



# Existence results for the radiation hydrodynamic equations with degenerate viscosity coefficients and vacuum

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Received 9 September 2017; revised 18 February 2018

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## Abstract

In this paper, we consider the compressible isentropic radiation hydrodynamic equations with density-dependent viscosity coefficients when the initial data are arbitrarily large and include vacuum at least appearing in the far field. Based on some reasonable assumptions for the radiation coefficients, we firstly establish the existence of a unique local regular solution, which implies the existence of the local strong solution. Moreover, we show a blow-up criterion for the regular solution that we obtained.

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*MSC:* primary 35Q30, 35D35 ; secondary 35B44, 35K65

*Keywords:* Navier–Stokes–Boltzmann equations; Regular solutions; Strong solutions; Vacuum; Degenerate viscosity; Blow-up criterion

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## 1. Introduction

The radiation hydrodynamic equations arise in high-temperature plasma [12] and in various astrophysical contexts [13]. The couplings between fluid field and radiation field involve momentum source and energy source depending on the specific radiation intensity driven by the

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<sup>1</sup> Zhigang Wang's research is supported by National Natural Science Foundation of China under grant 11401104 and China Postdoctoral Science Foundation under grant 2015M581579.

so-called radiation transfer equation [21]. If the matter satisfies the local thermodynamical equilibrium (LTE), then the coupled system of Navier–Stokes–Boltzmann (RHD) equations has the following form:

$$\begin{cases} \frac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \left( \rho u + \frac{1}{c^2} F_r \right)_t + \operatorname{div}(\rho u \otimes u + P_r) + \nabla P_m = \operatorname{div} \mathbb{T}, \end{cases} \quad (1.1)$$

where  $t \geq 0$  is the time;  $x \in \mathbb{R}^3$  is the spatial coordinate;  $v \in \mathbb{R}^+$  is the frequency of photon;  $\Omega \in S^2$  ( $S^2$  is the unit sphere in  $\mathbb{R}^3$ ) is the travel direction of photon;  $I(v, \Omega, t, x)$  is the specific radiation intensity;  $\rho(t, x)$  is the density;  $u(t, x)$  is the velocity of the fluid;  $P_m = A\rho^\gamma$  ( $A$  is a positive constant and  $\gamma$  is the adiabatic index.) is the material pressure;  $\mathbb{T}$  is the stress tensor given by

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\operatorname{div} u \mathbb{I}_3, \quad (1.2)$$

where  $\mathbb{I}_3$  is the  $3 \times 3$  unit matrix,  $\mu(\rho) = \alpha\rho$  is the shear viscosity,  $\lambda(\rho) = \beta\rho$  is the second viscosity, where the constant  $\alpha$  and  $\beta$  satisfy

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0. \quad (1.3)$$

The collision term  $A_r$ , the radiation flux  $F_r$  and the radiation pressure tensor  $P_r$  are given by

$$\begin{aligned} A_r &= S - \sigma_a I + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s I' - \sigma'_s I \right) d\Omega' dv', \\ F_r &= \int_0^\infty \int_{S^2} I(v, \Omega, t, x) \Omega d\Omega dv, \quad P_r = \frac{1}{c} \int_0^\infty \int_{S^2} I(v, \Omega, t, x) \Omega \otimes \Omega d\Omega dv, \end{aligned}$$

where  $I' = I(v', \Omega', t, x)$ ;  $S = S(v, \Omega, t, x, \rho) \geq 0$  describes the rate of energy emission because of spontaneous process;  $\sigma_a = \sigma_a(v, \Omega, t, x, \rho) \geq 0$  is the absorption coefficient;  $\sigma_s$  and  $\sigma'_s$  are the differential scattering coefficients with the form

$$\sigma_s = \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho) = O(\rho), \quad \sigma'_s = \sigma_s(v \rightarrow v', \Omega \cdot \Omega', \rho) = O(\rho).$$

In this paper, we look for the local strong solution with the initial data

$$(I, \rho, u)|_{t=0} = (I_0(v, \Omega, x), \rho_0(x), u_0(x)), \quad (v, \Omega, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^3, \quad (1.4)$$

and the far field behavior

$$(I, \rho, u) \rightarrow (0, 0, 0) \quad \text{as} \quad |x| \rightarrow \infty. \quad (1.5)$$

For convenience, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$\begin{aligned} D^{k,r} &= \{f \in L^1_{loc}(\mathbb{R}^3) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty\}, \quad D^k = D^{k,2} (k \geq 2), \\ D^1 &= \{f \in L^6(\mathbb{R}^3) : |f|_{D^1} = |\nabla f|_{L^2} < \infty\}, \quad \|(f, g)\|_X = \|f\|_X + \|g\|_X, \\ \|f\|_s &= \|f\|_{H^s(\mathbb{R}^3)}, \quad |f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \quad |f|_{D^k} = \|f\|_{D^k(\mathbb{R}^3)}, \\ \|f\|_{T;s} &= \|f(v, \Omega, t, x)\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s(\mathbb{R}^3))}. \end{aligned}$$

A detailed study of homogeneous Sobolev space can be found in [10].

Usually it is challenging to study the radiation hydrodynamic equations because of the mathematical difficulty and high complexity of the equations. For the Euler–Boltzmann equations, Jiang–Zhong [12] established the local well-posedness of  $C^1$  solutions to the Cauchy problem away from vacuum and showed that one  $C^1$  solution with the large initial data will blow up in finite time. Jiang–Wang [11] showed that some  $C^1$  solutions cannot exist for all time regardless of the size of the initial disturbance. Li–Zhu [18] established the existence of unique local regular solutions with vacuum on the basis of the theory of quasi-linear symmetric hyperbolic systems and the Minkowski’s inequality, and also showed that the regular solutions will blow up if the initial mass density vanishes in some local domain.

For the Navier–Stokes–Boltzmann equations, Chen–Wang [2] obtained the local well-posedness of classical solutions to the Cauchy problem away from vacuum under some reasonable assumptions. Ducomet–Nečasová [8,9] gained the global weak solutions and their large time behavior in one dimension. Li–Zhu [17] studied the formation of singularities for classical solutions in multi-dimensional space. Buet–Després [1] observed that the associated entropy equation may contain a negative production term for RHD system in contrast with the second law of thermodynamics. Moreover, Ducomet–Feireisl–Nečasová [7] obtained the existence of global weak solution for some RHD model, where the velocity field  $u$  may develop uncontrolled time oscillations on the hypothetical vacuum zones.

Recently, there is some progress on the local theory for strong solutions with the initial data including vacuum. For Navier–Stokes equations with constant viscosity coefficients, Cho and his collaborators [3–5] proved the existence of a local arbitrarily large strong solution by introducing the following initial layer compatibility condition:

$$-\operatorname{div} \mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g$$

for some  $g \in L^2$ . Li–Zhu [19,20] generalized these results to the Navier–Stokes–Boltzmann equations. For Navier–Stokes equations with density-dependent viscosity coefficients, the degeneration of the viscosity in the vacuum region makes the problem more difficult. In order to overcome this obstacle, Li–Pan–Zhu [14,15,24,25] proposed a new quantity  $\nabla \rho / \rho$ , which should belong to space  $L^6 \cap D^1 \cap D^2$ , to establish the local existence of classical solutions with far field vacuum. Some other results on degenerate viscosities and initial vacuum can be seen in [6,16,23].

The main purpose of this paper is to establish the existence of a unique regular solution for Navier–Stokes–Boltzmann equations with density-dependent viscosity coefficients when the initial data are arbitrarily large and include vacuum at least appearing in the far field. The rest of this paper is organized as follows. In Section 2, we introduce some assumptions on the radiation

coefficients and state the main results. In Section 3, we establish the well-posedness of the local regular solutions (see Definition 2.1) based on some a priori estimates independent of the lower bound of the density. In Section 4, we obtain the local existence of the strong solutions (see Definition 2.2) by the conclusions obtained in Section 3. Finally, in Section 5 we show a blow-up criterion for the regular solutions that we obtained.

## 2. Assumptions on the radiation coefficients and main results

In this section, we will introduce some assumptions on the radiation coefficients and state our main results.

### 2.1. Assumptions on the radiation coefficients

Since it is very difficult to evaluate these radiation quantities in quantum mechanics, we have to give the following two assumptions on the structure of the radiation coefficients  $S$ ,  $\sigma_a$ ,  $\sigma_s$  and  $\sigma'_s$ . For convenience, we set  $f(v, \Omega, t, x, \rho^i) = f(\cdot, \cdot, \cdot, \cdot, \rho^i)$ , and  $C > 0$  is a generic constant in this subsection.

**Assumption 2.1 (Emission and absorption coefficients).** Let

$$\begin{cases} S = S(v, \Omega, t, x, \rho) = \rho \bar{S}(v, \Omega, t, x, \rho), & \bar{S} \geq 0, \\ \sigma_a = \sigma_a(v, \Omega, t, x, \rho) = \rho \bar{\sigma}_a(v, \Omega, t, x, \rho), & \bar{\sigma}_a \geq 0. \end{cases}$$

For any  $\phi = \rho^{\gamma-1}$  and  $s = 0, 1, 2$ , we assume

$$\begin{cases} \|\bar{S}\|_{L^1 \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi\|_2)(1 + \|\phi\|_s), \\ \|\bar{S}_t\|_{L^1(\mathbb{R}^+ \times S^2; C([0, T]; L^2))} \leq M(\|\phi\|_2)(1 + |\phi_t|_2), \\ \|\bar{\sigma}_a\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi\|_2)(1 + \|\phi\|_s), \\ \|(\bar{\sigma}_a)_t\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T]; L^2))} \leq M(\|\phi\|_2)(1 + |\phi_t|_2), \end{cases}$$

and for  $s = 0, 1$ ,

$$\begin{cases} \|\bar{S}(\cdot, \cdot, \cdot, \cdot, \rho^2) - \bar{S}(\cdot, \cdot, \cdot, \cdot, \rho^1)\|_{L^1 \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi^1, \phi^2\|_2)\|\phi^2 - \phi^1\|_s, \\ \|\bar{\sigma}_a(\cdot, \cdot, \cdot, \cdot, \rho^2) - \bar{\sigma}_a(\cdot, \cdot, \cdot, \cdot, \rho^1)\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi^1, \phi^2\|_2)\|\phi^2 - \phi^1\|_s, \end{cases}$$

where  $M = M(\cdot) : [0, +\infty) \rightarrow [1, +\infty)$  is a strictly increasing continuous function.

**Remark 2.1.** Due to Assumption 2.1, it is easy to see that for  $s = 0, 1, 2, S$  and  $\sigma_a$  satisfy

$$\begin{cases} \|S\|_{L^1 \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi\|_2) \|\phi\|_2^{\frac{1}{\gamma-1}} (1 + \|\phi\|_s), \\ \|\sigma_a\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \leq M(\|\phi\|_2) \|\phi\|_2^{\frac{1}{\gamma-1}} (1 + \|\phi\|_s), \end{cases}$$

and for  $s = 0, 1$ ,

$$\begin{cases} \|S(\cdot, \cdot, \cdot, \cdot, \rho^2) - S(\cdot, \cdot, \cdot, \cdot, \rho^1)\|_{L^1 \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \\ \leq M(\|(\phi^1, \phi^2)\|_2) \|(\phi^1, \phi^2)\|_2^{\frac{1}{\gamma-1}} \|\phi^2 - \phi^1\|_s, \\ \|\sigma_a(\cdot, \cdot, \cdot, \cdot, \rho^2) - \sigma_a(\cdot, \cdot, \cdot, \cdot, \rho^1)\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^s))} \\ \leq M(\|(\phi^1, \phi^2)\|_2) \|(\phi^1, \phi^2)\|_2^{\frac{1}{\gamma-1}} \|\phi^2 - \phi^1\|_s. \end{cases}$$

**Assumption 2.2 (Differential scattering coefficients).** Let

$$\sigma_s = \rho \bar{\sigma}_s(v' \rightarrow v, \Omega' \cdot \Omega), \quad \text{and} \quad \sigma'_s = \rho \bar{\sigma}'_s(v \rightarrow v', \Omega \cdot \Omega'),$$

where  $\bar{\sigma}_s \geq 0$  and  $\bar{\sigma}'_s \geq 0$  satisfy

$$\begin{aligned} \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\bar{\sigma}_s|^2 d\Omega' dv' \right)^{\lambda_1} d\Omega dv &\leq C, \\ \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \bar{\sigma}'_s d\Omega' dv' \right)^{\lambda_2} d\Omega dv + \int_0^\infty \int_{S^2} \bar{\sigma}'_s d\Omega' dv' &\leq C, \end{aligned}$$

where  $\lambda_1 = 1$  or  $\frac{1}{2}$ , and  $\lambda_2 = 1$  or  $2$ .

**Remark 2.2.** In [21], the expressions of  $\sigma_a$  and  $\sigma_s$  describing Compton Scattering process are given by

$$\sigma_a(v, \Omega, t, x, \rho, \theta) = D_1 \rho \theta^{-\frac{1}{2}} \exp\left(-\frac{D_2}{\theta^{\frac{1}{2}}}\left(\frac{v-v_0}{v_0}\right)^2\right), \quad \sigma_s = \bar{\sigma}_s(v \rightarrow v', \Omega \cdot \Omega') \rho, \quad (2.1)$$

where  $v_0$  is the fixed frequency,  $D_i$  ( $i = 1, 2$ ) are positive constants. By view of the laws of Boyle and Gay–Lussac for ideal gas, it is easy to see that  $\sigma_a$  satisfies Assumption 2.1.

## 2.2. Main results

We introduce the new variable  $\phi(t, x) = \rho^{\gamma-1}$  and  $\psi = \frac{1}{\gamma-1} \nabla \phi / \phi = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})^\top$ , and problem (1.1)–(1.5) can be rewritten into

$$\begin{cases} \frac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\ \phi_t + u \cdot \nabla \phi + (\gamma - 1) \phi \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u + \frac{A\gamma}{\gamma-1} \nabla \phi + Lu = \psi \cdot Q(u) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv, \\ (I, \phi, u)|_{t=0} = (I_0, \phi_0, u_0), \quad (v, \Omega, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^3, \\ (I, \phi, u) \rightarrow (0, 0, 0) \quad \text{as } |x| \rightarrow \infty, \quad t > 0, \end{cases} \quad (2.2)$$

where  $L$  is the so-called Lamé operator given by

$$Lu = -\alpha \Delta u - (\alpha + \beta) \nabla \operatorname{div} u,$$

$Q(u)$  are given by

$$Q(u) = \alpha (\nabla u + (\nabla u)^\top) + \beta \operatorname{div} u \mathbb{I}_3,$$

and

$$\bar{A}_r = \bar{S} - \bar{\sigma}_a I + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \bar{\sigma}_s I' - \bar{\sigma}_s' I \right) d\Omega' dv'.$$

Similar to [20], the regular solutions of problem (1.1)–(1.4)–(1.5) can be defined as follows.

**Definition 2.1 (Regular solutions).** Let  $T > 0$  be a finite constant.  $(I, \phi, u)$  is called a regular solution to Cauchy problem (1.1)–(1.4)–(1.5) in  $\mathbb{R}^+ \times S^2 \times [0, T] \times \mathbb{R}^3$  if  $(I, \phi, u)$  satisfies

- (A)  $(I, \phi, u)$  satisfies (2.2) a.e. in  $(t, x) \in \mathbb{R}^+ \times S^2 \times (0, T] \times \mathbb{R}^3$ ;
- (B)  $I \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2))$ ,  $I_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1))$ ;
- (C)  $\phi \geq 0$ ,  $\phi \in C([0, T]; H^2)$ ,  $\phi_t \in C([0, T]; H^1)$ ;
- (D)  $\psi \in C([0, T]; D^1)$ ,  $\psi_t \in C([0, T]; L^2)$ ;
- (E)  $u \in C([0, T]; H^2) \cap L^2([0, T]; D^3)$ ,  $u_t \in C([0, T]; L^2) \cap L^2([0, T]; D^1)$ .

And the strong solution of problem (1.1)–(1.4)–(1.5) can be given as

**Definition 2.2 (Strong solutions).** Let  $T > 0$  be a finite constant.  $(I, \rho, u)$  is called a strong solution to Cauchy problem (1.1)–(1.4)–(1.5) in  $\mathbb{R}^+ \times S^2 \times [0, T] \times \mathbb{R}^3$  if  $(I, \rho, u)$  satisfies

- (A1)  $(I, \rho, u)$  satisfies (1.1)–(1.4)–(1.5) a.e. in  $(t, x) \in \mathbb{R}^+ \times S^2 \times (0, T] \times \mathbb{R}^3$ ;  
 (B1)  $I \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2))$ ,  $I_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1))$ ;  
 (C1)  $\rho \geq 0$ ,  $\rho \in C([0, T]; H^2)$ ,  $\rho_t \in C([0, T]; H^1)$ ;  
 (D1)  $u \in C([0, T]; H^2) \cap L^2([0, T]; D^3)$ ,  $u_t \in C([0, T]; L^2) \cap L^2([0, T]; D^1)$ ;  
 (E1)  $u_t + u \cdot \nabla u + Lu = (\nabla \rho / \rho) \cdot Q(u) - \frac{1}{c} \int_0^\infty \bar{A}_r \Omega d\Omega dv$  holds when  $\rho(t, x) = 0$ .

**Remark 2.3.** The conditions (C) or (C1) mean that the vacuum must appear at least in the far field.

Now we state the main existence results in this paper.

**Theorem 2.3 (Existence of the unique local regular solution).** *Let  $1 < \gamma \leq \frac{3}{2}$  or  $\gamma = 2$ . If the initial data  $(I_0, \phi_0, u_0)$  satisfies the regularity condition*

$$I_0 \in L^2(\mathbb{R}^+ \times S^2; H^2), \quad \phi_0 \geq 0, \quad (\phi_0, u_0) \in H^2, \quad \psi_0 \in D^1, \quad (2.3)$$

*then there exists a small time  $T_*$  and a unique regular solution  $(I, \phi, u)$  to Cauchy problem (1.1)–(1.4)–(1.5). Moreover, we also have  $\phi(t, x) \in C([0, T_*] \times \mathbb{R}^3)$ .*

**Remark 2.4.** This set of initial data contains a large class of functions, for example,

$$I_0 \in L^2(\mathbb{R}^+ \times S^2; C_0^2(\mathbb{R}^3)), \quad \rho_0(x) = \frac{1}{1 + |x|^{2\sigma}}, \quad u_0(x) \in C_0^2(\mathbb{R}^3),$$

where  $\sigma > \frac{3}{4(\gamma-1)}$ .

In addition, we use the variable  $\phi(t, x) = \rho^{\gamma-1}$  instead of the variable  $\phi(t, x) = \sqrt{A\gamma} \rho^{\frac{\gamma-1}{2}}$  in [24]. Although the range of  $\gamma$  becomes smaller (from  $1 < \gamma \leq 2$  or  $\gamma = 3$  to  $1 < \gamma \leq \frac{3}{2}$  or  $\gamma = 2$ ), the requirement for the initial density become weaker (from  $\rho_0^{\frac{\gamma-1}{2}} \in H^2, \nabla \rho_0 / \rho_0 \in D^1$  to  $\rho_0^{\gamma-1} \in H^2, \nabla \rho_0 / \rho_0 \in D^1$ ).

**Remark 2.5.** The above results obtained in Theorems 2.3 still work for function pairs  $(\mu(\rho) = \alpha\rho, \lambda(\rho) = \rho E(\rho))$  satisfying

$$\alpha > 0, \quad 2\alpha + 3E(\rho) \geq 0, \quad \text{and} \quad E(\rho) \in C^2(\bar{\mathbb{R}}^+).$$

Based on the results obtained in Theorem 2.3 and the standard quasi-linear hyperbolic equations theory, we quickly gain the following conclusion.

**Corollary 2.1 (Existence of strong solutions).** *Let  $1 < \gamma \leq \frac{3}{2}$  or  $\gamma = 2$ . If the initial data  $(I_0, \phi_0, u_0)$  satisfies the regularity condition (2.3), then there is at least a strong solution for problem (1.1)–(1.4)–(1.5).*

Finally, we show a blow-up criterion for regular solutions obtained in Theorem 2.3.

**Theorem 2.4 (Blow-up criterion for the local regular solution).** Assume that the conditions in Theorem 2.3 hold. If  $\bar{T} < +\infty$  is the maximal existence time of the local regular solution  $(I, \phi, u)$ , then we have

$$\limsup_{t \rightarrow \bar{T}} \left( \|I(t)\|_{L^2(\mathbb{R}^+ \times S^2; H^2)} + \|\phi(t)\|_2 + \|u(t)\|_1 + \|\psi(t)\|_{D^1} \right) = +\infty. \quad (2.4)$$

### 3. Existence of the unique regular solution

In this section, we will prove the existence of the unique regular solution for problem (1.1)–(1.4)–(1.5) (Theorem 2.3).

#### 3.1. Linearization

We now are concerned with the following linearized system

$$\begin{cases} \phi_t + \omega \cdot \nabla \phi + (\gamma - 1)\phi \operatorname{div} \omega = 0, \\ \frac{1}{c} I_t + \Omega \cdot \nabla I = \tilde{A}_r, \\ u_t + \omega \cdot \nabla u + A\gamma\theta \nabla \phi + Lu = \psi \cdot Q(\omega) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv, \end{cases} \quad (3.1)$$

which satisfies the following initial data

$$(I, \phi, \psi, u)|_{t=0} = (I_0, \phi_0, \psi_0, u_0), \quad x \in \mathbb{R}^3, \quad (3.2)$$

where  $\theta = \frac{1}{\gamma-1}$ ,

$$\tilde{A}_r = S - \sigma_a I + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s \chi - \sigma'_s I \right) d\Omega' dv',$$

$\chi$  is a known scalar function and  $\omega = (\omega^{(1)}, \omega^{(2)}, \omega^{(3)})^\top \in \mathbb{R}^3$  is a known vector satisfying  $(\chi, \omega)(t=0, x) = (I_0, u_0)$  and

$$\begin{aligned} \chi &\in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2)), \quad \chi_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1)); \\ \omega &\in C([0, T]; H^2) \cap L^2([0, T]; D^3), \quad \omega_t \in C([0, T]; L^2) \cap L^2([0, T]; D^1). \end{aligned} \quad (3.3)$$

Assume that

$$I_0 \in L^2(\mathbb{R}^+ \times S^2; H^2), \quad \phi_0 \geq 0, \quad (\phi_0 - \phi^\infty, u_0) \in H^2, \quad \psi_0 \in D^1 \quad (3.4)$$

where  $\phi^\infty \geq 0$  is a constant, then we can obtain the following existence of a strong solution  $(I, \phi, \psi, u)$  to (3.1)–(3.2) by the standard methods when the initial data is away from vacuum.



**Lemma 3.1.** *If the initial data (3.2) satisfy (3.4) and  $\phi_0 > \delta$  for some positive constant, then there exists a unique strong solution  $(I, \phi, \psi, u)$  to (3.1)–(3.3) satisfying*

$$\begin{aligned} I &\in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2)), \quad I_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1)); \\ \phi &\geq \underline{\delta}, \quad \phi - \phi^\infty \in C([0, T]; H^2), \quad \phi_t \in C([0, T]; H^1), \\ \psi &\in C([0, T]; D^1), \quad \psi_t \in C([0, T]; L^2), \\ u &\in C([0, T]; H^2) \cap L^2([0, T]; D^3), \quad u_t \in C([0, T]; L^2) \cap L^2([0, T]; D^1), \end{aligned} \quad (3.5)$$

where  $\underline{\delta}$  is a positive constant.

**Proof.** Firstly, by the standard hyperbolic theory, we can obtain the existence of the solution  $\phi$  to (3.1)<sub>1</sub>, see Lemma 6 in [5]. And  $\phi$  can be denoted by

$$\phi(t, x) = \phi_0(U(0; t, x)) \exp\left(-(\gamma - 1) \int_0^t \operatorname{div} \omega(s, U(s; t, x)) ds\right), \quad (3.6)$$

where  $U \in C([0, T] \times [0, T] \times \mathbb{R}^3)$  is the solution to the initial value problem

$$\begin{cases} \frac{d}{dt} U(t; s, x) = \omega(t, U(t; s, x)), & 0 \leq t \leq T, \\ U(s; s, x) = x, & 0 \leq s \leq T, \quad x \in \mathbb{R}^3. \end{cases} \quad (3.7)$$

Due to  $\phi_0 > \delta$ , (3.4) and (3.6), we can conclude that there exists a positive constant  $\underline{\delta}$  such that  $\phi \geq \underline{\delta}$ .

Secondly, the equation (3.1)<sub>2</sub> can be rewritten into

$$\frac{1}{c} I_t + \Omega \cdot \nabla I + \left( \sigma_a + \int_0^\infty \int_{S^2} \sigma'_s d\Omega' dv' \right) I = F(v, \Omega, t, x), \quad (3.8)$$

where

$$F(v, \Omega, t, x) = S + \int_0^\infty \int_{S^2} \frac{v}{v'} \sigma_s \chi d\Omega' dv' \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2)).$$

Then, it is easy to get the existence of a unique solution  $I$  satisfying

$$I \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2)), \quad I_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1)).$$

Thirdly, since  $\phi \geq \underline{\delta}$ , we quickly know that

$$\psi \in C([0, T]; D^1), \quad \psi_t \in C([0, T]; L^2).$$

Finally, by the following linear parabolic equations

$$u_t + \omega \cdot \nabla \omega + A\gamma\theta\nabla\phi + Lu = \psi \cdot \mathcal{Q}(\omega) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv,$$

and the classical Galerkin methods (see [4,5]), we can obtain the desired conclusions for the velocity  $u$ . Therefore, we complete the proof of this Lemma.  $\square$

### 3.2. A prior estimate

Let  $(I, \phi, \psi, u)$  be the unique strong solution to (3.1)–(3.3). We will prove some a prior estimates on  $(I, \phi, \psi, u)$  independent of the lower bound  $\delta$  of  $\phi_0$ . We choose a positive constant  $b_0$  large enough such that

$$2 + \phi^\infty + |\phi_0|_\infty + \|\phi_0 - \phi^\infty\|_2 + |\psi_0|_{D^1} + \|u_0\|_2 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; H^2)} \leq b_0, \quad (3.9)$$

and

$$\begin{aligned} \sup_{0 \leq t \leq T^*} \|\chi\|_{L^2(\mathbb{R}^+ \times S^2; H^2)}^2 &\leq b_1^2, \\ \sup_{0 \leq t \leq T^*} \|\chi_t\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 &\leq b_2^2, \\ \sup_{0 \leq t \leq T^*} |\omega(t)|_2^2 + \int_0^{T^*} |\nabla \omega(t)|_2^2 dt &\leq b_3^2, \\ \sup_{0 \leq t \leq T^*} |\omega(t)|_{D^1}^2 + \int_0^{T^*} (|\omega(t)|_{D^2}^2 + |\omega_t(t)|_2^2) dt &\leq b_4^2, \\ \sup_{0 \leq t \leq T^*} (|\omega(t)|_{D^2}^2 + |\omega_t(t)|_2^2) + \int_0^{T^*} (|\omega(t)|_{D^3}^2 + |\omega_t(t)|_{D^1}^2) dt &\leq b_5^2 \end{aligned} \quad (3.10)$$

for some time  $T^* \in (0, T)$  and constants  $b_i$  ( $i = 1, 2, 3, 4, 5$ ) such that  $1 < b_0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$ . We will determine  $b_i$  and  $T^*$  later (see (3.50)). Let  $C$  be a generic positive constant depending only on fixed constants  $\alpha, \beta, A, \gamma$  and  $T$ .

Firstly, we give some estimates of  $\phi$ .

**Lemma 3.2 (Estimates of  $\phi$ ).**

$$|\phi(t)|_\infty^2 + \|\phi(t) - \phi^\infty\|_2^2 \leq Cb_0^2, \quad |\phi_t(t)|_2 \leq Cb_0b_4, \quad |\phi_t|_{D^1} \leq Cb_0b_5,$$

for  $0 \leq t \leq T_1 = \min(T^*, (1 + b_5)^{-2})$ .

**Proof.** Using the standard energy estimate theories introduced in [5], it is easy to derive that

$$\|\phi(t) - \phi^\infty\|_2 \leq \left( \|\phi_0 - \phi^\infty\|_2 + \phi^\infty \int_0^t \|\nabla \omega(s)\|_2 ds \right) \exp \left( C \int_0^t \|\nabla \omega(s)\|_2 ds \right).$$

Since

$$\int_0^t \|\nabla \omega(s)\|_2 ds \leq t^{\frac{1}{2}} \left( \int_0^t \|\nabla \omega(s)\|_2^2 ds \right)^{\frac{1}{2}} \leq C(b_4 t + b_5 t^{\frac{1}{2}}),$$

we have

$$\|\phi - \phi^\infty\|_2 \leq C b_0^2$$

for  $0 \leq t \leq T_1 = \min(T^*, (1 + b_5)^{-2})$ .

By view of the following relation

$$\phi_t = -\omega \cdot \nabla \phi - (\gamma - 1)\phi \operatorname{div} \omega,$$

we also have, for  $0 \leq t \leq T_1$ ,

$$\begin{cases} |\phi_t(t)|_2 \leq C(|\omega(t)|_6 |\nabla \phi(t)|_3 + |\phi(t)|_\infty |\operatorname{div} \omega(t)|_2) \leq C b_0 b_4, \\ |\phi_t(t)|_{D^1} \leq C(|\omega(t)|_\infty |\phi(t)|_{D^2} + |\phi(t)|_\infty |\omega(t)|_{D^2} + |\nabla \phi(t)|_6 |\nabla \omega(t)|_3) \leq C b_0 b_5. \end{cases} \quad (3.11)$$

Therefore, we complete the proof.  $\square$

From [24], we know that  $\psi$  satisfies the following positive symmetric hyperbolic system:

$$\psi_t + \sum_{l=1}^3 A_l \partial_l \psi + B \psi + \nabla \operatorname{div} \omega = 0, \quad \psi_0 \in D^1, \quad (3.12)$$

where

$$A_l = (a_{ij}^l)_{3 \times 3}, \quad \text{for } i, j, l = 1, 2, 3$$

are symmetric with

$$a_{ij}^l = \omega^{(l)} \quad \text{for } i = j; \quad \text{otherwise } a_{ij}^l = 0,$$

and  $B = (\nabla \omega)^\top$ . By the standard energy estimate theory, we can obtain some a priori estimates of  $\psi$  in the following lemma, which are used to deal with the degenerate Lamé operator when vacuum appears.

**Lemma 3.3** (*Estimates of  $\psi$* ).

$$|\psi(t)|_{D^1}^2 \leq Cb_0^2, \quad |\psi(t)_t|_2^2 \leq Cb_5^4, \quad 0 \leq t \leq T_1.$$

**Proof.** The proof is similar as the proof of Lemma 3.3 in [24], we omit it here.  $\square$

Now, we establish the a prior estimates of  $I$ .

**Lemma 3.4** (*Estimates of  $I$* ).

$$\begin{aligned} \|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_2]; H^2))} &\leq Cb_0, \\ \|I_t\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_2]; H^1))} &\leq M(b_0)b_0^{\theta+1}b_1, \end{aligned} \quad (3.13)$$

for  $T_2 = \min(T_*, (1 + M(b_0)b_0^{2\theta}b_1^2)^{-1})$ .

**Proof.** Multiplying (3.1)<sub>2</sub> by  $2I$  and integrating over  $\mathbb{R}^3$  for  $x$ , we can obtain

$$\begin{aligned} \frac{d}{dt}|I|_2^2 &\leq C|S|_2|I|_2 + C|\phi^\theta|_\infty|I|_2\|\chi\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \left( \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 \bar{\sigma}_s^2 d\Omega' dv' \right)^{\frac{1}{2}} \\ &\leq C \left( |I|_2^2 + |S|_2^2 + b_0^{2\theta}b_1^2 \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 \bar{\sigma}_s^2 d\Omega' dv' \right), \end{aligned} \quad (3.14)$$

where we use the condition that  $\sigma_a \geq 0$  and  $\sigma'_s \geq 0$ .

Differentiating (3.1)<sub>2</sub>  $\zeta$ -times with respect to  $x$ , multiplying the resulting equation by  $2\partial_x^\zeta I$  and integrating over  $\mathbb{R}^3$  on  $x$ , we have

$$\begin{aligned} \frac{d}{dt} \int \frac{1}{c} |\partial_x^\zeta I|^2 dx &+ \int \left( \sigma_a + \int_0^\infty \int_{S^2} \sigma'_s d\Omega' dv' \right) (\partial_x^\zeta I)^2 dx \\ &= - \int \left( \partial_x^\zeta (\sigma_a I) - \sigma_a \partial_x^\zeta I \right) \partial_x^\zeta I dx \\ &- \int \int_0^\infty \int_{S^2} \bar{\sigma}'_s \left( \partial_x^\zeta (\phi^\theta I) - \phi^\theta \partial_x^\zeta I \right) d\Omega' dv' \cdot \partial_x^\zeta I dx \\ &+ \int \partial_x^\zeta S \partial_x^\zeta I dx + \int \left( \int_0^\infty \int_{S^2} \frac{v}{v'} \bar{\sigma}_s \partial_x^\zeta (\phi^\theta \chi) d\Omega' dv' \right) \partial_x^\zeta I dx \equiv: \sum_{i=1}^4 R_i. \end{aligned} \quad (3.15)$$

Next, we consider these terms on the right-hand side of (3.15) when  $|\zeta| \leq 2$ . For the term  $R_1$ , we have

$$\begin{aligned}
R_1 &= - \int \left( \partial_x^\zeta (\sigma_a I) - \sigma_a \partial_x^\zeta I \right) \partial_x^\zeta I dx \\
&\leq C |\nabla \sigma_a|_\infty |I|_2 |\nabla I|_2 \leq M(b_0) b_0^{\theta+1} \|I\|_1^2 \quad \text{when } |\zeta| = 1; \\
R_1 &\leq C (|\nabla^2 \sigma_a|_2 |I|_\infty + |\nabla \sigma_a|_3 |\nabla I|_6) |\nabla^2 I|_2 \\
&\leq M(b_0) b_0^{\theta+1} \|I\|_2^2 \quad \text{when } |\zeta| = 2.
\end{aligned} \tag{3.16}$$

For the term  $R_2$ , we have

$$\begin{aligned}
R_2 &= - \int \int_0^\infty \int_{S^2} \overline{\sigma}'_s \left( \partial_x^\zeta (\phi^\theta I) - \phi^\theta \partial_x^\zeta I \right) d\Omega' dv' \cdot \partial_x^\zeta I dx; \\
&\leq C |\nabla \phi^\theta|_3 |I|_6 |\nabla I|_2 \int_0^\infty \int_{S^2} \overline{\sigma}'_s d\Omega' dv' \leq C b_0^\theta |\nabla I|_2^2 \quad \text{when } |\zeta| = 1; \\
&\leq C \left( |\nabla^2 \phi^\theta|_2 |I|_\infty + |\nabla \phi^\theta|_3 |\nabla I|_6 \right) |\nabla^2 I|_2 \leq C b_0^\theta \|I\|_2^2 \quad \text{when } |\zeta| = 2.
\end{aligned} \tag{3.17}$$

For the term  $R_3$ , we have

$$R_3 = \int \partial_x^\zeta S \partial_x^\zeta I dx \leq C |\partial_x^\zeta S|_2 |\partial_x^\zeta I|_2 \leq C (|\partial_x^\zeta S|_2^2 + |\partial_x^\zeta I|_2^2). \tag{3.18}$$

For the term  $R_4$ , we have

$$\begin{aligned}
R_4 &= \int \left( \int_0^\infty \int_{S^2} \frac{v}{v'} \overline{\sigma}'_s \partial_x^\zeta (\phi^\theta \chi) d\Omega' dv' \right) \partial_x^\zeta I dx \\
&\leq \int_0^\infty \int_{S^2} \frac{v}{v'} \overline{\sigma}'_s \left( |\nabla \phi^\theta|_2 |\chi|_\infty + |\phi^\theta|_\infty |\chi|_2 \right) |\partial_x^\zeta I|_2 d\Omega' dv' \\
&\leq C b_0^\theta \left( \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\overline{\sigma}'_s|^2 d\Omega' dv' \right)^{\frac{1}{2}} \left( \int_0^\infty \int_{S^2} \|\chi\|_2^2 d\Omega' dv' \right)^{\frac{1}{2}} |\partial_x^\zeta I|_2 \\
&\leq C |\partial_x^\zeta I|_2^2 + b_0^{2\theta} b_1^2 \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\overline{\sigma}'_s|^2 d\Omega' dv' \quad \text{when } |\zeta| = 1; \\
R_4 &\leq C \int_0^\infty \int_{S^2} \frac{v}{v'} \overline{\sigma}'_s \left( |\nabla^2 \phi^\theta|_2 |\chi|_\infty + |\nabla \phi^\theta|_3 |\nabla \chi|_6 + |\phi^\theta|_\infty |\nabla^2 \chi|_2 \right) |\partial_x^\zeta I|_2 d\Omega' dv' \\
&\leq C |\partial_x^\zeta I|_2^2 + b_0^{2\theta} b_1^2 \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\overline{\sigma}'_s|^2 d\Omega' dv' \quad \text{when } |\zeta| = 2.
\end{aligned} \tag{3.19}$$

Thus, we can conclude that

$$\frac{d}{dt} \|I\|_2^2 \leq M(b_0)b_0^{\theta+1} \|I\|_2^2 + C \|S\|_2^2 + b_0^{2\theta} b_1^2 \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\bar{\sigma}_s|^2 d\Omega' dv'. \quad (3.20)$$

By Gronwall's inequality, we have

$$\begin{aligned} & \|I(v, \Omega, t, x)\|_{T_2;2}^2 \\ & \leq \exp(M(b_0)b_0^{\theta+1} T_2) \left( \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; H^2)}^2 + M(b_0)b_0^{2\theta} b_1^2 T_2 \right) \leq C b_0^2. \end{aligned} \quad (3.21)$$

Moreover, using the equation (3.1)<sub>2</sub>, we can derive the desired estimates for  $I_t$ .

$$\begin{aligned} \|I_t\|_{T_2;1} & \leq C \left( \|I\|_{T_2;2} + \|S\|_{T_2;1} + \|\sigma_a\|_2 \|I\|_{T_2;1} + \int_0^\infty \int_{S^2} |\bar{\sigma}'_s| d\Omega' dv' \|\phi^\theta\|_1 \|I\|_1 \right. \\ & \quad \left. + \left( \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 |\bar{\sigma}_s|^2 d\Omega' dv' \right)^{\frac{1}{2}} \|\phi^\theta\|_1 \|\chi\|_{L^2(\mathbb{R}^+ \times S^2; H^1)} \right) \leq M(b_0)b_0^{\theta+1} b_1. \quad \square \end{aligned} \quad (3.22)$$

Now we begin to estimate the lower order regularity of the velocity  $u$ .

**Lemma 3.5** (Lower order estimates of the velocity  $u$ ).

$$|u(t)|_2^2 + \int_0^t |\nabla u(s)|_2^2 ds \leq C b_0^2$$

for  $0 \leq t \leq T_3 = \min(T^*, (1 + M(b_0)b_5^4)^{-1})$ .

**Proof.** Multiplying (3.1)<sub>3</sub> by  $u$  and integrating over  $\mathbb{R}^3$ , we can obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u|_2^2 + \alpha |\nabla u|_2^2 + (\alpha + \beta) |\operatorname{div} u|_2^2 \\ & = \int_{\mathbb{R}^3} \left( -\omega \cdot \nabla \omega - A\gamma\theta \nabla \phi + \psi \cdot \mathcal{Q}(\omega) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv \right) \cdot u dx \equiv: \sum_{i=1}^4 I_i. \end{aligned} \quad (3.23)$$

According to Hölder's inequality, Gagliardo–Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}
I_1 &= - \int_{\mathbb{R}^3} \omega \cdot \nabla \omega \cdot u \, dx \leq C |\omega|_3 |\nabla \omega|_2 |u|_6 \leq C |\omega|_3^2 |\nabla \omega|_2^2 + \frac{\alpha}{10} |\nabla u|_2^2, \\
I_2 &= - \int_{\mathbb{R}^3} A \gamma \theta \nabla \phi \cdot u \, dx \leq C |\nabla \phi|_2 |u|_2 \leq C |u|_2^2 + C |\nabla \phi|_2^2, \\
I_3 &= \int_{\mathbb{R}^3} \psi \cdot Q(\omega) \cdot u \, dx \leq C |\psi|_6 |\nabla \omega|_3 |u|_2 \leq C |u|_2^2 + C |\psi|_6^2 |\nabla \omega|_3^2,
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} -\frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega \, d\Omega \, dv \cdot u \, dx \\
&= -\frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \left( \bar{S} - \bar{\sigma}_a I + \int_0^\infty \int_{S^2} \frac{v}{v'} \bar{\sigma}_s I' \, d\Omega' \, dv' \right) u \cdot \Omega \, dx \, d\Omega \, dv \\
&\quad + \frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \bar{\sigma}'_s I u \cdot \Omega \, d\Omega' \, dv' \, dx \, d\Omega \, dv := \sum_{j=1}^4 I_{4j},
\end{aligned}$$

where

$$\begin{aligned}
I_{41} &= -\frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \bar{S} u \cdot \Omega \, dx \, d\Omega \, dv \leq C \|\bar{S}\|_{L^1(\mathbb{R}^+ \times S^2; L^2)} |u|_2, \\
I_{42} &= \frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \bar{\sigma}_a I u \cdot \Omega \, dx \, d\Omega \, dv \leq C \|\bar{\sigma}_a\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} |u|_2, \\
I_{43} &= -\frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \frac{v}{v'} \bar{\sigma}_s I' \, d\Omega' \, dv' u \cdot \Omega \, dx \, d\Omega \, dv \\
&\leq C \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right|^2 \bar{\sigma}_s^2 \, d\Omega' \, dv' \right)^{\frac{1}{2}} \, d\Omega \, dv \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} |u|_2, \\
I_{44} &= \frac{1}{c} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \bar{\sigma}'_s I \, d\Omega' \, dv' u \cdot \Omega \, dx \, d\Omega \, dv \\
&\leq C \left( \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \bar{\sigma}'_s \, d\Omega' \, dv' \right)^2 \, d\Omega \, dv \right)^{\frac{1}{2}} \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} |u|_2.
\end{aligned}$$

Then we have

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 + \alpha |\nabla u|_2^2 \leq C |u|_2^2 + M(b_0) b_5^4. \quad (3.24)$$

Integrating (3.24) over  $(0, t)$ , we have

$$|u(t)|_2^2 + \int_0^t \alpha |\nabla u(s)|_2^2 ds \leq C \int_0^t |u(s)|_2^2 ds + C |u_0|_2^2 + M(b_0) b_5^4 t.$$

By view of Gronwall's inequality, we can derive that for  $0 \leq t \leq T_3$

$$|u(t)|_2^2 + \int_0^t \alpha |\nabla u(s)|_2^2 ds \leq C (|u_0|_2^2 + M(b_0) b_5^4 t) \exp(Ct) \leq C b_0^2. \quad (3.25)$$

Therefore, we complete the proof of this lemma.  $\square$

Next, in order to estimate the higher order regularity of the velocity  $u$ , we have to introduce the effective viscous flux  $F$  and vorticity  $\varpi$ , which can be given as

$$F = (2\alpha + \beta) \operatorname{div} u - A\gamma\theta(\phi - \phi^\infty), \quad \varpi = \nabla \times u. \quad (3.26)$$

By the momentum equations (3.1)<sub>3</sub>, we know that in the sense of distribution  $F$  and  $\varpi$  satisfy

$$\begin{cases} \Delta F = \operatorname{div} \left( u_t + \omega \cdot \nabla \omega - \psi \cdot Q(\omega) + \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv \right), \\ \Delta \varpi = \frac{1}{\alpha} \nabla \times \left( u_t + \omega \cdot \nabla \omega - \psi \cdot Q(\omega) + \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv \right). \end{cases} \quad (3.27)$$

So we immediately have

$$-\Delta u = \nabla \times \varpi - \nabla \operatorname{div} u = \nabla \times \varpi - \nabla \left( \frac{F + A\gamma\theta(\phi - \phi^\infty)}{2\alpha + \beta} \right). \quad (3.28)$$

**Lemma 3.6** (*Higher order estimates of the velocity  $u$* ).

$$|u(t)|_{D^1}^2 + \int_0^t \left( |u_t(s)|_2^2 + |u(s)|_{D^2}^2 \right) ds \leq C b_0^2,$$

$$|u(t)|_{D^2}^2 + |u_t(t)|_2^2 + \int_0^t \left( |u(s)|_{D^3}^2 + |u_t(s)|_{D^1}^2 \right) ds \leq M(b_0) b_4^3 b_5,$$

for  $0 \leq t \leq T_4 = \min(T^*, (1 + M(b_0) b_5^{2\theta+6})^{-1})$ .



**Proof.** Step 1. From (3.28), we use the standard elliptic estimate to obtain

$$\begin{aligned} |u|_{D^2} &\leq C(|\nabla \times \varpi|_2 + |\nabla F|_2 + |\nabla \phi|_2), \\ &\leq C(|\nabla \varpi|_2 + |\nabla F|_2 + b_0). \end{aligned}$$

From (3.27), again via the standard elliptic estimate, we have

$$\begin{aligned} |\nabla \varpi|_2 + |\nabla F|_2 &\leq C(|u_t|_2 + |\omega|_6 |\nabla \omega|_3 + |\psi|_6 |Q(\omega)|_3 + \int_0^\infty \int_{S^2} |\bar{A}_r|_2 \Omega d\Omega dv) \\ &\leq C(M(b_0) b_4^{\frac{3}{2}} b_5^{\frac{1}{2}} + |u_t|_2), \end{aligned} \quad (3.29)$$

where we also use the fact

$$\begin{aligned} \int_0^\infty \int_{S^2} |\bar{A}_r|_2 \Omega d\Omega dv \\ \leq \|\bar{S}\|_{L^1(\mathbb{R}^+ \times S^2; L^2)} + (\|\bar{\sigma}_a\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} + C) \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \leq M(b_0) b_0^2. \end{aligned}$$

Then, we can deduce that

$$|u|_{D^2} \leq C(|u_t|_2 + M(b_0) b_4^{\frac{3}{2}} b_5^{\frac{1}{2}}). \quad (3.30)$$

Step 2 (Estimate of  $|\nabla u|_2$ ). Multiplying (3.1)<sub>3</sub> by  $u_t$  and integrating over  $\mathbb{R}^3$ , we can see that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\alpha |\nabla u|^2 + (\alpha + \beta) |\operatorname{div} u|^2) dx + |u_t|_2^2 \\ &= \int_{\mathbb{R}^3} (-\omega \cdot \nabla \omega - A\gamma \theta \nabla \phi + \psi \cdot Q(\omega) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r \Omega d\Omega dv) \cdot u_t dx \equiv: \sum_{i=5}^8 I_i. \end{aligned} \quad (3.31)$$

According to Hölder's inequality, Gagliardo–Nirenberg inequality, Young's inequality, we have

$$\begin{aligned} I_5 &= - \int_{\mathbb{R}^3} (\omega \cdot \nabla \omega) \cdot u_t dx \leq C|\omega|_\infty |\nabla \omega|_2 |u_t|_2 \leq C|\nabla \omega|_2^3 |\nabla^2 \omega|_2 + \frac{1}{10} |u_t|_2^2, \\ I_6 &= - \int_{\mathbb{R}^3} A\gamma \theta \nabla \phi \cdot u_t dx \leq C|\nabla \phi|_2 |u_t|_2 \leq \frac{1}{10} |u_t|_2^2 + C|\nabla \phi|_2^2, \\ I_7 &= \int_{\mathbb{R}^3} \psi \cdot Q(\omega) \cdot u_t dx \leq C|u_t|_2 |\psi|_6 |Q(\omega)|_3 \leq \frac{1}{10} |u_t|_2^2 + C|\psi|_6^2 |\nabla \omega|_3^2, \end{aligned} \quad (3.32)$$

$$\begin{aligned}
I_8 &= -\frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \bar{A}_r u_t \cdot \Omega d\Omega dv dx \\
&\leq \frac{1}{10} |u_t|_2^2 + C(\|\bar{S}\|_{L^1(\mathbb{R} \times S^2; L^2)}^2 + \|\bar{\sigma}_a\|_{L^2(\mathbb{R} \times S^2; L^\infty)}^2 + \|I\|_{L^2(\mathbb{R} \times S^2; L^2)}^2)
\end{aligned}$$

By (3.31)–(3.32), we have

$$\frac{d}{dt} |\nabla u|_2^2 + |u_t|_2^2 \leq M(b_0) b_5^4. \quad (3.33)$$

Thus, we can conclude that for  $0 \leq t \leq T_4$

$$|\nabla u(t)|_2^2 + \int_0^t |u_t|_2^2 ds \leq C(|\nabla u_0|_2^2 + M(b_0) b_5^4 t) \leq C b_0^2. \quad (3.34)$$

Along with (3.30), we can also obtain for  $0 \leq t \leq T_4$

$$\int_0^t |u|_{D^2}^2 ds \leq C \int_0^t \left( |u_t|_2 + M(b_0) b_4^{\frac{3}{2}} b_5^{\frac{1}{2}} \right)^2 ds \leq C b_0^2.$$

**Step 3 (Estimate of  $|u_t|_2$ ).** We consider the estimate of  $|u_t|_2$ . Differentiating (3.1)<sub>3</sub> with respect to  $t$ , we have

$$u_{tt} + (Lu)_t = -(\omega \cdot \nabla \omega)_t - A\gamma\theta(\nabla\phi)_t + (\psi \cdot \mathcal{Q}(\omega))_t - \frac{1}{c} \int_0^\infty \int_{S^2} (\bar{A}_r)_t \Omega d\Omega dv. \quad (3.35)$$

Then, multiplying (3.35) by  $u_t$  and integrating over  $\mathbb{R}^3$  we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |u_t|_2^2 + \alpha |\nabla u_t|_2^2 + (\alpha + \beta) |\operatorname{div} u_t|_2^2 \\
&= \int_{\mathbb{R}^3} \left( (-\omega \cdot \nabla \omega)_t - A\gamma\theta(\nabla\phi)_t + (\psi \cdot \mathcal{Q}(\omega))_t - \frac{1}{c} \int_0^\infty \int_{S^2} (\bar{A}_r)_t \Omega d\Omega dv \right) \cdot u_t dx \\
&\equiv: \sum_{i=9}^{12} I_i.
\end{aligned} \quad (3.36)$$

By Hölder's inequality, Gagliardo–Nirenberg inequality and Young's inequality, we can obtain the following inequality.

$$\begin{aligned}
I_9 &= - \int_{\mathbb{R}^3} (\omega \cdot \nabla \omega)_t \cdot u_t \, dx \leq C(|\omega|_\infty |\nabla \omega_t|_2 |u_t|_2 + |\omega_t|_6 |\nabla \omega|_3 |u_t|_2) \\
&\leq \frac{1}{b_5^2} |\nabla \omega_t|_2^2 + C b_5^2 \|\nabla \omega\|_1^2 |u_t|_2^2, \\
I_{10} &= - \int_{\mathbb{R}^3} A \gamma \theta (\nabla \phi)_t \cdot u_t \, dx \leq C |\phi_t|_{D^1} |u_t|_2.
\end{aligned} \tag{3.37}$$

For  $I_{11}$ , we have

$$I_{11} = \int_{\mathbb{R}^3} \psi \cdot Q(\omega)_t \cdot u_t \, dx + \int_{\mathbb{R}^3} \psi_t \cdot Q(\omega) \cdot u_t \, dx = I_{11A} + I_{11B}. \tag{3.38}$$

We firstly consider the term:

$$\begin{aligned}
I_{11A} &\leq C |\psi|_6 |\nabla \omega_t|_2 |u_t|_3 \\
&\leq \frac{1}{b_5^2} |\nabla \omega_t|_2^2 + \frac{\alpha}{10} |\nabla u_t|_2^2 + C b_0^4 b_5^4 |u_t|_2^2,
\end{aligned} \tag{3.39}$$

where we have used the following inequality

$$|u_t|_3 \leq C |u_t|_2^{\frac{1}{2}} |\nabla u_t|_2^{\frac{1}{2}}. \tag{3.40}$$

And for the second term:

$$\begin{aligned}
I_{11B} &= - \int_{\mathbb{R}^3} \left( \nabla(\omega \cdot \psi) \cdot Q(\omega) \cdot u_t + \nabla \operatorname{div} \omega \cdot Q(\omega) \cdot u_t \right) dx \\
&\leq C \int_{\mathbb{R}^3} (|\omega| |\psi| |\nabla Q(\omega)| |u_t| + |\omega| |\psi| |Q(\omega)| |\nabla u_t| + |\nabla \operatorname{div} \omega| |Q(\omega)| |u_t|) dx \\
&\leq C |\omega|_\infty |\psi|_6 (|\nabla^2 \omega|_2 |u_t|_3 + |\nabla \omega|_3 |\nabla u_t|_2) + C |\nabla^2 \omega|_6 |\nabla \omega|_2 |u_t|_3 \\
&\leq \frac{1}{b_5^2} |\omega|_{D^{2,6}}^2 + \frac{\alpha}{10} |\nabla u_t|_2^2 + C b_5^8 |u_t|_2^2 + C b_5^6.
\end{aligned} \tag{3.41}$$

For  $I_{12}$ , we have

$$\begin{aligned}
I_{12} &= - \frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} (\bar{A}_r)_t u_t \cdot \Omega d\Omega dv dx \\
&\leq |u_t|_2^2 + C (\|\bar{S}_t\|_{L^1(\mathbb{R} \times S^2; L^2)}^2 + \|I\|_{L^2(\mathbb{R} \times S^2; L^\infty)}^2 \|(\bar{\sigma}_a)_t\|_{L^2(\mathbb{R} \times S^2; L^2)}^2 \\
&\quad + \|\bar{\sigma}_a\|_{L^2(\mathbb{R} \times S^2; L^\infty)}^2 \|I_t\|_{L^2(\mathbb{R} \times S^2; L^2)}^2 + \|I_t\|_{L^2(\mathbb{R} \times S^2; L^2)}^2) \\
&\leq |u_t|_2^2 + M(b_0) b_0^{2\theta+4} b_1^2
\end{aligned} \tag{3.42}$$

Combining (3.30), (3.34) and (3.36)–(3.42), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_t|_2^2 + \alpha |\nabla u_t|_2^2 + (\alpha + \beta) |\operatorname{div} u_t|_2^2 \\ & \leq C b_5^8 |u_t|_2^2 + M(b_0) b_5^{2\theta+6} + \frac{C}{b_5^2} (|\nabla \omega_t|_2^2 + |\omega|_{D^{2,6}}^2). \end{aligned} \quad (3.43)$$

Integrating (3.43) over  $(\tau, t)$  ( $\tau \in (0, t)$ ), we have

$$\begin{aligned} & |u_t(t)|_2^2 + \int_{\tau}^t \alpha |\nabla u_t(s)|_2^2 ds \\ & \leq |u_t(\tau)|_2^2 + M(b_0) b_5^{2\theta+6} t + \int_{\tau}^t C b_5^8 |u_t|_2^2 ds + C. \end{aligned} \quad (3.44)$$

From the momentum equations (3.1)<sub>3</sub>, we can obtain

$$\begin{aligned} |u_t(\tau)|_2 & \leq C \left( |\omega|_{\infty} |\nabla \omega|_2 + |\nabla \phi|_2 + |Lu|_2 + |\psi|_6 |Q(\omega)|_2 \right. \\ & \quad + \|\bar{S}\|_{L^1(\mathbb{R}^+ \times S^2; L^2)} + \|\bar{\sigma}_a\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^{\infty})} \\ & \quad \left. + \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \right) (\tau). \end{aligned} \quad (3.45)$$

Then, using the assumptions (3.3)–(3.5), it is easy to see that

$$\limsup_{\tau \rightarrow 0} |u_t(\tau)|_2 \leq M(b_0) b_0^2. \quad (3.46)$$

So letting  $\tau \rightarrow 0$  in (3.44), via Gronwall's inequality, we have

$$|u_t(t)|_2^2 + \int_0^t \alpha |\nabla u_t(s)|_2^2 ds \leq (M(b_0) b_5^{2\theta+6} t + M(b_0) b_0^2) \exp(C b_5^8 t) \leq M(b_0) b_0^2, \quad (3.47)$$

for  $0 \leq t \leq T_4$ .

**Step 4.** Finally, we consider the estimates of the higher order terms for the velocity  $u$ . From (3.30), (3.47) and (3.28), it is easy to see that for  $0 \leq t \leq T_4$ ,

$$|u(t)|_{D^2} \leq C(|u_t|_2 + M(b_0) b_4^{\frac{3}{2}} b_5^{\frac{1}{2}}) \leq M(b_0) b_4^{\frac{3}{2}} b_5^{\frac{1}{2}},$$

and

$$\begin{aligned} |u|_{D^3} & \leq C(|\nabla \times \varpi|_{D^1} + |\nabla F|_{D^1} + |\nabla \phi|_{D^1}), \\ & \leq C(|\nabla \varpi|_{D^1} + |\nabla F|_{D^1} + b_0). \end{aligned} \quad (3.48)$$

Thus, we use the standard elliptic estimate to obtain

$$\begin{aligned} |\nabla \varpi|_{D^1} + |\nabla F|_{D^1} &\leq C(|u_t|_{D^1} + |\omega \cdot \nabla \omega|_{D^1} + |\psi \cdot \mathcal{Q}(\omega)|_{D^1} + \int_0^\infty \int_{S^2} |\bar{A}_r|_{D^1} \Omega d\Omega dv) \\ &\leq C(|u_t|_{D^1} + M(b_0)b_5^2). \end{aligned}$$

Along with (3.47)–(3.48), we immediately gain the desired estimate for  $|u|_{D^3}$ .  $\square$

By Lemmas 3.2–3.6, we can conclude that for  $0 \leq t \leq T_4$

$$\begin{aligned} |\phi(t)|_\infty^2 + \|\phi(t) - \phi^\infty\|_2^2 + \|\phi_t(t)\|_1^2 &\leq Cb_5^4, \\ |\psi(t)|_{D^1}^2 + |\psi_t(t)|_2^2 &\leq Cb_5^4, \\ \|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_2]; H^2))} &\leq Cb_0, \\ \|I_t\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_2]; H^1))} &\leq M(b_0)b_0^{\theta+1}b_1, \\ \|u(t)\|_1^2 + \int_0^t (|u_t(s)|_2^2 + \|\nabla u(s)\|_1^2) ds &\leq Cb_0^2, \\ |u(t)|_{D^2}^2 + |u_t(t)|_2^2 + \int_0^t (|u(s)|_{D^3}^2 + |u_t(s)|_{D^1}^2) ds &\leq M(b_0)b_4^3b_5. \end{aligned} \quad (3.49)$$

Therefore, we can take the constants  $b_i$  ( $i = 1, 2, 3$ ) and  $T^*$  by

$$\begin{aligned} b_1 &= Cb_0, \quad b_2 = M(b_0)b_0^{\theta+1}b_1, \quad b_3 = b_4 = Cb_0, \quad b_5 = M(b_0)b_3^3, \\ \text{and } T^* &= \min(T, (1 + M(b_0)b_5^{2\theta+6})^{-1}), \end{aligned} \quad (3.50)$$

such that

$$\begin{aligned} \sup_{0 \leq t \leq T^*} \|I(t)\|_{L^2(\mathbb{R}^+ \times S^2; H^2)}^2 &\leq b_1^2, \\ \sup_{0 \leq t \leq T^*} \|I_t(t)\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 &\leq b_2^2, \\ \sup_{0 \leq t \leq T^*} |u(t)|_2^2 + \int_0^{T^*} |\nabla u(t)|_2^2 dt &\leq b_3^2, \\ \sup_{0 \leq t \leq T^*} |u(t)|_{D^1}^2 + \int_0^{T^*} (|u(t)|_{D^2}^2 + |u_t(t)|_2^2) dt &\leq b_4^2, \end{aligned} \quad (3.51)$$

$$\begin{aligned} \sup_{0 \leq t \leq T^*} (|u(t)|_{D^2}^2 + |u_t(t)|_2^2) + \int_0^{T^*} (|u(t)|_{D^3}^2 + |u_t(t)|_{D^1}^2) dt &\leq b_5^2, \\ \sup_{0 \leq t \leq T^*} (|\phi(t)|_\infty^2 + \|\phi(t) - \phi^\infty\|_2^2 + \|\phi_t(t)\|_1^2) &\leq Cb_5^4, \\ \sup_{0 \leq t \leq T^*} (|\psi(t)|_{D^1}^2 + |\psi_t(t)|_2^2) &\leq Cb_5^4. \end{aligned}$$

### 3.3. Unique solvability of the linearization with vacuum

Based on the a prior estimate (3.51), we can obtain the following existence result when  $\phi_0 \geq 0$ .

**Lemma 3.7.** *If the initial data (3.2) satisfy (3.4) and  $\phi_0 \geq 0$ , then there exists a unique regular solution  $(I, \phi, \psi, u)$  to (3.1)–(3.3) such that*

$$\begin{aligned} I &\in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2)), \quad I_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^1)); \\ \phi &\geq 0, \quad \phi \in C([0, T^*]; H^2), \quad \phi_t \in C([0, T^*]; H^1), \quad \psi \in C([0, T^*]; D^1), \\ \psi_t &\in C([0, T^*]; L^2), \quad u \in C([0, T^*]; H^2) \cap L^2([0, T^*]; H^3), \\ u_t &\in C([0, T^*]; L^2) \cap L^2([0, T^*]; D^1). \end{aligned} \quad (3.52)$$

And  $\partial_i \psi^{(j)} = \partial_j \psi^{(i)}$  in the distribution sense for  $(i, j = 1, 2, 3)$ . Moreover,  $(I, \phi, \psi, u)$  also satisfies the local a prior estimates (3.51).

**Proof.** Step 1. Existence. Firstly, we set

$$\phi_{\delta 0} = \phi_0 + \delta, \quad \text{and} \quad \psi_{\delta 0} = 2\theta \nabla \phi_0 / (\phi_0 + \delta)$$

for each  $\delta \in (0, 1)$ . By the assumption (3.9), for all sufficiently small  $\delta > 0$ , we have

$$1 + |\phi_{\delta 0}|_\infty + \|\phi_{\delta 0} - \delta\|_2 + |\psi_{\delta 0}|_{D^1} + \|u_0\|_2 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; H^2)} \leq Cb_0^2 = \bar{b}_0.$$

Therefore, for the linearized problem (3.1)–(3.3) with the initial data  $(I_0, \phi_{\delta 0}, u_0, \psi_{\delta 0})$ , we can obtain a unique strong solution  $(I^\delta, \phi^\delta, u^\delta, \psi^\delta)$  satisfying the local estimate (3.51).

Via this uniform estimates (3.51), it is well-known that there exists a subsequence of solutions

$$(I^\delta, \phi^\delta, u^\delta, \psi^\delta) \quad \text{converges to a limit} \quad (I, \phi, u, \psi) \quad \text{in weak or weak* sense.} \quad (3.53)$$

Due to the compact property (see [22]), for any  $R > 0$ , there exists a subsequence of solutions  $(I^\delta, \phi^\delta, u^\delta, \psi^\delta)$  satisfying

$$\begin{aligned} I^\delta &\rightarrow I \text{ in } L^2(\mathbb{R}^+ \times S^2; C([0, T^*]; H^1(B_R))) \\ (\phi^\delta, u^\delta) &\rightarrow (\phi, u) \text{ in } C([0, T^*]; H^1(B_R)), \quad \psi^\delta \rightarrow \psi \text{ in } C([0, T^*]; L^2(B_R)), \end{aligned} \quad (3.54)$$

where  $B_R$  is a ball centered at origin with radius  $R$ . Along with the lower semi-continuity of norms, (3.53) and (3.54), we can derive that  $(I, \phi, u, \psi)$  also satisfies the local estimates (3.51).

From [24], we also know that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^3} (\phi_0^\delta - \phi^0) \zeta(0, x) dx &= 0, \\ \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^3} (\psi_0^\delta - \psi^0) \xi(0, x) dx &= 0, \end{aligned} \quad (3.55)$$

for any  $\zeta(t, x) \in C_c^\infty([0, T^*) \times \mathbb{R}^3)$  and  $\xi(t, x) \in C_c^\infty([0, T^*) \times \mathbb{R}^3)^3$ . Then, combining (3.51), (3.53) and (3.54), we can conclude that  $(I, \phi, u, \psi)$  is a weak solution to the linearized problem (3.1)–(3.3) satisfying the regularity (3.52).

Step 2. We can use the same arguments used in Lemma 3.1 to obtain the uniqueness and time continuity for  $(I, \phi, \psi, u)$ .  $\square$

### 3.4. Proof of Theorem 2.3

we will use the classical iteration scheme and the existence results gained in Section 3.3 to prove Theorem 2.3. Assume that

$$2 + |\phi_0|_\infty + \|(\phi_0, u_0)\|_2 + |\psi_0|_{D^1} + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; H^2)} \leq b_0.$$

Let  $I^0 \in L^2(\mathbb{R} \times S^2; C([0, T^*]; H^2))$  be the solution to the linear hyperbolic problem

$$\begin{cases} f_t + c\Omega \cdot \nabla f = 0, & \text{in } \mathbb{R}^+ \times S^2 \times (0, +\infty) \times \mathbb{R}^3, \\ f(0) = I_0 & \text{in } \mathbb{R}^+ \times S^2 \times \mathbb{R}^3, \end{cases}$$

and  $u^0 \in C([0, T^*]; H^2) \cap L^2([0, T^*]; H^3)$  be the solution to the linear parabolic problem

$$\begin{cases} h_t - \Delta h = 0 & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ h(0) = u_0 & \text{in } \mathbb{R}^3. \end{cases}$$

We take a small time  $T^\epsilon \in (0, T^*)$  so that

$$\begin{aligned} \sup_{0 \leq t \leq T^\epsilon} \|I^0(t)\|_{L^2(\mathbb{R}^+ \times S^2; H^2)}^2 &\leq b_1^2, \\ \sup_{0 \leq t \leq T^\epsilon} \|I_t^0(t)\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 &\leq b_2^2, \\ \sup_{0 \leq t \leq T^\epsilon} |u^0(t)|_2^2 + \int_0^{T^\epsilon} |\nabla u^0(t)|_2^2 dt &\leq b_3^2, \end{aligned} \quad (3.56)$$

$$\sup_{0 \leq t \leq T^\epsilon} |u^0(t)|_{D^1}^2 + \int_0^{T^\epsilon} (|u^0(t)|_{D^2}^2 + |u_t^0(t)|_2^2) dt \leq b_4^2,$$

$$\sup_{0 \leq t \leq T^\epsilon} (|u^0(t)|_{D^2}^2 + |u_t^0(t)|_2^2) + \int_0^{T^\epsilon} (|u^0(t)|_{D^3}^2 + |u_t^0(t)|_{D^1}^2) dt \leq b_5^2.$$

**Proof.** Step 1. Existence. Let  $\chi = I^0$ ,  $\omega = u^0$ , we can get  $(I^1, \phi^1, \psi^1, u^1)$  as a strong solution to problem (3.1)–(3.3). Inductively, assuming that  $(I^k, \phi^k, \psi^k, u^k)$  was defined for  $k \geq 1$ , we construct approximate solutions  $(I^{k+1}, \phi^{k+1}, \psi^{k+1}, u^{k+1})$  as following:

$$\begin{cases} \phi_t^{k+1} + u^k \cdot \nabla \phi^{k+1} + (\gamma - 1) \phi^{k+1} \operatorname{div} u^k = 0, \\ \psi_t^{k+1} + \sum_{l=1}^3 A_l(u^k) \partial_l \psi^{k+1} + B(u^k) \psi^{k+1} + \nabla \operatorname{div} u^k = 0, \\ \frac{1}{c} I_t^{k+1} + \Omega \cdot \nabla I^{k+1} = A_r^k, \\ u_t^{k+1} + u^k \cdot \nabla u^k + A \gamma \theta \nabla \phi^{k+1} = -L u^{k+1} + \psi^{k+1} \cdot Q(u^k) - \frac{1}{c} \int_0^\infty \int_{S^2} \bar{A}_r^k \Omega d\Omega dv, \\ (I^{k+1}, \phi^{k+1}, \psi^{k+1}, u^{k+1})|_{t=0} = (I_0, \phi_0, \psi_0, u_0), \end{cases} \quad (3.57)$$

where

$$\begin{aligned} A_r^k &= S^{k+1} - \sigma_a^{k+1} I^{k+1} + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s^{k+1} I'^k - (\sigma_s')^{k+1} I^{k+1} \right) d\Omega' dv', \\ \bar{A}_r^k &= \bar{S}^{k+1} - \bar{\sigma}_a^{k+1} I^{k+1} + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \bar{\sigma}_s I'^{k+1} - \bar{\sigma}_s' I^{k+1} \right) d\Omega' dv', \\ S^{k+1} &= S(v, \Omega, t, x, (\phi^{k+1})^\theta), \quad \sigma_a^{k+1} = \sigma_a(v, \Omega, t, x, (\phi^{k+1})^\theta), \\ \bar{S}^{k+1} &= \bar{S}(v, \Omega, t, x, (\phi^{k+1})^\theta), \quad \bar{\sigma}_a^{k+1} = \bar{\sigma}_a(v, \Omega, t, x, (\phi^{k+1})^\theta), \\ \sigma_s^{k+1} &= \bar{\sigma}_s \cdot (\phi^{k+1})^\theta, \quad (\sigma_s')^{k+1} = \bar{\sigma}_s' \cdot (\phi^{k+1})^\theta. \end{aligned}$$

By view of the estimates shown in Section 3.3, it is easy to see that the sequences of solutions  $(I^k, \phi^k, \psi^k, u^k)$  satisfy the uniform a prior estimate (3.51). We will prove the strong convergence of the full sequence  $(I^k, \phi^k, \psi^k, u^k)$ . Set

$$\bar{I}^{k+1} = I^{k+1} - I^k, \quad \bar{\phi}^{k+1} = \phi^{k+1} - \phi^k, \quad \bar{\psi}^{k+1} = \psi^{k+1} - \psi^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k,$$

then from (3.57), we have



$$\left\{ \begin{array}{l} \bar{\phi}_t^{k+1} + u^k \cdot \nabla \bar{\phi}^{k+1} + \bar{u}^k \cdot \nabla \phi^k + (\gamma - 1)(\bar{\phi}^{k+1} \operatorname{div} u^k + \phi^k \operatorname{div} \bar{u}^k) = 0, \\ \bar{\psi}_t^{k+1} + \sum_{l=1}^3 A_l(u^k) \partial_l \bar{\psi}^{k+1} + B(u^k) \bar{\psi}^{k+1} + \nabla \operatorname{div} \bar{u}^k = \Upsilon_1^k + \Upsilon_2^k, \\ \frac{1}{c} \bar{I}_t^{k+1} + \Omega \cdot \nabla \bar{I}^{k+1} + \left( \sigma_a^{k+1} + \int_0^\infty \int_{S^2} (\sigma'_s)^{k+1} d\Omega' dv' \right) \bar{I}^{k+1} = L_1, \\ \bar{u}_t^{k+1} + u^k \cdot \nabla \bar{u}^k + \bar{u}^k \cdot \nabla u^{k-1} + A\gamma\theta \nabla \bar{\phi}^{k+1} + L\bar{u}^{k+1} \\ = \psi^{k+1} \cdot Q(\bar{u}^k) + \bar{\psi}^{k+1} \cdot Q(u^{k-1}) + L_2, \end{array} \right. \quad (3.58)$$

where

$$\Upsilon_1^k = - \sum_{l=1}^3 (A_l(u^k) \partial_l \psi^k - A_l(u^{k-1}) \partial_l \psi^k), \quad \Upsilon_2^k = -(B(u^k) \psi^k - B(u^{k-1}) \psi^k),$$

and

$$\begin{aligned} L_1 &= (S^{k+1} - S^k) - I^k(\sigma_a^{k+1} - \sigma_a^k) - \int_0^\infty \int_{S^2} \left( (\sigma'_s)^{k+1} - (\sigma'_s)^k \right) I^k d\Omega' dv' \\ &\quad + \int_0^\infty \int_{S^2} \frac{v}{v'} \left( \sigma_s^k \bar{I}^k + I'^k(\sigma_s^{k+1} - \sigma_s^k) \right) d\Omega' dv' \\ L_2 &= -\frac{1}{c} \int_0^\infty \int_{S^2} \left( (\bar{S}^{k+1} - \bar{S}^k) - \bar{\sigma}_a^{k+1} \bar{I}^{k+1} - I^k(\bar{\sigma}_a^{k+1} - \bar{\sigma}_a^k) \right) \Omega d\Omega dv \\ &\quad - \frac{1}{c} \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \bar{\sigma}_s \bar{I}'^{k+1} - \bar{\sigma}'_s \bar{I}^{k+1} \right) \Omega d\Omega' dv' d\Omega dv. \end{aligned}$$

Firstly multiplying (3.58)<sub>1</sub> by  $2\bar{\phi}^{k+1}$  and integrating over  $\mathbb{R}^3$ , we can see that

$$\begin{aligned} \frac{d}{dt} |\bar{\phi}^{k+1}|_2^2 &= -2 \int_{\mathbb{R}^3} \left( u^k \cdot \nabla \bar{\phi}^{k+1} + \bar{u}^k \cdot \nabla \phi^k + (\gamma - 1)(\bar{\phi}^{k+1} \operatorname{div} u^k + \phi^k \operatorname{div} \bar{u}^k) \right) \bar{\phi}^{k+1} dx, \\ &\leq C |\nabla u^k|_\infty |\bar{\phi}^{k+1}|_2^2 + C |\bar{\phi}^{k+1}|_2 |\bar{u}^k|_6 |\nabla \phi^k|_3 + C |\bar{\phi}^{k+1}|_2 |\nabla \bar{u}^k|_2 |\phi^k|_\infty. \end{aligned} \quad (3.59)$$

Next, differentiating (3.58)<sub>1</sub>  $\zeta$ -times ( $|\zeta| = 1$ ) with respect to  $x$ , multiplying the resulting equation by  $2D^\zeta \bar{\phi}^{k+1}$  and integrating over  $\mathbb{R}^3$ , we can obtain

$$\begin{aligned}
\frac{d}{dt} |D^\zeta \bar{\phi}^{k+1}|_2^2 &= -2 \int_{\mathbb{R}^3} D^\zeta (u^k \cdot \nabla \bar{\phi}^{k+1} + \bar{u}^k \cdot \nabla \phi^k \\
&\quad + (\gamma - 1)(\bar{\phi}^{k+1} \operatorname{div} u^k + \phi^k \operatorname{div} \bar{u}^k)) D^\zeta \bar{\phi}^{k+1} dx \\
&\leq C |\nabla u^k|_\infty |\nabla \bar{\phi}^{k+1}|_2^2 + C |\nabla \phi^k|_3 |\nabla \bar{u}^k|_6 |\nabla \bar{\phi}^{k+1}|_2 \\
&\quad + C |\nabla \bar{\phi}^{k+1}|_2 |\bar{u}^k|_\infty |\nabla^2 \phi^k|_2 + C |\nabla^2 u^k|_3 |\nabla \bar{\phi}^{k+1}|_2^2 \\
&\quad + C |\nabla \bar{u}^k|_6 |\nabla \bar{\phi}^{k+1}|_2 |\nabla \phi^k|_3 + C |\phi^k|_\infty |\nabla \operatorname{div} \bar{u}^k|_2 |\nabla \bar{\phi}^{k+1}|_2.
\end{aligned} \tag{3.60}$$

By (3.59)–(3.60) and Young's inequality, it is easy to conclude that

$$\begin{cases} \frac{d}{dt} \|\bar{\phi}^{k+1}(t)\|_1^2 \leq \Phi_\eta^k(t) \|\bar{\phi}^{k+1}(t)\|_1^2 + \eta \|\nabla \bar{u}^k(t)\|_1^2, \\ \int_0^t \Phi_\eta^k(s) ds \leq \hat{C} + \hat{C}_\eta t \quad \text{for } t \in [0, T^\epsilon], \end{cases} \tag{3.61}$$

where  $\eta \in (0, \min(\frac{1}{10}, \frac{\sigma}{10}))$  is a positive constant determined later, and  $\hat{C}_\eta$  is a positive constant depending on  $\eta$  and  $\hat{C}$ .

Secondly, multiplying (3.58)<sub>2</sub> by  $2\bar{\psi}^{k+1}$  and integrating over  $\mathbb{R}^3$ , we have

$$\begin{aligned}
\frac{d}{dt} |\bar{\psi}^{k+1}|_2^2 &\leq C \left( \sum_{l=1}^3 |\partial_l A_l(u^k)|_\infty + |B(u^k)|_\infty \right) |\bar{\psi}^{k+1}|_2^2 \\
&\quad + C (|\Upsilon_1^k|_2 + |\Upsilon_2^k|_2 + |\nabla^2 \bar{u}^k|_2) |\bar{\psi}^{k+1}|_2.
\end{aligned} \tag{3.62}$$

Via Hölder's inequality, we can deduce that

$$|\Upsilon_1^k|_2 \leq C |\nabla \psi^k|_2 |\bar{u}^k|_\infty, \quad |\Upsilon_2^k|_2 \leq C |\psi^k|_6 |\nabla \bar{u}^k|_3. \tag{3.63}$$

From (3.62)–(3.63), we have

$$\begin{cases} \frac{d}{dt} |\bar{\psi}^{k+1}(t)|_2^2 \leq \Psi_\eta^k(t) |\bar{\psi}^{k+1}(t)|_2^2 + \eta \|\nabla \bar{u}^k(t)\|_1^2, \\ \int_0^t \Psi_\eta^k(s) ds \leq \hat{C} + \hat{C}_\eta t \quad \text{for } t \in [0, T^\epsilon]. \end{cases} \tag{3.64}$$

Thirdly, multiplying (3.58)<sub>3</sub> by  $\bar{I}^{k+1}$  and integrating over  $\mathbb{R}^+ \times S^2 \times \mathbb{R}^3$ , we can see that

$$\begin{aligned}
&\frac{d}{dt} \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)}^2 \\
&\leq \int_0^\infty \int_{S^2} \left( |S^{k+1} - S^k|_2 |\bar{I}^{k+1}|_2 + |I^k|_\infty |\sigma_a^{k+1} - \sigma_a^k|_2 |\bar{I}^{k+1}|_2 \right) d\Omega dv
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \left( \bar{\sigma}'_s |I^k|_\infty |(\phi^{k+1})^\theta - (\phi^k)^\theta|_2 |\bar{I}^{k+1}|_2 \right. \\
& + \frac{v}{v'} \bar{\sigma}'_s \left( |(\phi^k)^\theta|_\infty |\bar{I}^k|_2 + |I'^k|_\infty |(\phi^{k+1})^\theta - (\phi^k)^\theta|_2 \right) |\bar{I}^{k+1}|_2 \Big) d\Omega' dv' d\Omega dv \\
& \leq C \eta^{-1} \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)}^2 + \|\bar{\phi}^{k+1}\|_2^2 + \eta \|\bar{I}^k\|_{L^2(\mathbb{R}^+ \times S^2; L^2)}^2.
\end{aligned} \tag{3.65}$$

Similarly, we can also deduce that

$$\begin{aligned}
& \frac{d}{dt} |\bar{I}^{k+1}|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 \\
& \leq C \eta^{-1} \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 + \|\bar{\phi}^{k+1}\|_1^2 + \eta \|\bar{I}^k\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2.
\end{aligned} \tag{3.66}$$

Now, multiplying (3.58)<sub>4</sub> by  $2\bar{u}^{k+1}$  and integrating over  $\mathbb{R}^3$ , we have

$$\begin{aligned}
& \frac{d}{dt} |\bar{u}^{k+1}|_2^2 + 2\alpha |\nabla \bar{u}^{k+1}|_2^2 + 2(\alpha + \beta) |\operatorname{div} \bar{u}^{k+1}|_2^2 \\
& = -2 \int_{\mathbb{R}^3} \left( u^k \cdot \nabla \bar{u}^k + \bar{u}^k \cdot \nabla u^{k-1} + A\gamma \theta \nabla \phi^{k+1} \right) \cdot \bar{u}^{k+1} dx \\
& + 2 \int_{\mathbb{R}^3} \left( \psi^{k+1} \cdot Q(\bar{u}^k) + \bar{\psi}^{k+1} \cdot Q(u^{k-1}) + L_2 \right) \cdot \bar{u}^{k+1} dx =: \sum_{i=1}^6 J_i,
\end{aligned} \tag{3.67}$$

where

$$\begin{aligned}
J_1 & = -2 \int_{\mathbb{R}^3} u^k \nabla \bar{u}^k \cdot \bar{u}^{k+1} dx \leq C |u^k|_\infty |\nabla \bar{u}^k|_2 |\bar{u}^{k+1}|_2, \\
J_2 & = -2 \int_{\mathbb{R}^3} \bar{u}^k \nabla u^{k-1} \cdot \bar{u}^{k+1} dx \leq C |\nabla u^{k-1}|_3 |\nabla \bar{u}^k|_2 |\bar{u}^{k+1}|_2, \\
J_3 & = -2 \int_{\mathbb{R}^3} A\gamma \theta \nabla \phi^{k+1} \cdot \bar{u}^{k+1} dx \leq C |\bar{u}^{k+1}|_2 |\nabla \bar{\phi}^{k+1}|_2, \\
J_4 & = 2 \int_{\mathbb{R}^3} \psi^{k+1} Q(\bar{u}^k) \cdot \bar{u}^{k+1} dx \leq C |\psi^{k+1}|_6 |\nabla \bar{u}^k|_2 |\bar{u}^{k+1}|_3, \\
J_5 & = 2 \int_{\mathbb{R}^3} \bar{\psi}^{k+1} \cdot Q(u^{k-1}) \cdot \bar{u}^{k+1} dx \leq C |\nabla u^{k-1}|_\infty |\bar{\psi}^{k+1}|_2 |\bar{u}^{k+1}|_2,
\end{aligned}$$

and

$$J_6 = 2 \int_{\mathbb{R}^3} L_2 \cdot \bar{u}^{k+1} dx \leq C \left( \|\bar{S}^{k+1} - \bar{S}^k\|_{L^1(\mathbb{R}^+ \times S^2; L^2)} \right)$$

$$\begin{aligned}
& + \|\bar{\sigma}_a^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \\
& + \|\bar{I}^k\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} \|\bar{\sigma}_a^{k+1} - \bar{\sigma}_a^k\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \\
& + \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} |\bar{u}^{k+1}|_2, \\
& \leq C(|\bar{u}^{k+1}|_2^2 + \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)}^2 + |\bar{\phi}^{k+1}|_2^2).
\end{aligned}$$

Next, differentiating (3.58)<sub>3</sub>  $\zeta$ -times ( $|\zeta| = 1$ ) with respect to  $x$ , multiplying it by  $D^\zeta \bar{u}^{k+1}$  and integrating over  $\mathbb{R}^3$ , we can see that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |D^\zeta \bar{u}^{k+1}|_2^2 + \alpha |\nabla D^\zeta \bar{u}^{k+1}|_2^2 + (\alpha + \beta) |\operatorname{div} D^\zeta \bar{u}^{k+1}|_2^2 \\
& = \int_{\mathbb{R}^3} D^\zeta (-u^k \cdot \nabla \bar{u}^k - \bar{u}^k \cdot \nabla u^{k-1} - A\gamma\theta \nabla \bar{\phi}^{k+1}) \cdot D^\zeta \bar{u}^{k+1} dx \\
& + \int_{\mathbb{R}^3} D^\zeta (\psi^{k+1} \cdot Q(\bar{u}^k) + \bar{\psi}^{k+1} \cdot Q(u^{k-1}) + L_2) \cdot D^\zeta \bar{u}^{k+1} dx := \sum_{i=7}^{12} J_i,
\end{aligned} \tag{3.68}$$

where

$$\begin{aligned}
J_7 &= \int_{\mathbb{R}^3} -D^\zeta (u^k \cdot \nabla \bar{u}^k) \cdot D^\zeta \bar{u}^{k+1} dx \\
&\leq C |\nabla u^k|_6 |\nabla \bar{u}^k|_2 |\nabla \bar{u}^{k+1}|_3 + C |u^k|_\infty |\bar{u}^k|_{D^2} |\nabla \bar{u}^{k+1}|_2, \\
J_8 &= \int_{\mathbb{R}^3} -D^\zeta (\bar{u}^k \cdot \nabla u^{k-1}) \cdot D^\zeta \bar{u}^{k+1} dx \\
&\leq C |\nabla \bar{u}^k|_2 |\nabla \bar{u}^{k+1}|_3 |\nabla u^{k-1}|_6 + C |\bar{u}^k|_6 |\nabla \bar{u}^{k+1}|_3 |\nabla^2 u^{k-1}|_2, \\
J_9 &= \int_{\mathbb{R}^3} -A\gamma\theta D^\zeta (\nabla \bar{\phi}^{k+1}) \cdot D^\zeta \bar{u}^{k+1} dx \leq C |\nabla^2 \bar{u}^{k+1}|_2 |\nabla \bar{\phi}^{k+1}|_2, \\
J_{10} &= \int_{\mathbb{R}^3} D^\zeta (\psi^{k+1} \cdot Q(\bar{u}^k)) \cdot D^\zeta \bar{u}^{k+1} dx \\
&\leq C (|\nabla \psi^{k+1}|_2 |\nabla \bar{u}^k|_6 |\nabla \bar{u}^{k+1}|_3 + |\psi^{k+1}|_6 |\bar{u}^k|_{D^2} |\nabla \bar{u}^{k+1}|_3) \\
J_{11} &= \int_{\mathbb{R}^3} D^\zeta (\bar{\psi}^{k+1} \cdot Q(u^{k-1})) \cdot D^\zeta \bar{u}^{k+1} dx \\
&\leq C |\bar{\psi}^{k+1}|_2 |\nabla u^{k-1}|_\infty |\nabla^2 \bar{u}^{k+1}|_2,
\end{aligned}$$

and

$$J_{12} = \int_{\mathbb{R}^3} D^\zeta L_2 \cdot D^\zeta \bar{u}^{k+1} dx$$

$$\begin{aligned}
&\leq C \left( \|D^\zeta (\bar{S}^{k+1} - \bar{S}^k)\|_{L^1(\mathbb{R}^+ \times S^2; L^2)} + \|D^\zeta \bar{\sigma}_a^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^3)} \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^6)} \right. \\
&\quad + \|\bar{\sigma}_a^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} \|D^\zeta \bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \\
&\quad + \|\bar{I}^k\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} \|D^\zeta (\bar{\sigma}_a^{k+1} - \bar{\sigma}_a^k)\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \\
&\quad + \|D^\zeta \bar{I}^k\|_{L^2(\mathbb{R}^+ \times S^2; L^3)} \|(\bar{\sigma}_a^{k+1} - \bar{\sigma}_a^k)\|_{L^2(\mathbb{R}^+ \times S^2; L^6)} \\
&\quad \left. + \|D^\zeta \bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)} \right) |D^\zeta \bar{u}^{k+1}|_2 \\
&\leq C (|D^\zeta \bar{u}^{k+1}|_2^2 + \|\nabla \bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; L^2)}^2 + \|\bar{\phi}^{k+1}\|_1^2).
\end{aligned}$$

By (3.67)–(3.68), we easily have

$$\left\{ \begin{aligned} &\frac{d}{dt} \|\bar{u}^{k+1}\|_1^2 + \alpha \|\nabla \bar{u}^{k+1}\|_1^2 \\ &\quad \leq \Theta_\eta^k(t) \|\bar{u}^{k+1}\|_1^2 + \Theta_2^k(t) \|\bar{\phi}^{k+1}\|_1^2 + \Theta_3^k(t) \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 \\ &\quad \quad + \Theta_4^k(t) |\bar{\psi}^{k+1}|_2^2 + \eta \|\nabla \bar{u}^k\|_1^2, \\ &\int_0^t (\Theta_\eta^k(s) + \Theta_2^k(s) + \Theta_3^k(s) + \Theta_4^k(s)) ds \leq \hat{C} + \hat{C}_\eta t \quad \text{for } t \in [0, T^\epsilon]. \end{aligned} \right. \quad (3.69)$$

Finally, set

$$\Gamma^{k+1} = \|\bar{I}^{k+1}\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 + \|\bar{\phi}^{k+1}\|_1^2 + |\bar{\psi}^{k+1}|_2^2 + \|\bar{u}^{k+1}\|_1^2,$$

then we have

$$\frac{d}{dt} \Gamma^{k+1} + \alpha \|\nabla \bar{u}^{k+1}\|_1^2 \leq \Pi_\eta^k \Gamma^{k+1} + C_\eta \|\nabla \bar{u}^k\|_1^2,$$

for some  $\Pi_\eta^k$  such that  $\int_0^t \Pi_\eta^k(s) ds \leq \hat{C} + \hat{C}_\eta t$ . By view of Gronwall's inequality, we can derive that

$$\Gamma^{k+1} + \int_0^t \alpha \|\nabla \bar{u}^{k+1}\|_1^2 ds \leq \left( C_\eta \int_0^t \|\nabla \bar{u}^k\|_1^2 ds \right) \exp(\hat{C} + \hat{C}_\eta t).$$

We can take  $\eta > 0$  and  $\dot{T} \in (0, T^\epsilon)$  small enough such that

$$C_\eta \exp \hat{C} = \frac{\alpha}{4}, \quad \text{and} \quad \exp(\hat{C}_\eta \dot{T}) = 2.$$

Then we can conclude that

$$\sum_{k=1}^{\infty} \left( \sup_{0 \leq t \leq \dot{T}} \Gamma^{k+1} + \int_0^{\dot{T}} \mu \|\nabla \bar{u}^{k+1}\|_1^2 ds \right) \leq \hat{C} < +\infty,$$

which implies that the consequence  $(I^k, \phi^k, \psi^k, u^k)$  converges to a limit  $(I, \phi, \psi, u)$  in the following strong sense:

$$\begin{aligned} I^k &\rightarrow I \text{ in } L^2(\mathbb{R}^+ \times S^2; L^\infty([0, \dot{T}]; H^1(\mathbb{R}^3))), \\ \phi^k &\rightarrow \phi \text{ in } L^\infty([0, \dot{T}]; H^1(\mathbb{R}^3)), \\ \psi^k &\rightarrow \psi \text{ in } L^\infty([0, \dot{T}]; L^2(B_R)), \\ u^k &\rightarrow u \text{ in } L^\infty([0, \dot{T}]; H^1(\mathbb{R}^3)) \cap L^2([0, \dot{T}]; D^2(\mathbb{R}^3)), \end{aligned} \quad (3.70)$$

where  $R > 0$  can be arbitrarily large, and  $B_R$  is a ball centered at origin with radius  $R$ .

Based on the local estimate (3.51) and the lower-continuity of norm for weak or weak\* convergence, we know that  $(I, \phi, \psi, u)$  satisfies the estimate (3.51). Due to the strong convergence in (3.70), we can conclude that  $(I, \phi, \psi, u)$  is a regular solution to problem (1.1)–(1.4)–(1.5). So we complete the proof of the existence.

**Step 2. Uniqueness.** Assume that  $(I_1, \phi_1, \psi_1, u_1)$  and  $(I_2, \phi_2, \psi_2, u_2)$  are two regular solutions to problem (1.1)–(1.4)–(1.5) satisfying the uniform a priori estimate (3.51). Set

$$\bar{I} = I_1 - I_2, \quad \bar{\phi} = \phi_1 - \phi_2, \quad \bar{\psi} = \psi_1 - \psi_2, \quad \bar{u} = u_1 - u_2.$$

Using (2.2), it is easy to see that  $(\bar{I}, \bar{\phi}, \bar{\psi}, \bar{u})$  satisfies the following system

$$\begin{cases} \bar{\phi}_t + u_1 \cdot \nabla \bar{\phi} + \bar{u} \cdot \nabla \phi_2 + (\gamma - 1)(\bar{\phi} \operatorname{div} u_2 + \phi_1 \operatorname{div} \bar{u}) = 0, \\ \bar{\psi}_t + \sum_{l=1}^3 A_l(u^1) \partial_l \bar{\psi} + B(u^1) \bar{\psi} + \nabla \operatorname{div} \bar{u}^k = \bar{\Upsilon}_1 + \bar{\Upsilon}_2, \\ \frac{1}{c} \bar{I}_t + \Omega \cdot \nabla \bar{I} + \left( \sigma_a^1 + \int_0^\infty \int_{S^2} (\sigma'_s)^1 d\Omega' dv' \right) \bar{I} = \bar{L}_1, \\ \bar{u}_t + u_1 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_2 + A\gamma\theta \nabla \bar{\phi} = -L\bar{u} + \psi_1 \cdot Q(\bar{u}) + \bar{\psi} \cdot Q(u_2) + \bar{L}_2, \end{cases} \quad (3.71)$$

where  $\bar{\Upsilon}_1$  and  $\bar{\Upsilon}_2$  are defined by

$$\bar{\Upsilon}_1 = - \sum_{l=1}^3 (A_l(u^1) \partial_l \psi^2 - A_l(u^2) \partial_l \psi^2), \quad \bar{\Upsilon}_2 = -(B(u^1) \psi^2 - B(u^2) \psi^2)$$

and

$$\begin{aligned} \bar{L}_1 &= (S^1 - S^2) - I^2(\sigma_a^1 - \sigma_a^2) - \int_0^\infty \int_{S^2} \left( (\sigma'_s)^1 - (\sigma'_s)^2 \right) I^2 d\Omega' dv' \\ &\quad + \int_0^\infty \int_{S^2} \frac{v}{v'} \left( \sigma_s^1 \bar{I}' + I'^2 (\sigma_s^1 - \sigma_s^2) \right) d\Omega' dv' \end{aligned}$$

$$\begin{aligned}\bar{L}_2 = & -\frac{1}{c} \int_0^\infty \int_{S^2} \left( (\bar{S}^1 - \bar{S}^2) - \bar{\sigma}_a^1 \bar{I} - I^2 (\bar{\sigma}_a^1 - \bar{\sigma}_a^2) \right) \Omega d\Omega dv \\ & - \frac{1}{c} \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \bar{\sigma}_s \bar{I}' - \bar{\sigma}_s' \bar{I} \right) \Omega d\Omega' dv' d\Omega dv.\end{aligned}$$

Let

$$\Phi(t) = \|\bar{I}\|_{L^2(\mathbb{R}^+ \times S^2; H^1)}^2 + \|\bar{\phi}(t)\|_1^2 + |\bar{\psi}(t)|_2^2 + \|\bar{u}(t)\|_1^2.$$

By the same method used in the derivation of (3.61)–(3.69), we can also conclude that

$$\begin{cases} \frac{d}{dt} \Phi(t) + C \|\nabla \bar{u}(t)\|_1^2 \leq G(t) \Phi(t), \\ \int_0^t G(s) ds \leq \widehat{C} \quad \text{for } 0 \leq t \leq \dot{T}. \end{cases} \quad (3.72)$$

Using Gronwall's inequality, we can obtain the uniqueness.

**Step 3.** The time-continuity of the classical solution. We can use the standard method used in the proof of Lemma 3.1 to obtain the time-continuity of the classical solution, see [5].  $\square$

#### 4. Existence of the local strong solution

In this section, we will prove the local existence of strong solutions to Cauchy problem (1.1)–(1.4)–(1.5) based on Theorem 2.3.

**Proof.** As  $\gamma = 2$ , we quickly have the relation  $\rho(t, x) = \phi(t, x)$ , then the regular solution is the strong solution.

We now consider the case  $1 < \gamma \leq \frac{3}{2}$ . From Theorem 2.3, it is known that there exists a time  $T_* > 0$  so that Cauchy problem (1.1)–(1.4)–(1.5) has a unique regular solution  $(I, \phi, u)$ . Set

$$\rho = \phi^\theta,$$

where  $\theta = \frac{1}{\gamma-1}$ . Since  $\theta \geq 2$  and

$$\phi \in C((0, T_*) \times \mathbb{R}^3) \cap C([0, T_*]; H^2),$$

we know that

$$\rho(t, x) \in C((0, T_*) \times \mathbb{R}^3) \cap C([0, T_*]; H^2).$$

Multiplying (2.2)<sub>2</sub> by  $\frac{\partial \rho}{\partial \phi}(t, x) = \theta \phi^{\theta-1} \in C((0, T_*) \times \mathbb{R}^3)$ , we can obtain the continuity equation (1.1)<sub>2</sub>:

$$\rho_t + u \cdot \nabla \rho + \rho \operatorname{div} u = 0. \quad (4.1)$$

Along with (4.1) and  $u(t, x) \in C([0, T_*], H^2) \cap C^1([0, T_*], H^1)$ , we use the linear quasi-linear hyperbolic equation theory to have

$$\rho \in C([0, T_*], H^2) \cap C^1([0, T_*], H^1).$$

Multiplying (2.2)<sub>3</sub> by  $\rho(t, x) \in C((0, T_*) \times \mathbb{R}^3)$ , we obtain the momentum equations (1.1)<sub>3</sub>:

$$\rho u_t + \rho u \cdot \nabla u + \nabla P = \operatorname{div} \mathbb{T} - \frac{1}{c} \int_0^\infty \int_{S^2} A_r \Omega d\Omega dv. \quad (4.2)$$

By the continuity equation and Lemma 6 in [5], we can derive that

$$\rho(t, x) = \rho_0(U(0; t, x)) \exp \left( - \int_0^t \operatorname{div} u(s, U(s; t, x)) ds \right),$$

which, together with  $\rho_0 \geq 0$ , immediately implies that

$$\rho(t, x) \geq 0, \quad \forall (t, x) \in [0, T_*] \times \mathbb{R}^3.$$

In summary, the Cauchy problem (1.1)–(1.4)–(1.5) has a strong solution  $(I, \rho, u)$ .  $\square$

## 5. Blow-up criterion of regular solutions

In this section, we will give a blow-up criterion for the regular solutions obtained in Theorem 2.3. Suppose that  $(I, \phi, u)$  is a regular solution of problem (1.1)–(1.4)–(1.5). We first define the following two auxiliary quantities:

$$\Phi(t) = 1 + \|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0, t]; H^2))} + \sup_{0 \leq s \leq t} \|\phi(s)\|_2 + \sup_{0 \leq s \leq t} \|u(s)\|_1 + \sup_{0 \leq s \leq t} |\psi(s)|_{D^1}$$

and

$$\begin{aligned} \Theta(t) = & 1 + \|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0, t]; H^2))} + \|I_t\|_{L^2(\mathbb{R}^+ \times S^2; C([0, t]; H^2))} \\ & + \|\phi(t)\|_2 + \|\phi_t(t)\|_1 + \|u(t)\|_2 + |u_t(t)|_2 + |\psi(t)|_{D^1} \\ & + |\psi_t(t)|_2 + \int_0^t (|u(s)|_{D^3}^2 + |u_t(s)|_{D^1}^2) ds. \end{aligned}$$

Since  $\overline{T}$  is the maximal existence time of the local regular solution, we know that

$$\lim_{t \rightarrow \overline{T}} \Theta(t) = +\infty. \quad (5.1)$$



In order to prove Theorem 2.4, we only need to show that

$$\lim_{t \rightarrow \bar{T}} \Phi(t) = +\infty. \quad (5.2)$$

From the similar arguments in Lemma 3.2, Lemma 3.3 and Lemma 3.4, it is easy to deduce that

$$\begin{aligned} \|\phi(t)\|_2 &\leq C\|\phi_0\|_2 \exp\left(\int_0^t \|\nabla u(s)\|_2 ds\right), \\ |\psi(t)|_{D^1} &\leq C\left(|\psi_0|_{D^1} + \int_0^t \|\nabla^2 u(s)\|_1 ds\right) \exp\left(\int_0^t \|\nabla u(s)\|_2 ds\right), \\ \|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0,t]; H^2))}^2 &\leq \exp(tM(\Phi(t)))(1 + tM(\Phi(t))), \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \|\phi_t(t)\|_1 &\leq C\|\phi\|_2 \|u\|_2, \\ |\psi_t(t)|_{D^1} &\leq C(|u|_\infty |\psi|_{D^1} + |\nabla u|_3 |\psi|_6 + |u|_{D^2}), \\ \|I_t\|_{L^2(\mathbb{R}^+ \times S^2; C([0,t]; H^2))}^2 &\leq M(\Phi(t)). \end{aligned} \quad (5.4)$$

For the higher order terms  $|u|_{D^2}$  and  $|u|_{D^3}$ , we can use the similar method in Lemma 3.6 to obtain

$$\begin{aligned} |u|_{D^2} &\leq C\left(|u_t|_2 + |u \cdot \nabla u|_2 + |\nabla \phi|_2 + \int_0^\infty \int_{S^2} |\bar{A}_r|_2 d\Omega dv\right) \\ &\leq C\left(|u_t|_2 + |u|_3 |u|_{D^2} + |\nabla \phi|_2 + M(|\phi|_\infty)(|\phi|_2 + \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^2)})\right), \\ |u|_{D^3} &\leq C\left(|\nabla u_t|_2 + |\nabla(u \cdot \nabla u)|_2 + |\nabla^2 \phi|_2 + \int_0^\infty \int_{S^2} |\nabla \bar{A}_r|_2 d\Omega dv\right) \\ &\leq C\left(|\nabla u_t|_2 + |\nabla u|_2 |u|_{D^2}^{\frac{1}{2}} |u|_{D^3}^{\frac{1}{2}} + |\nabla^2 \phi|_2 + M(|\phi|_\infty)(\|\phi\|_1 + \|I\|_{L^2(\mathbb{R}^+ \times S^2; H^1)})\right), \end{aligned} \quad (5.5)$$

which implies that

$$\begin{aligned} |u|_{D^2} &\leq C\left(|u_t|_2 + M(\Phi(t))\right), \\ |u|_{D^3} &\leq C\left(|\nabla u_t|_2 + M(\Phi(t))(1 + |u_t|_2)\right). \end{aligned} \quad (5.6)$$

Now, we consider the estimate for  $|u_t|_2$ . Differentiating (2.2)<sub>3</sub> with respect to  $t$ , we have

$$u_{tt} + Lu_t = -(u \cdot \nabla u)_t - A\gamma\theta(\nabla \phi)_t + (\psi \cdot \mathcal{Q}(u))_t - \frac{1}{c} \int_0^\infty \int_{S^2} (\bar{A}_r)_t \Omega d\Omega dv. \quad (5.7)$$

Multiplying (5.7) by  $u_t$  and integrating the resulting equations over  $\mathbb{R}^3$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_t|_2^2 + \alpha |\nabla u_t|_2^2 + (\alpha + \beta) |\operatorname{div} u_t|_2^2 \\ &= \int_{\mathbb{R}^3} \left( (-u \cdot \nabla u)_t - A\gamma \theta (\nabla \phi)_t + (\psi \cdot Q(u))_t - \frac{1}{c} \int_0^\infty \int_{S^2} (\bar{A}_r)_t \Omega d\Omega dv \right) \cdot u_t dx \\ &\equiv: \sum_{i=1}^4 K_i, \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} K_1 &= - \int_{\mathbb{R}^3} (u \cdot \nabla u)_t \cdot u_t dx \leq C |\nabla u|_2 |u_t|_3 |\nabla u_t|_2 \leq C |\nabla u|_2^4 |u_t|_2^2 + \frac{\alpha}{10} |\nabla u_t|_2^2, \\ K_2 &= - \int_{\mathbb{R}^3} A\gamma \theta \nabla \phi_t \cdot u_t dx \leq C \|\phi_t\|_1 |u_t|_2, \\ K_3 &= \int_{\mathbb{R}^3} (\psi \cdot Q(u))_t \cdot u_t dx \leq \int_{\mathbb{R}^3} (|\psi_t| |Q(u)| |u_t| + |\psi| |Q(u)_t| |u_t|) dx \\ &\leq C \int_{\mathbb{R}^3} (|u| |\nabla \psi| + |\nabla u| |\psi| + |\nabla \operatorname{div} u|) |Q(u)| |u_t| + |\psi| |Q(u)_t| |u_t| dx \\ &\leq C (|u|_\infty |\nabla \psi|_2 |\nabla u|_3 |u_t|_6 + |\nabla u|_\infty |\psi|_3 |\nabla u|_2 |u_t|_6 \\ &\quad + |\nabla \operatorname{div} u|_6 |\nabla u|_2 |u_t|_3 + |\psi|_6 |\nabla u_t|_2 |u_t|_3) \\ &\leq C \left( (|\nabla \psi|_2^2 |\nabla u|_2^2 + |\psi|_3^4 |\nabla u|_2^4) |u|_{D^2}^2 + (|\nabla u|_2^4 + |\psi|_6^4) |u_t|_2^2 \right) \\ &\quad + \frac{\alpha}{10} |\nabla u_t|_2^2 + \eta |u|_{D^3}^2, \\ K_4 &= - \frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} (\bar{A}_r)_t \Omega d\Omega dv \cdot u_t dv dx \leq CM(\Phi(t)) + |u_t|_2^2. \end{aligned}$$

Here  $\eta$  is a enough small constant. Thus, using (5.6) and (5.8) we can deduce that

$$|u_t|_2^2 + \int_\tau^t |\nabla u_t|_2^2 ds \leq C + |u_t(\tau)|_2^2 + C \int_\tau^t (1 + |u_t|_2^2) M(\Phi(t)) ds, \quad (5.9)$$

which implies that

$$|u_t|_2^2 + \int_\tau^t |\nabla u_t|_2^2 ds \leq C \Theta(\tau) \exp(\bar{T} M(\Phi(t))). \quad (5.10)$$

Therefore, by virtue of (5.3)–(5.4)–(5.6)–(5.10), we can deduce that

$$\Theta(t) \leq C(1 + \overline{T})(1 + \Theta(\tau))^2 M(\Phi(t)) \exp(\overline{T}M(\Phi(t))). \quad (5.11)$$

From (5.11), we can obtain the result (5.2).

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