \|\bar{y}_x\|_{L^2(0,L)}^{\frac{3}{2}} dt \le \varepsilon C \int_0^T \frac{1}{t^{\frac{3}{4}}} dt. \tag{3.84}$$

Then, by (3.63), (3.81) to (3.84), we obtain

$$\Delta_{\max} \le \varepsilon C \left(\|y_0 - \bar{y}_0\|_{L^2(0,L)} + \Delta_{\max}\right).$$

For the last case (vi), (3.79)–(3.84) hold, and (3.59) follows. Above all, we have proved (3.59) for all the cases (i)–(vi), which completes the proof of Proposition 3.4. □

3.2.2. *Smoothness of the semigroup*

Lemma 3.5. *Let $\varepsilon > 0$ and $T > 0$ be given. Then the nonlinear map $\tilde{S}(t)$ defined by the unique solution of (3.1) is of class C^3 from $L^2(0, L)$ to $C\left([0, T]; L^2(0, L)\right)$. Moreover, its derivative $\tilde{S}^{(1)}$ at $y_0 \in L^2(0, L)$ is given by*

$$\tilde{S}^{(1)}(y_0)(h) := \xi(y)(h), \quad \forall h \in L^2(0, L), \tag{3.85}$$

where $\xi(y)(h)$ is defined by the following system (3.86) with $y = \tilde{S}(y_0)$.

$$\begin{cases} \xi_t + \xi_x + \xi_{xxx} + \Phi'_\varepsilon(\|y\|_{L^2(0,L)}) \dfrac{\int_0^L y\xi\, dx}{\|y\|_{L^2(0,L)}} y y_x + \Phi_\varepsilon(\|y\|_{L^2(0,L)})(y\xi_x + \xi y_x) = 0, \\ \xi(t,0) = \xi(t,L) = 0, \\ \xi_x(t,L) = 0, \\ \xi(0,x) = h(x). \end{cases} \tag{3.86}$$

Proof. We refer to [44] and [1, Theorem 5.4] for a detailed argument in related circumstances. □

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3.2.3. Center manifold

Combining [42, Remark 2.3], Corollary 2.2, Proposition 3.4 and Lemma 3.5, we are in a position to apply [42, Theorem 2.19] and [42, Theorem 2.28]. To that end, let us first introduce some definitions from [42]. Let $\mathbb{X}$ be a Banach space, with a norm denoted by $\|\cdot\|_{\mathbb{X}}$.

Definition 3.1. (See [42, Definition 2.1].) A family of (possibly nonlinear) operators $U(t,s):\mathbb{X}\to\mathbb{X}$, $(t,s)\in D:=\{(t,s)\in\mathbb{R}\times\mathbb{R}, t\geq s\}$, is said to be an evolutionary process in $\mathbb{X}$ if the following conditions hold:

(i) $U(t,t)=\mathrm{Id}$, $\forall t\in\mathbb{R}$, where Id is the identity on $\mathbb{X}$;
(ii) $U(t,s)U(s,r)=U(t,r)$, $\forall(t,s)\in D$, $\forall(s,r)\in D$;
(iii) $\|U(t,s)(x)-U(t,s)(y)\|_{\mathbb{X}}\leq Ke^{\omega(t-s)}\|x-y\|_{\mathbb{X}}$, $\forall x\in\mathbb{X}$, $\forall y\in\mathbb{X}$, $\forall(t,s)\in D$, where K and ω are positive constants;
(iv) $U(t,s)(0)=0$, $\forall(t,s)\in D$.

An evolutionary process $U(t,s):\mathbb{X}\to\mathbb{X}$, $(t,s)\in D$ is said to be linear if $U(t,s):\mathbb{X}\to\mathbb{X}$ is a linear map for every $(t,s)\in D$. An evolutionary process $U(t,s):\mathbb{X}\to\mathbb{X}$, $(t,s)\in D$ is said to be periodic with period $\tau>0$ if

$$U(t+\tau,s+\tau)=U(t,s),\quad \forall(t,s)\in D.$$

In fact, in [42, Definition 2.1], $U(t,s)$ are maps from $\mathbb{X}_t$ to $\mathbb{X}_s$ where $\{\mathbb{X}_t, t\in\mathbb{R}\}$ is a family of Banach spaces. But in our application $\mathbb{X}_t$ is independent of t.

The next definition concerns the distance between a linear evolutionary process and a nonlinear evolutionary process.

Definition 3.2. (See [42, Definition 2.6].) Let $(U(t,s))_{t\geq s}$ be a linear evolutionary process and let δ be a positive constant. A nonlinear evolutionary process $(X(t,s))_{t\geq s}$ is said to be δ-close to $(U(t,s))_{t\geq s}$ if there are positive constants μ,η such that $\eta e^{\mu}<\delta$ and

$$\|\phi(t,s)(x)-\phi(t,s)(y)\|_{\mathbb{X}}\leq\eta e^{\mu(t-s)}\|x-y\|_{\mathbb{X}},\quad \forall(t,s)\in D,\ \forall x,y\in\mathbb{X},$$

where

$$\phi(t,s)(x):=X(t,s)(x)-U(t,s)(x),\quad \forall(t,s)\in D,\ \forall x\in\mathbb{X}.$$

Let us consider these definitions in the following framework:

$$\mathbb{X}:=L^2(0,L),\qquad U(t,s):=S(t-s),\qquad X(t,s):=\tilde{S}(t-s),\quad \forall(t,s)\in D. \tag{3.87}$$

Then, one has the following lemma.

Lemma 3.6. *For every $\tau>0$, $(U(t,s))_{t\geq s}$ is a linear τ-periodic evolutionary process and $(X(t,s))_{t\geq s}$ is a nonlinear τ-periodic evolutionary process. Moreover, for every $\delta>0$, there exists $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$,*

$$(X(t,s))_{t\geq s}\text{ is }\delta\text{-close to }(U(t,s))_{t\geq s}. \tag{3.88}$$

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Proof. The τ-periodicities of U and X are obvious. Let us use the notation of Section 3.2.1. We take $T := 1$ in Proposition 3.4. Using a simple induction argument, together with Lemma 2.1 and (3.57), one gets that, for every nonnegative integer n,

$$\|y(t,\cdot) - \bar{y}(t,\cdot)\|_{L^2(0,L)} \leq \left(\tilde{C}(\varepsilon) + 1\right)^{n+1} \|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall t \in [n, n+1], \tag{3.89}$$

$$\|\alpha(t,\cdot) - \bar{\alpha}(t,\cdot)\|_{L^2(0,L)} \leq \left(\left(\tilde{C}(\varepsilon) + 1\right)^{n+1} - 1\right) \|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall t \in [n, n+1]. \tag{3.90}$$

Since, for every $C > 0$ and for every nonnegative integer n, $(C+1)^{n+1} - 1 \leq (n+1)C(C+1)^n$, (3.90) gives

$$\begin{aligned} &\|\alpha(t,\cdot) - \bar{\alpha}(t,\cdot)\|_{L^2(0,L)} \\ &\quad \leq \tilde{C}(\varepsilon) \exp\left(t \ln\left(1 + \tilde{C}(\varepsilon)\right) + \ln(1+t)\right) \|y_0 - \bar{y}_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \end{aligned} \tag{3.91}$$

From Lemma 2.1, $(U(t,s))_{t\geq s}$ is a (linear) evolutionary process and, then, using (3.91), one gets that $(X(t,s))_{t\geq s}$ is a nonlinear evolutionary process. Finally, from (3.58) and (3.91), one gets that, for every $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, (3.88) holds. $\square$

Note that, by (3.22), (3.23) and (3.24), the linear evolutionary process $(U(t,s))_{t\geq s}$ defined in (3.87) has an exponential trichotomy in the sense of [42, Definition 2.2]. (Let us emphasize that, with the notation of this definition, one has, for every $t \geq 0$, $P_1(t) = \mathrm{Id} - P$, $P_2(t) = 0$, and $P_3(t) = P$.)

Then, using Lemma 3.5, Lemma 3.6, [42, Theorem 2.19] and [42, Theorem 2.28], we have the following theorem.

Theorem 3.1. *There exists a positive constant ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a (globally) Lipschitz continuous $g : M \to M^\perp$ of class C^3 in a neighborhood of $0 \in M$ such that, with $G := \{x_1 + g(x_1) : x_1 \in M\}$,*

1. *for every $y_0 \in G$ and for every $t \in [0, +\infty)$, $\tilde{S}(t)y_0 \in G$;*
2. *there exist positive constants K, η such that, for every $y_0 \in L^2(0, L)$,*

$$\mathrm{dist}(\tilde{S}(t)y_0, G) \leq K e^{-\eta t}\, \mathrm{dist}(y_0, G), \tag{3.92}$$

where $\mathrm{dist}(f, G)$ *denotes the distance of $f \in L^2(0, L)$ to G.*

Remark 3.2. The set G in Theorem 3.1 is called a center-unstable manifold in [42, Theorem 2.19]. However, as already mentioned above, in our situation, with the notation of [42], $P_2(t) = 0$ for every $t \in \mathbb{R}$ and then the name of center manifold can be adopted: see [42, Remark 2.20].

From Theorem 3.1, Theorem 1.1 holds if (and only if)

$$\tilde{S}(t)y_0 \to 0 \text{ as } t \to +\infty,\ \forall y_0 \in G \quad \text{such that} \quad \|y_0\|_{L^2(0,L)} \text{ is small enough,} \tag{3.93}$$

at least if ε is small enough. We prove (3.93) in the next section.

4. Dynamic on the center manifold

In this section, we prove (3.93), which concludes the proof of Theorem 1.1.

Proof. Let us fix $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is as in Theorem 3.1. Let $y_0 \in G$. Let, for $t \in [0, +\infty)$, $y(t)(x) := y(t,x) := (\tilde{S}(t)y_0)(x)$. We write

$$y(t,x) = p(t)\varphi(x) + y^\star(t,x), \tag{4.1}$$

where $\varphi(x)$ is defined in (3.20) and

$$y^\star(t,x) = g(p(t)\varphi(x)) \in M^\perp. \tag{4.2}$$

By (3.20) and (3.21), we have, at least if $\|y(t)\|_{L^2(0,L)}$ is small enough which will be always assumed in this proof,

$$\begin{aligned} \frac{dp(t)}{dt} &= \int_0^L y_t\,(t,x)\,\varphi(x)dx = \int_0^L (-y_x - yy_x - y_{xxx})\,\varphi(x)dx \\ &= \int_0^L y(t,x)\varphi_x(x)dx - \int_0^L y(t,x)y_x(t,x)\varphi(x)dx + \int_0^L y(t,x)\varphi_{xxx}(x)dx \\ &= \frac{1}{2}\int_0^L y^2(t,x)\varphi_x(x)dx. \end{aligned} \tag{4.3}$$

By (3.20) and (4.1), we have

$$\varphi_x + \varphi_{xxx} = 0,\ y_x + y_{xxx} = p(t)(\varphi_x(x) + \varphi_{xxx}(x)) + y^\star_x + y^\star_{xxx} = y^\star_x + y^\star_{xxx}.$$

Then by the definition for projection P, (3.21), and using integration by parts,

$$\begin{aligned} (I-P)(y_x + y_{xxx}) = (I-P)(y^\star_x + y^\star_{xxx}) &= y^\star_x + y^\star_{xxx} - \varphi(x)\int_0^L (y^\star_x + y^\star_{xxx})\varphi(x)\,dx \\ &= y^\star_x + y^\star_{xxx} + \varphi(x)\int_0^L y^\star(t,x)(\varphi_x + \varphi_{xxx})\,dx \\ &= y^\star_x + y^\star_{xxx}. \end{aligned}$$

Thus, we obtain the following system for $y^\star(t,x)$ by applying the operator $I-P$ to (1.1)

$$\begin{cases} y^\star_t + y^\star_x + (I-P)yy_x + y^\star_{xxx} = 0, \\ y^\star(t,0) = y^\star(t,L) = 0, \\ y^\star_x(t,L) = 0. \end{cases} \tag{4.4}$$

It follows from (4.1) that

$$\begin{aligned} yy_x &= \left(p(t)\varphi(x) + y^\star(t,x)\right)\left(p(t)\varphi_x(x) + y^\star_x(t,x)\right) \\ &= p^2(t)\varphi(x)\varphi_x(x) + p(t)y^\star(t,x)\varphi_x(x) + p(t)\varphi(x)y^\star_x(t,x) + y^\star(t,x)y^\star_x(t,x). \end{aligned}$$

Consequently, we have

$$\begin{aligned} (I-P)yy_x = {} & p^2(t)\varphi(x)\varphi_x(x) + p(t)y^\star(t,x)\varphi_x(x) + p(t)\varphi(x)y^\star_x(t,x) + y^\star(t,x)y^\star_x(t,x) \\ & - p^2(t)\varphi(x)\int_0^L \varphi^2(x)\varphi_x(x)dx - p(t)\varphi(x)\int_0^L y^\star(t,x)\varphi(x)\varphi_x(x)dx \\ & - p(t)\varphi(x)\int_0^L \varphi^2(x)y^\star_x(t,x)dx - \varphi(x)\int_0^L \varphi(x)y^\star(t,x)y^\star_x(t,x)dx. \end{aligned} \tag{4.5}$$

By using (3.20) and integrations by parts, we have

$$\int_0^L \varphi^2(x)\varphi_x(x)dx = 0, \tag{4.6}$$

$$\int_0^L y^\star(t,x)\varphi(x)\varphi_x(x)dx = -\frac{1}{2}\int_0^L \varphi^2(x)y^\star_x(t,x)dx, \tag{4.7}$$

$$\int_0^L \varphi(x)y^\star(t,x)y^\star_x(t,x)dx = -\frac{1}{2}\int_0^L \varphi_x(x)\left(y^\star(t,x)\right)^2dx. \tag{4.8}$$

It can be deduced from (4.5), (4.6), (4.7) and (4.8) that

$$\begin{aligned} (I-P)yy_x = {} & p^2(t)\varphi(x)\varphi_x(x) + p(t)y^\star(t,x)\varphi_x(x) + p(t)\varphi(x)y^\star_x(t,x) + y^\star(t,x)y^\star_x(t,x) \\ & - \frac{1}{2}p(t)\varphi(x)\int_0^L \varphi^2(x)y^\star_x(t,x)dx + \frac{1}{2}\varphi(x)\int_0^L \varphi_x(x)\left(y^\star(t,x)\right)^2dx. \end{aligned} \tag{4.9}$$

According to Theorem 3.1, using a power series expansion, (4.2) becomes

$$y^\star(t,x) = a(x)p(t) + b(x)p^2(t) + c(x)O(p^3(t)), \quad \text{as } |p(t)| \to 0. \tag{4.10}$$

It follows from (4.3) that

$$\frac{dp(t)}{dt} = O(p^2(t)). \tag{4.11}$$

Then, by using (4.1), (4.4), (4.9), (4.10), (4.11), and by comparing the coefficients of $p(t)$ and $p^2(t)$, we obtain

$$\begin{cases} a_x(x) + a_{xxx}(x) = 0, \\ a(0) = a(L) = 0, \\ a_x(L) = 0, \end{cases} \tag{4.12}$$

and

$$\begin{cases} b_x(x) + b_{xxx}(x) + \varphi(x)\varphi_x(x) = 0, \\ b(0) = b(L) = 0, \\ b_x(L) = 0. \end{cases} \tag{4.13}$$

The solution of (4.12) is

$$a(x) = C\varphi(x).$$

Note that $y^*(t,x) \in M^\perp$, it follows that

$$\int_0^L a(x)\varphi(x)dx = 0.$$

Thus,

$$C = 0,$$

i.e.,

$$a(x) = 0. \tag{4.14}$$

The solution of (4.13) is

$$b(x) = C_1 + C_2\cos x - \frac{1}{3}\sin x + \frac{1}{6k\pi}x\sin x + \frac{1}{36k\pi}\cos(2x),$$

where

$$C_1 + C_2 = -\frac{1}{36k\pi}. \tag{4.15}$$

Note that $y^\star(t,x) \in M^\perp$, then we have

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$$\int_0^L b(x)\varphi(x)dx = 0,$$

i.e.

$$C_1 \frac{2\pi}{\sqrt{3\pi}} + C_2 \frac{-\pi}{\sqrt{3\pi}} + \frac{1}{6k\pi} \times \frac{-3\pi}{2\sqrt{3\pi}} = 0,$$

which leads to

$$2\pi C_1 - \pi C_2 - \frac{1}{4k} = 0. \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$C_1 = \frac{2}{27k\pi}, \qquad C_2 = -\frac{11}{108k\pi}.$$

Therefore,

$$b(x) = \frac{2}{27k\pi} - \frac{11}{108k\pi}\cos x - \frac{1}{3}\sin x + \frac{1}{6k\pi} x \sin x + \frac{1}{36k\pi}\cos(2x). \tag{4.17}$$

Combining (4.1), (4.3), (4.10), (4.14) and (4.17), we obtain

$$\begin{aligned}
\frac{dp(t)}{dt} &= \frac{1}{2}\int_0^L \Big(p(t)\varphi(x) + b(x)p^2(t) + c(x)O(p^3(t))\Big)^2 \varphi_x(x)dx \\
&= p^3(t)\int_0^L b(x)\varphi(x)\varphi_x(x)dx + O(p^4(t)) \\
&= \frac{p^3(t)}{3\pi}\left(-\frac{1}{3}\pi + \frac{1}{6}\pi\right) + O(p^4(t)) \\
&= -\frac{p^3(t)}{18} + O(p^4(t)), \quad \text{as } |p(t)| \to 0.
\end{aligned}$$

This concludes the proof of (3.93) and the proof of Theorem 1.1. □

Remark 4.1. From our proof of Theorem 1.1, one has the following decay rate as t goes to infinity: for every $\kappa > 3$, there exists $\delta_2 > 0$ such that if $\|y_0\|_{L^2(0,L)} < \delta_2$, then

$$\|y(t,\cdot)\|_{L^2(0,L)} \le \frac{\kappa}{\sqrt{t}}, \quad \forall t > 0. \tag{4.18}$$

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