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Refuge versus dispersion in the logistic equation

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Abstract

In this paper we consider a logistic equation with nonlinear diffusion arising in population dynamics. In this model, there exists a refuge where the species grows following a Malthusian law and, in addition, there exists also a non-linear diffusion representing a repulsive dispersion of the species. We prove existence and uniqueness of positive solution and study the behavior of this solution with respect to the parameter λ, the growth rate of the species. Mainly, we use bifurcation techniques, the sub-supersolution method and a construction of appropriate large solutions.

MSC: 35J60; 35B32; 92B05

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1. Introduction

Reaction–diffusion models have been used to study the behavior of a population living in a habitat. Denoting by $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a bounded and regular domain, the habitat and by $u(x)$ the

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density of the individuals of the species at the location $x \in \Omega$, the classical model can be written as follows

$$\begin{cases} -\Delta(\varphi(x,u)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where φ and f are regular functions in $\Omega \times \mathbb{R}$. The term on the left side of (1) represents the diffusion of the species, that is, its spatial movement. In the model described by (1), the diffusion depends on the position $x \in \Omega$ and the population density u. The nonlinear diffusion function φ can have several different shapes, depending on the nature of behavioral interactions between organisms (see [1] and [2]). For instance, according to [2], if individuals move completely independently of each other, φ is characterized by a linear function of density u, that is, it increases with u at a constant rate. This case is called *simple* or *linear diffusion*. If interactions between moving individuals are repulsive, then the movement rate will increase with the population density, since at high densities organisms continuously come into contact and induce each other to disperse. In this case the diffusion rate φ_u will increase with the density. Similarly, if the movement is aggregative, the diffusivity will initially decline as u increases.

On the other hand, $f(x,u)$ is the *reaction term* and it represents the local rate of reproduction per individual, in other words, per capita population growth rate.

Specifically, in this paper we analyze the following elliptic equation

$$\begin{cases} -\Delta(u + a(x)u^r) = \lambda u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p, r > 1$, $b \in \mathcal{C}(\overline{\Omega}, \mathbb{R}_+)$ and $a \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}_+)$ are regular functions that can vanish on some subsets of Ω. In this specific case, $f(x,u) = \lambda u - b(x)u^p$, it is the well-known logistic reaction term and, in the context of population dynamics, λ is the intrinsic rate of natural increase of the species and

$$C(x) \equiv \frac{\lambda}{b(x)}$$

denotes the maximum density supported locally by available resources, that is, the carrying capacity. Thus, the region where $b(x) = 0$ can be understood as a refuge area for the species, i.e., the carrying capacity is infinite. For more details about problems with refuge areas see the pioneering papers [3] and [4], where the problem of the refuge was addressed by the first time, see also [1,5,6] and the recent book [7].

In the nonlinear diffusion term $\varphi(x,u) = u + a(x)u^r$, the function a denotes the type of diffusion movement of the species: linear when $a = 0$ and repulsive when $a > 0$. Thus, the set $\{x \in \Omega; a(x) > 0\}$ is a region where the species must avoid agglomeration. In our discussion, we will consider different configurations for the refuge area and the zone with repulsive movement to analyze how these sets affect the persistence of the species.

Let us recall the main known results about (2). For this, given a regular subdomain $D \subset \Omega$, we denote by $\lambda_1[-\Delta; D]$ the principal eigenvalue of the Laplacian in D under homogeneous Dirichlet boundary conditions, $\lambda_1[-\Delta; D] = \infty$ when $D = \emptyset$ and, by simplicity, $\lambda_1 = \lambda_1[-\Delta; \Omega]$.

When $a \equiv 0$ in Ω, denoting $\Omega_{b+} := \{x \in \Omega;\ b(x) > 0\}$ and $\Omega_{b0} := \Omega \setminus \overline{\Omega}_{b+}$, (2) becomes the classical logistic equation with linear diffusion and refuge. For instance, suppose that Ω_{b0} is connected and regular, it is well-known that there exists a unique positive solution u_λ if, and only if, $\lambda \in (\lambda_1, \lambda_1[-\Delta; \Omega_{b0}])$. Moreover, there exists a detailed study of the profile of this solution when $\lambda \to \lambda_1[-\Delta; \Omega_{b0}]$, see [8], [9] and [7]. In short, we have

$$\lim_{\lambda \uparrow \lambda_1[-\Delta;\Omega_{b0}]} u_\lambda(x) \begin{cases} = +\infty & \text{if } x \in \overline{\Omega}_{b0}, \\ < +\infty & \text{if } x \notin \overline{\Omega}_{b0}. \end{cases}$$

Actually, in our knowledge, [10] is the most pioneering paper analyzing the logistic equation with component refuge areas, as well as the construction of large solution in this case. We also refer to [7] which presents a most complete collection of the available results of (2) with $a \equiv 0$, including a detailed analysis of the global dynamics of the parabolic counterpart of the model.

When $a \not\equiv 0$ only some partial results are available. The case $a = \text{constant} > 0$, $b \equiv 0$ (or $b \equiv 1$) and $p = r = 2$ was studied in [11] and if $a(x), b(x) > 0$ in $\overline{\Omega}$ with $p \geq r$ is included in the hypothesis of Theorem 2.1 of [12]. Both papers show that there exists a unique positive solution of (2) if, and only if, $\lambda > \lambda_1$. In [13], the authors analyze an equation related to (2), including a combination of linear and non-linear diffusion.

Now, we state the main assumptions on the functions a and b:

(H) The open sets

$$\Omega_{b+} := \{x \in \Omega; b(x) > 0\}, \quad \Omega_{b0} := \Omega \setminus \overline{\Omega}_{b+}$$

are of class $\mathcal{C}^2$ and Ω_{b0} consists of finitely many connected components B_i, $1 \leq i \leq m$ such that

$$\overline{B}_i \subset \Omega, \ 1 \leq i \leq m, \qquad \overline{B}_j \cap \overline{B}_i = \emptyset \quad \text{if } j \neq i.$$

Similarly, we write

$$\Omega_{a+} := \{x \in \Omega; a(x) > 0\}, \quad \text{and} \quad \Omega_{a0} := \Omega \setminus \overline{\Omega}_{a+}.$$

For each $i = 1, ..., m$, if

$$\Omega_{0,i} := \Omega_{a0} \cap B_i \neq \emptyset,$$

we denote by $\lambda_{0,i}$ the principal eigenvalue of the following problem

$$\begin{cases} -\Delta u = \lambda \mathcal{X}_{\Omega_{0,i}} u & \text{in } B_i, \\ u = 0 & \text{on } \partial B_i. \end{cases}$$

We also adopt $\lambda_{0,i} = \infty$ if $\Omega_{a0} \cap B_i = \emptyset$.

Without loss of generality, we will assume that the labeling of these components has been already carried out so that either

$$\lambda_{0,1} = ... = \lambda_{0,m_1} < \lambda_{0,m_1+1} \leq ... \leq \lambda_{0,m} \tag{3}$$

for some $m_1 \in \{1, ..., m-1\}$ or

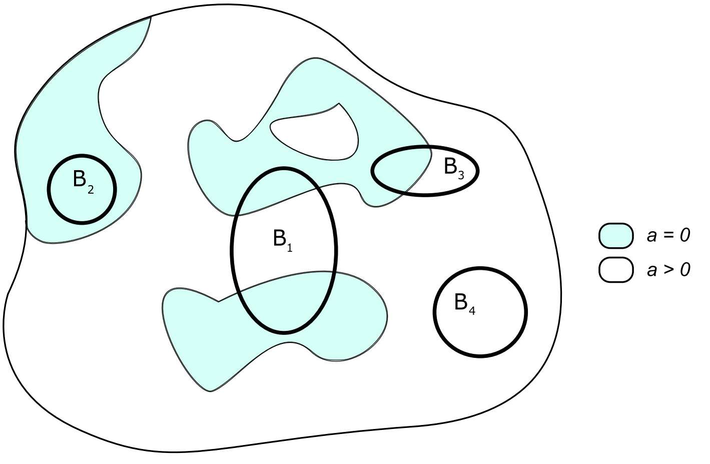

Fig. 1. An admissible configuration for Ω_{a0} and Ω_{b0} with $m = 4$ and $m_1 = 1$. The dark region represents the set Ω_{a0}.

$$\lambda_{0,1} = \ldots = \lambda_{0,m}.$$

See Fig. 1 for a possible disposition of these sets.

In the same way, we denote by λ_{a0} the principal eigenvalue of the following problem

$$\begin{cases} -\Delta u = \lambda \mathcal{X}_{\Omega_{a0}} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our first result deals with the case $b \equiv 0$ in Ω.

Theorem 1.1. *Suppose $b \equiv 0$ in Ω. Then, (2) possesses a positive solution if, and only if, $\lambda \in (\lambda_1, \lambda_{a0})$. Moreover, it is unique if it exists, and it will be denoted by u_λ. In addition, the map*

$$\lambda \in (\lambda_1, \lambda_{a0}) \mapsto u_\lambda \in \mathcal{C}_0^2(\overline{\Omega})$$

is increasing and of class $\mathcal{C}^1$. Furthermore,

$$\lim_{\lambda \downarrow \lambda_1} \|u_\lambda\|_0 = 0 \tag{4}$$

and

$$\lim_{\lambda \uparrow \lambda_{a0}} u_\lambda(x) = \infty \quad \text{for each } x \in \Omega. \tag{5}$$

In the context of population dynamics, this result shows that the presence of such nonlinear diffusion produces the following effect: when $\Omega_{a0} \neq \emptyset$ then the species persists if the growth of the species increases, λ, satisfies $\lambda > \lambda_1$. On the other hand, when there exists an area whose movement is linear (i.e., $\Omega_{a0} \neq \emptyset$) the species persists only for rates $\lambda \in (\lambda_1, \lambda_{a0})$. Moreover, in this model, the solution blows-up in the whole Ω when $\lambda \to \lambda_{a0}$.

Now, we consider the general case. We have

Theorem 1.2. *Suppose $b \not\equiv 0$ in Ω.*

a) *If $\overline{\Omega}_{a0} \cap \overline{\Omega}_{b0} = \emptyset$ then (2) possesses a positive solution if, and only if, $\lambda > \lambda_1$. Moreover, for $p \geq r$, it is unique if it exists, and it will be denoted by u_λ. In addition, the map*

$$\lambda \in (\lambda_1, \infty) \mapsto u_\lambda \in \mathcal{C}_0^2(\overline{\Omega})$$

is increasing and of class $\mathcal{C}^1$. Furthermore

$$\lim_{\lambda \downarrow \lambda_1} \|u_\lambda\|_0 = 0 \tag{6}$$

and

$$\lim_{\lambda \uparrow \infty} u_\lambda = \infty \quad \text{uniformly in } \Omega_{b0}. \tag{7}$$

b) *If $\Omega_0 := \Omega_{a0} \cap \Omega_{b0} \neq \emptyset$ and $p > r$, then (2) possesses a positive solution if, and only if, $\lambda \in (\lambda_1, \lambda_{0,1})$. Moreover, it is unique if it exists, and it will be denoted by u_λ. In addition, the map*

$$\lambda \in (\lambda_1, \lambda_{0,1}) \mapsto u_\lambda \in \mathcal{C}_0^2(\overline{\Omega})$$

is increasing and of class $\mathcal{C}^1$. Furthermore,

$$\lim_{\lambda \downarrow \lambda_1} \|u_\lambda\|_0 = 0, \tag{8}$$

and

$$\lim_{\lambda \uparrow \lambda_{0,1}} u_\lambda(x) \begin{cases} = \infty & \text{if } x \in \bigcup_{i=1}^{m_1} B_i, \\ < \infty & \text{if } x \in \Omega \setminus \bigcup_{i=1}^{m_1} \overline{B}_i. \end{cases} \tag{9}$$

The bifurcation diagrams associated to these cases are represented in Fig. 2.

In order to clarify the biological interpretation of the main result of the paper, let us consider the case where the refuge Ω_{b0} consists of two components B_1 and B_2 (i.e., $\Omega_{b0} = B_1 \cup B_2$). Then:

a) If the zone where the species avoids agglomeration (Ω_{a+}) contains the refuge (Ω_{b0}), that is $\Omega_{b0} \subset \Omega_{a+}$, then the species remains controlled for all $\lambda > \lambda_1$.
b) Assuming that a portion of the refuge, say Ω_0, the species diffuses linearly, that is, $\Omega_{b0} \cap \Omega_{a0} \neq \emptyset$, then the species blows up for values of $\lambda \geq \min\{\lambda_{0,1}, \lambda_{0,2}\}$. Moreover,
   (i) If $\lambda_{0,1} < \lambda_{0,2}$ (resp. $\lambda_{0,1} > \lambda_{0,2}$), then the species blows-up not only in $\Omega_0 \cap B_1$ (resp. $\Omega_0 \cap B_2$), but in all B_1 (resp. B_2) and remains bounded in $\Omega \setminus \overline{B}_1$ (resp. $\Omega \setminus \overline{B}_2$), including the other part of the refuge B_1 (resp. B_2).
   (ii) If $\lambda_{0,1} = \lambda_{0,2}$, then species blows up in the whole refuge Ω_{b0}, not only in Ω_0.

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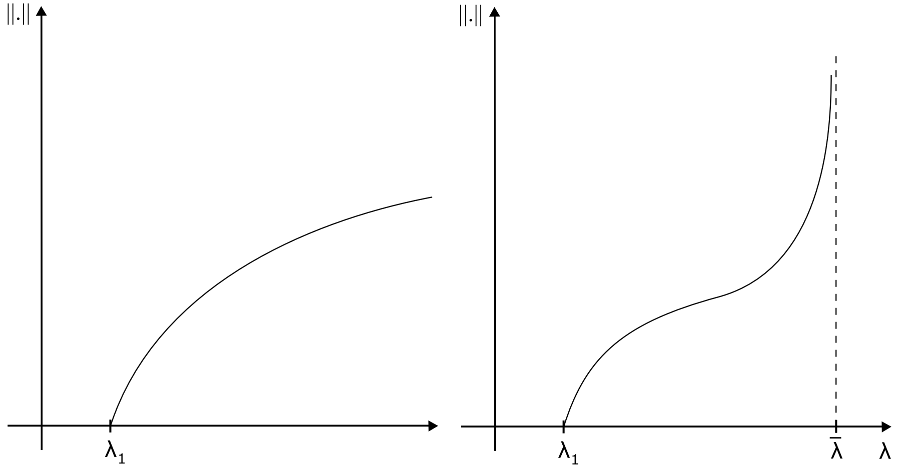

Fig. 2. Possible bifurcation diagrams. On the left hand side, we represent the cases $b \equiv 0$ and $\Omega_{a0} = \emptyset$ or $b \not\equiv 0$ and $\overline{\Omega}_{a0} \cap \overline{\Omega}_{b0} = \emptyset$. On the right hand side, we draw the cases $b \equiv 0$ and $\Omega_{a0} \neq \emptyset$ where $\overline{\lambda} = \lambda_{a0}$, or $b \not\equiv 0$ and $\overline{\Omega}_{a0} \cap \overline{\Omega}_{b0} \neq \emptyset$ where $\overline{\lambda} = \lambda_{0,1}$.

It is worth mentioning that in both cases λ_1 is the critical size of the rate of natural increase such that the habitat Ω can maintain the species with this dispersion movement. Thus, in all the cases above, u is driven to extinction if $\lambda \leq \lambda_1$.

The outline of this paper is as follows: first, in Section 2 we introduce an appropriate change of variables and we collect some results which will be useful throughout the paper. In Section 3 we prove Theorems 1.1 and 1.2 *a*) and *b*) partially. Section 4 is devoted to obtain some results about large solutions and in the last section we apply these results to complete the proof of Theorem 1.2, showing (9) when $x \in \Omega \setminus \cup_{i=1}^{m_1} \overline{B}_i$.

2. Preliminary results

In this section we present some preliminary results that will be used throughout the rest of the paper.

A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is said to be a (classical) solution of (2) if it satisfies (2) pointwise in $\overline{\Omega}$.

To study (2), we will introduce the following change of variable:

$$w = I(x,u) = u + a(x)u^r \Leftrightarrow q(x,w) = u. \tag{10}$$

Then, (2) is equivalent to

$$\begin{cases} -\Delta w = \lambda q(x,w) - b(x)q(x,w)^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Hence, $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution of (2) if, and only if, w is a solution of (11).

Since we are interested in positive solutions of (2), we can define

$$q(x,s) = 0, \quad \forall x \in \Omega,\ s \leq 0.$$

As a consequence, any solution of (11) is non-negative. In fact, by the Strong Maximum Principle any non-trivial solution w of (11) is strictly positive, that is, $w(x) > 0$ for all $x \in \Omega$. Also note that, by the Strong Maximum Principle, $\lambda > 0$ is a necessary condition for the existence of positive solutions.

The first result of this section shows some useful properties of $q(x, w)$.

Lemma 2.1.

a) *For each $x \in \Omega$, the map*

$$s \mapsto \frac{q(x,s)}{s} \quad s \geq 0$$

is non-increasing and satisfies

$$\lim_{s\to 0} \frac{q(x,s)}{s} = 1 \quad \textit{uniformly in } \Omega, \tag{12}$$

$$\lim_{s\to\infty} \frac{q(x,s)}{s} = \mathcal{X}_{\Omega_{a0}}(x) = \begin{cases} 0 & \textit{if } a(x) > 0, \\ 1 & \textit{if } a(x) = 0, \end{cases} \tag{13}$$

and

$$\mathcal{X}_{\Omega_{a0}}(x)s \leq q(x,s) \leq s \quad \forall x \in \Omega,\ s \geq 0. \tag{14}$$

b) *For each $x \in \Omega$, the map*

$$s \mapsto \frac{q(x,s)^p}{s} \quad s \geq 0$$

satisfies

$$\lim_{s\to 0} \frac{q(x,s)^p}{s} = 0, \tag{15}$$

$$\lim_{s\to+\infty} \frac{q(x,s)^p}{s} = \begin{cases} \infty & \textit{if } r < p \textit{ or } a(x) = 0, \\ \frac{1}{a(x)} & \textit{if } r = p \textit{ and } a(x) > 0, \\ 0 & \textit{if } r > p \textit{ and } a(x) > 0, \end{cases} \tag{16}$$

and it is an increasing map if $p \geq r$.

Proof. a) Since $q(x, \cdot)$ is the inverse function of $I(x, \cdot)$, we get that it is increasing and it verifies

$$s = q(x,s) + a(x)q(x,s)^r \quad \forall x \in \Omega. \tag{17}$$

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Thus,

$$\frac{q(x,s)}{s} = \frac{1}{1+a(x)q(x,s)^{r-1}} \leq 1 \quad \forall x \in \Omega.$$

Therefore, $s \mapsto q(x,s)/s$ is decreasing if $a(x) > 0$ and $q(x,s)/s = 1$ if $a(x) = 0$. Furthermore, since $q(x,0) = 0$ and $\lim_{s\to\infty} q(x,s) = \infty$, it follows that

$$\lim_{s\to 0} \frac{q(x,s)}{s} = \lim_{s\to 0} \frac{1}{1+a(x)q(x,s)^{r-1}} = 1,$$
$$\lim_{s\to \infty} \frac{q(x,s)}{s} = \lim_{s\to \infty} \frac{1}{1+a(x)q(x,s)^{r-1}} = \mathcal{X}_{\Omega_{a0}}(x).$$

b) Using (12), we have that

$$\lim_{s\to 0} \frac{q(x,s)^p}{s} = \lim_{s\to 0} \frac{q(x,s)}{s} q(x,s)^{p-1} = 0.$$

In view of (13) and (17), we obtain

$$\lim_{s\to \infty} \frac{q(x,s)^p}{s} = \lim_{s\to \infty} \frac{1}{q(x,s)^{1-p} + a(x)q(x,s)^{r-p}} = \begin{cases} \infty, & \text{if } r < p \text{ or } a(x) = 0, \\ \frac{1}{a(x)}, & \text{if } r = p \text{ and } a(x) > 0, \\ 0, & \text{if } r > p \text{ and } a(x) > 0, \end{cases}$$

and $q(x,s)^p/s$ is increasing if $p \geq r$. □

Throughout this paper, for any $V \in L^\infty(\Omega)$ called *potential*, we shall denote by $\lambda_1[-\Delta + V; \Omega]$ the principal eigenvalue of $-\Delta + V$ in Ω under homogeneous Dirichlet boundary conditions. By simplicity, we also use the convention $\lambda_1 := \lambda_1[-\Delta; \Omega]$.

Given D an open set and $\mathcal{O}$ a regular domain such that $D \subset \mathcal{O} \subset \Omega$, the following function will play a crucial role in our exposition

$$\mu(\lambda) = \mu_{D,\mathcal{O}}(\lambda) := \lambda_1[-\Delta - \lambda\mathcal{X}_D; \mathcal{O}], \quad \lambda \in \mathbb{R}. \tag{18}$$

Its useful properties for this work are summarized in the following result, whose proof is by-product of the general theory of Chapter 9 in [14].

Lemma 2.2. *The function μ defined in (18) possesses a unique zero, say $\lambda_{D,\mathcal{O}}$. Moreover, $\mu(\lambda) > 0$ if, and only if, $\lambda < \lambda_{D,\mathcal{O}}$. Furthermore, it satisfies*

$$\lambda_1 < \lambda_1[-\Delta; \mathcal{O}] < \lambda_{D,\mathcal{O}}, \tag{19}$$

and $\lambda_{D,\mathcal{O}}$ is the principal eigenvalue of the following problem

$$\begin{cases} -\Delta u = \lambda \mathcal{X}_D u & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{20}$$

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Specifically, we are interested in two particular eigenvalues: when $D = \Omega_{a0}$ and $\mathcal{O} = \Omega$, in this case we denote by $\lambda_{a0} := \lambda_{\Omega_{a0},\Omega}$; and when $D = \Omega_{a0} \cap B_i$ and $\mathcal{O} = B_i$, where we denote by $\lambda_{0,i} := \lambda_{\Omega_{0,i},B_i}$, for each $i \in \{1, ..., m\}$. We emphasize that

$$\lambda_1 < \lambda_{a0} \quad \text{and} \quad \lambda_1 < \lambda_{0,i}, \qquad i = 1, ..., m. \tag{21}$$

With these considerations, we can show the following non-existence result of (11).

Lemma 2.3.

a) *Suppose $b \equiv 0$ in Ω. If there exists a positive solution of (11), then $\lambda \in (\lambda_1, \lambda_{a0})$.*
b) *Suppose $b \not\equiv 0$. If there exists a positive solution of (11), then $\lambda \in (\lambda_1, \lambda_{0,1})$.*

Proof. We will prove *b*), the proof of *a*) is analogous.

b) Suppose that $w > 0$ is a positive solution of (11). By the properties of the map $s \mapsto q(x, s)$ (see Lemma 2.1), we deduce that

$$-\lambda < -\lambda \frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w}.$$

Thus, by the monotonicity of the principal eigenvalue with respect to the potential, we have

$$0 = \lambda_1 \left[-\Delta - \lambda \frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w}; \Omega \right] > \lambda_1[-\Delta - \lambda; \Omega] = \lambda_1 - \lambda.$$

On the other hand, using the monotonicity of the principal eigenvalue with respect to the domain, we get

$$\begin{aligned} 0 = \lambda_1 \left[-\Delta - \lambda \frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w}; \Omega \right] &< \lambda_1 \left[-\Delta - \lambda \frac{q(x,w)}{w}; B_1 \right] \\ &< \lambda_1[-\Delta - \lambda \mathcal{X}_{\Omega_{0,1}}, B_1] = \mu_{\Omega_{0,1},B_1}(\lambda), \end{aligned}$$

where (14) was used to obtain the last inequality. By the properties of the map $\mu_{\Omega_{0,1},B_1}$ we know that

$$0 < \mu_{\Omega_{0,1},B_1}(\lambda)$$

if, and only if,

$$\lambda < \lambda_{0,1}.$$

This completes the proof. □

Now, we will show that λ_1 is the only bifurcation point of positive solutions of (11) from the trivial solution. For this, let e_1 denote the unique (positive) solution of

$$\begin{cases} -\Delta e_1 = 1 & \text{in } \Omega, \\ e_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and let E be the Banach space consisting of all $u \in C(\overline{\Omega})$ for which there exists $\gamma = \gamma(u) > 0$ such that

$$-\gamma e_1 < u < \gamma e_1$$

endowed with the norm

$$\|u\|_E := \inf\{\gamma > 0; \; -\gamma e_1 < u < \gamma e_1\}$$

and the natural point-wise order. Then E is an ordered Banach space whose positive cone, say P, is normal and has nonempty interior. Thus, consider the map $\mathfrak{F} : \mathbb{R} \times E \longrightarrow E$ defined by

$$\mathfrak{F}(\lambda, w) = w - (-\Delta)^{-1}(\lambda q(x, w) - b(x) q(x, w)^p),$$

where $(-\Delta)^{-1}$ is the inverse of Laplacian operator under homogeneous Dirichlet boundary conditions. The operator $\mathfrak{F}$ is of class $\mathcal{C}^1$ and (11) can be written in the form

$$\mathfrak{F}(\lambda, w) = 0. \tag{22}$$

Moreover, by the Strong Maximum Principle any positive solution of (22) is strongly positive.

Proposition 2.1. *λ_1 is a bifurcation point of (11) from the trivial solution to a continuum of positive solutions of (11). Moreover, it is the unique bifurcation point to positive solutions from $(\lambda, 0)$. Let $\Sigma_0 \subset \mathcal{S}$ denote the component of positive solutions of (11) emanating from $(\lambda, 0)$. Then, Σ_0 is unbounded in $\mathbb{R} \times E$.*

Proof. Observe that (22) can be written as

$$\mathcal{L}(\lambda) w + \mathcal{N}(\lambda, w) = 0$$

where $\mathcal{L}(\lambda) = I_E - \lambda(-\Delta)^{-1}$ and $\mathcal{N}(\lambda, w) = -(-\Delta)^{-1}(\lambda(q(x, w) - w) - b(x) q(x, w)^p)$. Moreover, thanks to (12) and (15), we have

$$\lim_{s \to 0} \frac{\lambda(q(x, s) - s) - b(x) q(x, s)^p}{s} = 0,$$

and then $\mathcal{N}(\lambda, w) = o(\|w\|_E)$ as $\|w\|_E \to 0$.

Therefore, we can apply the unilateral bifurcation theorem for positive operators of [15] (see Theorem 6.5.5) to conclude the result. $\square$

The next result will be used extensively throughout this work.

Lemma 2.4. *Consider the problem*

$$\begin{cases} -\Delta u = f(\lambda, x, u) & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where $u_0 \geq 0$ is a function in $\mathcal{C}(\partial\Omega)$. Assume that $f : \mathbb{R} \times \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that

$$s \mapsto \frac{f(\lambda, x, s)}{s}$$

is non-increasing for all $x \in \Omega$ and there exists $x_1 \in \Omega$ such that $f(\lambda, x_1, s)/s$ is decreasing. Then:

a) *There exists at most a positive solution of (23).*
b) *Let $\underline{u}, \overline{u} \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ with $\underline{u} \geq 0$ and $\overline{u} > 0$ a sub and supersolution of (23), respectively. If there exists $\varepsilon > 0$ such that $\varepsilon \underline{u} \leq \overline{u}$, then*

$$\underline{u} \leq \overline{u}.$$

Proof. a) Since $f(\lambda, x, s)/s$ is non-increasing in s for all $x \in \Omega$, then if $w \neq v$ are two positive solutions of (23), then $w = cv$ for a some positive constant c (see Remark 1 in [16]). Hence, in x_1 we have

$$\left(\frac{f(\lambda, x_1, v)}{v} - \frac{f(\lambda, x_1, cv)}{cv} \right) v = 0. \tag{24}$$

Since $s \mapsto f(\lambda, x_1, s)/s$ is decreasing, if $c > 1$ we would have

$$\left(\frac{f(\lambda, x_1, v)}{v} - \frac{f(\lambda, x_1, cv)}{cv} \right) v < 0,$$

a contradiction with (24). Analogously, $c < 1$ cannot occur and, hence, $c = 1$ and $w = v$.

b) Let

$$\Lambda = \{t \in [0, 1];\ t\underline{u} \leq \overline{u}\}.$$

By hypothesis, $\varepsilon \in \Lambda$. We will prove that $1 \in \Lambda$. Indeed, otherwise we would have

$$0 < t_0 := \sup \Lambda < 1.$$

Choosing $K > 0$ large enough such that $f(\lambda, x, s) + Ks$ is increasing on $[0, \max \overline{u}]$, we obtain for $\underline{u} \neq 0$

$$
\begin{aligned}
-\Delta(\overline{u}-t_0\underline{u})+K(\overline{u}-t_0\underline{u}) &\geq f(\lambda,x,\overline{u})-t_0 f(\lambda,x,\underline{u})+K(\overline{u}-t_0\underline{u})\\
&\geq f(\lambda,x,\overline{u})+K\overline{u}-t_0 f(\lambda,x,\underline{u})-Kt_0\underline{u}\\
&\geq f(\lambda,x,t_0\underline{u})+Kt_0\underline{u}-t_0 f(\lambda,x,\underline{u})-Kt_0\underline{u}\\
&\geq t_0\underline{u}\left[\frac{f(\lambda,x,t_0\underline{u})}{t_0\underline{u}}-\frac{f(\lambda,x,\underline{u})}{\underline{u}}\right]>0,
\end{aligned}
$$

where in the last inequality we have used that $s\mapsto f(\lambda,x,s)/s$ is non-increasing and $t_0<1$. In the same way, if $\underline{u}=0$ we have $-\Delta(\overline{u}-t_0\underline{u})+K(\overline{u}-t_0\underline{u})\geq 0$. Thus, $w:=\overline{u}-t_0\underline{u}$ verifies

$$
\begin{cases}
-\Delta w+Kw>0 & \text{in } \Omega,\\
w\geq 0 & \text{on } \partial\Omega.
\end{cases}
$$

Therefore, by the Strong Maximum Principle, we obtain for $\delta>0$ sufficiently small that $t_0+\delta\in\Lambda$, which is a contradiction. □

As a consequence, we obtain the following result.

Proposition 2.2. *If either $b\equiv 0$, or $b\not\equiv 0$ and $p\geq r$, then (11) admits at most a positive solution.*

Proof. If $b\equiv 0$ in Ω, then $s\mapsto \lambda q(x,s)/s$ is non-increasing and decreasing for $x\in\Omega_{a+}$. If $b\not\equiv 0$ and $p\geq r$, then $s\mapsto(\lambda q(x,s)-b(x)q(x,s)^p)/s$ is non-increasing and decreasing for $x\in\Omega\setminus\Omega_{0,i}$. In both cases, by Lemma 2.4 *a*), we obtain the result. □

3. Existence of positive solutions

The goal of this section is to prove Theorem 1.1 and paragraphs *a*) and *b*) of Theorem 1.2. Some arguments used in these proofs are inspired in Theorem 4.1 of [8], see also [9] and [7].

Proof of Theorem 1.1. By Proposition 2.1, λ_1 is a bifurcation point of (11) from the trivial solution and it is the only one for positive solutions. Moreover, there exists an unbounded continuum Σ_0 of positive solutions emanating from $(\lambda_1,0)$. In order to prove the existence of a positive solution for every $\lambda\in(\lambda_1,\lambda_{a0})$, we consider two cases: $\Omega_{a0}=\emptyset$ and $\Omega_{a0}\neq\emptyset$.

a) Case $\Omega_{a0}=\emptyset$.

It suffices to show that, for every $\lambda_*>\lambda_1$, there exists a constant $C=C(\lambda_*)>0$ such that

$$
\|w\|_0\leq C \quad \forall(\lambda,w)\in\Sigma_0,\ \lambda\leq\lambda_*. \tag{25}
$$

Indeed, by the global nature of Σ_0, this estimate implies that $\mathrm{Proj}_{\mathbb{R}}\Sigma_0=(\lambda_1,\infty)$, where $\mathrm{Proj}_{\mathbb{R}}\Sigma_0$ is the projection of Σ_0 into $\mathbb{R}$. To prove (25), we will build a family $\overline{W}(\lambda)$ of supersolutions of (11) and apply Theorem 2.2 of [17]. Thus, we consider the continuous map $\overline{W}:[\lambda_1,\lambda_*]\to\mathcal{C}^2_0(\overline{\Omega})$ defined by $\overline{W}(\lambda)=K(\lambda)e$, where $K(\lambda)$ is a positive constant to be chosen later and e is the unique (positive) solution of

$$
\begin{cases}
-\Delta e=1 & \text{in } \widehat{\Omega},\\
e=0 & \text{on } \partial\widehat{\Omega},
\end{cases} \tag{26}
$$

for some regular domain $\Omega \subset\subset \widehat{\Omega}$. Then, $\overline{W}(\lambda) = K(\lambda)e$ is a supersolution of (11) if

$$1 \geq \lambda \frac{q(x, Ke)}{Ke} e \quad \text{in } \Omega.$$

Since $\Omega_{a0} = \emptyset$, $a(x) > 0$ a.e. in Ω, (13) gives

$$\lim_{s \to \infty} \frac{q(x, s)}{s} = 0 \quad \text{a.e. in } \Omega.$$

Consequently, for $K = K(\lambda) > 0$ sufficiently large, $\overline{W}(\lambda) = K(\lambda)e$ is a supersolution (but not a solution) of (11), for every $\lambda \in [\lambda_1, \lambda_*]$ and $W(\lambda_1) = K(\lambda_1)e > 0$ in Ω. Thus, by Theorem 2.2 of [17], it follows (25).

The convergence (4) is an immediate consequence of Proposition 2.1.

To prove the monotonicity of map $\lambda \mapsto u_\lambda$ we argue as follows. Given $\lambda, \mu > \lambda_1$ with $\lambda < \mu$, we have

$$-\Delta w_\mu = \mu q(x, w_\mu) > \lambda q(x, w_\mu),$$

implying

$$w_\lambda < w_\mu \Leftrightarrow u_\lambda < u_\mu \quad \text{in } \Omega.$$

Finally, in order to prove (5) let us show that $\varepsilon(\lambda)\varphi_1$ is a subsolution of (11), for $\varepsilon(\lambda) > 0$ to be determined and φ_1 is the positive eigenfunction associated to λ_1 such that $\|\varphi_1\|_0 = 1$. Indeed, it suffices to verify that

$$1 + a(x)q(x, \varepsilon(\lambda)\varphi_1)^{r-1} = \frac{\varepsilon(\lambda)\varphi_1}{q(x, \varepsilon(\lambda)\varphi_1)} \leq \frac{\lambda}{\lambda_1} \quad x \in \Omega. \tag{27}$$

Choosing

$$\varepsilon(\lambda) := \left(\frac{\lambda - \lambda_1}{\lambda_1 \max_{\overline{\Omega}} a(x)} \right)^{1/(r-1)},$$

it is easy to see that (27) is satisfied, once that $q(x, s) \leq s$, for all $x \in \Omega$. Moreover,

$$\varepsilon(\lambda) \to \infty \quad \text{as } \lambda \to \infty. \tag{28}$$

On the other hand, we have already shown that $K(\lambda)e$ is a supersolution of (11), where e stands for the unique (positive) solution of (26) and $K(\lambda) > 0$ is a constant large. Hence, for $K(\lambda) > 0$ such that

$$\varepsilon(\lambda)\varphi_1 \leq K(\lambda)e,$$

by the sub-supersolution method and the uniqueness of positive solution of (11), we infer

$$\varepsilon(\lambda)\varphi_1 \leq w_\lambda.$$

Using (28), we conclude that

$$\lim_{\lambda\uparrow\infty} w_\lambda(x) = \infty \quad \text{for each } \ x \in \Omega.$$

Therefore, once that $w_\lambda = u_\lambda + a(x)u_\lambda^r$, we obtain (5).

b) Case $\Omega_{a0} \neq \emptyset$.

Observe that in this case (22) can be written as $\mathcal{L}(\lambda)w + \mathcal{N}(\lambda, w) = 0$ where

$$\mathcal{L}(\lambda)w = w - \lambda(-\Delta)^{-1}(\mathcal{X}_{\Omega_{a0}}(x)w) \quad \text{and} \quad \mathcal{N}(\lambda, w) = -(-\Delta)^{-1}(\lambda(q(x, w) - \mathcal{X}_{\Omega_{a0}}(x)w)).$$

In view of (13), we can prove that

$$\lim_{s\to\infty} \frac{q(x, s) - \mathcal{X}_{\Omega_{a0}}(x)s}{s} = 0,$$

and then, $\mathcal{N}(\lambda, w) = o(\|w\|_E)$ as $\|w\|_E \to \infty$.

By the classical change of variable, see page 465 in [18], we can apply again Theorem 6.5.5 of [15] to conclude that λ_{a0} is a bifurcation point from infinity of positive solutions, and it is the unique for positive solutions. Moreover, there exists an unbounded continuum Σ_∞ of positive solutions emanating from infinity at λ_{a0}. Since these bifurcation points are unique, we get

$$\Sigma_\infty = \Sigma_0.$$

As a consequence, by the global nature of these continuum, we obtain that there exists a positive solution for all $\lambda \in (\lambda_1, \lambda_{a0})$.

In order to prove (5) we follow the arguments of [8]. Note that the map $\lambda \in (\lambda_1, \lambda_{a0}) \mapsto w_\lambda$, where w_λ is the unique solution of (2), is differentiable. Indeed, define $\mathcal{H} : \mathbb{R} \times \mathcal{C}_0^2(\overline{\Omega}) \longrightarrow \mathcal{C}_0(\overline{\Omega})$ given by

$$\mathcal{H}(\lambda, w) = -\Delta w - \lambda q(x, w).$$

For each $(\lambda, w) \in \Sigma_0$ we have $\mathcal{H}(\lambda, w) = 0$, and

$$\mathcal{H}_w(\lambda, w)\xi = \left[-\Delta - \lambda q_w(x, w)\right]\xi.$$

Since $s \mapsto q(x, s)/s$ is non-increasing and decreasing in a subdomain, we deduce that

$$\frac{q(x, w)}{w} > q_w(x, w) \quad \text{in } \Omega.$$

Thus, combining the above inequality, that (λ, w) is a solution of (11) and the monotonicity of principal eigenvalue with respect to the potential, we infer that

$$0 = \lambda_1\left[-\Delta - \lambda\frac{q(x, w)}{w}; \Omega\right] < \lambda_1[-\Delta - \lambda q_w(x, w); \Omega]. \tag{29}$$

Subsequently, $\mathcal{H}_w(\lambda, w)$ is no singular for all $(\lambda, w) \in \Sigma_0$. Hence, by the Implicit Function Theorem, we conclude the differentiability of the map $\lambda \mapsto w_\lambda$.

Therefore, we can differentiate (11) with respect to λ and get

$$(-\Delta - \lambda q_w(x,w))w'_\lambda = q(x,w) > 0 \quad \text{in } \Omega, \tag{30}$$

where $w'_\lambda = dw_\lambda/d\lambda$. In view of (29), the operator $-\Delta - \lambda q_w(x,w)$ satisfies the Strong Maximum Principle and we deduce from (30) that $w'_\lambda > 0$ in Ω.

On the other hand, since $q_w(x,w) = (1 + a(x)rq(x,w)^{r-1})^{-1} > \mathcal{X}_{\Omega_{a0}}$ in Ω, (30) gives

$$(-\Delta - \lambda \mathcal{X}_{\Omega_{a0}})w'_\lambda > q(x,w) \quad \text{in } \Omega. \tag{31}$$

Now, we fix $\overline{\lambda} \in (\lambda_1, \lambda_{a0})$ and let φ_1 be the positive eigenfunction associated to λ_{a0} with $\|\varphi_1\|_0 = 1$. For sufficiently small $c > 0$ we have

$$q(x, w_{\overline{\lambda}}) > c\mathcal{X}_{\Omega_{a0}}\varphi_1 \quad \text{in } \Omega.$$

Since $\lambda \mapsto w_\lambda$ is increasing, we get that

$$q(x, w_{\overline{\lambda}}) > c\mathcal{X}_{\Omega_{a0}}\varphi_1 \quad \text{in } \Omega, \quad \forall \lambda \in [\overline{\lambda}, \lambda_{a0}).$$

Let v_λ be the unique solution of the linear problem

$$\begin{cases} (-\Delta - \lambda \mathcal{X}_{\Omega_{a0}})u = c\mathcal{X}_{\Omega_{a0}}\varphi_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists because $\lambda < \lambda_{a0}$. In view of (31), the Maximum Principle implies that

$$w'_\lambda > v_\lambda \quad \text{in } \Omega.$$

However

$$v_\lambda = \frac{c\varphi_1}{\lambda_{a0} - \lambda},$$

and then

$$\lim_{\lambda \uparrow \lambda_{a0}} w'_\lambda(x) > \lim_{\lambda \uparrow \lambda_{a0}} v_\lambda(x) = \infty$$

for each $x \in \Omega$. Consequently

$$\lim_{\lambda \uparrow \lambda_{a0}} w_\lambda(x) = \infty \quad \forall x \in \Omega. \quad \square$$

Proof of Theorem 1.2. By Proposition 2.1, λ_1 is a bifurcation point of (11) from the trivial solution and it is the only one for positive solutions. Moreover, there exists an unbounded continuum Σ_0 of positive solutions emanating from $(\lambda_1, 0)$. Again we will distinguish two cases:

a) Case $\overline{\Omega}_{a0} \cap \overline{\Omega}_{b0} = \emptyset$.
To prove the existence of a positive solution for all $\lambda > \lambda_1$ we have to show (25). In order to prove this, let us consider the family $\overline{W} : [\lambda_1, \lambda_*] \to \mathcal{C}_0^2(\overline{\Omega})$ defined by $\overline{W}(\lambda) = K(\lambda)e$, where $K = K(\lambda)$ is a positive constant to be chosen and e is the unique positive solution of (26). Then $K(\lambda)e$ is a supersolution of (11) provided that

$$1 \geq \lambda \frac{q(x, Ke)}{Ke} e - b(x) \frac{q(x, Ke)^p}{Ke} e \quad \text{in } \Omega.$$

Since $\overline{\Omega}_{a0} \cap \overline{\Omega}_{b0} = \emptyset$, by (13)–(16), we have

$$\lambda \frac{q(x, s)}{s} e - b(x) \frac{q(x, s)^p}{s} e < 1 \quad \forall s > s_0, x \in \Omega$$

for a sufficiently large constant $s_0 > 0$. Hence, there exists $K > 0$ such that $\overline{W}(\lambda) = Ke$ is a supersolution (but not a solution) of (11), for every $\lambda \in [\lambda_1, \lambda_*]$ and $W(\lambda_1) = K(\lambda_1)e > 0$ in Ω. Thus, the result follows.
Now, in order to prove (7), let $\varphi_1^{B_i} > 0$ be the eigenvalue associated to $\lambda_1[-\Delta, B_i]$ such that $\|\varphi_1^{B_i}\|_0 = 1$ and consider

$$\Psi_i = \begin{cases} \varphi_1^{B_i} & \text{in } B_i, \\ 0 & \text{in } \Omega \setminus \overline{B_i}. \end{cases}$$

It is clear that $\Psi_i \in H_0^1(\Omega)$. We will show that for $\lambda > \lambda_1[-\Delta; B_i]$, $\varepsilon(\lambda)\Psi_i$ is a subsolution of (11) (in the sense of [19]) for a constant $\varepsilon(\lambda) > 0$ to be chosen. Indeed, since $b \equiv 0$ in B_i, $i \in \{1, ..., m\}$ and $\Psi_i = 0$ in $\Omega \setminus \overline{B_i}$, it suffices to verify that

$$1 + a(x)q(x, \varepsilon(\lambda)\varphi_1^{B_i})^{r-1} = \frac{\varepsilon(\lambda)\varphi_1^{B_i}}{q(x, \varepsilon(\lambda)\varphi_1^{B_i})} \leq \frac{\lambda}{\lambda_1[-\Delta; B_i]} \quad x \in B_i. \tag{32}$$

Choosing

$$\varepsilon(\lambda) := \left(\frac{\lambda - \lambda_1[-\Delta; B_i]}{\lambda_1[-\Delta; B_i] \max_{\overline{B_i}} a(x)} \right)^{1/(r-1)}, \qquad \lambda > [-\Delta; B_i],$$

it is easy to see that (32) is satisfied, once that $q(x, s) \leq s$, for all $x \in \Omega$. Moreover,

$$\varepsilon(\lambda) \to \infty \quad \text{as } \lambda \to \infty. \tag{33}$$

On the other hand, we have already shown that $K(\lambda)e$ is a supersolution of (11), where e is the unique positive solution of (26) and $K(\lambda) > 0$ is a sufficiently large constant. Hence, there exists $K(\lambda) > 0$ such that

$$\varepsilon(\lambda)\Psi_i \leq K(\lambda)e \quad \text{in } \Omega,$$

by the sub-supersolution method and the uniqueness of positive solution of (11), we infer

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$$\varepsilon(\lambda)\varphi_1^{B_i} \le w_\lambda \quad \text{in } B_i.$$

Using (33), we conclude that

$$\lim_{\lambda\uparrow\infty} w_\lambda(x) = \infty \quad \text{uniformly in } B_i.$$

Therefore, once that $w_\lambda = u_\lambda + a(x)u_\lambda^r$, we obtain (7).

b) Case $\Omega_{a0} \cap \Omega_{b0} \neq \emptyset$.

Thanks to Lemma 2.3, (11) does not possess positive solutions for $\lambda \notin (\lambda_1, \lambda_{0,1})$ and, as a consequence, by the global nature of Σ_0, there exists a sequence of positive solutions of (11), (λ_n, w_n), such that $\|w_n\|_0 \to \infty$ and $\lambda_n \to \lambda^* \le \lambda_{0,1}$. We will prove that $\lambda^* = \lambda_{0,1}$. To this aim, we follow the arguments of Lemma 2.4 of [20]. First note that $|w_n|_2 \to \infty$, where $|\cdot|_2$ stands for the $L^2(\Omega)$ norm. Otherwise, multiplying (11) by w_n, integrating by parts and using (14) we have

$$\|w_n\|^2_{H_0^1} = \lambda_n \int_\Omega q(x, w_n)w_n - \int_\Omega b(x)q(x, w_n)^p \le \lambda_{0,1}|w_n|_2^2$$

and, hence, $\|w_n\|_{H_0^1}$ is bounded. By elliptic regularity, $\|w_n\|_{W^{2,m}}$ is also bounded for all $m > 1$. Thus, the Morrey's embedding gives

$$\|w_n\|_0 \le C\|w_n\|_{W^{2,m}} \le \overline{C}$$

which is a contradiction.

Define $z_n = w_n|w_n|_2^{-1}$, $n \ge 1$. Multiplying (11) by $z_n|w_n|_2^{-1}$, using integration by parts and (14), we infer

$$\|z_n\|^2_{H_0^1} = \lambda_n \int_\Omega \frac{q(x, w_n)}{|w_n|_2} z_n - \int_\Omega b(x)\frac{q(x, w_n)^p}{|w_n|_2} z_n \le \lambda_n|z_n|_2^2 \le \lambda_{0,1}.$$

This shows that z_n is bounded in $H_0^1(\Omega)$. Thus, up to subsequence if necessary, $z_n \to z$ in $L^2(\Omega)$ with $|z|_2 = 1$ and $z > 0$.

Let us prove that $z = 0$ in $\Omega \setminus \Omega_{b0}$. Indeed, if $z(x) > 0$ in a subset of $\Omega \setminus \overline{\Omega}_{b0}$ with Lebesgue measure non-zero, we take D an arbitrary regular domain such that $\overline{D} \subset \Omega \setminus \overline{\Omega}_{b0}$ and $z(x) > 0$ a.e. in D. Then, for any $\phi \in C_0^\infty(D)$, multiplying (11) by $\phi|w_n|_2^{-1}$, integrating in D and applying the formula of integration by parts, it yields

$$\begin{aligned} -\int_D z_n\Delta\phi &= \lambda_n \int_D \frac{q(x, w_n)}{|w_n|_2}\phi - \int_D b(x)\frac{q(x, w_n)^p}{|w_n|_2}\phi \\ &= \lambda_n \int_D \frac{q(x, z_n|w_n|_2)}{z_n|w_n|_2} z_n\phi - \int_D b(x)\frac{q(x, w_n|w_n|_2)^p}{z_n|w_n|_2} z_n\phi. \end{aligned} \tag{34}$$

Since $z(x) > 0$ a.e. $x \in D$, then $w_n(x) = z_n(x)|w_n|_2 \to \infty$ a.e. $x \in D$. Once that $p > r$ and $b > 0$ in $\overline{D}$, passing to the limit as $n \to \infty$ in (34) we obtain

$$-\int_D z\Delta\phi - \lambda^* \int_D \mathcal{X}_{\Omega_{a0}} z\phi = -\infty$$

which is a contradiction. This shows that $z = 0$ a.e. in $\Omega \setminus \Omega_{b0}$, hence, $z \in H_0^1(\Omega_{b0})$.
Thus, for any $\varphi \in C_0^\infty(\Omega_{b0})$, since that $b \equiv 0$ in Ω_{b0}, multiplying (11) by $\varphi|w_n|_2^{-1}$, integrating in Ω_{b0} and applying the formula of integration by parts, it follows that

$$-\int_{\Omega_{b0}} z_n\Delta\varphi = \lambda_n \int_{\Omega_{b0}} \frac{q(x,w_n)}{|w_n|_2}\phi = \lambda_n \int_{\Omega_{b0}} \frac{q(x,z_n|w_n|_2)}{z_n|w_n|_2} z_n\phi.$$

Letting $n \to \infty$ in above equality yields

$$-\int_{\Omega_{b0}} z\Delta\varphi = \lambda^* \int_{\Omega_{b0}} \mathcal{X}_{\Omega_{a0}} z\varphi \quad \forall\varphi \in C_0^\infty(\Omega_{b0}).$$

Since $z \in H_0^1(\Omega_{b0})$, $z > 0$ in Ω_{b0} with $|z|_2 = 1$ and $\lambda^* \le \lambda_{0,1}$, it follows that $\lambda^* = \lambda_{0,1}$.
The uniqueness is given by Proposition 2.2.
In both cases, (6)–(8) and the monotonicity of the map $\lambda \mapsto u_\lambda$ is a consequence of Proposition 2.1 and Lemma 2.4 b), respectively.

To prove (9), we need show that the map $\lambda \in (\lambda_1, \lambda_{0,1}) \mapsto w_\lambda$ is of class $\mathcal{C}^1$. Indeed, consider $\mathcal{H} : \mathbb{R} \times \mathcal{C}_0^2(\overline{\Omega}) \longrightarrow \mathcal{C}_0(\overline{\Omega})$ given by

$$\mathcal{H}(\lambda, w) = -\Delta w - \lambda q(x,w) + b(x)q(x,w)^p.$$

For each (λ, w) pair of solutions of (11) we have $\mathcal{H}(\lambda, w) = 0$ and

$$\mathcal{H}_w(\lambda, w)\xi = -\Delta\xi - \lambda q_w(x,w)\xi + b(x)pq(x,w)^{p-1}q_w(x,w)\xi. \tag{35}$$

Since $s \mapsto q(x,s)/s$ is non-increasing and $s \mapsto q(x,s)^p/s$ is non-decreasing, we deduce that

$$\frac{q(x,w)}{pw} < q_w(x,w) \le \frac{q(x,w)}{w},$$

implying that

$$-\lambda\frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w} < -\lambda q_w(x,w) + b(x)pq(x,w)^{p-1}q_w(x,w).$$

Since (λ, w) is a positive solution of (11), the above inequality gives

$$\begin{aligned} 0 &= \lambda_1\left[-\Delta - \lambda\frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w}; \Omega\right] \\ &< \lambda_1[-\Delta - \lambda q_w(x,w) + b(x)pq(x,w)^{p-1}q_w(x,w); \Omega]. \end{aligned} \tag{36}$$

Therefore, (35) and (36) imply that $\mathcal{H}_u(\lambda, w)$ is no singular for all (λ, w) pair of solutions of (11). Hence, by the Implicit Function Theorem, $\lambda \mapsto w_\lambda$ is of class $\mathcal{C}^1$ and we can differentiate (11) with respect to λ and get

$$(-\Delta - \lambda q_w(x, w_\lambda) + b(x) p q(x, w_\lambda)^{p-1} q_w(x, w_\lambda)) w'_\lambda = q(x, w_\lambda) > 0 \quad \text{in } \Omega \tag{37}$$

where $w'_\lambda = dw_\lambda / d\lambda$. In view of (36), the operator

$$-\Delta - \lambda q_w(x, w_\lambda) + b(x) p q(x, w_\lambda)^{p-1} q_w(x, w_\lambda)$$

satisfies the Strong Maximum Principle and we deduce from (37) that $w'_\lambda > 0$ in Ω.

On the other hand, in B_i, $1 \le i \le m_1$ we have $b(x) = 0$ and (37) gives us

$$-\Delta w'_\lambda = q(x, w) + \lambda q_w(x, w) w'_\lambda \quad \text{in } B_i.$$

Since $q_w(x, w) > \mathcal{X}_{\Omega_{a0} \cap B_i}$ in B_i, the above equality implies

$$(-\Delta - \lambda \mathcal{X}_{\Omega_{a0} \cap B_i}) w'_\lambda > q(x, w) \quad \text{in } B_i. \tag{38}$$

Now, we fix $\overline{\lambda} \in (\lambda_1, \lambda_{0,1})$ and let φ_1^i be the positive eigenfunction associated to $\lambda_{0,i}$ with $\|\varphi_1^i\|_0 = 1$. For a constant $c > 0$ sufficiently small, we have

$$q(x, w_{\overline{\lambda}}) > c \mathcal{X}_{\Omega_{0,i}} \varphi_1^i \quad \text{in } B_i.$$

Once that $\lambda \mapsto w_\lambda$ is increasing,

$$q(x, w_{\overline{\lambda}}) > c \mathcal{X}_{\Omega_{0,i}} \varphi_1^i \quad \text{in } B_i, \ \forall \lambda \in [\overline{\lambda}, \lambda_{0,1}).$$

Let v_λ^i be the unique solution of the linear problem

$$\begin{cases} (-\Delta - \lambda \mathcal{X}_{\Omega_{0,i}}) u = c \mathcal{X}_{\Omega_{0,i}} \varphi_1^i & \text{in } B_i, \\ u = 0 & \text{on } \partial B_i, \end{cases}$$

which exists because $\lambda < \lambda_{0,i}$. In view of (38), the Maximum Principle implies that

$$w'_\lambda > v_\lambda^i \quad \text{in } B_i.$$

However

$$v_\lambda^i = \frac{c \varphi_1^i}{\lambda_{0,i} - \lambda}$$

and once that $\lambda_{0,i} = \lambda_{0,1}$ for each $i \in \{1, ..., m_1\}$, we deduce

$$\lim_{\lambda \uparrow \lambda_{0,1}} w'_\lambda(x) > \lim_{\lambda \uparrow \lambda_{0,1}} v_\lambda^i(x) = \infty$$

for each $x \in B_i$. Consequently

$$\lim_{\lambda \uparrow \lambda_{0,1}} w_\lambda(x) = \infty \quad \forall x \in B_i,\ 1 \le i \le m_1,$$

which ends the proof of the theorem. □

4. Large solutions

To complete the proof of Theorem 1.2, we only have to prove that $\lim_{\lambda \uparrow \lambda_{0,1}} u_\lambda(x) < \infty$ for $x \in \Omega \setminus (\cup_{i=1}^{m_1} \overline{B}_i)$. Following the general framework of [10] (see also [7]), the idea is to show that there exists a function U defined in $\Omega \setminus (\cup_{i=1}^{m_1} \overline{B}_i)$ such that $u_\lambda \le U$, for all $\lambda \in (\lambda_1, \lambda_{0,1})$. Thus, the main purpose of this section is to build such a function. For this, we recall that a solution of the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

is usually known as a *large solution* of

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \tag{39}$$

that means a classical solution $u \in \mathcal{C}^2(\Omega)$ of (39) such that

$$\lim_{\text{dist}(x,\partial\Omega)\downarrow 0} u(x) = \infty.$$

There are many results about large solutions, see for example [10,21,7,22] and their references. The following lemma is a consequence of these works.

Lemma 4.1. *Consider the problem*

$$\begin{cases} -\Delta w = \lambda w - b(x) q_d(w)^p & \text{in } \Omega, \\ w = \infty & \text{on } \partial\Omega, \end{cases} \tag{40}$$

where $d \ge 0$, $\lambda > 0$ and q_d is the inverse function of $s \mapsto s + ds^r$. Assume $b(x) \ge b_0 > 0$ in Ω and $p > r$. Then (40) possesses a non-negative solution.

Proof. We are going to apply Theorem 1.1 of [21]. Therefore, it is sufficient to verify the following hypotheses:

(A_1) $q_d^p \in \mathcal{C}^1([0,+\infty))$ with $q_d^p \ge 0$ and $\frac{q_d(s)^p}{s}$ is increasing in $(0,+\infty)$, and the Keller–Osserman condition

(A_2) $\displaystyle\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty$, where $F(t) = \displaystyle\int_0^t q_d(s)^p ds$.

Firstly, we note that if $d = 0$ we have $q_d(s) = s$ and it is easy to see that (A_1)–(A_2) are satisfied.

Suppose $d > 0$. Since $s = q_d(s) + dq_s(s)^r$, we can write

$$\frac{q_d(s)^p}{s} = \frac{1}{q_d(s)^{1-p} + dq_d(s)^{r-p}}.$$

Once that $p > r$, $s \mapsto \frac{q_d(s)^p}{s}$ is increasing in $(0, +\infty)$ and (A_1) follows.

To verify (A_2), first we will show that, for $p > r$, there exists a constant $C > 0$ such that

$$Cs^k \le q_d(s)^p \quad \forall s \ge 1,\ k \in (1, p/r).$$

Indeed, define

$$h(s) := \frac{s^k}{q_d(s)^p} = \left[q_d(s)^{1-\frac{p}{k}} + dq_d(s)^{r-\frac{p}{k}}\right]^k \quad s \ge 1.$$

Since $k \in (1, p/r)$, h is decreasing and therefore

$$\frac{s^k}{q_d(s)^p} \le \frac{1}{q_d(1)^p} = \frac{1}{C} \quad s \ge 1.$$

Thus, defining

$$g(s) = \begin{cases} Cs^k & 1 \le s, \\ 0 & 0 \le s \le 1, \end{cases}$$

we have that $g(s) \le q_d^p(s)$ in $(0, \infty)$. Then

$$\int_1^\infty \left(\int_0^s q_d(t)^p dt\right)^{-\frac{1}{2}} dtds \le \int_1^\infty \left(\int_0^s g(t)dt\right)^{-\frac{1}{2}} dtds = C_0 \int_1^\infty \frac{1}{s^{(k+1)/2}} ds < \infty,$$

which ends the proof. □

Now, we will analyze another problem of large solutions, which is the principal result of this section. Specifically, denoting by

$$\Omega_1 := \Omega \setminus \bigcup_{i=1}^{m_1} \overline{B}_i,$$

we will establish the existence of positive solution for the singular value problem

$$\begin{cases} -\Delta w = \lambda q(x, w) - b(x)q(x, w)^p & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ w = \infty & \text{on } \partial\Omega_1 \setminus \partial\Omega. \end{cases} \tag{41}$$

To prove it, we will follow Section 3 of [23]. Thus, we consider the inhomogeneous problem associated to (41)

$$\begin{cases} -\Delta w = \lambda q(x,w) - b(x)q(x,w)^p & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ w = M & \text{on } \partial\Omega_1 \setminus \partial\Omega, \end{cases} \tag{42}$$

for each $M > 0$. Denoting $\lambda_{0,m+1} = \infty$, the next result characterizes the existence of positive solutions of (42).

Lemma 4.2. *If $p > r$, then (42) possesses a positive solution if, and only if, $\lambda < \lambda_{0,m_1+1}$. Moreover, it is unique if exists, and we denote it by $w_{[\lambda,M]}$. Furthermore, the maps $\lambda \mapsto w_{[\lambda,M]}$ and $M \mapsto w_{[\lambda,M]}$ are non-decreasing.*

Proof. If $w > 0$ is a solution of (42), then it is a strict supersolution of the associated homogeneous problem. Therefore, in the case $\lambda_{0,m_1+1} < \infty$ we have

$$\begin{aligned} 0 &< \lambda_1\left[-\Delta - \lambda\frac{q(x,w)}{w} + b(x)\frac{q(x,w)^p}{w}; \Omega_1\right] \\ &< \lambda_1[-\Delta - \lambda\mathcal{X}_{\Omega_{0,m_1+1}}; B_{m_1+1}] = \mu_{\Omega_{0,m_1+1}, B_{m_1+1}}(\lambda). \end{aligned}$$

By Lemma 2.2, the above inequality implies that $\lambda < \lambda_{0,m_1+1}$ is a necessary condition for the existence of positive solutions of (42).

Assuming $\lambda < \lambda_{0,m_1+1}$ and since $\underline{w} \equiv 0$ is a subsolution (but not a solution) of (42), to prove the existence of positive solution it suffices to construct a positive supersolution of (42). To this aim we argue as follows.

The construction in the cases $m_1 < m$ and $m_1 = m$ are similar, then we will deal only with the case $m_1 < m$. Fix $\lambda < \lambda_{0,m_1+1}$. Note that $\partial\Omega_1 \setminus \partial\Omega$ consists of the (finite) components of ∂B_i, $i \in \{1, ..., m_1\}$. Thus, for $\delta > 0$ small, let

$$\mathcal{U}_i^\delta := \{x \in \Omega; \operatorname{dist}(x, \partial B_i) < \delta\}$$

be a regular subdomain and $\varphi_i^\delta > 0$ the principal eigenfunction of $-\Delta$ in $\mathcal{U}_i^\delta$ under homogeneous Dirichlet boundary conditions with $\|\varphi_i^\delta\|_0 = 1$. Denoting by $\lambda_i^\delta = \lambda_1[-\Delta; \mathcal{U}_i^\delta]$ the corresponding principal eigenvalue and once that

$$|\mathcal{U}_i^\delta| \to 0 \quad \text{as } \delta \to 0,$$

where $|\cdot|$ denotes the Lebesgue measure, the monotonicity of the principal eigenvalue with respect the domain implies that we can choose δ sufficiently small such that

$$\lambda_i^\delta > \lambda, \tag{43}$$

for all (finite) components of ∂B_i, $i = 1, ..., m$.

On the other hand, consider

$$B_i^\delta := \{x \in \Omega_1; \operatorname{dist}(x, B_i) < \delta\} \subset\subset \Omega_1, \tag{44}$$

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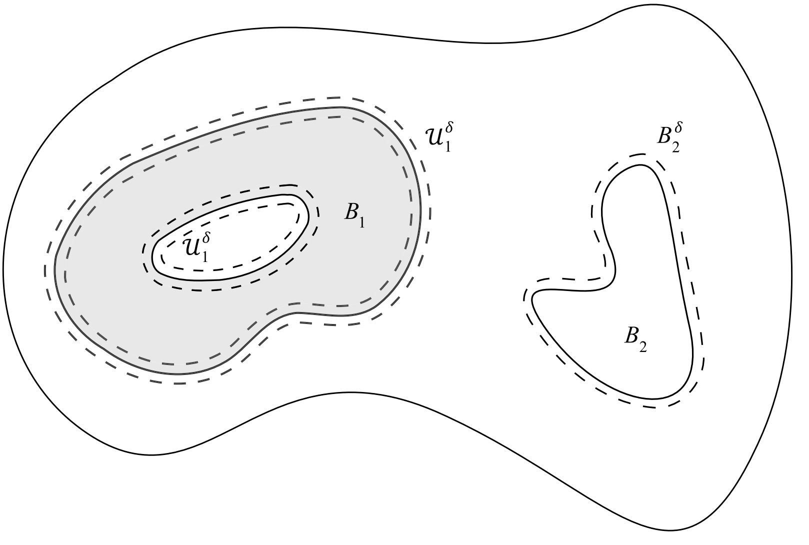

Fig. 3. A typical configuration of the sets $\mathcal{U}_i^\delta$ and B_i^δ where $\Omega_1 = \Omega \setminus \overline{B}_1$. The dashed lines around of ∂B_1 delimit the two components of set $\mathcal{U}_1^\delta$ and the dashed line around of B_2 delimits the set B_2^δ.

for each $i \in \{m_1 + 1, ..., m\}$, i.e., the B_i's which are contained in Ω_1. The Fig. 3 sketches the construction of $\mathcal{U}_i^\delta$ and B_i^δ for a particular case. Denote by $\lambda_{0,i}^\delta$ the principal eigenvalue of

$$\begin{cases} -\Delta w = \lambda \mathcal{X}_{\Omega_{a0} \cap B_i^\delta} w & \text{in } B_i^\delta, \\ w = 0 & \text{on } \partial B_i^\delta. \end{cases} \tag{45}$$

Observe that $\lambda_{0,i}^\delta$ is the zero of the map $\lambda \mapsto \rho(\lambda, \delta)$ where

$$\rho(\lambda, \delta) := \lambda_1[-\Delta - \lambda \mathcal{X}_{\Omega_{a0} \cap B_i^\delta}; B_i^\delta].$$

On the one hand, following the proof of Theorem 9.1 of [14], it can be derived that the map $\lambda \mapsto \rho(\lambda, \delta)$ is analytic and its zero is simple. On the other hand, from Section 8.5 of [14], it follows that the map $\delta \mapsto \rho(\lambda, \delta)$ varies continuously. Hence, we can deduce that

$$\lambda_{0,i}^\delta \to \lambda_{0,i} \quad \text{as } \delta \to 0.$$

Thus, since $\lambda_{0,i} > \lambda_{0,1} > \lambda$, $i \in \{m_1 + 1, ...m\}$, for δ sufficiently small, we have

$$\lambda_{0,i}^\delta > \lambda \tag{46}$$

and, hence, Theorem 1.1 provides us with a (unique) positive solution, say w_i^δ, of

$$\begin{cases} -\Delta w = \overline{\lambda}_i q(x, w) & \text{in } B_i^\delta, \\ w = 0 & \text{on } \partial B_i^\delta, \end{cases} \tag{47}$$

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for some $\overline{\lambda}_i \in (\lambda, \lambda_{0,i}^{\delta})$. Therefore, for $\delta > 0$ satisfying (43) and (46), consider a positive function

$$\phi : \Omega_1 \setminus \left(\left[\bigcup_{i=1}^{m_1} (\mathcal{U}_i^{\delta/2} \cap \overline{\Omega}_1) \right] \cup \left[\bigcup_{i=m_1+1}^{m} \overline{B}_i^{\delta/2} \right] \right) \to \mathbb{R}$$

such that

$$\Phi(x) = \begin{cases} w_i^{\delta}(x) & \text{if } x \in \overline{B}_i^{\delta/2},\ i = \{m_1+1, \dots, m\}, \\ \varphi_i^{\delta} & \text{if } x \in \mathcal{U}_i^{\delta/2} \cap \overline{\Omega}_1,\ i = \{1, \dots, m_1\}, \\ \phi(x) & \text{otherwise,} \end{cases} \tag{48}$$

is a $\mathcal{C}^2(\overline{\Omega}_1)$ function. We claim that $K\Phi$ is a supersolution of (42) for a sufficiently large positive constant K. Indeed, it is easy to see that $K\Phi = K\phi > 0$ in $\partial\Omega_1 \cap \partial\Omega$ and for K large, $K\Phi = K\varphi_i^{\delta/2} > M$ in each component of $\partial\Omega_1 \setminus \partial\Omega$. Then, $K\Phi$ is a supersolution of (42) provided that

$$-\Delta(K\Phi) \geq \lambda q(x, K\Phi) - b(x) q(x, K\Phi)^p \quad \text{in } \Omega_1. \tag{49}$$

In $\Omega_1 \setminus \left(\left[\bigcup_{i=1}^{m_1} (\mathcal{U}_i^{\delta/2} \cap \overline{\Omega}_1) \right] \cup \left[\bigcup_{i=m_1+1}^{m} \overline{B}_i^{\delta/2} \right] \right)$, (49) is equivalent to

$$-\Delta\phi \geq \lambda \frac{q(x, K\phi)}{K\phi} \phi - b(x) \frac{q(x, K\phi)^p}{K\phi} \phi.$$

Since $b(x) > b_0 > 0$ in $\Omega_1 \setminus \left(\left[\bigcup_{i=1}^{m_1} (\mathcal{U}_i^{\delta} \cap \overline{\Omega}_1) \right] \cup \left[\bigcup_{i=m_1+1}^{m} \overline{B}_i^{\delta} \right] \right)$ and in view of (13)–(16) we have

$$\lim_{s \to \infty} \left[\lambda \frac{q(x, s)}{s} - b(x) \frac{q(x, s)^p}{s} \right] = -\infty,$$

uniformly in $\Omega_1 \setminus \left(\left[\bigcup_{i=1}^{m_1} (\mathcal{U}_i^{\delta/2} \cap \overline{\Omega}_1) \right] \cup \left[\bigcup_{i=m_1+1}^{m} \overline{B}_i^{\delta/2} \right] \right)$. Thus, for sufficiently large K enough, (49) is satisfied.

Now, in each $\mathcal{U}_i^{\delta/2} \cap \Omega_1$, $i \in \{1, \dots, m_1\}$, (49) is equivalent to

$$\lambda_i^{\delta} \geq \lambda \frac{q(x, K\varphi_i^{\delta})}{K\varphi_i^{\delta}} - b(x) \frac{q(x, K\varphi_i^{\delta})^p}{K\varphi_i^{\delta}}.$$

By the properties of $q(w, \cdot)$ and in view of (43), the above inequality follows for any $K > 0$.

Finally, in $\overline{B}_i^{\delta/2}$, $i \in \{m_1+1, \dots, m\}$, since $\overline{\lambda}_i > \lambda$, (49) holds if

$$\lambda \left(\frac{q(x, w_i^{\delta})}{w_i^{\delta}} - \frac{q(x, K w_i^{\delta})}{K w_i^{\delta}} \right) w_i^{\delta} \geq -b(x) \frac{q(x, K w_i^{\delta})^p}{K}. \tag{50}$$

Once that the map $s \mapsto q(x, s)/s$ is non-increasing, (50) is satisfied for $K > 1$ and, hence, $K\Phi$ is a supersolution of (11). This proves the existence of a positive solution of (11).

The uniqueness follows by a similar argument of Proposition 2.2. The monotonicity of maps $\lambda \mapsto w_{[\lambda,M]}$ and $M \mapsto w_{[\lambda,M]}$ is a consequence of that $w_{[\lambda,M]}$ is a subsolution of (42) with $\widehat{\lambda} \geq \lambda$ and $\widehat{M} \geq M$. $\square$

Note that in the previous proof the domain Ω_1 does not play a crucial rule. It is only important that $B_i \subset\subset \Omega_1$, $i \in \{m_1+1, ..., m\}$. Then, as a consequence of this proof, we obtain

Corollary 4.1. *Consider $\delta > 0$ small such that, denoting by*

$$D_\delta := \{x \in \Omega_1;\ dist(x, \partial\Omega_1 \setminus \partial\Omega) \leq \delta\} \quad and \quad \Omega_1^\delta = \Omega_1 \setminus D_\delta,$$

we have

$$\bigcup_{i=m_1+1}^{m} \overline{B}_i \subset \Omega_1^\delta.$$

Then, for $p > r$ and $M > 0$, the problem

$$\begin{cases} -\Delta w = \lambda q(x,w) - b(x)q(x,w)^p & in\ \Omega_1^\delta, \\ w = 0 & on\ \partial\Omega_1^\delta \cap \partial\Omega, \\ w = M & on\ \partial\Omega_1^\delta \setminus \partial\Omega, \end{cases}$$

has a positive solution if, and only if, $\lambda < \lambda_{0,m_1+1}$. Moreover, it is unique if exists.

Proof. Just note that the same arguments of the previous Lemma can be applied, including the construction of the supersolution. $\square$

A crucial step in order to prove existence of large solution is to obtain suitable a priori estimates. The following result establishes this bound.

Lemma 4.3. *For each compact subset $K \subset \{x \in \Omega_1;\ b(x) > 0\}$, there exists a constant $C := C(K) > 0$ such that*

$$\|w_{[\lambda,M]}\|_0 \leq C \quad \forall \lambda < \lambda_{0,m_1+1},\ M > 0.$$

Proof. Let $K \subset \{x \in \Omega_1;\ b(x) > 0\}$ be compact and $\delta > 0$ sufficiently small such that

$$K_\delta := \{x \in \Omega_1;\ \mathrm{dist}(x,K) < \delta\} \subset \{x \in \Omega_1;\ b(x) > 0\}.$$

Thus, we have $b(x) \geq \min_{K_\delta} b(x) > 0$ and, by Lemma 4.1, there exists a large solution, say W_d, of

$$-\Delta w = \lambda w - b(x)q_d(w)^p \quad \text{in } K_\delta,$$

where $d = \max_{K_\delta} a(x) \geq 0$. For all $M > 0$, W_d is a supersolution of

$$\begin{cases} -\Delta w = \lambda q(x,w) - b(x)q(x,w)^p & \text{in } K_{\delta/2}, \\ w = w_{[\lambda,M]} & \text{on } \partial K_{\delta/2}. \end{cases} \tag{51}$$

Indeed, on one hand

$$\lambda s \geq \lambda q(x,s) \quad \text{and} \quad q(x,s) \geq q_d(s) \quad \forall s \geq 0,$$

which implies

$$-\Delta W_d = \lambda W_d - b(x)q_d(W_d)^p \geq \lambda q(x,W_d) - b(x)q(x,W_d)^p \quad \text{in } K_{\delta/2}.$$

Moreover, $W_d \geq w_{[\lambda,M]}$ on $\partial K_{\delta/2}$ because $W_d = \infty$ in ∂K_δ. Therefore, $w_{[\lambda,M]}|_{K_{\delta/2}}$ is a solution of (42). Since

$$\frac{f(\lambda,x,s)}{s} := \lambda \frac{q(x,s)}{s} - b(x)\frac{q(x,s)^p}{s}$$

is decreasing in $\{x \in \Omega_1;\ b(x) > 0\}$, by Lemma 2.4 b) we obtain

$$w_{[\lambda,M]} \leq W_d \quad \text{in } K_{\delta/},\ \forall M > 0.$$

Consequently,

$$\|w_{[\lambda,M]}\|_{\mathcal{C}(K)} \leq \max_{\overline{K}_{\delta/2}} W_d,$$

which ends the proof. □

The main result of this section can be stated as follows.

Proposition 4.1. *If $\lambda < \lambda_{0,m_1+1}$, then (41) possesses a positive solution. Moreover, in such case, the point-wise limit*

$$W_{[\lambda,\infty]} := \lim_{M\uparrow\infty} w_{[\lambda,M]},$$

provides us with the minimal positive large solution of (41), that is, any positive large solution Θ of (41) satisfies

$$W_{[\lambda,\infty]} \leq \Theta.$$

Proof. We fix $\lambda < \lambda_{0,m_1+1}$. By Lemma 4.2, the map $M \mapsto w_{[\lambda,M]}$ is non-decreasing, hence the point-wise limit is well defined. To show that it solves (41), we proceed as follows. First we prove that it is finite. In the set $\{x \in \Omega_1; b(x) > 0\}$ is finite by Lemma 4.3. If $m_1 = m$ then $\Omega_1 = \{x \in \Omega_1; b(x) > 0\}$ and the result is complete.

On the other hand, if $m_1 < m$, for each $\overline{B}_i \subset \Omega_1,\ i \in \{m_1+1, ..., m\}$, we can choose $\delta > 0$ small such that

$$D_\delta := \{x \in \Omega_1;\ \mathrm{dist}(x, \partial\Omega_1 \setminus \partial\Omega) \le \delta\} \subset \Omega_1 \setminus \bigcup_{i=m_1+1}^{m} \overline{B}_i \quad \text{and} \quad \Omega_1^\delta = \Omega_1 \setminus D_\delta. \tag{52}$$

For each of these δ's, there exists an open set $\mathcal{O}_\delta$ satisfying

$$\partial\Omega_1^\delta \subset \mathcal{O}_\delta \subset\subset \Omega_1 \setminus \cup_{i=m_1+1}^{m} \overline{B}_i.$$

Fix one of those δ's. Then, thanks to Lemma 4.3, there exists a constant $C > 0$ such that

$$\|w_{[\lambda,M]}\|_{C(\partial\Omega_1^\delta)} \le \|w_{[\lambda,M]}\|_{C(\overline{\mathcal{O}}_\delta)} \le C \quad \forall M > 0. \tag{53}$$

Hence, $w_{[\lambda,M]}|_{\Omega_1^\delta}$ is a subsolution of

$$\begin{cases} -\Delta w = \lambda q(x,w) - b(x) q(x,w)^p & \text{in } \Omega_1^\delta, \\ w = 0 & \text{on } \partial\Omega_1^\delta \cap \partial\Omega, \\ w = C & \text{on } \partial\Omega_1^\delta \setminus \partial\Omega, \end{cases} \tag{54}$$

and, as a consequence

$$w_{[\lambda,M]} \le w_{[\lambda,C,\delta]} \quad \text{in } \Omega_1^\delta \quad \forall M > 0, \tag{55}$$

where $w_{[\lambda,C,\delta]}$ stands for the unique solution of (54), whose existence is guaranteed by Corollary 4.1. This shows that the point-wise limit $W_{[\lambda,\infty]}$ is finite in Ω_1.

Now, we take two open sets $\mathcal{O}$, $\mathcal{O}_1$ and a sufficiently small δ such that

$$\mathcal{O} \subset\subset \mathcal{O}_1 \subset\subset \Omega_1^\delta \subset\subset \Omega_1.$$

By the elliptic L^p-estimates and Morrey's Theorem, there exists a constant $C > 0$ such that, for each $M > 0$,

$$\|w_{[\lambda,M]}\|_{\mathcal{C}^1(\overline{\mathcal{O}}_1)} \le C.$$

Therefore, by the compact embedding $\mathcal{C}^1(\overline{\mathcal{O}}_1)$ into $\mathcal{C}^{0,\alpha}(\overline{\mathcal{O}}_1)$ and the uniqueness of the point-wise limit, $W_{[\lambda,\infty]}$ is Hölder continuous. Consequently, by elliptic regularity, we obtain that it solves (41).

It remains to prove that $W_{[\lambda,\infty]}$ is a minimal positive solution of (41). Indeed, let Θ be any positive solution of (41) and $C > 0$ be the constant that satisfies (53). Since $\Theta = \infty$ on $\partial\Omega_1 \setminus \partial\Omega$, we have for $\delta > 0$ small that

$$C \le \Theta \quad \text{on } \partial\Omega_1^\delta \setminus \partial\Omega.$$

Therefore, $\Theta|_{\Omega_1^\delta}$ is supersolution of (54) and consequently

$$w_{[\lambda,C,\delta]} \le \Theta \quad \text{in } \Omega_1^\delta. \tag{56}$$

Combining (55) and (56), we obtain

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$$w_{[\lambda,M]} \leq \Theta \quad \text{in } \Omega_1.$$

Thus, letting $M \uparrow \infty$ yields

$$W_{[\lambda,\infty]} \leq \Theta,$$

which establishes the minimality of $W_{[\lambda,\infty]}$. □

5. Profile of positive solution

If $\Omega_{a0} \cap \Omega_{b0} \neq \emptyset$, we already know that

$$\lim_{\lambda \uparrow \lambda_{0,1}} w_\lambda = \infty \quad \text{uniformly in } \bigcup_{i=1}^{m_1} B_i.$$

To complete the study of the behavior of the solution w_λ as $\lambda \uparrow \lambda_{0,1}$, it remains to show what happens with the points $x \in \Omega_1 := \Omega \setminus \cup_{i=1}^{m_1} \overline{B}_i$.

Completing the proof of Theorem 1.2 *b*). Let $W := W_{[\lambda_{0,1},\infty]}$ be a positive solution of (41), given by Proposition 4.1. Considering Ω_1^δ as in (52) with $\delta > 0$ sufficiently small, it satisfies

$$-\Delta W = \lambda_{0,1} q(x, W) - b(x) q(x, W)^p \geq \lambda q(x, W) - b(x) q(x, W)^p \quad \text{in } \Omega_1^\delta,$$

for all $\lambda \in (\lambda_1, \lambda_{0,1})$. Thus, W is a supersolution of

$$\begin{cases} -\Delta w = \lambda q(x, w) - b(x) q(x, w)^p & \text{in } \Omega_1^\delta, \\ w = w_\lambda|_{\partial \Omega_1^\delta} & \text{on } \partial \Omega_1^\delta, \end{cases}$$

whose (unique) solution is $w_\lambda|_{\Omega_1^\delta}$. By Lemma 2.4 b), we obtain

$$w_\lambda \leq W \quad \text{in } \Omega_1^\delta,$$

since δ is arbitrarily small,

$$w_\lambda \leq W \quad \text{in } \Omega_1.$$

Thus, letting $\lambda \uparrow \lambda_{0,1}$ yields

$$\lim_{\lambda \uparrow \lambda_{0,1}} w_\lambda(x) < \infty \quad \forall x \in \Omega_1,$$

which completes the proof of theorem. □

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