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Doubly Nonlinear Degenerate Parabolic Systems with Coupled Nonlinear Boundary Conditions[1]

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In this paper, we study the global existence and the global nonexistence of doubly nonlinear degenerate parabolic systems with nonlinear boundary conditions. We first prove a local existence result by the regularization method. Next, we construct a weak comparison principle. Then we discuss the large time behavior of solutions by using a modified upper and lower solution methods and constructing various upper and lower solutions. Necessary and sufficient conditions on the global existence of all positive (weak) solutions are obtained. © 2002 Elsevier Science (USA)

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence and nonexistence of global solutions of the following problem:

$$
\begin{array}{lll}
(|u|^{m_1-1}u)_t = (|u_x|^{p_1-1}u_x)_x, & & 0<x<1, \quad t>0, \\
(|v|^{m_2-1}v)_t = (|v_x|^{p_2-1}v_x)_x, & & 0<x<1, \quad t>0, \\
u_x|_{x=0} = 0, & u_x|_{x=1} = \lambda u^{l_{11}} v^{l_{12}}|_{x=1}, & t>0, \\
v_x|_{x=0} = 0, & v_x|_{x=1} = \lambda u^{l_{21}} v^{l_{22}}|_{x=1}, & t>0, \\
u(x,0) = u_0(x), & v(x,0) = v_0(x), & 0\leqslant x\leqslant 1,
\end{array}
\tag{1.1}
$$

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where $m_i, p_i > 0,\ i = 1, 2,\ l_{12}, l_{21} > 0, l_{11}, l_{22} \geqslant 0$ and $\lambda > 0$ are all constants. According to physical settings, $\phi_i(s) = |s|^{m_i - 1}s$ stands for the mass concentration, the term $A_i(s) = |s_x|^{p_i - 1}s_x$ reflects the diffusion effect which do not conform to Darcy's law [15, 30], $u^{l_{i1}}v^{l_{i2}}$ the forced flux at the boundary $x = 1$. The differential equations in (1.1), which is the typical example of the non-Newtonian filtration equations, have been suggested as some models, see [1, 7–9, 15, 17, 36, 37, 39] and the references therein. For example, the gas flow equation through a porous medium and the completely turbulent flow fall in the class of equations we consider. The nonlinear boundary conditions in (1.1) can be physically interpreted as a nonlinear radiation law, which here is actually an absorption law, see [3, 19, 34].

In the recent years, the questions like blow-up and global solvability for semilinear parabolic equation or systems with nonlinear boundary condition have been intensively studied, see [3–8, 14, 16, 18–22, 24, 27–29, 34, 37, 40–48] and references therein. The Dirichlet or Cauchy Problem for the p-Laplacian has also been studied in that extent, see [2, 25, 26, 33] and references therein.

For problem (1.1) with $p_1 = p_2 = 1$ or more general form of (1.1) with the Darcy's law diffusion, there were many results on the local existence, global existence and blow-up in finite time, see [3, 4, 16, 23, 37, 38, 42, 44, 46]. Amann [3,4] considered the (classical) local solvability and the geometric theory of the following quasilinear problem:

$$\begin{aligned} &u_t + \mathscr{A}(t, u)u = f(t, u), && x \in \Omega, \quad t > 0, \\ &\mathscr{B}(t, u)u = g(t, u), && x \in \partial\Omega, \\ &u(x, 0) = u_0(x), && x \in \bar{\Omega} \end{aligned} \tag{1.2}$$

together with certain results concerning the continuous dependence of the solutions upon the data of the problem, by means of semigroups methods, and proved that the semilinear parabolic problem (1.2) possesses a unique maximal classical solution. Escher [16] considered the global existence and global nonexistence of problem (1.2) and obtained some sufficient conditions on which the solution of (1.2) blows up in finite time by means of "concavity method" of abstract Cauchy problem. Pao [37] established the upper and lower solutions method, and gave some sufficient conditions on the global solutions and blow-up in finite time for a more general problem. Samarskii et al. [38] considered global solvability and blow-up of solutions to semilinear parabolic systems with Dirichlet boundary conditions and gave optimal conditions of global existence of arbitrary solutions to the initial boundary value problem of the porous medium flows. Galaktionov et al. [23] studied blow-up phenomena for nonlinear parabolic equations and systems with nonlinear diffusivity, source terms and boundary conditions. Wang et al. [46] and M.X. Wang et al. [42, 44] considered the following porous medium flow problem:

$$
\begin{aligned}
&u_t = \nabla(u^m \nabla u) + \lambda u^{l_{11}} v^{l_{12}}, && x \in \Omega, && t > 0,\\
&v_t = \nabla(u^n \nabla u) + \lambda u^{l_{21}} v^{l_{22}}, && x \in \Omega, && t > 0,\\
&\frac{\partial u}{\partial \eta} = u^{\alpha} v^{p}, \frac{\partial v}{\partial \eta} = u^{q} v^{\beta}, && x \in \partial\Omega, && t > 0,\\
&u(x,0) = u_0(x), v(x,0) = v_0(x), && x \in \Omega,
\end{aligned}
$$

respectively, and obtained necessary and sufficient conditions on the global existence of all positive (classical) solutions.

Recently, S. Wang et al. [47] considered the single doubly nonlinear parabolic equation with nonlinear boundary conditions, namely,

$$
\begin{aligned}
&(|u|^{m-1}u)_t = (|u_x|^{p-1}u_x)_x, && 0<x<1, && t>0,\\
&u_x|_{x=0} = 0, \qquad u_x|_{x=1} = \lambda u^{\alpha}|_{x=1}, && && t>0,\\
&u(x,0) = u_0(x), && 0 \leqslant x \leqslant 1
\end{aligned}
\tag{1.3}
$$

and obtained necessary and sufficient conditions on the global existence of all positive (weak) solutions by using upper and lower methods. Filo [20] considered the multidimensional variant of (1.3) and a local existence result of weak solution was obtained by using the semidiscretization method. Filo and Kalur [21] also considered the local existence of a more general scalar case of (1.3) and gave some sufficient conditions on the global existence of weak solutions under some assumptions of general nonlinear term by using L^{∞}-estimate methods.

In this paper we will consider large time behaviors of weak solutions to (1.1).

Throughout the paper we assume that (A)

(i) $u_0(x), v_0(x) \in C^{2+\mu}([0,1])$ for some $0<\mu<1$, $u_0(x), v_0(x) > 0$;

(ii) $(|u_{0x}|^{p_1-1}u_{0x})_x, (|v_{0x}|^{p_2-1}v_{0x})_x \in L^2([0,1])$ on $[0,1]$;

(iii) $u_0(x), v_0(x)$ satisfy the compatibility conditions:

$$
\begin{aligned}
&u_{0x}(0) = 0, \qquad u_{0x}(1) = \lambda u_0^{l_{11}}(1) v_0^{l_{12}}(1),\\
&v_{0x}(0) = 0, \qquad v_{0x}(1) = \lambda u_0^{l_{21}}(1) v_0^{l_{22}}(1).
\end{aligned}
$$

Our main result is

THEOREM. *All positive (weak) solutions of* (1.1) *exist globally if and only if*

$$l_{11} < \min\left\{\frac{m_1}{p_1}, \frac{m_1+1}{p_1+1}\right\}, \qquad l_{22} < \min\left\{\frac{m_2}{p_2}, \frac{m_2+1}{p_2+1}\right\},$$

$$l_{12} l_{21} \leqslant \left(\min\left\{\frac{m_1}{p_1}, \frac{m_1+1}{p_1+1}\right\} - l_{11}\right)\left(\min\left\{\frac{m_2}{p_2}, \frac{m_2+1}{p_2+1}\right\} - l_{22}\right). \quad (1.4)$$

The rest of the paper is organized as follows: In Section 2, the definition of weak solution is given and a weak comparison principle is established to serve as the basis of the study. Local existence and continuation results are proved by the regularization method in Section 3. In Sections 4 and 5, our main theorem is proved by constructing various upper and lower solutions.

2. PRELIMINARIES. COMPARISON PRINCIPLE

Due to the gradient degeneration or singularity of the equation, we cannot expect problem (1.1) to be solvable in the classical sense even for smooth data. Therefore, a notion of weak solution to problem (1.1) is needed.

DEFINITION. For any $T > 0$, denote $Q_T = (0, 1) \times (0, T]$. A pair of nonnegative function $(u(x, t), v(x, t)) \in C(\bar{Q}_T) \times C(\bar{Q}_T)$ is called a weak upper (or lower) solution of problem (1.1) in Q_T with λ if

(i) $u, v \in L^\infty(0, T; W^{1,\infty}((0, 1))) \cap W^{1,2}(0, T; L^2((0, 1)))$, $(u(x, 0), v(x, 0)) \geqslant (\leqslant)(u_0(x), v_0(x))$;

(ii) for any nonnegative functions $w_1(x, t), w_2(x, t) \in L^1(0, T; W^{1,2}((0, 1))) \cap L^2(Q_T)$, we have

$$\iint_{Q_T} [w_1(u^{m_1})_t + w_{1x}|u_x|^{p_1-1}u_x]\, dx\, dt \geqslant (\leqslant) \int_0^T w_1(\lambda u^{l_{11}} v^{l_{12}})^{p_1}|_{x=1}\, dt$$

and

$$\iint_{Q_T} [w_2(v^{m_2})_t + w_{2x}|v_x|^{p_2-1}v_x]\, dx\, dt \geqslant (\leqslant) \int_0^T w_2(\lambda u^{l_{21}} v^{l_{22}})^{p_2}|_{x=1}\, dt.$$

$(u(x, t), v(x, t))$ is called a (weak) solution of problem (1.1) if it is both a (weak) upper solution and lower solution of problem (1.1) in Q_T with λ.

The main purpose of this paper is to give the necessary and sufficient conditions on the existence of global positive (weak) solutions. The basic technique used in the present paper is constructing various upper and lower solutions and comparing them with the solutions of problem (1.1). To this aim, we first prove a weak comparison principle.

PROPOSITION 2.1 (Comparison Principle). *Assume that* $\underline{\lambda} < \lambda < \bar{\lambda}$. *Let* $(\underline{u}, \underline{v}) \geqslant (\delta, \delta)$ *and* $(\bar{u}, \bar{v})$ *be a lower and upper solution of* (1.1) *in* Q_T *with* $\underline{\lambda}$ *and* $\bar{\lambda}$, *respectively. Then* $(\underline{u}, \underline{v}) \leqslant (u, v) \leqslant (\bar{u}, \bar{v})$ *on* $\bar{Q}_T$.

Proof. For small $\delta > 0$. Let

$$H_\delta(z) = \min\Big(1, \max\Big(\frac{z}{\delta}, 0\Big)\Big)$$

and set $w_1 = H_\delta(\underline{u} - u)$, then according to the definitions of solution and lower solution we have

$$\begin{aligned}
&\iint_{Q_\tau} \{H_\delta(\underline{u} - u)(\underline{u}^{m_1} - u^{m_1})_t + (H_\delta(\underline{u} - u))_x[|\underline{u}_x|^{p_1-1}\underline{u}_x - |u_x|^{p_1-1}u_x\} \, dx\, dt \\
&\quad \leqslant \int_0^\tau H_\delta(\underline{u} - u)[(\underline{\lambda}\underline{u}^{l_{11}}\underline{v}^{l_{12}})^{p_1} - (\lambda u^{l_{11}} v^{l_{12}})^{p_1}]|_{x=1} \, dt, \qquad \tau \in [0, T].
\end{aligned}$$

As in [2], by letting $\delta \to 0$ we get

$$\begin{aligned}
&\iint_{Q_\tau} (\underline{u}^{m_1} - u^{m_1})_t \chi[\underline{u} > u] \, dx\, dt \\
&\quad \leqslant \int_0^\tau [(\underline{\lambda}\underline{u}^{l_{11}}\underline{v}^{l_{12}})^{p_1} - (\lambda u^{l_{11}} v^{l_{12}})^{p_1}]\chi[\underline{u} > u]|_{x=1} \, dt \\
&\quad \leqslant \int_0^\tau \{\underline{v}^{p_1 l_{12}}[(\underline{\lambda}\underline{u}^{l_{11}})^{p_1} - (\lambda u^{l_{11}})^{p_1}] \\
&\qquad + \lambda^{p_1} u^{p_1 l_{11}} p_1 l_{12} \theta_1^{p_1 l_{12} - 1}(\underline{v} - v)\}\chi[\underline{u} > u]|_{x=1} \, dt
\end{aligned}$$

for some $\theta_1 > 0$ lying between $\underline{v}(1, t)$ and $v(1, t)$. If

$$f(x, t) = \underline{v}^{p_1 l_{12}}[(\underline{\lambda}\underline{u}^{l_{11}})^{p_1} - (\lambda u^{l_{11}})^{p_1}] + \lambda^{p_1} u^{p_1 l_{11}} p_1 l_{12} \theta_1^{p_1 l_{12} - 1}(\underline{v} - v),$$

then

$$\iint_{Q_\tau} (\underline{u}^{m_1} - u^{m_1})_t \chi[\underline{u} > u] \, dx\, dt \leqslant \int_0^\tau f(x, t)\chi[\underline{u} > u]|_{x=1} \, dt, \qquad \tau \in [0, T]. \tag{2.1}$$

Similarly, we have

$$\iint_{Q_\tau} (\underline{v}^{m_2} - v^{m_2})_t \chi[\underline{v} > v]\, dx\, dt \quad \leqslant \int_0^\tau g(x,t)\chi[\underline{v} > v]|_{x=1}\, dt, \qquad \tau \in [0, T], \tag{2.2}$$

where

$$g(x,t) = \underline{u}^{p_2 l_{21}}[(\underline{\lambda}\underline{v}^{l_{22}})^{p_2} - (\lambda v^{l_{22}})^{p_2}] + \lambda^{p_2} v^{p_2 l_{22}} p_2 l_{21} \theta_2^{p_2 l_{21}-1}(\underline{u} - u)$$

for some $\theta_2 > 0$ lying between $\underline{u}(1,t)$ and $u(1,t)$.

Since $(0,0) < (\delta,\delta) \leqslant (\underline{u}(1,0), \underline{v}(1,0)) \leqslant (u(1,0), v(1,0))$ and $\underline{\lambda} < \lambda$, we follow from the continuity of $\underline{u}, \underline{v}, u$ and v that there exists a time $\tau_1 > 0$ such that $f(1,t) \leqslant 0$ and $g(1,t) \leqslant 0$ for all $t \in [0, \tau_1]$. Therefore, we have that $(\underline{u}, \underline{v}) \leqslant (u, v)$ on $\bar{Q}_{\tau_1}$. Define

$$\tau^* = \sup\{\tau \in [0,T]\colon\ \underline{u}(1,t) \leqslant u(1,t) \text{ and } \underline{v}(1,t) \leqslant v(1,t) \text{ for all } t \in [0,\tau]\}.$$

We claim that $\tau^* = T$. Otherwise, from the continuity of $\underline{u}, \underline{v}, u$ and v there exists $\varepsilon > 0$ such that $\tau^* + \varepsilon < T$, $f(1,t) \leqslant 0$ and $g(1,t) \leqslant 0$ for all $t \in [0, \tau^* + \varepsilon]$. By (2.1) and (2.2) we have that $(\underline{u}, \underline{v}) \leqslant (u, v)$ on $\bar{Q}_{\tau^*+\varepsilon}$, which contradicts the definition of τ^*. Hence, $(\underline{u}, \underline{v}) \leqslant (u, v)$ on $\bar{Q}_T$.

Obviously, (δ, δ) is a lower solution of (1.1) on $\bar{Q}_T$ with $\underline{\lambda}$. Therefore, $(u, v) \geqslant (\delta, \delta) > (0, 0)$ on $\bar{Q}_T$. Using this fact, as in the above proof we can prove that $(u, v) \leqslant (\bar{u}, \bar{v})$ on $\bar{Q}_T$.

The proof of Proposition 2.1 is completed. ∎

Remark 2.1. The above comparison principle do not imply the uniqueness. The uniqueness of solutions to problem (1.1) is open.

3. LOCAL EXISTENCE

In this section, we prove the following local existence result.

PROPOSITION 3.1. *Let (A) be satisfied. Then there exists a time T: $0 < T \leqslant +\infty$ such that* (1.1) *has a weak solution* (u, v) *satisfying* $(u, v) \geqslant (\delta, \delta) > 0$ *on* $\bar{Q}_T$. *Moreover, as a solution* (u, v) *can be extended to* Q_T *with a maximum* $T > 0$, *that is, either* $T = +\infty$, *or* $T < +\infty$ *so that*

$$\lim_{t \to T^-} \sup_{x \in [0,1]} (\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty) = +\infty.$$

To prove the above local existence result, we need the following lemmas from Sobolev's space.

Let Ω be a bounded domain in R^N of boundary $\partial\Omega$ and for $0<T<\infty$ let Q_T denote the cylindrical domain $\Omega \times (0, T]$.

Let h be any real number and for a function u consider the truncations of u given by

$$(u-h)^+ = \max\{u-h, 0\}.$$

LEMMA 3.2 (Ladyzenskaja et al. [31]; Ladyzenskaja and Ural'ceva [32]). *Let $u \in L^1(Q_T)$ satisfy*

$$\int_0^T \int_\Omega (u-h)^+ \, dx\, dt \leqslant F|Q_T \cap \{u>h\}|^{1+\beta}, \quad \forall h \geqslant k_0 > 0,$$

where F, β are positive constants. Then

$$\operatorname{esssup}_{Q_T} u^+ \leqslant k_0 + \left(1+\frac{1}{\beta}\right) F|\Omega|^\beta T^\beta.$$

LEMMA 3.3 (Dibenedetto [15]). *Let $m, p \geqslant 1$ and $u \in V^{m,p}(Q_T) = L^\infty(0,T ; L^m(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$. Then there exists a constant c depending only upon N, p, m and the structure of $\partial\Omega$ such that*

$$\|u\|_{L^l(Q_T)} \leqslant c\left(1+\frac{T}{|\Omega|^{(N(p-m)+mp)/(Nm)}}\right)^{1/l} \|u\|_{V^{m,p}(Q_T)},$$

where $l = p\frac{N+m}{N}$ and $\|u\|_{V^{m,p}(Q_T)} = \operatorname{esssup}_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{m,\Omega} + \|Du\|_{p,Q_T}$.

We shall prove Proposition 3.1 in two steps: First, we consider a regularized problem and derive some necessary estimates; then, by an approximation process and passing to limit a solution to the original problem is obtained. To this end, put

$$M = \|u_0(x)\|_\infty + \|v_0(x)\|_\infty, \qquad K = \|u_{0x}(x)\|_\infty + \|v_{0x}(x)\|_\infty$$

and consider the problem

$$\begin{aligned}
&\Phi_1(u_\varepsilon)u_{\varepsilon t} = ((u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}} u_{\varepsilon x})_x, && 0<x<1, \quad 0<t<\tau,\\
&\Phi_2(v_\varepsilon)v_{\varepsilon t} = ((v_{\varepsilon x}^2+\varepsilon)^{\frac{p_2-1}{2}} v_{\varepsilon x})_x, && 0<x<1, \quad 0<t<\tau,\\
&u_{\varepsilon x}|_{x=0} = 0, \qquad u_{\varepsilon x}|_{x=1} = G_1(u_\varepsilon, v_\varepsilon)|_{x=1}, && 0<t<\tau,\\
&v_{\varepsilon x}|_{x=0} = 0, \qquad v_{\varepsilon x}|_{x=1} = G_2(u_\varepsilon, v_\varepsilon)|_{x=1}, && 0<t<\tau,\\
&u_\varepsilon(x,0) = u_0(x), \qquad v_\varepsilon(x,0) = v_0(x), && 0\leqslant x\leqslant 1
\end{aligned} \tag{3.1}$$

for any $\tau \in (0, +\infty)$ and any fixed $\varepsilon \in (0, 1)$. We define $\Phi_i(w)$ and $G_i(w, z)$ as follows: $\Phi_i(w) \in C^\infty(R), G_i(w, z) \in C^\infty(R^2)$, $\Phi_i(w) = m_i w^{m_i - 1}, G_i(w, z) = \lambda w^{l_{i1}} z^{l_{i2}}$ for $\delta \leqslant w, z \leqslant M + 1$, there exist positive constants l, L such that

$$0 < l \leqslant \Phi_i(w), G_i(w, z) \leqslant L < +\infty, \qquad i = 1, 2$$

for any $w, z \in R$ and $\frac{\partial G_1(w,z)}{\partial z} \geqslant 0$ and $\frac{\partial G_2(w,z)}{\partial w} \geqslant 0$ for any $w, z \in R$. Moreover, let $a_{i\varepsilon}(z) \in C^\infty(R), 0 < \rho_\varepsilon \leqslant a'_{i\varepsilon}(z) \leqslant \rho_\varepsilon^{-1}$ on R for some $0 < \rho_\varepsilon < 1$ and

$$a_{i\varepsilon}(z) = (z^2 + \varepsilon)^{\frac{p_i - 1}{2}} z \qquad \text{for } |z| \leqslant K + L + 1$$

and consider the following problem:

$$\begin{array}{llll}
\Phi_1(u_\varepsilon) u_{\varepsilon t} = (a_{1\varepsilon}(u_{\varepsilon x}))_x, & 0 < x < 1, & 0 < t < \tau, & \\
\Phi_2(v_\varepsilon) v_{\varepsilon t} = (a_{2\varepsilon}(v_{\varepsilon x}))_x, & 0 < x < 1, & 0 < t < \tau, & \\
u_{\varepsilon x}|_{x=0} = 0, & u_{\varepsilon x}|_{x=1} = G_1(u_\varepsilon, v_\varepsilon)|_{x=1}, & 0 < t < \tau, & (3.2) \\
v_{\varepsilon x}|_{x=0} = 0, & v_{\varepsilon x}|_{x=1} = G_2(u_\varepsilon, v_\varepsilon)|_{x=1}, & 0 < t < \tau, & \\
u_\varepsilon(x, 0) = u_0(x), & v_\varepsilon(x, 0) = v_0(x), & 0 \leqslant x \leqslant 1. &
\end{array}$$

Now, the standard parabolic theory (see [31, 37]) guarantees the existence of a unique solution $(u_\varepsilon, v_\varepsilon)$ to (3.2) in the class $H^{2+\beta, 1+\beta/2}(\bar{Q}_\tau)$ for some $\beta \in (0, 1)$. Obviously, Comparison principle holds for (3.2). Therefore,

$$u_\varepsilon(x, t) \geqslant \delta > 0, \qquad v_\varepsilon(x, t) \geqslant \delta > 0 \text{ and } u_{\varepsilon t}, v_{\varepsilon t} \geqslant 0 \text{ on } \bar{Q}_\tau.$$

Firstly, we have

$$||u_{\varepsilon x}||_\infty \leqslant K + L, ||v_{\varepsilon x}||_\infty \leqslant K + L \qquad \text{on } \bar{Q}_\tau.$$

In fact, let $w = u_{\varepsilon x}, z = v_{\varepsilon x}$ then (w, z) satisfies

$$\begin{array}{lll}
\Phi_1(u_\varepsilon) w_t - a'_{1\varepsilon}(u_{\varepsilon x}) w_{xx} - a''_{1\varepsilon}(u_{\varepsilon x}) w_x^2 + \Phi'_1(u_\varepsilon) u_{\varepsilon t} w = 0, & & 0 < t < \tau, \\
\Phi_2(v_\varepsilon) z_t - a'_{2\varepsilon}(v_{\varepsilon x}) z_{xx} - a''_{2\varepsilon}(v_{\varepsilon x}) z_x^2 + \Phi'_2(v_\varepsilon) v_{\varepsilon t} z = 0, & & 0 < t < \tau, \\
w|_{x=0} = 0, & w|_{x=1} = G_1(u_\varepsilon, v_\varepsilon)|_{x=1}, & 0 < t < \tau, \\
z|_{x=0} = 0, & z|_{x=1} = G_2(u_\varepsilon, v_\varepsilon)|_{x=1}, & 0 < t < \tau, \\
w(x, 0) = u_{0x}(x), & z(x, 0) = v_{0x}(x), & 0 \leqslant x \leqslant 1.
\end{array}$$

The maximum principle yields that

$$||w||_\infty \leqslant \max\{||u_{0x}||_\infty, ||v_{0x}||_\infty, L\}$$

and

$$\|z\|_\infty \leqslant \max\{\|u_{0x}\|_\infty, \|v_{0x}\|_\infty, L\}.$$

Therefore, we have

$$\|u_{\varepsilon x}\|_\infty \leqslant K + L, \|v_{\varepsilon x}\|_\infty \leqslant K + L \qquad \text{on } \bar{Q}_\tau.$$

Thus,

$$a_{1\varepsilon}(u_{\varepsilon x}) = (u_{\varepsilon x}^2 + \varepsilon)^{\frac{p_1-1}{2}} u_{\varepsilon x}, \qquad a_{2\varepsilon}(v_{\varepsilon x}) = (v_{\varepsilon x}^2 + \varepsilon)^{\frac{p_2-1}{2}} v_{\varepsilon x}$$

on $\bar{Q}_\tau$. And hence $(u_\varepsilon, v_\varepsilon)$ is a solution of (3.1) in $\bar{Q}_\tau$.

Secondly, we shall show that there exists $T \in (0, \tau)$, independent of ε, such that

$$u_\varepsilon(x,t) \leqslant M + 1 \quad \text{and} \quad v_\varepsilon(x,t) \leqslant M + 1 \qquad \text{on } \bar{Q}_T.$$

Let $s \in (0,1)$ and fix $t \in [0,s]$. Denote $k_0 = \|u_0\|_\infty + \|v_0\|_\infty, z^+ = \max\{z, 0\}$, $A_h(t) = \{x \in (0,1)\colon u_\varepsilon(x,t) > h\}$ and $\mu(h) = \int_0^t \int_{A_h(s)} dx\,ds$.

We may think that $\mu(h) \leqslant 1$.

From (3.1) we easily derive

$$\begin{aligned} &\int_0^t\int_0^1 \Phi_1(u_\varepsilon)u_{\varepsilon t}(u_\varepsilon - h)^+ \,dx\,dt + \int_0^t\int_0^1 (u_{\varepsilon x}^2 + \varepsilon)^{\frac{p_1-1}{2}} u_{\varepsilon x}(u_\varepsilon - h)_x^+ \,dx\,dt \\ &\quad = \int_0^t (G_1^2(u_\varepsilon, v_\varepsilon) + \varepsilon)^{\frac{p_1-1}{2}} G_1(u_\varepsilon, v_\varepsilon)(u_\varepsilon - h)^+|_{x=1}\,dt. \end{aligned}$$

Owing to the fact that $0 < l \leqslant \Phi_1(u_\varepsilon), G_1(u_\varepsilon, v_\varepsilon) \leqslant L$, we see that

$$\int_0^1 (u_\varepsilon - h)^{+^2}\,dx + \int_0^t\int_0^1 |u_{\varepsilon x}|^{p_1-1}(u_\varepsilon - h)_x^{+^2}\,dx\,dt \leqslant c \int_0^t (u - h)^+|_{x=1}\,dt$$

for some positive constant c independent of ε. By using trace embedding theorem we have for any $\tau > 1$ and a.e. $t \in [0,s]$

$$(u_\varepsilon - h)^+|_{x=1} \leqslant c\|(u_\varepsilon - h)^+(\cdot,t)\|_{W^{1,\frac{p_1}{\tau}+1}((0,1))}.$$

Therefore, the above inequality and Holder's inequality yield

$$\begin{aligned} &\int_0^1 (u_\varepsilon - h)^{+^2}\,dx + \int_0^t\int_0^1 |u_{\varepsilon x}|^{p_1-1}(u_\varepsilon - h)_x^{+^2}\,dx\,dt \\ &\quad \leqslant c \int_0^t \|(u_\varepsilon - h)^+(\cdot,t)\|_{W^{1,\frac{p_1}{\tau}+1}((0,1))}\,dt \end{aligned}$$

$$\leqslant c\left(\int_0^t\int_0^1 |(u_\varepsilon - h)^+|^{\frac{p_1}{\tau}+1}\,dx\,dt\right)^{1/(\frac{p_1}{\tau}+1)}$$

$$+\,c\left(\int_0^t\int_0^1 |(u_\varepsilon - h)_x^+|^{\frac{p_1}{\tau}+1}\,dx\,dt\right)^{1/(\frac{p_1}{\tau}+1)}.$$

Young's inequality yields

$$\int_0^1 (u_\varepsilon - h)^{+^2}\,dx + \int_0^t\int_0^1 |u_{\varepsilon x}|^{p_1-1}(u_\varepsilon - h)_x^{+^2}\,dx\,dt$$

$$\leqslant c\|(u_\varepsilon - h)^+\|_{L^{\frac{p_1}{\tau}+1}(Q_t)} + c\mu^{\alpha_1},$$

where

$$\alpha_1 = \frac{1}{\frac{p_1}{\tau}+1}\left(1 - \frac{\frac{p_1}{\tau}+1}{p_1+1}\right)\times\frac{1}{1-\frac{1}{p_1}+1}.$$

By using Lemma 3.3, Holder's inequality and Young's inequality we have

$$\|(u_\varepsilon - h)^+\|_{L^{3(p_1+1)}(Q_t)}\leqslant c\|(u_\varepsilon - h)^+\|_{V^{2,p_1+1}(Q_t)}$$

$$\leqslant c(\|(u_\varepsilon - h)^+\|_{L^{\frac{p_1}{\tau}+1}(Q_t)} + \mu^{\alpha_1}(h))^{\frac{1}{2}} + c(\|(u_\varepsilon - h)^+\|_{L^{\frac{p_1}{\tau}+1}(Q_t)} + \mu^{\alpha_1})^{\frac{1}{p_1+1}}$$

$$\leqslant c\|(u_\varepsilon - h)^+\|^{\frac{1}{2}}_{L^{\frac{p_1}{\tau}+1}(Q_t)} + c\mu^{\frac{\alpha_1}{2}}(h) + c\|(u_\varepsilon - h)^+\|^{\frac{1}{p_1+1}}_{L^{\frac{p_1}{\tau}+1}(Q_t)} + c\mu^{\frac{\alpha_1}{p_1+1}}(h)$$

$$\leqslant c(\|(u_\varepsilon - h)^+\|^{\frac{p_1}{\tau}+1}_{L^{3(p_1+1)}(Q_t)}\mu^{1-(\frac{p_1}{\tau}+1)/3(p_1+1))}(h))^{\frac{1}{2((p_1/\tau)+1)}} + c\mu^{\frac{\alpha_1}{2}}(h)$$

$$+\ c(\|(u_\varepsilon - h)^+\|^{\frac{p_1}{\tau}+1}_{L^{3(p_1+1)}(Q_t)}\mu^{1-(\frac{p_1}{\tau}+1)/3(p_1+1)}(h))^{\frac{1}{(p_1+1)((p_1/\tau)+1)}} + c\mu^{\frac{\alpha_1}{(p_1+1)}}(h)$$

$$\leqslant\frac{1}{2}\|(u_\varepsilon - h)^+\|_{L^{3(p_1+1)}(Q_t)} + c\mu^{\frac{\alpha_1}{2}}(h) + c\mu^{\alpha_2}(h) + c\mu^{\alpha_3}(h) + c\mu^{\frac{\alpha_1}{(p_1+1)}}(h),$$

where

$$\alpha_2 = \frac{1}{\frac{p_1}{\tau}+1}\left(1 - \frac{\frac{p_1}{\tau}+1}{3(p_1+1)}\right),$$

$$\alpha_3 = \frac{1}{(p_1+1)(\frac{p_1}{\tau}+1)}\left(1 - \frac{\frac{p_1}{\tau}+1}{3(p_1+1)}\right)\times\frac{1}{1-\frac{1}{p_1+1}}.$$

Hence,

$$\|(u_\varepsilon - h)^+\|_{L^{3(p_1+1)}(Q_t)} \leqslant c\mu^{\frac{\alpha_1}{2}}(h) + c\mu^{\alpha_2}(h) + c\mu^{\alpha_3}(h) + c\mu^{\frac{\alpha_1}{p_1+1}}(h).$$

By using Holder's inequality we have

$$\begin{aligned}\int_0^t\int_0^1 (u_\varepsilon - h)^+\,dx\,dt &\leqslant \|(u_\varepsilon - h)^+\|_{L^{3(p_1+1)}(Q_t)} \cdot \mu^{1-\frac{1}{(3p_1+1)}}(h)\\ &\leqslant c[\mu^{1-\frac{1}{3(p_1+1)}+\frac{\alpha_1}{2}}(h) + \mu^{1-\frac{1}{3(p_1+1)}+\alpha_2}(h)\\ &\quad + \mu^{1-\frac{1}{3(p_1+1)}+\alpha_3}(h) + \mu^{1-\frac{1}{3(p_1+1)}+\frac{\alpha_1}{(p_1+1)}}(h)].\end{aligned}$$

Choose $\tau > 1$ to be sufficiently large such that

$$\frac{\alpha_1}{2} - \frac{1}{3(p_1+1)} > 0, \qquad \max\{\alpha_2, \alpha_3\} - \frac{1}{3(p_1+1)} > 0, \quad \frac{\alpha_1}{p_1+1} - \frac{1}{3(p_1+1)} > 0.$$

Then there exists $\beta > 0$ such that

$$\int_0^t\int_0^1 (u_\varepsilon - h)^+\,dx\,dt \leqslant c(\mu(h))^{1+\beta}, \qquad \forall h \geqslant k_0 > 0.$$

By Lemma 3.2 we have

$$u_\varepsilon(x,t) \leqslant k_0 + \left(1 + \frac{1}{\beta}\right)ct^\beta.$$

Therefore, there exists $T > 0$ independent of ε such that

$$u_\varepsilon(x,t) \leqslant M + 1 \qquad \text{on } \bar{Q}_T.$$

Similarly, we have

$$v_\varepsilon(x,t) \leqslant M + 1 \qquad \text{on } \bar{Q}_T.$$

Thus, we have the following:

LEMMA 3.4. *There exist T and a positive constant C independent of ε such that*

$$\delta \leqslant u_\varepsilon(x,t), v_\varepsilon(x,t) \leqslant C, \qquad \|u_x\|_\infty, \|v_x\|_\infty \leqslant C \quad \text{on } \bar{Q}_T. \tag{3.3}$$

Next, we will get an L^2-estimates.

LEMMA 3.5. *There exists a positive constant C independent of ε such that*

$$\int_0^t\int_0^1 (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)\,dx\,dt \leqslant C < +\infty. \tag{3.4}$$

Proof of Proposition 3.1. Differentiate the first equation of (3.1) with respect to t and multiply it by $\Phi_1(u_\varepsilon)u_{\varepsilon t}$. Then integrate it over $[0,1]\times[0,T]$ to get

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,t)\,dx + \int_0^T\int_0^1 (u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}(p_1u_{\varepsilon x}^2+\varepsilon)u_{\varepsilon xt}^2\,dx\,dt\\
&\quad = \frac{1}{2}\int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,0)\,dx - \int_0^T\int_0^1 (u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}(p_1u_{\varepsilon x}^2+\varepsilon)u_{\varepsilon xt}\\
&\qquad \times \Phi_1'(u_\varepsilon)u_{\varepsilon x}u_{\varepsilon t}\,dx\,dt + \int_0^T ((u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}u_{\varepsilon x})_t\Phi_1(u_\varepsilon)u_{\varepsilon t}(1,t)\,dt.
\end{aligned}$$

Using Holder's inequality, we have

$$\begin{aligned}
&\int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,t)\,dx + \int_0^T\int_0^1 (u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}(p_1u_{\varepsilon x}^2+\varepsilon)u_{\varepsilon xt}^2\,dx\,dt\\
&\quad\leqslant \int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,0)\,dx\\
&\qquad + \int_0^T\int_0^1 (u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}(p_1u_{\varepsilon x}^2+\varepsilon)(\Phi_1'(u_\varepsilon)u_{\varepsilon x}u_{\varepsilon t})^2\,dx\,dt\\
&\qquad + 2\int_0^T ((u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}u_{\varepsilon x})_t\Phi_1(u_\varepsilon)u_{\varepsilon t}(1,t)\,dt.
\end{aligned}\tag{3.5}$$

Using $u_\varepsilon(x,t), v_\varepsilon(x,t)\geqslant\delta$ and the boundary conditions in (3.1), we know that there exists $x_0\in[0,1)$ such that

$$u_{\varepsilon x}(x,t)\geqslant\frac{\lambda\delta^{l_{11}l_{12}}}{2},\qquad v_{\varepsilon x}(x,t)\geqslant\frac{\lambda\delta^{l_{21}l_{22}}}{2},\quad (x,t)\in[x_0,1]\times[0,T].$$

Hence, we have

$$\int_0^T\int_0^1 (u_{\varepsilon x}^2+\varepsilon)^{\frac{p_1-1}{2}-1}(p_1u_{\varepsilon x}^2+\varepsilon)u_{\varepsilon xt}^2\,dx\,dt \geqslant c\int_0^T\int_{x_0}^1 u_{\varepsilon xt}^2\,dx \tag{3.6}$$

for some positive constant c independent of ε.

Then, (3.5), together with (3.6), gives

$$\int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,t)\,dx + c\int_0^T\int_{x_0}^1 u_{\varepsilon xt}^2\,dx\,dt \leqslant \int_0^1 (\Phi_1(u_\varepsilon)u_{\varepsilon t})^2(x,0)\,dx + c\int_0^T\int_0^1 u_{\varepsilon t}^2\,dx\,dt + c\int_0^T (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)(1,t)\,dt. \tag{3.7}$$

with the help of the boundary conditions and Young's inequality.

Similarly, we have

$$\int_0^1 (\Phi_2(v_\varepsilon)v_{\varepsilon t})^2(x,t)\,dx + c\int_0^T\int_{x_0}^1 v_{\varepsilon xt}^2\,dx\,dt \leqslant \int_0^1 (\Phi_2(v_\varepsilon)v_{\varepsilon t})^2(x,0)\,dx + c\int_0^T\int_0^1 v_{\varepsilon t}^2\,dx\,dt + c\int_0^T (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)(1,t)\,dt. \tag{3.8}$$

Using Sobolev's inequalities, we have

$$C\int_0^T (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)(1,t)\,dt \leqslant \tau_1\int_0^T\int_{x_0}^1 (u_{\varepsilon xt}^2 + v_{\varepsilon xt}^2)\,dx\,dt + c(\tau_1)\int_0^T\int_{x_0}^1 (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)\,dx\,dt \tag{3.9}$$

for any positive constant τ_1 independent of ε.

Combining (3.7)–(3.9), we have

$$\int_0^1 ((\Phi_1(u_\varepsilon)u_{\varepsilon t})^2 + (\Phi_2(v_\varepsilon)v_{\varepsilon t})^2)(x,t)\,dx \leqslant \int_0^1 ((\Phi_1(u_\varepsilon)u_{\varepsilon t})^2 + (\Phi_2(v_\varepsilon)v_{\varepsilon t})^2)(x,0)\,dx + \int_0^T\int_0^1 (u_{\varepsilon t}^2 + v_{\varepsilon t}^2)\,dx\,dt. \tag{3.10}$$

Gronwall's inequality leads to the desired results.

Thus, we need only let $\varepsilon \to 0$. From (3.3) (3.4) and (3.10) we know that $u_\varepsilon, v_\varepsilon, u_{\varepsilon x}, v_{\varepsilon x}$ are weakly $*$ compact in $L^\infty(Q_T)$ and $u_{\varepsilon t}, v_{\varepsilon t}, \Phi_1(u_\varepsilon)u_{\varepsilon t}$, $\Phi_2(v_\varepsilon)v_{\varepsilon t}$ are weakly compact in $L^2(Q_T)$.

By Aubin's Lemma (see [10, 11, 35]), it turns out that (for some subsequence, also denoted by $(u_\varepsilon, v_\varepsilon)$ in the sequel for simplicity)

$$(u_\varepsilon, v_\varepsilon) \to (u,v) \text{ in } (C(\bar{Q}_T))^2 \text{ uniformly on compact subsets of } \bar{Q}_T.$$

Therefore, $(u,v)\in(C(\bar{Q}_T))^2,\ u_x,v_x\in L^\infty(Q_T),\ u_t,v_t\in L^2(Q_T)$ and

$$\Phi_1(u_\varepsilon)u_{\varepsilon t}\to\Phi_1(u)u_t \qquad \text{weakly in } L^2(Q_T),$$
$$\Phi_2(v_\varepsilon)v_{\varepsilon t}\to\Phi_1(v)v_t \qquad \text{weakly in } L^2(Q_T).$$

By the standard monotonicity argument it follows that

$$a_{1\varepsilon}(u_{\varepsilon x})\to|u_x|^{p_1-1}u_x \qquad \text{weak} * - L^\infty(Q_T),$$
$$a_{2\varepsilon}(v_{\varepsilon x})\to|v_x|^{p_2-1}v_x \qquad \text{weak} * - L^\infty(Q_T).$$

Therefore, it is easy to verify that function $(u(x,t),v(x,t))$ is a solution of (1.1).

The proof of Proposition 3.1 is completed. ∎

4. PROOF OF THE SUFFICIENCY

Assume that (1.4) holds, we will prove that all positive solutions of (1.1) exist globally.

We will divide proof of the sufficiency into some lemmas.

Throughout this section we denote

$$\tau_j=\frac{p_j}{p_j+1},\qquad \phi_j(x)=\frac{1}{2}(1-x^{\frac{1}{\tau_j}}),\quad j=1,2$$

and use $\bar{\lambda}$ to denote a positive constant satisfying $\bar{\lambda}>\lambda$.

LEMMA 4.1. *Assume that* $m_i<p_i,\ i=1,2$. *If* $l_{11}<\frac{m_1}{p_1},\ l_{22}<\frac{m_2}{p_2}$ *and* $l_{12}l_{21}\leqslant(\frac{m_1}{p_1}-l_{11})(\frac{m_2}{p_2}-l_{22})$, *then the solution* (u,v) *of* (1.1) *exists globally.*

Proof. First, it is easy to prove that there exist $l_1,l_2>1$ such that

$$\frac{\frac{m_1}{p_1}-l_{11}}{1-\frac{m_1}{p_1}}l_1-\frac{l_{12}}{1-\frac{m_2}{p_2}}l_2\geqslant 0,$$
$$-\frac{l_{21}}{1-\frac{m_1}{p_1}}l_1+\frac{\frac{m_2}{p_2}-l_{22}}{1-\frac{m_2}{p_2}}l_2\geqslant 0. \tag{4.1}$$

By (1.4) and $l_{12},l_{21}>0$ we have that $\frac{m_1}{p_1}-l_{11},\frac{m_2}{p_2}-l_{22},1-\frac{m_1}{p_1},1-\frac{m_2}{p_2}$ are positive. Set $y_1(x,t)=a(1+x^{\frac{1}{\tau_1}})+e^{ll_1(t+1)}$, $y_2(x,t)=b(1+x^{\frac{1}{\tau_2}})+e^{ll_2(t+1)}$ and take

$$w=y_1^{\frac{p_1}{p_1-m_1}},\qquad z=y_2^{\frac{p_2}{p_2-m_2}},$$

where $l_1, l_2 > 1$ satisfy (4.1) and

$$a = \frac{(p_1 - m_1)\tau_1}{p_1} 2^{\frac{p_2 l_{12}}{p_2 - m_2}}, \qquad b = \frac{(p_2 - m_2)\tau_2}{p_2} 2^{\frac{p_1 l_{21}}{p_1 - m_1}},$$

$$\begin{aligned} l = \max\{&\log(2a)/l_1, \log(2b)/l_2, \\ &\left(\frac{p_1}{p_1 - m_1}\right)^{p_1 - 1} \frac{2a^{p_1}}{m_1 l_1 \tau_1^{p_1}} \left(1 + \frac{a p_1 m_1}{p_1 - m_1}\right), \\ &\left(\frac{p_2}{p_2 - m_2}\right)^{p_2 - 1} \frac{2a^{p_2}}{m_2 l_2 \tau_2^{p_2}} \left(1 + \frac{a p_2 m_2}{p_2 - m_2}\right), \\ &(p_1 - m_1)\log(\max u_0(x))/(l_1 p_1), \\ &(p_2 - m_2)\log(\max v_0(x))/(l_2 p_2)\}. \end{aligned}$$

By the choices of a, b and l we have

$$y_1 \leqslant 2e^{ll_1(t+1)}, \qquad y_2 \leqslant 2e^{ll_2(t+1)}. \tag{4.2}$$

By direct computations and (4.2) we have, for $(x, t) \in (0, 1) \times [0, +\infty)$,

$$\begin{aligned} (w^{m_1})_t &= \frac{p_1 m_1 l l_1}{p_1 - m_1} y_1^{\frac{p_1 m_1}{p_1 - m_1} - 1} e^{ll_1(t+1)}, \\ w_x &= \frac{a p_1}{(p_1 - m_1)\tau_1} y_1^{\frac{m_1}{p_1 - m_1}} x^{\frac{1}{\tau_1} - 1}, \\ ((w_x)^{p_1})_x &= \left(\frac{a p_1}{(p_1 - m_1)\tau_1}\right)^{p_1} \left(y_1^{\frac{p_1 m_1}{p_1 - m_1}} x\right)_x \\ &= \left(\frac{a p_1}{(p_1 - m_1)\tau_1}\right)^{p_1} \left\{ y_1^{\frac{p_1 m_1}{p_1 - m_1}} + \frac{a p_1 m_1}{p_1 - m_1} y_1^{\frac{p_1 m_1}{p_1 - m_1} - 1} x^{\frac{1}{\tau_1}} \right\} \\ &= \left(\frac{a p_1}{(p_1 - m_1)\tau_1}\right)^{p_1} y_1^{\frac{p_1 m_1}{p_1 - m_1} - 1} \left\{ y_1 + \frac{a p_1 m_1}{p_1 - m_1} x^{\frac{1}{\tau_1}} \right\} \\ &\leqslant \left(\frac{a p_1}{(p_1 - m_1)\tau_1}\right)^{p_1} y_1^{\frac{p_1 m_1}{p_1 - m_1} - 1} 2e^{ll_1(t+1)} \left(1 + \frac{a p_1 m_1}{p_1 - m_1}\right) \\ &\leqslant \frac{p_1 m_1 l l_1}{p_1 - m_1} y_1^{\frac{p_1 m_1}{p_1 - m_1} - 1} e^{ll_1(t+1)} \\ &= (w^{m_1})_t, \end{aligned}$$

i.e.

$$(w_x^{p_1})_x \leqslant (w^{m_1})_t. \tag{4.3}$$

Similarly, we have

$$(z_x^{p_2})_x \leqslant (z^{m_2})_t. \tag{4.4}$$

And

$$w_x|_{x=0} = 0, \qquad z_x|_{x=0} = 0,$$

$$\begin{aligned}
w_x &= \frac{a p_1}{(p_1 - m_1)\tau_1} y_1^{\frac{m_1}{p_1 - m_1}} \\
&= w^{l_{11}} z^{l_{12}} \frac{a p_1}{(p_1 - m_1)\tau_1} y_1^{\frac{m_1 - p_1 l_{11}}{p_1 - m_1}} y_2^{-\frac{p_2 l_{12}}{p_2 - m_2}} \\
&\geqslant w^{l_{11}} z^{l_{12}} \frac{a p_1}{(p_1 - m_1)\tau_1} 2^{-\frac{p_2 l_{12}}{p_2 - m_2}} \exp\left\{ \left[\frac{\frac{m_1}{p_1} - l_{11}}{1 - \frac{m_1}{p_1}} l_1 - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l_2 \right] l(t+1) \right\} \\
&\geqslant w^{l_{11}} z^{l_{12}} \frac{a p_1}{(p_1 - m_1)\tau_1} 2^{-\frac{p_2 l_{12}}{p_2 - m_2}} = \bar{\lambda} w^{l_{11}} z^{l_{12}}, \qquad x = 1, \qquad t > 0.
\end{aligned} \tag{4.5}$$

Similarly, we have

$$z_x \geqslant \bar{\lambda} w^{l_{21}} z^{l_{22}}, \qquad x = 1, \qquad t > 0. \tag{4.6}$$

For $x \in [0, 1]$, we have

$$\begin{aligned}
w(x, 0) &= (a(1 + x^{\frac{1}{\tau_1}}) + e^{l l_1})^{\frac{p_1}{p_1 - m_1}} \geqslant e^{l l_1 \frac{p_1}{p_1 - m_1}} \\
&\geqslant \max u_0(x) \geqslant u_0(x),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
z(x, 0) &= (a(1 + x^{\frac{1}{\tau_2}}) + e^{l l_2})^{\frac{p_2}{p_2 - m_2}} \geqslant e^{l l_2 \frac{p_2}{p_2 - m_2}} \\
&\geqslant \max v_0(x) \geqslant v_0(x).
\end{aligned} \tag{4.8}$$

From (4.3)–(4.8) we see that (w, z) is an upper solution of (1.1) with $\bar{\lambda}$. Therefore, $(u, v) \leqslant (w, z)$, and hence (u, v) exists globally. ∎

Lemma 4.2. *Assume* $m_1 = p_1$, $m_2 < p_2$. *If* $l_{11} < 1$, $l_{22} < \frac{m_2}{p_2}$ *and* $l_{12} l_{21} \leqslant (1 - l_{11})(\frac{m_2}{p_2} - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. First, it is easy to prove that there exist $l_1, l_2 > 1$ such that

$$(1 - l_{11})l_1 - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l_2 \geqslant 0,$$
$$- l_{21}l_1 + \frac{\frac{m_2}{p_2} - l_{22}}{1 - \frac{m_2}{p_2}} l_2 \geqslant 0. \tag{4.9}$$

Set

$$w = a(1 + x^{\frac{1}{\tau_1}})e^{ll_1(t+1)}, \qquad z = [b(1 + x^{\frac{1}{\tau_2}}) + e^{ll_2(t+1)}]^{\frac{p_2}{p_2 - m_2}},$$

where l_1, l_2 satisfy (4.9).

Similar to the proof of Lemma 4.1 we can prove that there exist $a, b, l > 0$ such that (w, z), defined by this manner, is an upper solution of (1.1) with $\bar{\lambda}$, therefore (u, v) exists globally. ∎

LEMMA 4.3. *Assume* $m_1 < p_1$, $m_2 = p_2$. *If* $l_{11} < \frac{m_1}{p_1}$, $l_{22} < 1$ *and* $l_{12}l_{21} \leqslant (\frac{m_1}{p_1} - l_{11})(1 - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. The proof is similar to that of Lemma 4.2. ∎

LEMMA 4.4. *Assume* $m_1 = p_1$, $m_2 = p_2$. *If* $l_{11} < 1$, $l_{22} < 1$ *and* $l_{12}l_{21} \leqslant (1 - l_{11})(1 - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. First, it is easy to prove that there exist $l_1, l_2 > 1$ such that

$$(1 - l_{11})l_1 - l_{12}l_2 \geqslant 0,$$
$$-l_{21}l_1 + (1 - l_{22})l_2 \geqslant 0. \tag{4.10}$$

Set

$$w = a(1 + x^{\frac{1}{\tau_1}})e^{ll_1(t+1)}, \qquad z = b(1 + x^{\frac{1}{\tau_2}})e^{ll_2(t+1)},$$

where l_1, l_2 satisfy (4.10).

Similar to the above arguments it is easy to prove that there exist $a, b, l > 0$ such that (w, z), defined by this manner, is an upper solution of (1.1) with $\bar{\lambda}$, therefore (u, v) exists globally. ∎

LEMMA 4.5. *Assume that* $m_i > p_i$, $i = 1, 2$. *If* $l_{11} < \frac{m_1+1}{p_1+1}$, $l_{22} < \frac{m_2+1}{p_2+1}$ *and* l_{12} $l_{21} \leqslant (\frac{m_1+1}{p_1+1} - l_{11})(\frac{m_2+1}{p_2+1} - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. It suffices to prove that for any $T > 0$ there exists $C(T) > 0$ such that

$$u(x, t), v(x, t) \leqslant C(T) < +\infty, \qquad (x, t) \in \bar{Q}_T. \tag{4.11}$$

To this aim, denote $\theta = \frac{p_1}{m_1 - p_1}$, $l = 1$, $\sigma = \frac{p_2}{m_2 - p_2}$ and

$$M = \max\left\{\frac{k^{p_1+1}\theta^{p_1-1}(2\tau_1 + p_1(\theta+2))}{8^{p_1}\tau_1^{p_1+1}m_1},\ \frac{\sigma^{p_2-1}(2\tau_2 + p_2(\sigma+2))}{8^{p_2}\tau_2^{p_2+1}m_2}\right\},$$

$$k = 2^{3+2(l_{11}\theta+\sigma l_{12})+\frac{[3+2(l_{21}\theta+\sigma l_{22})]l_{12}\sigma}{(1-l_{22})\sigma+\tau_2}}\frac{\bar{\lambda}\tau_1}{\theta}\left(\frac{\tau_2\bar{\lambda}}{\sigma}\right)^{\frac{\sigma l_{12}}{(1-l_{22})\sigma+\tau_2}}.$$

It is obvious that for any given T: $0 < T < +\infty$ there exists natural number $N = N(T)$ such that $\frac{N \log 2}{M} > T > \frac{(N-1)\log 2}{M}$. Take

$$w_i(x,t) = \{\varepsilon_i[e^{-M(t-T_{i-1})} - \frac{1}{4}(1-\phi_1(x))^{\frac{k}{\varepsilon_i^{\tau_1}}}]\}^{-\theta} = \varepsilon_i^{-\theta}y_1^{-\theta}$$

and

$$z_i(x,t) = \{\eta_i[e^{-M(t-T_{i-1})} - \frac{1}{4}(1-\phi_2(x))^{\frac{l}{\eta_i^{\tau_2}}}]\}^{-\sigma} = \eta_i^{-\sigma}y_2^{-\sigma},$$

where $T_0 = 0, T_i = \frac{i\log 2}{M}$, $i = 1,\ldots,N$ and $\varepsilon_i, \eta_i > 0$ are all constants to be determined later.

It is obvious that for any given $i = 1,\ldots,N$, $y_j(x,t)$, $j = 1,2$ are well defined on $Q_i \coloneqq [0,1]\times[T_{i-1},T_i]\cap[0,1]\times[0,T]$ and $\frac{1}{4}\leqslant y_j\leqslant 1$, $j = 1,2$.

By the direct calculations we have for $(x,t)\in Q_i$

$$(w_i^{m_1})_t = M\theta m_1\varepsilon_i\varepsilon_i^{-m_1\theta}y_1^{-m_1\theta-1}\geqslant\frac{M\theta m_1}{2}\varepsilon_i^{-m_1\theta},$$

$$w_{ix} = \frac{k\theta}{8\tau_1}\varepsilon_i^{-(\theta+\tau_1)}y_1^{-\theta-1}(1-\phi_1(x))^{\frac{k}{\varepsilon_i^{\tau_1}}-1}x^{\frac{1}{\tau_1}-1}$$

$$(w_{ix}^{p_1})_x = \left(\frac{k\theta}{8\tau_1}\right)^{p_1}\varepsilon_i^{-p_1(\theta+\tau_1)}(y_1^{-p_1(\theta+1)}(1-\phi_1(x))^{p_1\left(\frac{k}{\varepsilon_i^{\tau_1}}-1\right)}x)_x$$

$$= \left(\frac{k\theta}{8\tau_1}\right)^{p_1}\varepsilon_i^{-p_1(\theta+\tau_1)}\left\{y_1^{-p_1(\theta+1)}(1-\phi_1(x))^{p_1\left(\frac{k}{\varepsilon_i^{\tau_1}}-1\right)}\right.$$

$$+\frac{p_1}{2\tau_1}\left(\frac{k}{\varepsilon_i^{\tau_1}}-1\right)y_1^{-p_1(\theta+1)}(1-\phi_1(x))^{p_1\left(\frac{k}{\varepsilon_i^{\tau_1}}-1\right)-1}x^{\frac{1}{\tau_1}}$$

$$\left.+\frac{p_1k(\theta+1)}{8\tau_1\varepsilon_i^{\tau_1}}y_1^{-p_1(\theta+1)-1}(1-\phi_1(x))^{(p_1+1)\left(\frac{k}{\varepsilon_i^{\tau_1}}-1\right)}x^{\frac{1}{\tau_1}}\right\}$$

$$\leqslant \left(\frac{k\theta}{8\tau_1}\right)^{p_1} \varepsilon_i^{-p_1(\theta+\tau_1)} y_1^{-p_1(\theta+1)} \left\{1 + \frac{p_1 k}{2\tau_1 \varepsilon_i^{\tau_1}} + \frac{p_1 k(\theta+1)}{2\tau_1 \varepsilon_i^{\tau_1}}\right\}$$
$$\leqslant \left(\frac{k\theta}{8\tau_1}\right)^{p_1} \varepsilon_i^{-\theta m_1} 4^{p_1(\theta+1)} \left\{\varepsilon_i^{\tau_1} + \frac{p_1 k}{2\tau_1} + \frac{p_1 k(\theta+1)}{2\tau_1}\right\}$$
$$\leqslant \left(\frac{k\theta}{8\tau_1}\right)^{p_1} \varepsilon_i^{-\theta m_1} 4^{p_1(\theta+1)} \left\{k + \frac{p_1 k}{2\tau_1} + \frac{p_1 k(\theta+1)}{2\tau_1}\right\}$$
$$\leqslant \frac{M\theta m_1}{2} \varepsilon_i^{-m_1\theta} \leqslant (w_i^{m_1})_t.$$

Similarly, we have

$$(z_i^{m_2})_t \geqslant (z_{ix}^{p_2})_x, \qquad (x,t) \in Q_i.$$

Obviously,

$$w_{ix}|_{x=0} = 0, \qquad z_{ix}|_{x=0} = 0,$$

$$\begin{aligned} w_{ix} &= \varepsilon_i^{-(\theta+\tau_1)} \tfrac{k\theta}{8\tau_1} y_1^{-\theta-1} \\ &= w_i^{l_{11}} z_i^{l_{12}} \varepsilon_i^{-(\theta+\tau_1)+l_{11}\theta} \eta_i^{l_{12}\sigma} \tfrac{k\theta}{8\tau_1} \left(e^{-M(t-T_{i-1})} - \frac{1}{4}\right)^{(l_{11}-1)\theta+\sigma l_{12}-1} \\ &\geqslant \bar{\lambda} w_i^{l_{11}} z_i^{l_{12}}, \qquad x = 1, \qquad T_{i-1} \leqslant t \leqslant T_i \end{aligned}$$

holds if the following inequality holds:

$$\varepsilon_i^{-(\theta+\tau_1)+l_{11}\theta} \eta_i^{l_{12}\sigma} \frac{k\theta}{8\tau_1} 4^{-(l_{11}\theta+\sigma l_{12})} \geqslant \bar{\lambda}. \tag{4.12}$$

Similarly, we have

$$z_{ix} \geqslant \bar{\lambda} w_i^{l_{21}} z_i^{l_{22}}$$

holds if the following inequality holds:

$$\varepsilon_i^{l_{21}\theta} \eta_i^{-(\sigma+\tau_2)+l_{22}\sigma} \frac{l\sigma}{8\tau_2} 4^{-(l_{21}\theta+\sigma l_{22})} \geqslant \bar{\lambda}. \tag{4.13}$$

By the choices of k, l one see that (4.12) and (4.13) hold if

$$\varepsilon_i^{-[(1-l_{11})\theta+\tau_1]+\frac{l_{12}l_{21}\theta\sigma}{(1-l_{22})\sigma+\tau_2}} \geqslant 1,$$

$$\eta_i = \left(\frac{\sigma 4^{-(l_{21}\theta+\sigma l_{22})}}{8\tau_2\bar{\lambda}}\right)^{\frac{1}{(1-l_{22})\sigma+\tau_2}} \varepsilon_i^{\frac{l_{21}\theta}{(1-l_{22})\sigma+\tau_2}}. \tag{4.14}$$

Set $\varepsilon_i = a(\frac{1}{4})^i,\ \eta_i = b(\frac{1}{4})^i,\ i = 1, \ldots, N$, then (4.14) holds if

$$\left(a\left(\frac{1}{4}\right)^i\right)^{-[(1-l_{11})\theta+\tau_1]+\frac{l_{12}l_{21}\theta\sigma}{(1-l_{22})\sigma+\tau_2}} \geqslant 1$$

$$b = 4^i\left(\frac{\sigma 4^{-(l_{21}\theta+\sigma l_{22})}}{8\tau_2\bar{\lambda}}\right)^{\frac{1}{(1-l_{22})\sigma+\tau_2}}\left(a\left(\frac{1}{4}\right)^i\right)^{\frac{l_{21}\theta}{(1-l_{22})\sigma+\tau_2}}. \tag{4.15}$$

A direct calculation shows

$$\begin{aligned}
&-[(1-l_{11})\theta+\tau_1]+\frac{l_{12}l_{21}\theta\sigma}{(1-l_{22})\sigma+\tau_2}\\
&\quad=\frac{\sigma\theta}{(1-l_{22})\sigma+\tau_2}\left[l_{12}l_{21}-\left(\frac{m_1+1}{p_1+1}-l_{11}\right)\left(\frac{m_2+1}{p_2+1}-l_{22}\right)\right]\\
&\quad\leqslant 0.
\end{aligned}$$

Hence, for fixed i we can choose a and b sufficiently small such that (4.15) holds and

$$w_1(x,0)=\left\{\varepsilon_1\left[1-\frac{1}{4}(1-\phi_1)^{\frac{k}{\varepsilon_1^{\tau_1}}}\right]\right\}^{-\theta}\geqslant\varepsilon_1^{-\theta}=\left(\frac{a}{4}\right)^{-\theta}\geqslant\max_{0\leqslant x\leqslant 1}u_0(x)\geqslant u_0(x),$$

$$z_1(x,0)=\left\{\eta_1\left[1-\frac{1}{4}(1-\phi_2)^{\frac{l}{\eta_1^{\tau_2}}}\right]\right\}^{-\sigma}\geqslant\eta_1^{-\sigma}=\left(\frac{b}{4}\right)^{-\sigma}\geqslant\max_{0\leqslant x\leqslant 1}v_0(x)\geqslant v_0(x).$$

For the above fixed a and b, when $i = 2, \ldots, N$, we have

$$\begin{aligned}
w_{i-1}(x,T_{i-1})&=\left\{\varepsilon_{i-1}\left[e^{-M(T_{i-1}-T_{i-2})}-\frac{1}{4}(1-\phi_1)^{\frac{k}{\varepsilon_{i-1}^{\tau_1}}}\right]\right\}^{-\theta}\\
&\leqslant\left(\varepsilon_{i-1}\frac{1}{4}\right)^{-\theta}=\varepsilon_i^{-\theta}\leqslant w_i(x,T_{i-1})
\end{aligned}$$

and

$$\begin{aligned}
z_{i-1}(x,T_{i-1})&=\left\{\eta_{i-1}\left[e^{-M(T_{i-1}-T_{i-2})}-\frac{1}{4}(1-\phi_1)^{\frac{l}{\eta_{i-1}^{\tau_2}}}\right]\right\}^{-\sigma}\\
&\leqslant\left(\eta_{i-1}\tfrac{1}{4}\right)^{-\sigma}=\eta_i^{-\sigma}\leqslant z_i(x,T_{i-1}).
\end{aligned}$$

This shows that (w,z) is an upper solution of (1.1) with $\bar{\lambda}$ on Q_i. By comparison principle we have $(u,v)\leqslant(w_i,z_i)$ on Q_i, $i=1,\dots,N$. It is obvious that there exist $C(T)$: $0<C(T)<+\infty$ such that (4.11) holds. This completes the proof of Lemma 4.5. ∎

LEMMA 4.6 *Assume* $m_1<p_1,\ m_2>p_2$. *If* $l_{11}<\frac{m_1}{p_1},\ l_{22}<\frac{m_2+1}{p_2+1}$ *and* $l_{12}l_{21}\leqslant(\frac{m_1}{p_1}-l_{11})(\frac{m_2+1}{p_2+1}-l_{22})$, *then the solution* (u,v) *of* (1.1) *exists globally.*

Proof. First, we consider the following cases:

(1) $p_1\leqslant 1$; (2) $p_1>1$ and $m_1\leqslant 1$; (3) $p_1>1,\ m_1>1$ and $l_{11}\geqslant\frac{m_1-1}{p_1-1}$.

As in Lemma 4.5, it suffices to prove that for any $T>0$ there exists $C(T)>0$ such that (4.11) holds. To this aim, denote $\theta=\frac{1}{(1-l_{11})}>1,\ \sigma=\frac{p_2}{m_2-p_2},\quad \alpha=p_1-m_1,\quad k=\frac{1}{p_1-(p_1-m_1)\theta}\geqslant 1,\quad c=\max\{1,(\frac{\theta}{2\tau_1})^{p_1}(1+\frac{2p_1(\theta-1)}{\tau_1})/(km_1\theta)\}$, $M=\max\{\frac{1,(l^{p_2+1}\sigma^{p_2-1}(2\tau_2+p_2(\sigma+2))}{8^{p_2}\tau_2^{p_2+1}m_2}\}$ and

$$l=2^{3+\frac{2\sigma l_{22}+(2\sigma_{12}+1)[(1-l_{22})\sigma+\tau_2]}{\sigma l_{12}}}\frac{\bar{\lambda}\tau_2[1+c(\log 2+1)]^{\theta}l_{21}}{\sigma}$$
$$\times\left(\frac{\bar{\lambda}\tau_1}{\theta}\right)^{\frac{(1-l_{22})\sigma+\tau_2}{\sigma l_{12}}}.$$

It is obvious that for any given T: $0<T<+\infty$ there exists natural number $N=N(T)$ such that $\frac{N\log 2}{M}>T>\frac{(N-1)\log 2}{M}$. Take

$$w_i(x,t)=\varepsilon_i\{[1+c\varepsilon_i^{\alpha}(t+1-T_{i-1})]^k-\phi_1(x)\}^{\theta}=\varepsilon_i y_1^{\theta}$$

and

$$z_i(x,t)=\left\{\eta_i\left[e^{-M(t-T_{i-1})}-\frac{1}{4}(1-\phi_2(x))^{\eta_i^{\frac{l}{\tau_2}}}\right]\right\}^{-\sigma}=\eta_i^{-\sigma}y_2^{-\sigma}$$

where $T_0=0,T_i=\frac{i\log 2}{M},\ i=1,\dots,N$ and $\varepsilon_i,\eta_i>0$ are all constants to be determined later.

It is also obvious that for any given $i=1,\dots,N,\ y_j(x,t),\ j=1,2$ are well defined on $Q_i:=[0,1]\times[T_{i-1},T_i]\cap[0,1]0\times[0,T]$ and

$$\frac{1}{4}\leqslant y_1\leqslant\varepsilon_i^{k\alpha}[1+c(1+\log 2)],\qquad \frac{1}{4}\leqslant y_2\leqslant 1 \tag{4.16}$$

if ε_i is sufficiently large.

Thus, by direct calculations we have for $(x,t) \in Q_i$, $i = 1, \ldots, N$

$$(w_i^{m_1})_t = \varepsilon_i^{m_1+\alpha} ckm_1\theta y_1^{m_1\theta-1}[1 + c\varepsilon_i^\alpha(t+1-T_{i-1})]^{k-1},$$

$$w_{ix} = \varepsilon_i \frac{\theta}{2\tau_1} x^{\frac{1}{\tau_1}-1} y_1^{\theta-1},$$

$$\begin{aligned}
(w_{ix}^{p_1})_x &= \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{p_1} (y_1^{p_1(\theta-1)}x)_x \\
&= \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{p_1} \left\{ y_1^{p_1(\theta-1)} + \frac{p_1(\theta-1)}{2\tau_1} y_1^{p_1(\theta-1)-1} x^{\frac{1}{\tau_1}} \right\} \\
&\leqslant \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{p_1} y_1^{p_1(\theta-1)} \left(1 + \frac{2p_1(\theta-1)}{\tau_1}\right) \\
&= \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{m_1+\alpha} y_1^{m_1\theta-1} y_1^{(p_1-m_1)\theta-p_1+1} \left(1 + \frac{2p_1(\theta-1)}{\tau_1}\right).
\end{aligned}$$

For the present case, a direct calculation shows that $(p_1 - m_1)\theta - p_1 + 1 \geqslant 0$ and $k > 1$ if $(p_1 - m_1)\theta - p_1 + 1 > 0$ while $k = 1$ if $(p_1 - m_1)\theta - p_1 + 1 = 0$. Therefore,

$$\begin{aligned}
(w_{ix}^{p_1})_x &\leqslant \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{m_1+\alpha} y_1^{m_1\theta-1} \\
&\quad \times [1 + c\varepsilon_i^\alpha(t+1-T_{i-1})]^{k[(p_1-m_1)\theta-p_1+1]} \left(1 + \frac{2p_1(\theta-1)}{\tau_1}\right) \\
&= \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{m_1+\alpha} y_1^{m_1\theta-1}[1 + c\varepsilon_i^\alpha(t+1-T_{i-1})]^{k-1} \left(1 + \frac{2_1(\theta-1)}{\tau_1}\right) \\
&\leqslant \varepsilon_i^{m_1+\alpha} y_1^{m_1\theta-1}[1 + c\varepsilon_i^\alpha(t+1-T_{i-1})]^{k-1} ckm_1\theta \\
&= (w_i^{m_1})_t.
\end{aligned}$$

As in Lemma 4.5, we have

$$(z_i^{m_2})_t \geqslant (z_{ix}^{p_2})_x, \qquad (x,t) \in Q_i.$$

Obviously,

$$w_{ix}|_{x=0} = 0, \qquad z_{ix}|_{x=0} = 0.$$

Using $\theta(1-l_{11})-1=0$ we see that

$$\begin{aligned} w_{ix} &= \varepsilon_i \frac{\theta}{2\tau_1} y_1^{\theta-1} = w_i^{l_{11}} z_i^{l_{12}} \varepsilon_i^{1-l_{11}} \eta_i^{l_{12}\sigma} \frac{\theta}{2\tau_1} y_1^{\theta(1-l_{11})-1} y_2^{\sigma l_{12}} \\ &\geqslant w_i^{l_{11}} z_i^{l_{12}} \varepsilon_i^{1-l_{11}} \eta_i^{l_{12}\sigma} \frac{\theta}{2\tau_1}\left(\frac{1}{4}\right)^{\sigma l_{12}} \geqslant \bar{\lambda} w_i^{l_{11}} z_i^{l_{12}}, \qquad x=1, \quad T_{i-1}\leqslant t\leqslant T_i \end{aligned}$$

holds if the following inequality holds:

$$\varepsilon_i^{1-l_{11}} \eta_i^{l_{12}\sigma} \frac{\theta}{\tau_1 2^{2\sigma l_{12}+1}} \geqslant \bar{\lambda}. \tag{4.17}$$

Using (4.16) we know that

$$\begin{aligned} z_{ix} &= \eta_i^{-(\sigma+\tau_2)} \frac{\sigma l}{8\tau_2} y_2^{-\sigma-1} \\ &= w_i^{l_{21}} z_i^{l_{22}} \varepsilon_i^{-l_{21}} \eta_i^{-(\sigma+\tau_2)+l_{22}\sigma} \frac{l\sigma}{8\tau_2} y_1^{-\theta l_{21}} y_2^{(l_{22}-1)\sigma-1} \\ &\geqslant w_i^{l_{21}} z_i^{l_{22}} \varepsilon_i^{-(1+\alpha k\theta)l_{21}} \eta_i^{-\frac{(\sigma+\tau_2)+l_{22}\sigma l\sigma[1+c(\log 2+1)]^{-\theta l_{21}}}{\tau_2 2^{2l_{22}\sigma+3}}} \\ &\geqslant \bar{\lambda} w_i^{l_{21}} z_i^{l_{22}}, \qquad x=1, \quad T_{i-1}\leqslant t\leqslant T_i \end{aligned}$$

holds if the following inequality holds:

$$\varepsilon_i^{-(1+\alpha k\theta)l_{21}} \eta_i^{-(\sigma+\tau_2)+l_{22}\sigma} \frac{l\sigma[1+c(\log 2+1)]^{-\theta l_{21}}}{\tau_2 2^{2l_{22}\sigma+3}} \geqslant \bar{\lambda}. \tag{4.18}$$

By the choice of l one see that (4.17) and (4.18) hold if

$$\varepsilon_i^{1-l_{11}-\frac{l_{12}l_{21}(1+\alpha k\theta)\sigma}{(1-l_{22})\sigma+\tau_2}} \geqslant 1,$$

$$\begin{aligned} \eta_i &= \left(\frac{l\sigma}{\tau_2 2^{2l_{22}\sigma+3}[1+c(\log 2+1)]^{\theta l_{21}}\bar{\lambda}}\right)^{\frac{1}{(1-l_{22})\sigma+\tau_2}} \\ &\quad \times \varepsilon_i^{-\frac{l_{21}(1+\alpha k\theta)}{(1-l_{22})\sigma+\tau_2}}. \end{aligned} \tag{4.19}$$

Set $\varepsilon_i = a(2+\log 2)^{\frac{k\theta i}{1+\alpha k\theta}}$, $\eta_i = b(\frac{1}{4})^i$, $i=1,\dots,N$, then (4.19) holds if

$$\left(a(2+\log 2)^{\frac{k\theta i}{1+\alpha k\theta}}\right)^{1-l_{11}-\frac{l_{12}l_{21}(1+\alpha k\theta)\sigma}{(1-l_{22})\sigma+\tau_2}} \geqslant 1,$$

$$b = 4^i \left(\frac{l\sigma}{\tau_2 2^{2l_{22}\sigma+3}[1 + c(\log 2 + 1)]^{\theta l_{21}} \bar{\lambda}} \right)^{\frac{1}{(1-l_{22})\sigma+\tau_2}}$$

$$\times \varepsilon_i^{-\frac{l_{21}(1+\alpha k\theta)}{(1-l_{22})\sigma+\tau_2}}. \tag{4.20}$$

A direct calculation shows that

$$1 - l_{11} - \frac{l_{12}l_{21}(1 + \alpha k\theta)\sigma}{(1 - l_{22})\sigma + \tau_2}$$

$$= -\frac{\sigma(1 + \alpha k\theta)}{\sigma(1 - l_{22}) + \tau_2} \left[l_{12}l_{21} - \left(\frac{m_1}{p_1} - l_{11} \right) \left(\frac{m_2 + 1}{p_2 + 1} - l_{22} \right) \right] \geqslant 0.$$

Hence, for fixed i we can choose a sufficiently large and then choose b sufficiently small to satisfy (4.20) and

$$w_1(x,0) = \varepsilon_1\{[1 + c\varepsilon_1^{\alpha}]^k - \phi_1(x)\}^{\theta} \geqslant \varepsilon_1(\tfrac{1}{4})^{\theta} = a(2 + \log 2)^{k\theta/(1+\alpha k\theta)}(\tfrac{1}{4})^{\theta}$$

$$\geqslant \max_{0\leqslant x\leqslant 1} u_0(x) \geqslant u_0(x),$$

$$z_1(x,0) = \{\eta_1[1 - \tfrac{1}{4}(1 - \phi_2)^{\frac{l}{\eta_1^{\tau_2}}}]\}^{-\sigma} \geqslant (\eta_1)^{-\sigma} = (\tfrac{4}{b})^{\sigma} \geqslant \max_{0\leqslant x\leqslant 1} v_0(x) \geqslant v_0(x).$$

For the above fixed a, b and $i = 2, \ldots, N$, we have

$$w_{i-1}(x, T_{i-1}) \leqslant \varepsilon_{i-1}^{1+\alpha k\theta}[1 + c(\log 2 + 1)]^{k\theta}$$

$$\leqslant c^{k\theta}\varepsilon_{i-1}^{1+\alpha k\theta}[2 + \log 2]^{k\theta} = c^{k\theta}\varepsilon_i^{1+\alpha k\theta} \leqslant w_i(x, T_{i-1}),$$

$$z_{i-1}(x, T_{i-1}) = \left\{ \eta_{i-1} \left[e^{-M(T_{i-1}-T_{i-2})} - \tfrac{1}{4}(1 - \phi_1)^{\frac{l}{\eta_{i-1}^{\tau_2}}} \right] \right\}^{-\sigma}$$

$$\leqslant (\eta_{i-1}\tfrac{1}{4})^{-\sigma} = \eta_i^{-\sigma} \leqslant z_i(x, T_{i-1}).$$

This shows that (w_i, z_i) is an upper solution of (1.1) with $\bar{\lambda}$ on Q_i. By comparison principle we have $(u, v) \leqslant (w_i, z_i)$ on Q_i, $i = 1, \ldots, N$. Therefore, there exist $C(T)$: $0 < C(T) < +\infty$ such that (4.11) holds. And hence the solution (u, v) of (1.1) exists globally.

Secondly, we consider the following case:

(4) $p_1 > 1$, $m_1 > 1$ and $l_{11} < \frac{m_1 - 1}{p_1 - 1}$.

As above, it suffices to prove that there exist $C(T)$: $0<C(T)<+\infty$ such that (4.11) holds. To this aim, denote $\theta=\frac{p_1-1}{p_1-m_1}>1$, $\alpha=\frac{(1-m_1/p_1)}{(m_1/p_1-l_{11})}\cdot\frac{p_1-m_1}{p_1-1}>0$, $\sigma=\frac{p_2}{m_2-p_2}$, $c=\max\{2,(\frac{\theta}{\tau_1})^{p_1}(1+\frac{p_1(\theta-1)}{\tau_1})/(m_1\theta)\}$, $\tau=\frac{\alpha-(p_1-m_1)}{p_1}$ and M,l as above.

For the present case, a direct calculation shows that $0<\alpha<p_1-m_1,\tau<0$, $\tau+\alpha[(1-l_{11})\theta-1]=0$ and $\frac{1-l_{11}}{1+\alpha\theta}=\frac{m_1}{p_1}-l_{11}$. Take

$$w_i(x,t)=\varepsilon_i\{1+c\varepsilon_i^{\alpha}(t+1-T_{i-1})+(1-\phi_1(x))^{\varepsilon_i^{\tau}}\}^{\theta}=\varepsilon_i y_1^{\theta}.$$

and

$$z_i(x,t)=\{\eta_i[e^{-M(t-T_{i-1})}-\tfrac{1}{4}(1-\phi_2(x))^{\frac{l}{\eta_i^{\tau_2}}}]\}^{-\sigma}=\eta_i^{\sigma}y_2^{-\sigma}.$$

where $T_0=0,T_i=\frac{i\log 2}{M}$, $i=1,\ldots,N$ and $\varepsilon_i,\eta_i>0$ are all constants to be determined later.

It is obvious that for any given $i=1,\ldots,N$, $y_j(x,t)$, $j=1,2$, are well defined on $Q_i:=[0,1]\times[T_{i-1},T_i]\cap[0,1]\times[0,T]$ and

$$1\leqslant c\varepsilon_i^{\alpha}\leqslant y_1\leqslant\varepsilon_i^{\alpha}[1+c(1+\log 2)],\qquad \frac{1}{4}\leqslant y_2\leqslant 1 \tag{4.21}$$

if ε_i is sufficiently large.

By direct calculations we have for $(x,t)\in Q_i$

$$\begin{aligned}
(w_i^{m_1})_t&=\varepsilon_i^{m_1+\alpha}cm_1\theta y_1^{m_1\theta-1},\\
w_{ix}&=\varepsilon_i^{1+\tau}\frac{\theta}{2\tau_1}(1-\phi_1(x))^{\varepsilon_i^{\tau}-1}x^{1/\tau_1-1}y_1^{\theta-1},\\
(w_{ix}^{p_1})_x&=\left(\frac{\theta}{2\tau_1}\right)^{p_1}\varepsilon_i^{p_1(1+\tau)}(y_1^{p_1(\theta-1)}(1-\phi_1(x))^{p_1(\varepsilon_i^{\tau}-1)}x)_x\\
&=\left(\frac{\theta}{2\tau_1}\right)^{p_1}\varepsilon_i^{p_1(1+\tau)}\Bigg\{y_1^{p_1(\theta-1)}(1-\phi_1(x))^{p_1(\varepsilon_i^{\tau}-1)}\\
&\quad+\frac{p_1(\varepsilon_i^{\tau}-1)}{2\tau_1}y_1^{p_1(\theta-1)}(1-\phi_1(x))^{p_1(\varepsilon_i^{\tau}-1)-1}x^{1/\tau_1}\\
&\quad+\frac{p_1(\theta-1)\varepsilon_i^{\tau}}{2\tau_1}y_1^{p_1(\theta-1)-1}(1-\phi_1(x))^{(p_1+1)(\varepsilon_i^{\tau}-1)}x^{1/\tau_1}\Bigg\}
\end{aligned}$$

$$\leqslant \left(\frac{\theta}{2\tau_1}\right)^{p_1} \varepsilon_i^{p_1(1+\tau)} y_1^{p_1(\theta-1)} \left(2^{p_1} + \frac{p_1(\theta-1)}{2\tau_1} 2^{p_1+1}\right)$$
$$= \left(\frac{\theta}{\tau_1}\right)^{p_1} \varepsilon_i^{m_1+\alpha} y_1^{m_1\theta-1} y_1^{(p_1-m_1)\theta-p_1+1} \left(1 + \frac{p_1(\theta-1)}{\tau_1}\right)$$
$$\times\, \varepsilon_i^{m_1+\alpha} c m_1 \theta y_1^{m_1\theta-1} \leqslant (w_i^{m_1})_t.$$

(Here, we have used that $\varepsilon_i^{\tau} \leqslant 1, y_1 \geqslant 1, (p_1 - m_1)\theta - p_1 + 1 = 0.$)

As in Lemma 4.5, we have

$$(z_i^{m_2})_t \geqslant (z_{ix}^{p_2})_x, \qquad (x,t) \in Q_i.$$

Obviously,

$$w_{ix}|_{x=0} = 0, \qquad z_{ix}|_{x=0} = 0.$$

Using $\tau + \alpha[\theta(1 - l_{11}) - 1] = 0$, $\frac{1-l_{11}}{1+\alpha\theta} = \frac{m_1}{p_1} - l_{11}$ and (4.21) we have

$$w_{ix} = \varepsilon_i^{1+\tau} \frac{\theta}{2\tau_1} y_1^{\theta-1} = w_i^{l_{11}} z_i^{l_{12}} \varepsilon_i^{1+\tau-l_{11}} \eta_i^{l_{12}\sigma} \frac{\theta}{2\tau_1} y_1^{(1-l_{11})\theta-1} y_2^{\sigma l_{12}}$$
$$\geqslant w_i^{l_{11}} z_i^{l_{12}} \varepsilon_i^{1-l_{11}+\tau+\alpha((1-l_{11})\theta-1)} 8\eta_i^{l_{12}\sigma} \frac{\theta}{2\tau_1} \left(\frac{1}{4}\right)^{\sigma l_{12}}$$
$$= w^{l_{11}} z^{l_{12}} \varepsilon_i^{1-l_{11}} \eta_i^{l_{12}\sigma} \frac{\theta}{2\tau_1} \left(\frac{1}{4}\right)^{\sigma l_{12}}, \qquad x = 1, \qquad T_{i-1} \leqslant t \leqslant T_i.$$

Therefore, if (4.17) and (4.18) with $k = 1$ hold, then for $x = 1, T_{i-1} \leqslant t \leqslant T_i$

$$w_{ix} \geqslant \bar{\lambda} w^{l_{11}} z^{l_{12}}, \qquad z_{ix} \geqslant \bar{\lambda} w^{l_{21}} z^{l_{22}}$$

hold.

Thus, as above we can get our results.

This completes the proof of Lemma 4.6. ∎

LEMMA 4.7. *Assume* $m_1 > p_1, m_2 < p_2$. *If* $l_{11} < \frac{m_1+1}{p_1+1}$, $l_{22} < \frac{m_2}{p_2}$ *and* $l_{12} l_{21} \leqslant (\frac{m_1+1}{p_1+1} - l_{11})(\frac{m_2}{p_2} - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. The proof is similar to that of Lemma 4.6. ∎

LEMMA 4.8. *Assume* $m_1 = p_1, m_2 > p_2$. *If* $l_{11} < 1, l_{22} < \frac{m_2+1}{p_2+1}$ *and* $l_{12} l_{21} \leqslant (1 - l_{11})(\frac{m_2+1}{p_2+1} - l_{22})$, *then the solution* (u, v) *of* (1.1) *exists globally.*

Proof. As in Lemma 4.5, it suffices to prove that there exist $C(T)$: $0 < C(T) < +\infty$ such that (4.11) holds. To this aim, take

$$w_i(x,t) = \varepsilon_i \{[1 + c(t + 1 - T_{i-1})]^k - \phi_1(x)\}^\theta = \varepsilon_i y_1^\theta$$

and

$$z_i(x,t) = \{\eta_i[e^{-M(t-T_{i-1})} - \tfrac{1}{4}(1-\phi_2(x))^{\frac{l}{\eta_i^{\tau_2}}}]\}^{-\sigma} = \eta_i^{\sigma} y_2^{-\sigma},$$

where $T_0 = 0, T_i = \frac{i \log 2}{M}$, $i = 1, \ldots, , N$, $\theta = \frac{1}{1-l_{11}}, \sigma, l, M, c$ as above, and

$$k = \begin{cases} 1, & m_1 = p_1 \geqslant 1, \\ \frac{1}{m_1}, & m_1 = p_1 < 1. \end{cases}$$

Similar to the above arguments we can show that there exist positive constants ε_i and η_i such that (w_i, z_i) is an upper solution of (1.1) with $\bar{\lambda}$ on Q_i. By comparison principle we have $(u, v) \leqslant (w_i, z_i)$ on Q_i, $i = 1, \ldots, N$. Therefore, there exist $C(T)$: $0 < C(T) < +\infty$ such that (4.11) holds. And hence the solution (u, v) of (1.1) exists globally.

This completes the proof of Lemma 4.8. ∎

LEMMA 4.9. *Assume $m_1 > p_1, m_2 = p_2$. If $l_{11} < \frac{m_1+1}{p_1+1}, l_{22} < 1$ and $l_{12} l_{21} \leqslant (\frac{m_1+1}{p_1+1} - l_{11})(1 - l_{22})$, then the solution (u, v) of* (1.1) *exists globally.*

Proof. The proof is similar to that of Lemma 4.8.

By Lemmas 4.1–4.9 we get the proof of the sufficiency. ∎

5. PROOF OF THE NECESSITY

We will complete the proof of the necessity by a series of lemmas.

In this section we denote $\tau_i = \frac{p_i}{p_i+1}$, $i = 1, 2$ and use $\underline{\lambda}$ to denote a positive constant satisfying $\underline{\lambda} < \lambda$.

First we will prove a result, which are useful as we proceed, from linear algebra.

LEMMA 5.1. *Assume that $a_{11}, a_{22} \geqslant 0$, $a_{12}, a_{21} > 0$ and $a_{12} a_{21} > a_{11} a_{22}$. Then for any given $a_1, a_2 \in R^1$, there exist $k, l > 1$ such that*

$$\begin{aligned} a_{11} k - a_{12} l + a_1 &\leqslant 0, \\ -a_{21} k + a_{22} l + a_2 &\leqslant 0. \end{aligned} \tag{5.1}$$

Proof. When $a_{11} > 0$, $a_{22} > 0$, by $a_{12} a_{21} > a_{11} a_{22}$ we have $0 < a_{11}/a_{12} < a_{21}/a_{22}$. Therefore, there exists $\mu > 0$ such that $0 < a_{11}/a_{12} < \mu < a_{21}/a_{22}$.

Set

$$k=\max\left\{2,\frac{2}{\mu},\frac{a_1}{a_{12}}/(\mu-\frac{a_{11}}{a_{12}}),-\frac{a_2}{a_{22}}/\left(\mu-\frac{a_{21}}{a_{22}}\right)\right\},\qquad l=k\mu,$$

then $k,l>1$ satisfy (5.1).

When $a_{11}=0,\ a_{22}>0$, set

$$l=\max\left\{2,\frac{ax_1}{a_{12}}\right\},\qquad k=\max\{2,(a_{22}l+a_2)/a_{21}\}.$$

When $a_{11}=0,\ a_{22}=0$, set

$$l=\max\left\{2,\frac{a_1}{a_{12}}\right\},\qquad k=\max\left\{2,\frac{a_2}{a_{21}}\right\}.$$

When $a_{11}>0, a_{22}=0$, set

$$k=\max\left\{2,\frac{a_2}{a_{21}}\right\},\qquad l=\max\{2,(a_{11}k+a_1)/a_{12}\}.$$

Then $k,l>1$ satisfy (5.1). ∎

LEMMA 5.2. *If* $l_{11}>\min\{\frac{m_1}{p_1},\frac{m_1+1}{p_1+1}\}$ *or* $l_{22}>\min\{\frac{m_2}{p_2},(m_2+1)/(p_2+1)\}$. *Then the solution* (u,v) *blows up in finite time.*

Proof. When $l_{11}>\min\{\frac{m_1}{p_1},\frac{m_1+1}{p_1+1}\}$, the local solution (u,v) of (1.1) satisfies $(u,v)\geqslant(\delta,\delta)$, it follows that

$$u_x=\lambda u^{l_{11}}v^{l_{12}}\geqslant\lambda\delta^{l_{12}}u^{l_{11}},\qquad x=1,\quad t>0.$$

Consider the following single parabolic problem:

$$\begin{aligned}&(|w|^{m_1-1}w)_t=(|w_x|^{p_1-1}w_x)_x, && 0<x<1,\quad t>0,\\ &w_x|_{x=0}=0,\qquad w_x|_{x=1}=\underline{\lambda}\delta^{l_{12}}w^{l_{11}}|_{x=1}, && t>0,\\ &w(x,0)=u_0(x)\geqslant\delta>0, && 0\leqslant x\leqslant 1.\end{aligned}$$

Then $w(x,t)$ blows up in finite time (see [47]). Choose $(\underline{u},\underline{v})=(w,\delta)$, by using $(u,v)\geqslant(\delta,\delta)$, it is easy to verify that $(\underline{u},\underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}(<\lambda)$. By the comparison principle we have that $(u,v)\geqslant(\underline{u},\underline{v})$, and hence (u,v) blows up in finite time.

When $l_{22}>\min\{\frac{m_2}{p_2},\frac{m_2+1}{p_2+1}\}$, the proof is similar. ∎

LEMMA 5.3. *Assume that* $m_i<p_i,\ i=1,2$. *If* $l_{11}\leqslant\frac{m_1}{p_1}, l_{22}\leqslant\frac{m_2}{p_2}$ *and* $l_{12}l_{21}>(\frac{m_1}{p_1}-l_{11})(\frac{m_2}{p_2}-l_{22})$, *then the solution* (u,v) *blows up in finite time.*

Proof. By Lemma 5.1 we see that there exist $k, l > 1$ satisfying

$$s = \frac{\frac{m_1}{p_1} - l_{11}}{1 - \frac{m_1}{p_1}} k - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l - \frac{l_{12}}{p_2 - m_2} + \frac{1 - l_{11}}{p_1 - m_1} \leqslant 0,$$
$$r = -\frac{l_{21}}{1 - \frac{m_1}{p_1}} k + \frac{\frac{m_2}{p_2} - l_{22}}{1 - \frac{m_2}{p_2}} l - \frac{l_{21}}{p_1 - m_1} + \frac{1 - l_{22}}{p_2 - m_2} \leqslant 0. \tag{5.2}$$

Let k, l satisfy (5.2), define $y(x,t) = a(1 + x^{\frac{1}{\tau_1}}) + (b - ct)^{-k}$, $z(x,t) = a(1 + x^{\frac{1}{\tau_2}}) + (b - ct)^{-l}$, $\theta = (p_1 + \frac{1}{k})/(p_1 - m_1)$, $\sigma = (p_2 + \frac{1}{l})/(p_2 - m_2)$ and take

$$\underline{u}(x,t) = y^{\theta}, \qquad \underline{v}(x,t) = z^{\sigma},$$

where

$$b = \max\{(\delta^{-1}2^{\theta})^{1/(k\theta)}, (\delta^{-1}2^{\sigma})^{1/(l\sigma)}\},$$

$$a = \min\{2^{-1}b^{-k}, 2^{-1}b^{-l}, \underline{\lambda}\tau_1\theta^{-1}b^{s}2^{1+l_{11}\theta-\theta}, \underline{\lambda}\tau_2\sigma^{-1}b^{r}2^{1+l_{22}\sigma-\sigma}\},$$

$$c = \min\left\{\frac{a^{p_1}\theta^{p_1-1}}{km_1\tau_1^{p_1}}, \frac{a^{p_2}\sigma^{p_2-1}}{lm_2\tau_2^{p_2}}\right\}.$$

By direct computations we have for $x \in (0,1), 0 < t < b/c$

$$(\underline{u}^{m_1})_t = m_1\theta cky^{m_1\theta-1}(b - ct)^{-k-1}, \qquad \underline{u}_x = \frac{a\theta}{\tau_1}y^{\theta-1}x^{\frac{1}{\tau_1}-1},$$

$$\begin{aligned}((\underline{u}_x)^{p_1})_x &= \left(\frac{a\theta}{\tau_1}\right)^{p_1}(y^{p_1(\theta-1)}x)_x \\ &= \left(\frac{a\theta}{\tau_1}\right)^{p_1}\left(y^{p_1(\theta-1)} + \frac{ap_1(\theta-1)}{\tau_1}y^{p_1(\theta-1)-1}x^{\frac{1}{\tau_1}}\right).\end{aligned}$$

Since $\theta = (p_1 + \frac{1}{k})/(p_1 - m_1) > \frac{p_1}{p_1 - m_1} > 1$, by $y \geqslant (b - ct)^{-k}$ we have

$$\begin{aligned}((\underline{u}_x)^{p_1})_x &\geqslant \left(\frac{a\theta}{\tau_1}\right)^{p_1} y^{p_1(\theta-1)} = \left(\frac{a\theta}{\tau_1}\right)^{p_1} y^{m_1\theta-1}y^{(p_1-m_1)\theta-p_1+1} \\ &= \left(\frac{a\theta}{\tau_1}\right)^{p_1} y^{m_1\theta-1}y^{\frac{k+1}{k}} \geqslant \left(\frac{a\theta}{\tau_1}\right)^{p_1} y^{m_1\theta-1}(b - ct)^{-(k+1)} \\ &\geqslant m_1ck\theta y^{m_1\theta-1}(b - ct)^{-(k+1)} = (\underline{u}^{m_1})_t, \qquad x \in (0,1), \quad 0 < t < b/c,\end{aligned}$$

i.e.,

$$((\underline{u}_x)^{p_1})_x \geqslant (\underline{u}^{m_1})_t, \qquad x \in (0,1), \quad 0 < t < b/c. \tag{5.3}$$

Similarly, we have

$$((\underline{v}_x)^{p_2})_x \geqslant (\underline{v}^{m_2})_t, \qquad x \in (0,1), \quad 0 < t < b/c. \tag{5.4}$$

Obviously,

$$\underline{u}_x|_{x=0} = \underline{u}_x|_{x=0} = 0, \qquad 0 < t < b/c. \tag{5.5}$$

By the choices of a, b and c we have

$$(b - ct)^{-k} \leqslant y \leqslant 2(b - ct)^{-k}, \qquad (b - ct)^{-l} \leqslant z \leqslant 2(b - ct)^{-l}. \tag{5.6}$$

By the expressions of θ, σ, the assumptions of this lemma and (5.2) we have

$$\theta(1 - l_{11}) > 1, \qquad -k[(1 - l_{11})\theta - 1] + l_{12} l \sigma = -s \geqslant 0. \tag{5.7}$$

Using (5.6) and (5.7) we get

$$\begin{aligned} \underline{u}_x &= \frac{a\theta}{\tau_1} y^{\theta - 1} = \underline{u}^{l_{11}} \underline{v}^{l_{12}} \frac{a\theta}{\tau_1} y^{\theta(1-l_{11})-1} z^{-l_{12}\sigma} \\ &\leqslant \underline{u}^{l_{11}} \underline{v}^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1-l_{11})-1} (b - ct)^{-k[\theta(1-l_{11})-1] + l_{12} l \sigma} \\ &= \underline{u}^{l_{11}} \underline{v}^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1-l_{11})-1} (b - ct)^{-s} \\ &\leqslant \underline{u}^{l_{11}} \underline{v}^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1-l_{11})-1} b^{-s} \leqslant \underline{\lambda} \underline{u}^{l_{11}} \underline{v}^{l_{12}}, \qquad x = 1, \qquad 0 < t < b/c, \end{aligned}$$

i.e.,

$$\underline{u}_x \leqslant \underline{\lambda} u^{l_{11}} \underline{v}^{l_{12}}, \qquad x = 1, \quad 0 < t < b/c. \tag{5.8}$$

Similarly, we have

$$\underline{v}_x \leqslant \underline{\lambda} u^{l_{21}} \underline{v}^{l_{22}}, \qquad x = 1, \quad 0 < t < b/c. \tag{5.9}$$

By the choices of a and b we have that for $x \in [0,1]$,

$$\underline{u}(x,0) = (a(1+x^{\frac{1}{\tau_1}}) + b^{-k})^{\theta} \leqslant (2b^{-k})^{\theta} \leqslant \delta \leqslant u_0(x),$$

$$\underline{v}(x,0) = (a(1+x^{\frac{1}{\tau_1}}) + b^{-l})^{\sigma} \leqslant (2b^{-l})^{\sigma} \leqslant \delta \leqslant v_0(x). \tag{5.10}$$

From (5.3)–(5.5), (5.8)–(5.10) we see that $(\underline{u}, \underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}$. Therefore, $(u, v) \geqslant (\underline{u}, \underline{v})$. Obviously, $(\underline{u}, \underline{v})$ blows up in finite time. And hence (u, v) blows up in finite time. ∎

Lemma 5.4. *Assume $m_1 < p_1$, $m_2 = p_2$. If $l_{11} \leqslant \frac{m_1}{p_1}$, $l_{22} \leqslant 1$ and $l_{12} l_{21} > (\frac{m_1}{p_1} - l_{11})(1 - l_{22})$, then the solution (u, v) of* (1.1) *blows up in finite time.*

Proof. It follows from Lemma 5.1 that there exist $k, l > 1$ such that

$$s = \frac{\frac{m_1}{p_1} - l_{11}}{1 - \frac{m_1}{p_1}} k - l_{12} l + \frac{1 - l_{11}}{p_1 - m_1} \leqslant 0,$$

$$r = -\frac{l_{21}}{1 - \frac{m_1}{p_1}} k + (1 - l_{22}) l - \frac{l_{21}}{p_1 - m_1} + 1 \leqslant 0. \tag{5.11}$$

Set $y(x,t) = a(1 + x^{\frac{1}{\tau_1}}) + (b - ct)^{-k}$, $z(x,t) = (b^{-1} + d(x^2 + x - 2))^{-1} - ct$, $\theta = (p_1 + \frac{1}{k})/(p_1 - m_1)$ and let

$$\underline{u}(x,t) = y^{\theta}(x,t), \qquad \underline{v}(x,t) = z^{-l}(x,t),$$

where k, l satisfy (5.11) and

$$b = \max\{\delta^{-1/l}, (\delta^{-1} 2^{\theta})^{1/(k\theta)}\}, \qquad a = \min\left\{\frac{1}{2} b^{-k}, \frac{\underline{\lambda}\tau}{\theta} b^{s} 2^{1-(1-l_{11})\theta}\right\}$$

$$d = \min\left\{\frac{1}{4b}, \frac{\underline{\lambda} b^{r-2}}{3l}\right\}, \qquad c = \min\left\{\frac{(dl)^{p_2} b^{p_2+2} d p_2 (l-1)}{2^{p_2} l m_2}, \frac{a^{p_1} \theta^{p_1 - 1}}{k m_1 \tau_1^{p_1}}\right\}.$$

By the expression of d we have for $0 \leqslant x \leqslant 1$, $0 < t < b/c$

$$\frac{1}{2b} \leqslant \frac{1}{b} + d(x^2 + x - 2) \leqslant \frac{1}{b}, \qquad 0 \leqslant z \leqslant 2b. \tag{5.12}$$

Using (5.12), by direct computations we have

$$(\underline{v}^{m_2})_t = c l m_2 z^{-m_2 l - 1}, \qquad \underline{v}_x = d l z^{-l-1} (b^{-1} + d(x^2 + x - 2))^{-2} (2x + 1),$$

$$
\begin{aligned}
(\underline{v}_x^{p_2})_x &= (dl)^{p_2}(z^{-p_2(l+1)}(b^{-1}+d(x^2+x-2))^{-2p_2}(2x+1)^{p_2})_x \\
&= (dl)^{p_2}\{z^{-p_2(l+1)}(b^{-1}+d(x^2+x-2))^{-2p_2}2p_2(2x+1)^{p_2-1}+z^{-p_2(l+1)-1} \\
&\quad \times (b^{-1}+d(x^2+x-2))^{-2p_2-2}(2x+1)^{p_2+1}dp_2 \\
&\quad \times [l+1-2z(b^{-1}+d(x^2+x-2))]\} \\
&\geqslant (dl)^{p_2}z^{-p_2l-1}(2b)^{-p_2}b^{2p_2+2}dp_2(l-1) \\
&= (dl)^{p_2}z^{-m_2l-1}(2b)^{-p_2}b^{2p_2+2}dp_2(l-1) \\
&\geqslant clm_2z^{-m_2l-1} = (\underline{v}^{m_2})_t, \qquad x\in(0,1), \quad 0<t<b/c,
\end{aligned}
$$

i.e.,

$$((\underline{v}_x)^{p_2})_x \geqslant (\underline{u}^{m_2})_t, \qquad x\in(0,1), \quad 0<t<b/c. \tag{5.13}$$

As in Lemma 5.3, we have

$$((\underline{u}_x)^{p_1})_x \geqslant (\underline{u}^{m_1})_t, \quad x\in(0,1), \quad 0<t<b/c. \tag{5.14}$$

Obviously,

$$\underline{u}_x|_{x=0} \geqslant 0, \qquad \underline{v}_x|_{x=0} \geqslant 0, \quad 0<t<b/c. \tag{5.15}$$

Using (5.11) we get for $x=1$, $0<t<b/c$

$$
\begin{aligned}
\underline{u}_x &= \frac{a\theta}{\tau_1}y^{\theta-1} \leqslant \underline{u}^{l_{11}}\underline{v}^{l_{12}}\frac{a\theta}{\tau_1}y^{\theta(1-l_{11})-1}z^{l_{12}l} \\
&\leqslant \frac{a\theta}{\tau_1}2^{\theta(1-l_{11})-1}(b-ct)^{-k[\theta(1-l_{11})-1]+l_{12}l}\underline{u}^{l_{11}}\underline{v}^{l_{12}} \\
&= \frac{a\theta}{\tau_1}2^{\theta(1-l_{11})-1}(b-ct)^{-s}\underline{u}^{l_{11}}\underline{v}^{l_{12}} \\
&\leqslant \frac{a\theta}{\tau_1}2^{\theta(1-l_{11})-1}b^{-s}\underline{u}^{l_{11}}\underline{v}^{l_{12}} \\
&\leqslant \underline{\lambda}\underline{u}^{l_{11}}\underline{v}^{l_{12}}, \qquad x=1, \quad 0<t<b/c,
\end{aligned}
$$

i.e.,

$$\underline{u}_x \leqslant \underline{\lambda}u^{l_{11}}\underline{v}^{l_{12}}, \qquad x=1, \quad 0<t<b/c. \tag{5.16}$$

And

$$\begin{aligned}\underline{v}_x &= 3ldb^2(b-ct)^{-l-1}\\ &= \underline{u}^{l_{21}}\underline{v}^{l_{22}}3ldb^2y^{-\theta l_{21}}(b-ct)^{l_{22}l-l-1}\\ &\leqslant \underline{u}^{l_{21}}\underline{v}^{l_{22}}3ldb^2(b-ct)^{l_{22}l-l-1+\theta k l_{21}}\\ &= \underline{u}^{l_{21}}\underline{v}^{l_{22}}3ldb^2(b-ct)^{-r}\\ &\leqslant \underline{u}^{l_{21}}\underline{v}^{l_{22}}3ldb^2b^{-r}\leqslant \underline{\lambda}\underline{u}^{l_{21}}\underline{v}^{l_{22}},\end{aligned}$$

i.e.,

$$\underline{v}_x \leqslant \underline{\lambda}\underline{u}^{l_{21}}\underline{v}^{l_{22}}, \qquad x=1, \quad 0<t<b/c. \tag{5.17}$$

For $x \in [0,1]$, we have

$$\begin{aligned}\underline{u}(x,0) &= (a(1+x^{\frac{1}{\tau_1}})+b^{-k})^{\theta} \leqslant (2b^{-k})^{\theta} \leqslant \delta \leqslant u_0(x),\\ \underline{v}(x,0) &= (b^{-1}+d(x^2+x-2))^{l} \leqslant b^{-l} \leqslant \delta \leqslant v_0(x).\end{aligned} \tag{5.18}$$

From (5.13)–(5.18) we see that $(\underline{u},\underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}$. Therefore, $(u,v) \geq (\underline{u},\underline{v})$. Obviously, $(\underline{u},\underline{v})$ blows up in finite time. And hence (u,v) blows up in finite time. ∎

LEMMA 5.5. *Assume* $m_1 = p_1$, $m_2 < p_2$. *If* $l_{11} \leqslant 1, l_{22} \leqslant \dfrac{m_2}{p_2}$ *and* $l_{12}l_{21} > (1-l_{11})(\frac{m_2}{p_2}-l_{22})$, *then the solution* (u,v) *of* (1.1) *blows up in finite time.*

Proof. The proof is similar to that of Lemma 5.4. ∎

LEMMA 5.6. *Assume* $m_1 = p_1$, $m_2 = p_2$. *If* $l_{11} \leqslant 1, l_{22} \leqslant 1$ *and* $l_{12}l_{21} > (1-l_{11})(1-l_{22})$, *then the solution* (u,v) *of* (1.1) *blows up in finite time.*

Proof. The proof is similar to that of Lemma 5.4. ∎

LEMMA 5.7. *Assume that* $m_i > p_i$, $i = 1,2$. *If* $l_{11} \leqslant \frac{m_1+1}{p_1+1}, l_{22} \leqslant \frac{m_2+1}{p_2+1}$ *and* $l_{12}l_{21} > (\frac{m_1+1}{p_1+1}-l_{11})(\frac{m_2+1}{p_2+1}-l_{22})$, *then the solution* (u,v) *blows up in finite time.*

Proof. By Lemma 5.1 we see that there exist $k, l > 1$ satisfying

$$s = \frac{\frac{m_1+1}{p_1+1} - l_{11}}{\frac{m_1+1}{p_1+1} - 1} k - \frac{l_{12}}{\frac{m_2+1}{p_2+1} - 1} l + \frac{l_{12}}{m_2 - p_2} - \frac{1 - l_{11}}{m_1 - p_1} \leqslant 0,$$

$$r = -\frac{l_{21}}{\frac{m_1+1}{p_1+1} - 1} k + \frac{\frac{m_2+1}{p_2+1} - l_{22}}{\frac{m_2+1}{p_2+1} - 1} l - \frac{l_{21}}{p_1 - m_1} + \frac{1 - l_{22}}{p_2 - m_2} \leqslant 0. \tag{5.19}$$

Define $y(x,t) = a(1-x) + (b-ct)^k$, $z(x,t) = a(1-x) + (b-ct)^l$, $\theta = (p_1 + \frac{k-1}{k})/(m_1 - p_1)$, $\sigma = (p_2 + \frac{l-1}{l})/(m_2 - p_2)$, $\underline{u}(x,t) = y^{-\theta}$, $\underline{v}(x,t) = z^{-\sigma}$, where k, l satisfy (5.19) and

$$b = \max\{(\delta 2^{\theta})^{-1/(k\theta)}, (\delta 2^{\sigma})^{-1/(l\sigma)}\},$$

$$a = \min\left\{b^k, b^l, \frac{\underline{\lambda} b^s}{\theta}, \frac{\underline{\lambda} b^r}{\sigma}\right\},$$

$$c = \min\left\{\frac{(a\theta)^{p_1} a p_1(\theta+1)}{m_1 \theta k}, \frac{(a\sigma)^{p_2} a p_2(\sigma+1)}{m_2 \sigma l}\right\}.$$

By direct computations we have

$$(\underline{u}^{m_1})_t = m_1 \theta c k y^{-m_1\theta - 1}(b-ct)^{k-1}, \quad \underline{u}_x = a\theta y^{-\theta-1},$$

$$\begin{aligned}
((\underline{u}_x)^{p_1})_x &= (a\theta)^{p_1}(y^{p_1(-\theta-1)})_x \\
&= (a\theta)^{p_1} a p_1(\theta+1) y^{-m_1\theta-1} y^{(m_1-p_1)\theta - p_1} \\
&\geqslant (a\theta)^{p_1} a p_1(\theta+1) y^{-m_1\theta-1}(b-ct)^{k[(m_1-p_1)\theta - p_1]} \\
&= (a\theta)^{p_1} a p_1(\theta+1) y^{-m_1\theta-1}(b-ct)^{k-1} \\
&\geqslant m_1\theta k c y^{-m_1\theta-1}(b-ct)^{k-1} = (\underline{u}^{m_1})_t, \qquad x \in (0,1), \quad 0 < t < b/c
\end{aligned}$$

(here we have used that $\theta > \frac{p_1}{m_1 - p_1}$)
i.e.,

$$((\underline{u}_x)^{p_1})_x \geqslant (\underline{u}^{m_1})_t, \qquad x \in (0,1), \quad 0 < t < b/c. \tag{5.20}$$

Similarly, we have

$$((\underline{v}_x)^{p_2})_x \geqslant (\underline{v}^{m_2})_t, \qquad x \in (0,1), \quad 0 < t < b/c. \tag{5.21}$$

Obviously,

$$\underline{u}_x|_{x=0} = \underline{u}_x|_{x=0} = 0, \qquad 0<t<b/c. \tag{5.22}$$

By the expressions of θ, σ, the assumptions of this lemma and (5.19) we have

$$l_{12} l\sigma + k(l_{11}\theta - 1 - \theta) = -s \geqslant 0. \tag{5.23}$$

Using (5.23) we get

$$\begin{aligned}\underline{u}_x &= a\theta(b-ct)^{-k(\theta+1)} = \underline{u}^{l_{11}}\underline{v}^{l_{12}} a\theta(b-ct)^{-k(\theta+1)+kl_{11}\theta+l_{12}\sigma l}\\ &= \underline{u}^{l_{11}}\underline{v}^{l_{12}} a\theta(b-ct)^{-s} \leqslant \underline{u}^{l_{11}}\underline{v}^{l_{12}} a\theta b^{-s} \leqslant \underline{\lambda}\underline{u}^{l_{11}}\underline{v}^{l_{12}}, \qquad x=1, \quad 0<t<b/c,\end{aligned}$$

i.e.,

$$\underline{u}_x \leqslant \underline{\lambda} \underline{u}^{l_{11}}\underline{v}^{l_{12}}, \qquad x=1, \quad 0<t<b/c. \tag{5.24}$$

Similarly, we have

$$\underline{v}_x \leqslant \underline{\lambda} \underline{u}^{l_{21}}\underline{v}^{l_{22}}, \qquad x=1, \quad 0<t<b/c. \tag{5.25}$$

By the choices of a and b we have that for $x \in [0,1]$,

$$\begin{aligned}\underline{u}(x,0) &= (a(1-x)+b^k)^{-\theta} \leqslant (2b^k)^{-\theta} \leqslant \delta \leqslant u_0(x),\\ \underline{v}(x,0) &= (a(1-x)+b^{-l})^{-\sigma} \leqslant (2b^l)^{-\sigma} \leqslant \delta \leqslant v_0(x).\end{aligned} \tag{5.26}$$

From (5.20)–(5.22), (5.24)–(5.26) we see that $(\underline{u}, \underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}$. Therefore $(u,v) \geq (\underline{u}, \underline{v})$. Obviously, $(\underline{u}, \underline{v})$ blows up in finite time. And hence (u,v) blows up in finite time. ∎

LEMMA 5.8. *Assume that $m_1 = p_1$ and $m_2 > p_2$. If $l_{11} \leqslant 1$, $l_{22} \leqslant \frac{m_2+1}{p_2+1}$, and $l_{12}l_{21} > (1-l_{11})(\frac{m_2+1}{p_2+1} - l_{22})$, then the solution (u,v) blows up in finite time.*

Proof. By Lemma 5.1, we see that there exist $k, l > 1$ satisfying

$$\begin{aligned} s &= (1-l_{11})k - \frac{l_{12}}{\frac{m_2+1}{p_2+1} - 1} l + \frac{l_{12}}{m_2 - p_2} + 1 \leqslant 0,\\ r &= -l_{21}k + \frac{\frac{m_2+1}{p_2+1} - l_{22}}{\frac{m_2+1}{p_2+1} - 1} l + \frac{1-l_{22}}{p_2 - m_2} \leqslant 0.\end{aligned}$$

Let k, l be as above, define $y(x,t) = (b^{-1} + d(x^2 + x - 2))^{-1} - ct$, $z(x,t) = a(1-x) + (b-ct)^l$, $\sigma = (p_2 + \frac{1-1}{l})/(m_2 - p_2)$, $\underline{u}(x,t) = y^{-k}, \underline{v}(x,t) = z^{-\sigma}$.

As in Lemmas 5.4 and 5.7, it is easy to verify that there exist positive constants a, b, c, d such that $(\underline{u}, \underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}$. Therefore, $(u, v) \geqslant (\underline{u}, \underline{v})$. Obviously, $(\underline{u}, \underline{v})$ blows up in finite time. And hence (u, v) blows up in finite time. ∎

Lemma 5.9. *Assume that $m_1 > p_1$ and $m_2 = p_2$. If $l_{11} \leqslant \frac{m_1+1}{p_1+1}, l_{22} \leqslant 1$ and $l_{12}l_{21} > (\frac{m_1+1}{p_1+1} - l_{11})(1 - l_{22})$, then the solution (u, v) blows up in finite time.*

Proof. The proof is similar to that of Lemma 5.8. ∎

Lemma 5.10. *Assume that $m_1 > p_1$ and $m_2 < p_2$. If $l_{11} \leqslant \frac{m_1+1}{p_1+1}$, $l_{22} \leqslant \frac{m_2}{p_2}$ and $l_{12}l_{21} > (\frac{m_1+1}{p_1+1} - l_{11})(\frac{m_2}{p_2} - l_{22})$, then the solution (u, v) blows up in finite time.*

Proof. By Lemma 5.1 we see that there exist $k, l > 1$ satisfying

$$s = \frac{\frac{m_1+1}{p_1+1} - l_{11}}{\frac{m_1+1}{p_1+1} - 1} k - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l + \frac{l_{12}}{m_2 - p_2} - \frac{1 - l_{11}}{m_1 - p_1} \leqslant 0,$$

$$r = -\frac{l_{21}}{\frac{m_1+1}{p_1+1} - 1} k + \frac{\frac{m_2}{p_2} - l_{22}}{1 - \frac{m_2}{p_2}} l - \frac{l_{21}}{p_1 - m_1} + \frac{1 - l_{22}}{p_2 - m_2} \leqslant 0.$$

Let k, l be as above, define $y(x, t) = a(1 - x) + (b - ct)^k$, $z(x, t) = a(1 + x^{\frac{1}{\tau_2}}) + (b - ct)^{-l}$, $\sigma = (p_2 + \frac{1}{l})/(p_2 - m_2)$, $\theta = (p_1 + 1 - \frac{1}{k})/(m_1 - p_1)$, $\underline{u}(x, t) = y^{-\theta}$, $\underline{v}(x, t) = z^{\sigma}$.

As in Lemmas 5.3 and 5.7, it is easy to verify that there exist positive constants a, b, c such that $(\underline{u}, \underline{v})$ is a lower solution of (1.1) with $\underline{\lambda}$. Therefore, $(u, v) \geqslant (\underline{u}, \underline{v})$. Obviously, $(\underline{u}, \underline{v})$ blows up in finite time. And hence (u, v) blows up in finite time.

Lemma 5.11. *Assume that $m_1 < p_1$ and $m_2 > p_2$. If $l_{11} \leqslant \frac{m_1}{p_1}$, $l_{22} \leqslant \frac{m_2+1}{p_2+1}$ and $l_{12}l_{21} > (\frac{m_1}{p_1} - l_{11})(\frac{m_2+1}{p_2+1} - l_{22})$, then the solution (u, v) blows up in finite time.*

Proof. The proof is similar to that of Lemma 5.10. ∎

By Lemmas 5.2–5.11 we get that the necessity holds.

Remark 5.1. The above blow up results turn out to be valid for the following N-dimensional form of problem (1.1):

$$(u^{m_1})_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_1 - 1} u_{x_i})_{x_i}, \qquad x \in \Omega, \quad t > 0,$$

$$(v^{m_2})_t = \sum_{i=1}^{N} (|v_{x_i}|^{p_2 - 1} v_{x_i})_{x_i}, \qquad x \in \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial \eta} = u^{l_{11}} v^{l_{12}}, \qquad x \in \partial\Omega, \quad t > 0$$
$$\frac{\partial v}{\partial \eta} = u^{l_{21}} v^{l_{22}}, \qquad x \in \partial\Omega, \quad t > 0$$
$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \bar{\Omega},$$

where $\Omega \subset R^N$ is a bounded open set with smooth boundary $\partial\Omega$, η stands the unit outward normal of $\partial\Omega$ to Ω.

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