



Regularity results on a class of doubly nonlocal problems

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Abstract

The purpose of this article is twofold. First, an issue of regularity of weak solution to the problem (P) (see below) is addressed. Secondly, we investigate the question of H^s versus C^0 -weighted minimizers of the functional associated to problem (P) and then give applications to existence and multiplicity results. © 2019 Elsevier Inc. All rights reserved.

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1. Introduction

In this article we will study the following problem:

$$(P) \quad \begin{cases} (-\Delta)^s u &= g(x, u) + \left(\int_{\Omega} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u) \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega \end{cases}$$

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where Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 2$, $s \in (0, 1)$, $\mu < N$, $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ Carathéodory function, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and F is the primitive of f . Here the operator $(-\Delta)^s$ is the fractional Laplacian defined up to a positive multiplicative constant as

$$(-\Delta)^s u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

where P.V. denotes the Cauchy principal value.

The existence and regularity of weak solutions have been a fascinating topic for the researchers for a long time. The work on Choquard equations was started with the quantum theory of a polaron model given by S. Pekar [28]. In 1976, in the modeling of a one component plasma, P. Choquard [20] used the following equation with $\mu = 1$, $p = 2$ and $N = 3$:

$$-\Delta u + u = \left(\frac{1}{|x|^\mu} * F(u) \right) f(u) \text{ in } \mathbb{R}^3 \quad (1.1)$$

where $f(u) = |u|^{p-2}u$ and $F' = f$. In [25], Moroz and Schaftingen established the existence of a ground state solution and the regularity of weak solutions of the problem (1.1) in higher dimensions $N \geq 3$, $\mu \in (0, N)$ and with more general functions $F \in C^1(\mathbb{R}, \mathbb{R})$ satisfying certain growth conditions. For more results on the existence of solutions we refer to [26,27] and the references therein. In [14], Yang and Gao studied the Brezis-Nirenberg type result for the following equation

$$-\Delta u = \lambda u + \left(\int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x - y|^\mu} dy \right) |u|^{2_\mu^*-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain having smooth boundary $\partial\Omega$, $\lambda > 0$, $0 < \mu < N$ and $2_\mu^* = \frac{2N-\mu}{N-2}$. Later, many researchers studied the Choquard equation for the existence and multiplicity of solutions, for instance see [4,15,23] and references therein.

On the other hand, in recent years, the subject of nonlocal elliptic equations involving fractional Laplacian has gained more popularity because of many applications such as continuum mechanics, game theory and phase transition phenomena. For an extensive survey on fractional Laplacian and its applications, one may refer to [1,34] and references therein. The nonlocal equations with Hartree-type nonlinearities were used to model the dynamics of pseudo-relativistic boson stars. In fractional quantum mechanics, fractional Schrödinger equations play an important role, for instance see [13,22,23]. For the existence and multiplicity results on fractional Laplacian, readers can refer to [24] and references therein. For the doubly nonlocal problem, precisely, the nonlocal elliptic equation involving fractional Laplacian and Choquard type nonlinearity, there are articles which discuss the existence and multiplicity of solutions, we cite [3, 11,29,37] and references therein, with no attempt to provide a complete list.

Regularity results about problem involving fractional diffusion are also attracting a large number of researchers. Consider the following nonlocal problem

$$(-\Delta)^s u = g \text{ in } \Omega, \quad u = h \text{ in } \mathbb{R}^N \setminus \Omega. \quad (1.2)$$

The interior regularity of solutions to (1.2) is primarily determined by Caffarelli and Silvestre. In [8], authors developed the $C^{1+\alpha}$ interior regularity for viscosity solutions to nonlocal equations with bounded measurable coefficients. For the convex equation, authors proved $C^{2s+\alpha}$ regularity in [9] while in [10], authors established a perturbative theory for non translation invariant equations. In [32], Silvestre studied regularity of weak solutions to free boundary problem. For the boundary regularity, Ros-Oton and Serra [30] studied the regularity of weak solutions to (1.2) with $h = 0$ and $g \in L^\infty(\Omega)$. By using a suitable upper barrier and the interior regularity results for the fractional Laplacian they prove that $u \in C^s(\mathbb{R}^N)$ and $\|u\|_{C^s} \leq c\|g\|_{L^\infty(\Omega)}$ for some constant c . Moreover, authors established a fractional analog of the Krylov boundary Harnack method to further prove $u \in C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. In [31], authors proved the high integrability of the weak solution by using the regularity of Riesz potential established in [33]. In [2], authors discussed the existence and regularity of weak solution to the following problem

$$(-\Delta)^s u = u^{-q} + f(u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega$$

where $q > 0$ and the function f is of subcritical growth. When f has critical growth then the question of existence and regularity have been answered in [18].

Despite the ample amount of research on doubly nonlocal problems, there is very little done in respect of regularity of weak solutions to these problems. For instance, in [11], authors proved the regularity of a ground state solution of doubly nonlocal equation with subcritical growth in the sense of Hardy-Littlewood-Sobolev inequality, by generalizing the idea of [26] in fractional framework. In [36], authors establish the $L^\infty(\mathbb{R})$ bound of the nonnegative ground state solution of doubly local problem with critical growth in the sense of Hardy-Littlewood-Sobolev inequality under the assumption that $\mu < \min\{N, 4s\}$.

In [16], Gao and Yang studied the Dirichlet problem involving Choquard nonlinearity with Laplacian operator. Here authors aim to prove the regularity for weak solutions. The boot-strap techniques as it is developed in [16] work for the subcritical growth and seems to fail in handling the critical non linearity in the sense of Hardy-Littlewood-Sobolev inequality. For the critical case, Moroz and Schaftingen [25], studied problem (1.1) and prove the $W_{\text{loc}}^{2,p}(\mathbb{R}^N)$, $p > 1$, regularity of the weak solution for problems in the whole space without a perturbation term $g(x, u)$. The techniques given in [25] cannot be straightforward carried to problem (P) in a general setting. The regularity of positive solution to the following singular problem

$$-\Delta u = u^{q-1} + \left(\int_{\Omega} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \quad 0 < q < 1 \quad (1.3)$$

was also an open problem.

Motivated by the above discussion and the stated issues, the first part of the present article is intended to address the question of $L^\infty(\Omega)$ bound for weak solutions of the problem (P) covering large classes of f and g . Since once $L^\infty(\Omega)$ is there then one can use the result given by Ros-Oton [30,32] coupled with Hardy-Littlewood-Sobolev inequality, to prove the desired regularity results. To prove the $L^\infty(\Omega)$ bound, we develop an unified approach handling both subcritical and critical case of the perturbation g . In this article we also provide an answer to the regularity of weak solutions to doubly nonlocal equation involving singular nonlinearity, particularly problem (1.3). The existence and multiplicity of solutions to problem (1.3), is specially address in [17]. The novelty of the obtained results here is that they hold true for all $\mu < N$,

contrasting to previous regularity results in literature. The techniques and tools which are used here to prove the $L^\infty(\Omega)$ estimate are contemporary and new. Precisely, we extend further the classical Brezis-Kato techniques [7] to improve the integrability of weak solutions to (P). In addition, we mention that to the best of our knowledge, there is no article which establish the proof of $L^\infty(\Omega)$ bound to problem involving singular nonlinearity. The results in this article can be used similarly to Laplacian operator (that is, $s = 1$) and are also new to the literature.

The second part of this article is destined to prove the H^s versus C^0 -weighted minimizers. That is, we show that the local minima with respect to $C_d^0(\overline{\Omega})$ topology will also be local minima with respect to X_0 topology. In variational problems this result illustrates a significant role as it helps to prove that the solutions to constraint minimization of the energy functional emerge as solutions to unconstraint local minimization of the energy functional. This procedure of constraint minimizations has ample amount of applications such as to prove the existence and multiplicity of solutions to elliptic problems, for instance see Theorem 6.1.

In case of local framework this result was first done by Brezis and Nirenberg [6]. Here authors prove that local minima in C^1 will remain so in H^1 topology despite of the fact that latter one is weaker than the former one. In fractional framework, this result is proved by Iannizzotto, Mosconi and Squassina [19]. But in case of nonlocal nonlinearity, in particular, Choquard equation, a particular case to our result had been answered by [16] for the Laplacian operator. For the general nonlinearity, this issue is recently posed as an open problem in [23]. In this article, we also provide a full answer to this open problem. Since there is significant amount of difference in handling doubly nonlocal problem, so we cannot stick around the tools given in [6,19] to establish the result.

Remark 1.1. We would like to remark that the results of our article can be adapted to the following fractional Schrödinger problem

$$(-\Delta)^s u + Vu = g(x, u) + \left(\int_{\Omega} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u) \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $V \in L^2(\Omega)$ and $(-\Delta)^s + V$ should be coercive in the energy space X_0 .

2. Functional framework and main results

This section of the article is intended to provide the fractional Sobolev space setting. For the complete and rigid details, one can refer [12,24]. Further in this section we state the main results of current article with a short sketch of proof.

For $0 < s < 1$, the fractional Sobolev space is defined as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < +\infty \right\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + [u]_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

Consider the space

$$X_0 := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equipped with the norm

$$\langle u, v \rangle = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy$$

where $Q = \mathbb{R}^N \setminus (\Omega^c \times \Omega^c)$. From the embedding results ([24]), the space X_0 is continuously embedded into $L^r(\mathbb{R}^N)$ with $r \in [1, 2_s^*]$ where $2_s^* = \frac{2N}{N-2s}$. The best constant S_s is defined

$$S_s = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy}{\left(\int_{\Omega} |u|^{2_s^*} dx \right)^{2/2_s^*}}. \quad (2.1)$$

Let $d : \overline{\Omega} \rightarrow \mathbb{R}_+$ by $d(x) := \text{dist}(x, \mathbb{R}^N \setminus \Omega)$, $x \in \overline{\Omega}$. The best constant S_H is defined as

$$S_H = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy}{\int_{\Omega} \frac{|u|^2}{d^{2s}} dx}. \quad (2.2)$$

Now we define the weighted Hölder-type spaces

$$C_d^0(\overline{\Omega}) := \left\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a continuous extension to } \overline{\Omega} \right\},$$

$$C_d^{0,\alpha}(\overline{\Omega}) := \left\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a } \alpha\text{-Hölder continuous extension to } \overline{\Omega} \right\}$$

endowed with the norms

$$\|u\|_{0,d} := \|u/d^s\|_{\infty}, \quad \|u\|_{\alpha,d} := \|u\|_{0,d} + \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x)/(x)^{d^s} - u(y)/(y)^s|}{|x - y|^\alpha}$$

respectively. We assume that f satisfies the following growth conditions throughout the current article.

(\mathcal{F}) $F \in C^1(\mathbb{R}, \mathbb{R})$, $F' = f$ and there exists $C > 0$ such that for all $t \in \mathbb{R}$,

$$|tf(t)| \leq C(|t|^{\frac{2N-\mu}{N}} + |t|^{\frac{2N-\mu}{N-2s}}).$$

Definition 2.1. A function $u \in X_0$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ is said to be a solution to (P) if

$$\int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy = \lambda \int_{\Omega} g(x, u) \phi dx + \iint_{\Omega \times \Omega} \frac{F(u)f(u)}{|x - y|^{\mu}} \phi dx dy$$

for all $\phi \in X_0$.

Let $G(x, u) = \int_0^u g(x, \tau) d\tau$ then functional associated with problem (P) is defined as

$$J(u) = \frac{\|u\|^2}{2} - \int_{\Omega} G(x, u) dx - \frac{1}{2} \iint_{\Omega \times \Omega} \frac{F(u)F(u)}{|x - y|^{\mu}} dx dy, \text{ for all } u \in X_0.$$

With this functional framework, we state the main results of the article. First we state the result about the regularity of weak solution to problem (P).

Theorem 2.2. Let $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

$$g(x, u) = O(|u|^{2_s^* - 1}), \text{ if } |u| \rightarrow \infty$$

uniformly for all $x \in \overline{\Omega}$. Then any solution $u \in X_0$ of (P) belongs to $L^\infty(\mathbb{R}^N) \cap C^s(\mathbb{R}^N)$. Furthermore, there exists positive constant C depending on $N, \mu, s, |\Omega|$ such that $|u|_\infty \leq C(1 +$

$$|u|_{2_s^*})^{\frac{2}{(2_s^* - 1)(2_s^* - 2)}} \left(1 + \left((1 + |u|_{2_s^*}) \left(|u|_{2_s^*}^{2_s^*} + R^{2_s^*} |u|_{2_s^*}^{2_s^* - 1} \right) \right)^{\frac{2_s^*}{2}} \right)^{\frac{2}{2_s^* (2_s^* - 1)}} \text{ and } R > 0 \text{ large enough}$$

$$\text{such that } \left(\int_{|u| > R} |u|_{2_s^*}^{2_s^*} dx \right)^{\frac{2_s^* - 2}{2_s^*}} \leq \frac{1}{2C(1 + |u|_{2_s^*})}.$$

Next we consider the regularity for singular problems.

Theorem 2.3. Let $q \in (0, 1)$ and $g(x, u) = u^{q-1}$. Then any positive solution $u \in X_0$ of (P) belongs to $L^\infty(\mathbb{R}^N) \cap C^s(\mathbb{R}^N)$. Moreover, there exists $C > 0$ depending on N, μ, s and $|\Omega|$ and a positive constant C_1 s.t.

$$|u|_\infty \leq 1 + C_1 S_1^{\frac{2}{(2_s^* - 1)(2_s^* - 2)}} \left(1 + \left(S_1 \left(|(u - 1)^+|_{2_s^*}^{2_s^*} + R^{2_s^*} |(u - 1)^+|_{2_s^*}^{2_s^* - 1} \right) \right)^{\frac{2_s^*}{2}} \right)^{\frac{2}{2_s^* (2_s^* - 1)}}$$

$$\text{with } S_1 = \max\{1, C(N, \mu, |\Omega|)|u|_{2_s^*}\}, R > 0 \text{ such that } \left(\int_{|u| > R} |(u - 1)^+|_{2_s^*}^{2_s^*} dx \right)^{\frac{2_s^* - 2}{2_s^*}} \leq \frac{1}{2(2_s^* + 1)S_1}.$$

Remark 2.4. Replacing u^{q-1} by $g(x, u)$ with $g : \Omega \times \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ satisfying $g(x, t)t^{1-q}$ uniformly bounded as $t \rightarrow 0^+$ and $t \rightarrow g(x, t)$ nonincreasing for a.e. $x \in \Omega$, then Theorem 2.3 holds.

To achieve the intended goal in the above results, we first prove the non local version of Brezis-Kato estimates (see Lemma 3.2 and 3.3) in a similar manner as in [7,25]. Subsequently we construct a sequence of coercive, bilinear maps. This sequence allows us to further construct a sequence of function u_n will converge weakly to u (weak solution to (P)). Then we inherit some classical technique of Brezis-Kato [7,25]. We prove that $u_n \in L^p(\Omega)$ with $2_s^* < p < p_0$ for some p_0 . Consequently, $u \in L^p(\Omega)$ with $2_s^* < p < p_0$. Using these estimates, we establish

$$\int_{\Omega} \frac{F(u(y))}{|x-y|^\mu} dy \in L^\infty(\Omega).$$

Then by Moser iterations proved established in Lemma 4.7, we prove that $u \in L^\infty(\Omega)$. For the $C^{0,\alpha}(\overline{\Omega})$ regularity we can conclude by using Ros-Oton and Serra [30] mentioned above. We mention here that the construction of the bilinear forms for the Theorem 2.3 is most sensitive part and require more technicality. We remark that if we use Moser iterations without employing the method we present above then we can achieve $L^\infty(\Omega)$ bound of weak solutions to (P) under the additional assumption $\mu < \min\{N, 4s\}$ and $f = |u|^{\frac{N-\mu+2}{N-2s}}$, see for instance [17]. To incorporate the case $\mu \geq \min\{N, 4s\}$, we develop the above stated unified course of steps.

The second main aim of this paper is to give an application of $L^\infty(\Omega)$ estimate. In that direction we have the following.

Theorem 2.5. *Let $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying*

$$g(x, u) = O(|u|^{2_s^*-1}), \text{ if } |u| \rightarrow \infty$$

uniformly for all $x \in \overline{\Omega}$. Let $v_0 \in X_0$. Then the following assertions holds are equivalent:

- (i) *there exists $\varepsilon > 0$ such that $J(v_0 + v) \geq J(v_0)$ for all $c \in X_0$, $\|v\| \leq \varepsilon$;*
- (ii) *there exists $\rho > 0$ such that $J(v_0 + v) \geq J(v_0)$ for all $v \in X_0 \cap C_d^0(\overline{\Omega})$, $\|v\|_{0,\delta} \leq \rho$.*

To prove the above result we have modified the techniques which have been developed by [6,19].

As an application of the H^s versus C^0 -weighted minimizers, in section 6, we proved the existence of weak solution to Choquard equation, which is also a local minimizer in X_0 topology (see Theorem 6.1). To prove the desired result, instead by trapping the nonlinearity between sub and supersolution, we generalize Perron's method for the doubly nonlocal problem [35, Theorem 2.4]. An advantage to proceed by this alternative method is that we don't need strong assumptions on sub and supersolution except the fact, they belong to X_0 .

For simplicity of illustration, we set some notations. We denote $\|u\|_{L^p(\Omega)}$ by $|u|_p$ and $\|u\|_{X_0}$ by $\|u\|$. $B_\rho^X(u)$, $\bar{B}_\rho^X(u)$ ($B_\rho^d(u)$, $\bar{B}_\rho^d(u)$) denote the open and closed ball, centered at u with radius ρ , respectively in X_0 ($C_d^0(\overline{\Omega})$). The positive constant C values change case by case.

Rest of the paper organized as follows: In section 3, we give some preliminary results. In section 4, we give some technical lemmas which will help us to prove the main theorems of the paper. In section 5, we prove the Theorem 2.2 and 2.3. In section 6, we give the proof of Theorem 2.5 and provide an application to Theorem 2.5.

3. Preliminary results

In this section we contribute some preliminary results, though rather straightforward, do not appear explicitly in former literature, and are worthy to archive them here.

The Hardy-Littlewood-Sobolev Inequality, foundational in study of Choquard equation is stated here.

Proposition 3.1. [21] *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, r, \mu, N)$ independent of f, h , such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dy dx \leq C(t, r, \mu, N) \|f\|_t \|h\|_r.$$

Lemma 3.2. *If $V \in L^\infty(\Omega) + L^{N/2s}(\Omega)$ then for every $\varepsilon > 0$ there exists C_ε such that for every $u \in X_0$, we have*

$$\int_{\Omega} V |u|^2 dx \leq \varepsilon^2 \|u\|^2 + C_\varepsilon \int_{\Omega} |u|^2 dx.$$

Proof. Let $V = V_1 + V_2$ where $V_1 \in L^\infty(\Omega)$ and $V_2 \in L^{N/2s}(\Omega)$. For each $k > 0$ we have

$$\begin{aligned} \int_{\Omega} V |u|^2 dx &\leq \|V_1\|_{L^\infty(\Omega)} \int_{\Omega} |u|^2 dx + k \int_{|V_2| \leq k} |u|^2 dx + \int_{|V_2| > k} |V_2| |u|^2 dx \\ &\leq \|V_1\|_{L^\infty(\Omega)} \int_{\Omega} |u|^2 dx + k \int_{|V_2| \leq k} |u|^2 dx + S_s^{-1} \left(\int_{|V_2| > k} |V_2|^{N/2s} dx \right)^{2s/N} \|u\|^2 \end{aligned}$$

where S_s is the best constant of the embedding X_0 into $L^{\frac{2N}{N-2s}}$. For a given $\varepsilon > 0$, choose $k > 0$ such that

$$S_s^{-1} \left(\int_{|V_2| > k} |V_2|^{N/2s} dx \right)^{2s/N} < \varepsilon^2.$$

It implies that

$$\int_{\Omega} V |u|^2 dx \leq \varepsilon^2 \|u\|^2 + C_\varepsilon \int_{\Omega} |u|^2 dx. \quad \square$$

Lemma 3.3. [25, Lemma 3.3] *Let $p, q, r, t \in [1, \infty)$ and $\lambda \in [0, 2]$ such that*

$$1 + \frac{N - \mu}{N} - \frac{1}{p} - \frac{1}{t} = \frac{\lambda}{q} + \frac{2 - \lambda}{r}.$$

If $\theta \in (0, 2)$ satisfies

$$\min\{q, r\} \left(\frac{N - \mu}{N} - \frac{1}{p} \right) < \theta < \max\{q, r\} \left(1 - \frac{1}{p} \right)$$

$$\min\{q, r\} \left(\frac{N - \mu}{N} - \frac{1}{t} \right) < 2 - \theta < \max\{q, r\} \left(1 - \frac{1}{t} \right)$$

then for $H \in L^p(\mathbb{R}^N)$, $K \in L^t(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * (H|u|^\theta)) K |u|^{2-\theta} dx \leq C \|H\|_{L^p(\mathbb{R}^N)} \|K\|_{L^t(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u|^q \right)^{\lambda/q} \left(\int_{\mathbb{R}^N} |u|^r \right)^{\frac{(2-\lambda)}{r}}.$$

Lemma 3.4. Let $N \geq 2s$, $0 < \mu < N$ and $\theta \in (0, 2)$. If $H, K \in L^{\frac{2N}{N-\mu+2s}}(\mathbb{R}^N) + L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ and $1 - \frac{\mu}{N} < \theta < 1 + \frac{\mu}{N}$ then for every $\varepsilon > 0$ there exists $C_{\varepsilon, \theta} \in \mathbb{R}$ such that for every $u \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * (H|u|^\theta)) K |u|^{2-\theta} dx \leq \varepsilon^2 \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^2 + C_{\varepsilon, \theta} \int_{\mathbb{R}^N} |u|^2 dx.$$

Proof. We follow the proof of [25, Lemma 3.2] in the nonlocal framework. Let $H = H_1 + H_2$ and $K = K_1 + K_2$ with $H_1, K_1 \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ and $H_2, K_2 \in L^{\frac{2N}{N-\mu+2s}}(\mathbb{R}^N)$. Now using Lemma 3.3 iteratively with appropriate values of p, q, r, t, θ and λ (see [25, Lemma 3.2]), we have

$$\begin{aligned} & \int_{\mathbb{R}^N} (|x|^{-\mu} * (H|u|^\theta)) K |u|^{2-\theta} dx \\ & \leq C \left(|H_2|_{\frac{2N}{N-\mu+2s}} + |K_2|_{\frac{2N}{N-\mu+2s}} \right)^2 \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^2 \\ & \quad + C \left(|H_1|_{\frac{2N}{N-\mu}} + |K_1|_{\frac{2N}{N-\mu}} \right)^2 \int_{\mathbb{R}^N} |u|^2 dx. \end{aligned}$$

For given $\varepsilon > 0$, choose H_2, K_2 such that

$$|H_2|_{\frac{2N}{N-\mu+2s}}, |K_2|_{\frac{2N}{N-\mu+2s}} < \frac{\varepsilon}{2\sqrt{C}}.$$

Therefore, the result holds. \square

Lemma 3.5. For $a, b \in \mathbb{R}$, $r \geq 2$, $k \geq 0$, we have

$$\frac{4(r-1)}{r^2} \left(|a_k|^{r/2} - |b_k|^{r/2} \right)^2 \leq (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2})$$

where

$$a_k = \max\{-k, \min\{a, k\}\} = \begin{cases} -k, & \text{if } a \leq -k, \\ a, & \text{if } -k < a < k, \\ k, & \text{if } a \geq k. \end{cases}$$

Proof. From [19, Lemma 3.1], we have

$$\frac{4(r-1)}{r^2} \left(a|a_k|^{\frac{r}{2}-1} - b|b_k|^{\frac{r}{2}-1} \right)^2 \leq (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}). \quad (3.1)$$

By symmetry of the inequality, it is enough to show that result hold for $a \leq b$. For this, let $a = a_k$ and $b = b_k$ in (3.1), we have

$$\frac{4(r-1)}{r^2} \left(a_k|a_k|^{\frac{r}{2}-1} - b_k|b_k|^{\frac{r}{2}-1} \right)^2 \leq (a_k - b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}).$$

Case 1: $0 \leq b < a$

Clearly $0 \leq b_k < a_k$ and $a_k - b_k \leq a - b$. This implies

$$(a_k - b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}) \leq (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}).$$

Case 2: $b \leq 0 \leq a$

Again notice that $b_k \leq 0 \leq a_k$, $a_k - b_k \leq a - b$ and $a_k b_k \leq |a_k b_k|$ we have

$$(a_k - b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}) \leq (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2})$$

and

$$\left(|a_k|^{r/2} - |b_k|^{r/2} \right)^2 \leq \left(a_k|a_k|^{\frac{r}{2}-1} - b_k|b_k|^{\frac{r}{2}-1} \right)^2.$$

Hence the proof. \square

4. Technical results

This section is devoted to the study of weak solutions to the following problem

$$(P_1) \begin{cases} (-\Delta)^s u = g(x, u) + \left(\int_{\Omega} \frac{H(y)u(y)}{|x-y|^\mu} dy \right) K(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $H, K \in L^{\frac{2N}{N-\mu+2s}}(\Omega) + L^{\frac{2N}{N-\mu}}(\Omega)$. Here we use the results, established in last section to improve the integrability regularity of weak solutions to the above mentioned problem.

Proposition 4.1. Let $H, K \in L^{\frac{2N}{N-\mu+2s}}(\Omega) + L^{\frac{2N}{N-\mu}}(\Omega)$. Let $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying

$$g(x, u) = O(|u|^{2_s^*-1}), \text{ if } |u| \rightarrow \infty$$

uniformly for all $x \in \overline{\Omega}$. Then any solution $u \in X_0$ of the problem (P_1) belongs to $L^r(\Omega)$ where $r \in [2, \frac{2N^2}{(N-\mu)(N-2s)})$.

Proof. For $\theta = 1$ in Lemma 3.4, there exists $\alpha > 0$ such that for every $\phi \in X_0$,

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|H(y)\phi(y)K(x)\phi(x)|}{|x-y|^{\mu}} dx dy &\leq \frac{1}{2} \left(\int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy \right)^2 \\ &\quad + \frac{\alpha}{2} \int_{\Omega} |\phi|^2 dx. \end{aligned} \quad (4.1)$$

If $3 \leq 2_s^* \leq 2_s^*$ then $|u|^{2_s^*-2} \in L^{N/2s}(\Omega)$. If $2 < 2_s^* < 3$ then choose $p > 1$ such that $1 \leq \frac{p(2_s^*-2)N}{2s} \leq 2_s^*$ then using Hölder's inequality gives us

$$\left(\int_{\Omega} |u|^{\frac{(2_s^*-2)N}{2s}} dx \right)^{2s/N} \leq C \left(\int_{\Omega} |u|^{\frac{p(2_s^*-2)N}{2s}} dx \right)^{2s/Np} < \infty.$$

Choose $L_1 > 0$ such that $\left(\int_{|u|>L_1} |u|^{\frac{(2_s^*-2)N}{2s}} dx \right)^{2s/N} \leq \frac{S_s}{2}$ where S_s is the best Sobolev constant defined in (2.1). Since $g(x, u) = O(|u|^{2_s^*-1})$ for u large enough, there exist $L/2 > L_1 > 0$ such that $g(x, u) \leq |u|^{2_s^*-1}$ uniformly for $x \in \overline{\Omega}$ and $|u| > L/2$. Define $\eta \in C_c^\infty[0, \infty)$ such that $0 \leq \eta \leq 1$ and

$$\eta(u) = \begin{cases} 1, & \text{if } |u| < L/2, \\ 0, & \text{if } |u| > L. \end{cases}$$

Define $V := (1 - \eta) \frac{g(x, u)}{u}$ and $T := \eta g(x, u) + \alpha u$. By the choice of η , we obtain

$$|V|_{N/2s} < S_s/2 \text{ and } T \in X'_0. \quad (4.2)$$

Observe that u is the unique solution to the following problem

$$(-\Delta)^s u + \alpha u = Vu + \left(\int_{\Omega} \frac{H(y)u(y)}{|x-y|^{\mu}} dy \right) K + T \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

Choose sequence $\{H_n\}_{n \in \mathbb{N}}$ and $\{K_n\}_{n \in \mathbb{N}}$ in $L^{\frac{2N}{N-\mu}}(\Omega)$ such that $|H_n| \leq |H|$, $|K_n| \leq |K|$ and $H_n \rightarrow H$, $K_n \rightarrow K$ a.e. in Ω . For each $n \in \mathbb{N}$, V_n denotes the truncated potential defined as $V_n = V$ if $|V| \leq n$ and $V_n = n$ if $|V| > n$. Now we introduce the bilinear form

$$B_n(v, w) = \int_Q \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy + \alpha \int_{\Omega} v w dx \\ - \int_{\Omega} \int_{\Omega} \frac{H_n(y)v(y)K_n(x)w(x)}{|x - y|^{\mu}} dx dy - \int_{\Omega} V_n v w dx.$$

In view of Hölder's inequality, Sobolev embedding, (4.2) and (4.1), one can easily conclude that B_n is continuous coercive bilinear form. Hence by Lax-Milgram Lemma (see [5, Corollary 5.8]) there exists a unique $u_n \in X_0$ such that for all $w \in X_0$ we have

$$B_n(u_n, w) = \int_{\Omega} T w dx. \quad (4.3)$$

Subsequently, u_n is a unique solution to the problem

$$(-\Delta)^s u_n + \alpha u_n = \left(\int_{\Omega} \frac{H_n(y)u_n(y)}{|x - y|^{\mu}} dy \right) K_n + V_n u_n + T \text{ in } \Omega, u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (4.4)$$

Furthermore, using (4.3) we can easily prove that u_n is a bounded sequence in X_0 . It implies that up to a subsequence, $u_n \rightharpoonup u$ weakly in X_0 . Let $u_{n,\tau} = \max\{-\tau, \min\{u_n, \tau\}\}$ for $\tau > 0$ and $x \in \Omega$. Testing Problem (4.4) with $\phi = |u_{n,\tau}|^{r-2} u_{n,\tau} \in X_0$ ($2 \leq r < \frac{2N}{N-\mu}$), with the help of Lemma 3.5, we get

$$\frac{4(r-1)}{r^2} \| |u_{n,\tau}|^{r/2} \|^2 + \alpha \int_{\Omega} |u_{n,\tau}|^{r/2} dx \\ \leq \int_Q \frac{(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dx dy + \alpha \int_{\Omega} u_n \phi dx \\ = \int_{\Omega} \int_{\Omega} \frac{H_n(y)u_n(y)K_n(x)|u_{n,\tau}|^{r-2} u_{n,\tau}}{|x - y|^{\mu}} dy + \int_{\Omega} V_n u_n |u_{n,\tau}|^{r-2} u_{n,\tau} dx \\ + \int_{\Omega} T |u_{n,\tau}|^{r-2} u_{n,\tau} dx. \quad (4.5)$$

Using Lemma 3.4 with $\varepsilon^2 = \frac{(r-1)}{r^2}$, we obtain

$$\begin{aligned}
 \int_{\Omega} \int_{\Omega} \frac{H_n(y)u_n(y)K_n(x)|u_{n,\tau}|^{r-2}u_{n,\tau}}{|x-y|^\mu} dx dy &\leq \int_{\Omega} \int_{\Omega} \frac{|H_n(y)u_{n,\tau}(y)||K_n(x)||u_{n,\tau}(x)|^{r-1}}{|x-y|^\mu} dx dy \\
 &+ \int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x-y|^\mu} dx dy \\
 &\leq \frac{(r-1)}{r^2} \| |u_{n,\tau}|^{r/2} \|^2 + C_r \int_{\Omega} |u_{n,\tau}|^r dx \\
 &+ \int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x-y|^\mu} dx dy
 \end{aligned} \tag{4.6}$$

where $E_{n,\tau} = \{x \in \mathbb{R}^N : |u_n(x)| \geq \tau\}$. By Hardy-Littlewood-Sobolev inequality and Hölder's inequality, we have

$$\begin{aligned}
 &\int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x-y|^\mu} dx dy \\
 &\leq C \left(\int_{\mathbb{R}^N} \left| |K_n||u_n|^{r-1} \right|^j d\xi \right)^{\frac{1}{j}} \left(\int_{E_{n,\tau}} |H_n u_n|^l d\xi \right)^{\frac{1}{l}}
 \end{aligned} \tag{4.7}$$

where j and l satisfy the relation $\frac{1}{j} = 1 + \frac{N-\mu}{2N} - \frac{1}{r}$ and $\frac{1}{l} = \frac{N-\mu}{2N} + \frac{1}{r}$. Using the fact that $H_n, K_n \in L^{\frac{2N}{N-\mu}}(\Omega)$ and again the Hölder's inequality, $u_n \in L^r(\mathbb{R}^N)$ implies that $|K_n||u_n|^{r-1} \in L^j(\mathbb{R}^N)$ and $|H_n u_n| \in L^l(\mathbb{R}^N)$. Therefore, as $\tau \rightarrow \infty$, (4.7) gives

$$\lim_{\tau \rightarrow \infty} \int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x-y|^\mu} dx dy = 0. \tag{4.8}$$

Using the Sobolev inequality, (4.5), (4.6) and (4.8), we have

$$\begin{aligned}
 &\frac{3(r-1)S_s}{r^2} \left(\int_{\Omega} |u_{n,\tau}|^{\frac{rN}{N-2s}} dx \right)^{\frac{N-2s}{N}} \\
 &\leq C_r \int_{\Omega} |u_n|^r + \int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x-y|^\mu} dx dy \\
 &\quad + \int_{\Omega} V_n u_n |u_{n,\tau}|^{r-2} u_{n,\tau} dx + \int_{\Omega} g |u_{n,\tau}|^{r-2} u_{n,\tau} dx.
 \end{aligned} \tag{4.9}$$

Employing the fact that g is a Carathéodory function,

$$\begin{aligned} \int_{\Omega} T|u_{n,\tau}|^{r-2}u_{n,\tau} \, dx &\leq \int_{|u|\leq L} g(x,u)|u_n|^{r-1} \, dx + \alpha \int_{\Omega} u|u_{n,\tau}|^{r-1} \, dx \\ &\leq C(L_1) \left(\int_{\Omega} |u|^r \, dx + \int_{\Omega} |u_n|^r \, dx \right). \end{aligned} \quad (4.10)$$

By Lemma 3.2 for $\varepsilon^2 = \frac{r-1}{r^2}$, we have

$$\begin{aligned} \int_{\Omega} V_n u_n |u_{n,\tau}|^{r-2} u_{n,\tau} \, dx &\leq 2 \int_{E_{n,\tau}} V_n |u_n|^r \, dx + \int_{\Omega} V_n |u_{n,\tau}|^r \, dx \\ &\leq \frac{(r-1)}{r^2} \| |u_{n,\tau}|^{r/2} \|^2 + C_r \int_{\Omega} |u_{n,\tau}|^r \, dx + 2 \int_{E_{n,\tau}} V_n |u_n|^r \, dx. \end{aligned} \quad (4.11)$$

Using Dominated Convergence theorem, one can easily show that $\lim_{\tau \rightarrow \infty} \int_{E_{n,\tau}} V_n |u_n|^r \, dx = 0$. Now taking into account (4.9), (4.10), (4.11) and letting $\tau \rightarrow \infty$, we have

$$\left(\int_{\Omega} |u_n|^{\frac{rN}{(N-2s)}} \, dx \right)^{\frac{N-2s}{N}} \leq C_r \left(\int_{\Omega} |u_n|^r \, dx + \int_{\Omega} |u|^r \, dx \right).$$

Therefore,

$$\limsup_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^{\frac{rN}{(N-2s)}} \, dx \right)^{\frac{N-2s}{N}} \leq C_r \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^r \, dx + \int_{\Omega} |u|^r \, dx \right).$$

Hence, by iterating a finite number of times, we infer that $u \in L^q(\Omega)$ for all $q \in [2, \frac{2N^2}{(N-\mu)(N-2s)})$. Moreover, there exists a positive constant $C(q, N, \mu, |\Omega|)$ such that $|u|_q \leq C(q, N, \mu, |\Omega|)|u|_{2_s^*}$. \square

Definition 4.2. For $\phi \in C^0(\overline{\Omega})$ with $\phi > 0$ in Ω , the set $C_\phi(\Omega)$ is defined as

$$C_\phi(\Omega) = \{u \in C^0(\overline{\Omega}) : \text{there exists } c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \text{ for all } x \in \Omega\},$$

endowed with the natural norm $\left\| \frac{u}{\phi} \right\|_{L^\infty(\Omega)}$.

Definition 4.3. The positive cone of $C_\phi(\Omega)$ is the open convex subset of $C_\phi(\Omega)$ defined as

$$C_\phi^+(\Omega) = \left\{ u \in C_\phi(\Omega) : \inf_{x \in \Omega} \frac{u(x)}{\phi(x)} > 0 \right\}.$$

Proposition 4.4. [2, Theorem 1.2] Let $\phi_1 \in C^s(\mathbb{R}^N) \cap C_{d^s}^+(\Omega)$ be the normalized eigenvalue of $(-\Delta)^s$ in X_0 . If $q \in (0, 1)$ then there exists a unique positive $\underline{u} \in X_0 \cap C_{\phi_1}^+(\Omega) \cap C_0(\overline{\Omega})$ classical solution to the following problem

$$(-\Delta)^s u = u^{q-1}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (4.12)$$

Proposition 4.5. Let $q \in (0, 1)$, $g(x, u) = u^{q-1}$ and $0 \leq H, K \in L^{\frac{2N}{N-\mu+2s}}(\Omega) + L^{\frac{2N}{N-\mu}}(\Omega)$. Let $u \in X_0$ be a positive weak solution of problem (P_1) . Then $u \in L^p(\Omega)$ where $p \in [2, \frac{2N^2}{(N-\mu)(N-2s)})$.

Proof. Since $0 \leq H, K$, we see that $\underline{u} \in X_0$ is a subsolution to problem (P_1) .

Claim. $\underline{u} \leq u$ a.e. in Ω .

Assuming by contradiction, assume that the Claim is not true. Since for any $u \in X_0$ we have

$$\|u^+\|^2 \leq \int_Q \frac{(u(x) - u(y))(u^+(x) - u^+(y))}{|x - y|^{N+2s}} dx dy.$$

Testing $(-\Delta)^s \underline{u} - (-\Delta)^s u \leq \underline{u}^{q-1} - u^{q-1}$ with $(\underline{u} - u)^+$, we obtain

$$\begin{aligned} 0 \leq \|(\underline{u} - u)^+\|^2 &\leq \int_Q \frac{((\underline{u} - u)^+(x) - (\underline{u} - u)^+(y))((\underline{u} - u)(x) - (\underline{u} - u)(y))}{|x - y|^{N+2s}} dx dy \\ &\leq \int_\Omega (\underline{u}^{q-1} - u^{q-1})(\underline{u} - u)^+ dx \leq 0. \end{aligned}$$

It implies $|\{x \in \Omega : \underline{u} \geq u \text{ a.e. in } \Omega\}| = 0$. It provides the expected contradiction. Hence $\underline{u} \leq u$ a.e. in Ω .

Observe that using Proposition 4.4, for all $\beta > 0$, we have

$$\chi_{\{u < \beta\}} u^{q-1} \leq \chi_{\{u < \beta\}} \frac{u}{\underline{u}^2} u^q < \chi_{\{u < \beta\}} \frac{u}{C_1^2 \phi_1^2} \beta^q \leq \chi_{\{u < \beta\}} \frac{u}{C_1^2 C_2^2 d^{2s}} \beta^q,$$

where C_1 and C_2 are appropriate positive constants. Hence we can choose $\delta := \delta(\beta) > 0$ such that $\chi_{\{u < \beta\}} u^{q-1} = \delta(\beta) \chi_{\{u < \beta\}} \frac{u}{d^{2s}}$. Now choose $\beta > 0$ such that $\gamma_1 := \frac{1}{2} - S_H \delta(\beta) > 0$ and $\gamma_2 := \frac{3(r-1)}{r^2} - S_H \delta(\beta) > 0$ for $2 \leq r < \frac{2N}{N-\mu}$ and with S_H defined on (2.2). The choice of $\beta, \delta(\beta)$ and Lax-Milgram Lemma, imply that u is the unique solution of the following problem:

$$\begin{aligned} &(-\Delta)^s u + \alpha u - \delta(\beta) \chi_{\{u < \beta\}} \frac{u}{d^{2s}} \\ &= \left(\int_\Omega \frac{H(y)u(y)}{|x - y|^\mu} dy \right) K + \chi_{\{u \geq \beta\}} u^{q-1} + \alpha u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \end{aligned}$$

where $\alpha > 0$ is chosen as in Proposition 4.1. Now we will follow the same arguments as in Proposition 4.1 to achieve the result. Notice that $T = \chi_{\{u \geq \beta\}} u^{q-1} + \alpha u \in X'_0$. For each $n \in \mathbb{N}$, we define the bilinear form

$$\begin{aligned} B_n(v, w) = & \int_Q \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy + \alpha \int_{\Omega} v w dx \\ & - \int_{\Omega} \int_{\Omega} \frac{H_n(y)v(y)K_n(x)w(x)}{|x - y|^{\mu}} dx dy - \int_{\Omega} \delta(\beta)\chi_{\{u < \beta\}} \frac{vw}{d^{2s}} dx. \end{aligned}$$

Using as the arguments as in Proposition 4.1, there exist unique $u_n \in X_0$ such that for all $w \in X_0$ we have

$$B_n(u_n, w) = \int_{\Omega} T w dx.$$

Moreover, u_n is a unique solution to the problem

$$(-\Delta)^s u_n + \alpha u_n = \left(\int_{\Omega} \frac{H_n(y)u_n(y)}{|x - y|^{\mu}} dy \right) K_n + \delta(\beta)\chi_{\{u < \beta\}} \frac{u_n}{d^{2s}} + T \text{ in } \Omega, u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

Clearly, $u_n \rightharpoonup u$ weakly in X_0 . Let $u_{n,\tau} = \max\{-\tau, \min\{u_n, \tau\}\}$ for $\tau > 0$ and $x \in \Omega$. Choose $\phi = |u_{n,\tau}|^{r-2} u_{n,\tau} \in X_0$ ($2 \leq r < \frac{2N}{N-\mu}$) as the test function in (4.4). Using the same arguments as in Proposition 4.1, we have

$$\begin{aligned} & \frac{3(r-1)S_s}{r^2} \left(\int_{\Omega} |u_{n,\tau}|^{\frac{rN}{(N-2s)}} dx \right)^{\frac{N-2s}{N}} \\ & \leq C_r \int_{\Omega} |u_n|^r + \int_{E_{n,\tau}} \int_{\Omega} \frac{|H_n(y)u_n(y)||K_n(x)||u_n(x)|^{r-1}}{|x - y|^{\mu}} dx dy \\ & \quad + \int_{\Omega} \delta(\beta)\chi_{\{u < \beta\}} \frac{u_n}{d^{2s}} |u_{n,\tau}|^{r-2} u_{n,\tau} dx + \int_{\Omega} g |u_{n,\tau}|^{r-2} u_{n,\tau} dx. \end{aligned} \quad (4.13)$$

Consider

$$\begin{aligned} \int_{\Omega} T |u_{n,\tau}|^{r-2} u_{n,\tau} dx &= \int_{\Omega} \chi_{\{u \geq \beta\}} (u^{q-1} + \alpha u) |u_{n,\tau}|^{r-2} u_{n,\tau} dx \\ &\leq C(N, \mu, r, |\Omega|) \left(\int_{\Omega} |u|^r dx + \int_{\Omega} |u_n|^r dx \right). \end{aligned} \quad (4.14)$$

With the help of Hardy inequality, we have

$$\begin{aligned}
\int_{\Omega} \delta(\beta) \chi_{\{u < \beta\}} \frac{u_n}{d^{2s}} |u_{n,\tau}|^{r-2} u_{n,\tau} dx &\leq 2 \int_{E_{n,\tau}} \delta(\beta) \frac{|u_n|^r}{d^{2s}} dx + \int_{\Omega} \frac{|u_{n,\tau}|^r}{d^{2s}} dx \\
&\leq S_H \delta(\beta) \| |u_{n,\tau}|^{r/2} \|^2 + 2 \int_{E_{n,\tau}} \delta(\beta) \frac{|u_n|^r}{d^{2s}} dx.
\end{aligned} \tag{4.15}$$

Using Dominated Convergence theorem, it follows that $\lim_{\tau \rightarrow \infty} \int_{E_{n,\tau}} \frac{\delta(\beta) |u_n|^r}{d^{2s}} dx = 0$. Now taking into account (4.13), (4.14), (4.15), definition of γ_2 and letting $\tau \rightarrow \infty$, we have

$$\left(\int_{\Omega} |u_n|^{\frac{rN}{(N-2s)}} dx \right)^{\frac{N-2s}{N}} \leq C(N, \mu, r, |\Omega|) \left(\int_{\Omega} |u_n|^r dx + \int_{\Omega} |u|^r dx \right).$$

Therefore,

$$\limsup_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^{\frac{rN}{(N-2s)}} dx \right)^{\frac{N-2s}{N}} \leq C(N, \mu, r, |\Omega|) \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^r dx + \int_{\Omega} |u|^r dx \right).$$

Hence, $u \in L^r(\Omega)$ for all $r \in \left[2, \frac{2N^2}{(N-\mu)(N-2s)}\right)$. As earlier we remark that there exists a positive constant $C(N, \mu, q, |\Omega|)$ such that $|u|_q \leq C(N, \mu, q, |\Omega|) |u|_{2_s^*}$. \square

Remark 4.6. We highlight here that the next lemma investigates the $L^\infty(\Omega)$ bound for the fractional Laplacian with critical Sobolev exponent. In [24] authors already proved this type of result for a positive solution. Here we used the ideas from [19,24] to extend the result of [24] to any weak solution.

Lemma 4.7. *Let u be any weak solution to the following problem*

$$(-\Delta)^s u = k(x, u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \tag{4.16}$$

where $|k(x, u)| \leq C(1 + |u|^{2_s^*-1})$ and $C > 0$. Then $u \in L^\infty(\Omega)$.

Proof. Let $u \in X_0$ be any weak solution to (4.16). Let $u_\tau = \max\{-\tau, \min\{u, \tau\}\}$ for $\tau > 0$. Let $\phi = u |u_\tau|^{r-2} \in X_0$ ($r \geq 2$) be a test function to problem (4.16), then by inequality (3.1), we deduce that

$$\begin{aligned}
|u|u_\tau|^{\frac{r}{2}-1}|_{2_s^*}^2 &\leq C \|u|u_\tau|^{\frac{r}{2}-1}\|^2 \leq \frac{Cr^2}{r-1} \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dx dy \\
&\leq Cr \int_\Omega |k(x,u)||u|u_\tau|^{r-2} dx \\
&\leq Cr \int_\Omega |u||u_\tau|^{r-2} + |u|^{2_s^*}|u_\tau|^{r-2} dx.
\end{aligned} \tag{4.17}$$

Claim. Let $r_1 = 2_s^* + 1$. Then $u \in L^{\frac{2_s^* r_1}{2}}(\Omega)$.

For this, consider

$$\begin{aligned}
\int_\Omega |u|^{2_s^*}|u_\tau|^{r_1-2} dx &= \int_{|u| \leq R} |u|^{2_s^*}|u_\tau|^{r_1-2} dx + \int_{|u| > R} |u|^{2_s^*}|u_\tau|^{r-2} dx \\
&\quad + \int_{|u| \leq R} R^{2_s^*}|u_\tau|^{r_1-2} dx \\
&\quad + \left(\int_\Omega (u^2|u_\tau|^{r-2})^{\frac{2_s^*}{2}} dx \right)^{\frac{2}{2_s^*}} \left(\int_{|u| > R} |u|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}}.
\end{aligned} \tag{4.18}$$

Choose $R > 0$ large enough such that

$$\left(\int_{|u| > R} |u|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2Cr_1}. \tag{4.19}$$

Taking into account (4.17), (4.18) jointly with (4.19), we obtain

$$|u|u_\tau|^{\frac{r_1}{2}-1}|_{2_s^*}^2 \leq Cr_1 \left(\int_\Omega |u|^{2_s^*} dx + \int_\Omega R^{2_s^*}|u|^{2_s^*-1} dx \right).$$

Appealing Fatou's Lemma as $\tau \rightarrow \infty$, we obtain

$$||u|^{\frac{r_1}{2}}|_{2_s^*}^2 \leq Cr_1 \left(\int_\Omega |u|^{2_s^*} dx + \int_\Omega R^{2_s^*}|u|^{2_s^*-1} dx \right) < \infty. \tag{4.20}$$

This establishes the Claim. Now let $\tau \rightarrow \infty$ in (4.17), we deduce that

$$\|u\|_{2_s^*}^{\frac{r}{2}} \leq Cr \int_{\Omega} |u|^{r-1} + |u|^{r+2_s^*-2} dx \leq 2Cr(1 + |\Omega|) \left(1 + \int_{\Omega} |u|^{r+2_s^*-2} \right).$$

It implies that

$$\left(1 + \int_{\Omega} |u|^{\frac{2_s^* r}{2}} \right)^{\frac{2}{2_s^*(r-2)}} \leq C_r^{\frac{1}{(r-2)}} \left(1 + \int_{\Omega} |u|^{r+2_s^*-2} \right)^{\frac{1}{(r-2)}} \quad (4.21)$$

where $C_r = 4Cr(1 + |\Omega|)$. For $j \geq 1$, we define r_{j+1} iteratively as $r_{j+1} + 2_s^* - 2 = \frac{2_s^* r_j}{2}$. It implies

$$(r_{j+1} - 2) = \left(\frac{2_s^*}{2} \right)^j (r_1 - 2).$$

From (4.21), we get

$$\left(1 + \int_{\Omega} |u|^{\frac{2_s^* r_{j+1}}{2}} \right)^{\frac{2}{2_s^*(r_{j+1}-2)}} \leq C_{j+1}^{\frac{1}{(r_{j+1}-2)}} \left(1 + \int_{\Omega} |u|^{\frac{2_s^* r_j}{2}} \right)^{\frac{2}{2_s^*(r_j-2)}}$$

where $C_{j+1} := 4Cr_{j+1}(1 + |\Omega|)$. Denote $D_j = \left(1 + \int_{\Omega} |u|^{\frac{2_s^* r_j}{2}} \right)^{\frac{2}{2_s^*(r_j-2)}}$, for $j \geq 1$. By limiting arguments, one can easily prove that, for $j > 1$,

$$D_{j+1} \leq \prod_{k=2}^{j+1} C_k^{\frac{1}{(r_k-2)}} D_1 \leq C_0 D_1.$$

It implies that $\|u\|_{\infty} \leq C_0 D_1$ where D_1 is explicitly given in (4.20). \square

5. Proof of Theorem 2.2 and 2.3

In this section we will conclude the proofs of Theorem 2.2 and Theorem 2.3. Before this we recall the following result, which can be consulted in [30].

Proposition 5.1. *Let Ω be a bounded Lipschitz domain satisfying the exterior ball condition, $g \in L^{\infty}(\Omega)$ and u be a solution of (1.2). Then $u \in C^s(\mathbb{R}^N)$ and*

$$\|u\|_{C^s(\mathbb{R}^N)} \leq C \|g\|_{L^{\infty}(\Omega)}$$

where C is a constant depending on Ω and s .

Proof of Theorem 2.2. Let $u \in X_0$ be a positive weak solution to problem (P) and $H = F(u)/u$ and $K = f$. Then From Proposition 4.1, we get $u \in L^r(\Omega)$ for all $r \in \left[2, \frac{2N^2}{(N-\mu)(N-2s)}\right)$. It implies $F(u) \in L^r(\Omega)$ for all $r \in \left[\frac{2N}{2N-\mu}, \frac{2N^2}{(N-\mu)(2N-\mu)}\right)$. Observe that $\frac{2N}{2N-\mu} < \frac{N}{N-\mu} < \frac{2N^2}{(N-\mu)(2N-\mu)}$ and there exists a constant $C(N, \mu, |\Omega|) > 0$ such that $|F(u)|_{\frac{N}{N-\mu}} \leq C(N, \mu, |\Omega|)|u|_{2_s^*}$. Therefore, we infer that $\int_{\Omega} \frac{F(u)}{|x-y|^\mu} dy \in L^\infty(\Omega)$ and

$$\left| \int_{\Omega} \frac{F(u)}{|x-y|^\mu} dy \right|_{\infty} \leq C(N, \mu, |\Omega|)|u|_{2_s^*}.$$

Using the assumptions on f and g , we obtain

$$\begin{aligned} (-\Delta)^s u &= g(x, u) + \left(\int_{\Omega} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f \\ &\leq C(N, \mu, |\Omega|)(1 + |u|_{2_s^*})(1 + |u|_{2_s^*}^{2_s^*-1}) = \mathcal{S}_0(1 + |u|_{2_s^*}^{2_s^*-1}) \text{ (say)}. \end{aligned}$$

From Lemma 4.7, we have $u \in L^\infty(\Omega)$. Furthermore, there exists a function $C_0 > 0$ independent of N, μ, s and $|\Omega|$ such that

$$\begin{aligned} |u|_{\infty} &\leq C_0 \mathcal{S}_0^{\frac{2}{(2_s^*-1)(2_s^*-2)}} D_1 \\ \text{with } D_1 &\leq \left(1 + \left((2_s^* + 1) \mathcal{S}_0 \left(\int_{\Omega} |u|^{2_s^*} dx + \int_{\Omega} R^{2_s^*} |u|^{2_s^*-1} dx \right) \right)^{\frac{2_s^*}{2}} \right)^{\frac{2}{2_s^*(2_s^*-1)}} \end{aligned}$$

and $R > 0$ chosen large enough such that

$$\left(\int_{|u|>R} |u|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2(2_s^* + 1)\mathcal{S}_0}. \quad (5.1)$$

Now using Proposition 5.1, we obtain that $u \in C^s(\mathbb{R}^N)$. \square

Proof of Theorem 2.3. From Proposition 4.5, and the assumption on f , we have

$$\int_{\Omega} \frac{F(u)}{|x-y|^\mu} dy \in L^\infty(\Omega).$$

Furthermore, there exists a constant $C(N, \mu, |\Omega|) > 0$ such that $|F(u)|_{\frac{N}{N-\mu}} \leq C(N, \mu, |\Omega|)|u|_{2_s^*}$. Therefore, we infer that

$$(-\Delta)^s u \leq u^{q-1} + C(N, \mu, |\Omega|)|u|_{2_s^*}|f| \leq u^{q-1} + C(N, \mu, |\Omega|)|u|_{2_s^*}(1 + |u|^{2_s^*-1}).$$

Let $\psi \in \mathbb{R} \rightarrow [0, 1]$ be a $C^\infty(\mathbb{R})$ convex increasing function such that $\psi'(t) \leq 1$ for all $t \in [0, 1]$ and $\psi'(t) = 1$ when $t \geq 1$. Define $\psi_\varepsilon(t) = \varepsilon \psi(\frac{t}{\varepsilon})$ then using the fact that ψ_ε is smooth, we obtain $\psi_\varepsilon \rightarrow (t-1)^+$ uniformly as $\varepsilon \rightarrow 0$. It implies

$$\begin{aligned} (-\Delta)^s \psi_\varepsilon(u) &\leq \psi'_\varepsilon(u)(-\Delta)^s u \leq \chi_{\{u>1\}}(-\Delta)^s u \\ &\leq \chi_{\{u>1\}}(u^{q-1} + C(N, \mu, |\Omega|)|u|_{2_s^*}(1 + |u|^{2_s^*-1})) \\ &\leq \max\{1, C(N, \mu, |\Omega|)|u|_{2_s^*}\}(1 + ((u-1)^+)^{2_s^*-1}) \\ &= \mathcal{S}_1(1 + ((u-1)^+)^{2_s^*-1}) \text{ (say)}. \end{aligned}$$

Hence, as $\varepsilon \rightarrow 0$, we deduce that

$$(-\Delta)^s (u-1)^+ \leq \mathcal{S}_1(1 + ((u-1)^+)^{2_s^*-1}).$$

Employing Lemma 4.7, we deduce that $(u-1)^+ \in L^\infty(\Omega)$, that is, $u \in L^\infty(\Omega)$. Furthermore, since u is a positive solution, there exists $C_1 > 0$ such that independent of N, μ, s and $|\Omega|$ such that

$$|u|_\infty \leq 1 + C_1 \mathcal{S}_1^{\frac{2}{(2_s^*-1)(2_s^*-2)}} D_1$$

with

$$D_1 \leq \left(1 + \left((2_s^* + 1) \mathcal{S}_1 \left(\int_\Omega |(u-1)^+|^{2_s^*} dx + \int_\Omega R^{2_s^*} |(u-1)^+|^{2_s^*-1} dx \right) \right)^{\frac{2_s^*}{2}} \right)^{\frac{2}{2_s^*(2_s^*-1)}}$$

and $R > 0$ chosen large enough such that

$$\left(\int_{|u|>R} |(u-1)^+|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2(2_s^* + 1) \mathcal{S}_1}.$$

Let \bar{u} be the unique solution (see [2, Theorem 1.2, Remark 1.5]) to the following problem

$$(-\Delta)^s \bar{u} = \bar{u}^{-q} + c, u > 0 \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega$$

where $c = C_1 |F(u)f(u)|_\infty$ with $C_1 = \left| \int_\Omega \frac{dy}{|x-y|^\mu} \right|_\infty$. Then following similar lines as in the proof of Claim in Proposition 4.5, we have $\underline{u} \leq u \leq \bar{u}$ a.e. in Ω where \underline{u} is the unique solution to (4.12). Therefore, $u \in X_0 \cap L^\infty(\Omega) \cap C_{\phi_1}^+(\Omega)$. Now from [18, Theorem 1.3], we have the desired result. \square

6. Applications

The purpose of this section is to derive applications from the uniform estimates given in Theorems 2.2 and 2.3. Precisely, here, we prove the Theorem 2.5 which deals with H^s versus $C_d^0(\overline{\Omega})$ weighted minimizers. Furthermore, we provide an application of this result, concerning the existence and multiplicity of solutions.

Proof of Theorem 2.5: (i) implies (ii). Assume by contradiction that there exists a sequence $v_n \rightarrow v_0$ in $C_d^0(\overline{\Omega})$ and $J(v_n) < J(v_0)$. It follows that

$$\int_{\Omega} G(x, v_n) dx \rightarrow \int_{\Omega} G(x, v_0) dx \text{ and } \iint_{\Omega \times \Omega} \frac{F(v_n)F(v_n)}{|x-y|^\mu} dx dy \rightarrow \iint_{\Omega \times \Omega} \frac{F(v_0)F(v_0)}{|x-y|^\mu} dx dy.$$

Taking into account above statements, we infer that $\limsup_{n \rightarrow \infty} \|v_n\|^2 \leq \|v_0\|^2$. Hence up to a subsequence v_n converges to v_0 weakly in X_0 . By Fatou's Lemma and above conclusion one obtains $\|v_n\| \rightarrow \|v\|$. This settles the proof.

Proof of Theorem 2.5: (ii) implies (i). To show the result, we will first consider the case $v_0 = 0$. It implies that

$$\inf_{v \in X_0 \cap \bar{B}_\rho^d(0)} J(v) = J(v_0) = 0. \quad (6.1)$$

Assume that (i) doesn't hold. Then we can choose $\varepsilon_n \in (0, \infty)$, $\varepsilon_n \rightarrow 0$ such that there exist $z_n \in \bar{B}_{\varepsilon_n}^X(0)$ with $J(z_n) < 0$. For each $m \in \mathbb{N}$, define the functions g_m , $G_m: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ and f_m , $F_m: \mathbb{R} \rightarrow \mathbb{R}^+$ as

$$g_m(x, t) = \max\{g(x, -m), \min\{g(x, t), g(x, m)\}\}, \quad G_m(x, t) := \int_0^t g_m(x, \tau) d\tau$$

$$\text{and } f_m(t) := \max\{f(-m), \min\{f(t), f(m)\}\}, \quad F_m(t) := \int_0^t f_m(\tau) d\tau.$$

Subsequently, we define the truncated functional J_m as

$$J_m(v) = \frac{\|v\|^2}{2} - \int_{\Omega} G_m(x, v) dx - \frac{1}{2} \iint_{\Omega \times \Omega} \frac{F_m(v)F_m(v)}{|x-y|^\mu} dx dy \text{ for all } v \in X_0.$$

Notice that $J_m \in C^1(X_0)$ and by appealing Dominated convergence theorem, we infer that $J_m(v) \rightarrow J(v)$ as $m \rightarrow \infty$ and for all $v \in X_0$. Thus, for every $n \in \mathbb{N}$ we pick $m_n \in \mathbb{N}$ such that $J_{m_n}(z_n) < 0$. Observe that $|G_m(x, v)| \leq (|g(x, -m)| + |g(x, m)|)|v|$ and $|F_m(v)| \leq (f(-m) + f(m))|v|$. That is, G_m and F_m has subcritical growth in the sense of Sobolev inequality and Hardy-Littlewood-Sobolev inequality respectively. Therefore, J_m is weakly lower

semicontinuous functional. Since $\bar{B}_{\varepsilon_n}^X(0)$ is a closed convex set, it implies that there exists $w_n \in \bar{B}_{\varepsilon_n}^X(0)$ such that

$$J_{m_n}(w_n) = \inf_{v \in \bar{B}_{\varepsilon_n}^X(0)} J_{m_n}(v) \leq J_{m_n}(w_n).$$

With the help of Lagrange multiplier's rule, one can easily prove that there exists $\lambda_n \in (0, 1]$ such that w_n is a weak solution of

$$\begin{cases} (-\Delta)^s u = \lambda_n \left(g_{k_n}(x, u) + \left(\int_{\Omega} \frac{F_{k_n}(u)(y)}{|x-y|^\mu} dy \right) f_{k_n}(u) \right) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since $\|w_n\| \in \bar{B}_{\varepsilon_n}^X(0)$, $\|w_n\| \rightarrow 0$ as $|\varepsilon_n| \rightarrow 0$. It implies $|w_n|_{2_s^*} \rightarrow 0$ and hence for n large enough we can choose $R = 0$ in (5.1). Subsequently there exists $K > 0$ such that $|w_n|_\infty \leq K$ for all n . By appealing [30, Theorem 1.2], we obtain that for all n , $w_n \in C_d^{0,\alpha}(\bar{\Omega})$ and $\|w_n\|_{C_d^{0,\alpha}(\bar{\Omega})} \leq K_1$ for some suitable $K_1 > 0$. Since $C_d^{0,\alpha}(\bar{\Omega})$ is compactly embedded into $C_d^0(\bar{\Omega})$, w_n is strongly convergent in $C_d^0(\bar{\Omega})$. Consequently, taking in account the fact that $w_n \rightarrow 0$ a.e. in Ω , we get $w_n \rightarrow 0$ in $C_d^0(\bar{\Omega})$. We conclude that for n large enough, $\|w_n\|_{C_d^0(\bar{\Omega})} \leq \rho$ and $|w_n|_\infty < 1$. From this we infer that

$$J(w_m) = J_{m_n}(w_m) < 0$$

and we obtain the desired contradiction to the assumption (6.1). Now we will consider the case $v \neq 0$. By given assumption (ii), it follows that $J'(v_0)(v) = 0$ for all $v \in C_c^\infty(\Omega)$ and applying the standard density arguments we infer that

$$J'(v_0)(v) = 0 \text{ for all } v \in X_0. \quad (6.2)$$

In view of Theorem 2.2, we have $u \in L^\infty(\Omega) \cap C_d^0(\bar{\Omega})$. For all $v \in X_0$, let

$$\widehat{F}(x, v) := \left(\int_{\Omega} \frac{F(v_0 + v)(y)}{|x-y|^\mu} dy \right) F(v_0 + v)(x) - \left(\int_{\Omega} \frac{F(v_0)(y)}{|x-y|^\mu} dy \right) (F(v_0) + 2f(v_0)v)(x)$$

$$\text{and } \widehat{G}(x, v) := G(x, (v_0 + v)(x)) - G(x, v_0(x)) - g(x, v_0(x))v(x).$$

Set

$$\widehat{J}(v) = \frac{\|v\|^2}{2} - \int_{\Omega} \widehat{G}(x, v) dx - \frac{1}{2} \int_{\Omega} \widehat{F}(x, v) dx \text{ for all } v \in X_0.$$

Note that $\widehat{J} \in C^1(X_0)$. Employing (6.2) and the definition of \widehat{F} and \widehat{G} , we have

$$\begin{aligned}
\widehat{J}(v) &= \frac{\|v_0 + v\|^2}{2} - \frac{\|v_0\|^2}{2} - \int_{\Omega} G(x, (v_0 + v)(x)) - G(x, v_0(x)) \, dx \\
&\quad - \frac{1}{2} \iint_{\Omega \times \Omega} \frac{F(v_0 + v)F(v_0 + v)}{|x - y|^\mu} \, dx dy + \frac{1}{2} \iint_{\Omega \times \Omega} \frac{F(v_0)F(v_0)}{|x - y|^\mu} \, dx dy \\
&= \widehat{J}(v_0 + v) - \widehat{J}(v_0).
\end{aligned}$$

We may deduce that $\widehat{J}(0) = 0$. Therefore given assumptions can be expressed as

$$\inf_{v \in X_0 \cap \widetilde{B}_\rho^d(0)} \widehat{J}(v) = 0.$$

Now by using above case we get the desired result and hence the proof of (ii) implies (i). \square

Theorem 6.1. Let $G : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

$$g(x, u) = O(|u|^{2_s^* - 1}), \text{ if } |u| \rightarrow \infty$$

uniformly for all $x \in \overline{\Omega}$. Let f satisfy (\mathcal{F}) . Let $f(\cdot)$ and $G(x, \cdot)$ be non decreasing functions for all $x \in \Omega$. Suppose $\underline{w}, \overline{w} \in X_0$ are a weak subsolution and a weak supersolution, respectively to (P) , which are not solutions. Then, there exists a solution $w_0 \in X_0$ to (P) such that $\underline{w} \leq w_0 \leq \overline{w}$ a.e. in Ω and w_0 is a local minimizer of J on X_0 .

Proof. Consider a closed convex set W of X_0 as

$$W := \{w \in X_0 : \underline{w} \leq w_0 \leq \overline{w} \text{ a.e. in } \Omega\}.$$

Using the definition of W , one can easily prove that

$$J(w) \geq \frac{\|w\|^2}{2} - c_1 - c_2$$

for appropriate positive constants c_1 and c_2 . This implies J is coercive on W . J is weakly lower semi continuous on W . Indeed, let $\{v_n\} \subset W$ such that $v_n \rightharpoonup v$ weakly in X_0 as $n \rightarrow \infty$. For each n ,

$$\begin{aligned}
\int_{\Omega} G(x, v_n) \, dx &\leq \int_{\Omega} G(x, v) \, dx < +\infty, \\
\iint_{\Omega \times \Omega} \frac{F(v_n)F(v_n)}{|x - y|^\mu} \, dx dy &\leq \iint_{\Omega \times \Omega} \frac{F(\overline{w})F(\overline{w})}{|x - y|^\mu} \, dx dy < +\infty.
\end{aligned}$$

Now we may invoke Dominated convergence theorem and the weak lower semicontinuity of norms to get that J is weakly lower semi continuous on W . Hence, there exists $w_0 \in X_0$ such that

$$\inf_{w \in W} J(w) = J(w_0). \quad (6.3)$$

Claim. w_0 is a weak solution to (P).

Let $\phi \in C_c^\infty(\Omega)$ and $\varepsilon > 0$. Define

$$u_\varepsilon = \min\{\overline{w}, \max\{\underline{w}, w_0 + \varepsilon\phi\}\} = v_0 + \varepsilon\phi - \phi^\varepsilon + \phi_\varepsilon$$

where $\phi^\varepsilon = \max\{0, w_0 + \varepsilon\phi - \overline{w}\}$ and $\phi_\varepsilon = \max\{0, \underline{w} - w_0 - \varepsilon\phi\}$. Observe that $\phi_\varepsilon, \phi^\varepsilon \in X_0 \cap L^\infty(\Omega)$. In view of the fact that $w_0 + t(u_\varepsilon - w_0) \in W$ for all $t \in (0, 1)$ and (6.3), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} (-\Delta)^s w_0 (u_\varepsilon - w_0) dx - \int_{\Omega} g(x, w_0) (u_\varepsilon - w_0) dx \\ & - \iint_{\Omega \times \Omega} \frac{F(w_0)f(w_0)(u_\varepsilon - w_0)}{|x - y|^\mu} dx dy \geq 0. \end{aligned}$$

Set

$$\begin{aligned} A^\varepsilon &= \int_{\mathbb{R}^N} (-\Delta)^s (w_0 - \overline{w}) \phi^\varepsilon dx + \int_{\mathbb{R}^N} (-\Delta)^s \overline{w} \phi^\varepsilon dx - \int_{\Omega} g(x, w_0) \phi^\varepsilon dx \\ & - \iint_{\Omega \times \Omega} \frac{F(w_0)f(w_0)\phi^\varepsilon}{|x - y|^\mu} dx dy, \\ A_\varepsilon &= \int_{\mathbb{R}^N} (-\Delta)^s (w_0 - \underline{w}) \phi_\varepsilon dx + \int_{\mathbb{R}^N} (-\Delta)^s \underline{w} \phi_\varepsilon dx - \int_{\Omega} g(x, w_0) \phi_\varepsilon dx \\ & - \iint_{\Omega \times \Omega} \frac{F(w_0)f(w_0)\phi_\varepsilon}{|x - y|^\mu} dx dy. \end{aligned}$$

Then by simple computations, we get

$$\int_{\mathbb{R}^N} (-\Delta)^s w_0 \phi dx - \int_{\Omega} g(x, w_0) \phi dx - \iint_{\Omega \times \Omega} \frac{F(w_0)f(w_0)\phi}{|x - y|^\mu} dx dy \geq \frac{1}{\varepsilon} (A^\varepsilon - A_\varepsilon). \quad (6.4)$$

Using the assertions as in [18, Proposition 3.2] with \overline{w} in spite of $u_{\lambda'}$, we have

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^N} (-\Delta)^s (w_0 - \overline{w}) \phi^\varepsilon dx \geq o(1) \text{ as } \varepsilon \rightarrow 0^+.$$

To this end, employing the fact that \overline{w} , we deduce that

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\mathbb{R}^N} (-\Delta)^s \bar{w} \phi^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Omega} g(x, w_0) \phi^\varepsilon dx - \frac{1}{\varepsilon} \iint_{\Omega \times \Omega} \frac{F(w_0) f(w_0) \phi^\varepsilon}{|x - y|^\mu} dx dy \\
& \geq \frac{1}{\varepsilon} \int_{\Omega} (g(x, \bar{w}) - g(x, w_0)) \phi^\varepsilon dx + \frac{1}{\varepsilon} \iint_{\Omega \times \Omega} \frac{(F(\bar{w}) f(\bar{w}) - F(w_0) f(w_0)) \phi^\varepsilon}{|x - y|^\mu} dx dy \\
& \geq \frac{1}{\varepsilon} \int_{\Omega} (g(x, \bar{w}) - g(x, w_0)) \phi^\varepsilon dx = o(1) \text{ as } \varepsilon \rightarrow 0^+.
\end{aligned}$$

Hence we infer that $\frac{1}{\varepsilon} A^\varepsilon \geq o(1)$ as $\varepsilon \rightarrow 0^+$. On the similar lines, one can prove that $\frac{1}{\varepsilon} A_\varepsilon \leq o(1)$ as $\varepsilon \rightarrow 0^+$. From (6.4), for all $\phi \in C_c^\infty(\Omega)$, it follows that

$$\int_{\mathbb{R}^N} (-\Delta)^s w_0 \phi dx - \int_{\Omega} g(x, w_0) \phi dx - \iint_{\Omega \times \Omega} \frac{F(w_0) f(w_0) \phi}{|x - y|^\mu} dx dy \geq 0 \text{ as } \varepsilon \rightarrow 0^+. \quad (6.5)$$

As $\phi \in C_c^\infty(\Omega)$ was arbitrarily chosen, it implies that w_0 is weak solution to (P). From this, we follow that there exists a solution $w_0 \in X_0$ to (P) such that $\underline{w} \leq w_0 \leq \bar{w}$ a.e. in Ω . To prove that w_0 is a local minimizer in X_0 , we proceed as follows. Using Theorem 2.2 and [30, Theorem 1.2], we deduce $w_0 \in C_d^{0,\alpha}(\bar{\Omega})$. Now consider

$$\begin{aligned}
(-\Delta)^s (w_0 - \underline{w}) & \geq (g(x, w_0) - g(x, \underline{w})) + \left(\int_{\Omega} \frac{F(w_0)}{|x - y|^\mu} dy \right) f(w_0) \\
& \quad - \left(\int_{\Omega} \frac{F(\underline{w})}{|x - y|^\mu} dy \right) f(\underline{w}) \\
& \geq 0.
\end{aligned}$$

Using the fact that \underline{w} is not solution to (P), we have $w_0 \neq \underline{w}$ and by definition, $w_0 - \underline{w} \geq 0$ in $\mathbb{R}^N \setminus \Omega$. From [19, Lemma 2.7], we infer that $w_0 - \underline{w} > C d^s$ for some $C > 0$. On a similar note $\bar{w} - w_0 > C d^s$ for some $C > 0$. For each $w \in \bar{B}_{C/2}^d(w_0)$, we have

$$\frac{w_0 - \underline{w}}{d^s} = \frac{w_0 - w}{d^s} + \frac{w - \underline{w}}{d^s} \geq \frac{C}{2}.$$

From above, it can read that $w_0 - \underline{w} > 0$ in Ω . Likewise, $\bar{w} - w_0 > 0$ in Ω . Therefore, w_0 emerge as a local minimizer of J on $X_0 \cap \bar{B}_{C/2}^d(w_0)$ and this completes the proof. \square

Remark 6.2. Consider the following problem

$$\begin{cases} (-\Delta)^s u = \lambda \left(|u|^{q-2} u + \left(\int_{\Omega} \frac{F(u)(y)}{|x - y|^\mu} dy \right) f(u) \right), & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (6.6)$$

where $\lambda > 0$, $1 < q < 2$ and f is a non decreasing function and satisfies (\mathcal{F}) . Let \underline{v} denote the solution to

$$(-\Delta)^s u = \lambda |u|^{q-2} u, \quad u > 0 \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

and let \bar{v} be a solution to

$$(-\Delta)^s u = 1, \quad u > 0 \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

Then for all $\lambda > 0$, \underline{v} is a subsolution to (6.6). And for λ small enough, \bar{v} is a supersolution to (6.6). Now using Theorem 6.1, there exists a solution to (6.6), which is a local minimizer in X_0 . The mountain pass lemma provides then the existence of a second solution.

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