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# Null controllability of the $N$-dimensional Stokes system with $N-1$ scalar controls

Jean-Michel Coron [a,b,1], Sergio Guerrero [b,*,2]

[a] *Institut universitaire de France, France*
[b] *Université Pierre et Marie Curie – Paris 6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris F-75005, France*

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ABSTRACT

In this paper we deal with the $N$-dimensional Stokes system in a bounded domain with Dirichlet boundary conditions. The main result establishes the null controllability with internal controls having one vanishing component. This result improves the one in [E. Fernández-Cara, S. Guerrero, O.Yu. Imanuvilov, J.-P. Puel, Some controllability results for the $N$-dimensional Navier–Stokes and Boussinesq systems with $N-1$ scalar controls, SIAM J. Control Optim. 45 (1) (2006) 146–173], since no condition is imposed on the control domain.



## 1. Introduction

Let $\Omega$ be a nonempty bounded connected open subset of $\mathbf{R}^N$ ($N=2$ or $3$) of class $C^\infty$. Let $T>0$ and let $\omega\subset\Omega$ be a (small) nonempty open subset which is the *control domain*. We will use the notation $Q=\Omega\times(0,T)$ and $\Sigma=\partial\Omega\times(0,T)$.

Let us recall the definition of some usual spaces in the context of Stokes equations (see, for instance, [13]):

$$V=\left\{y\in H_0^1(\Omega)^N\colon\ \nabla\cdot y=0 \text{ in } \Omega\right\}$$

* Corresponding author.
*E-mail addresses:* coron@ann.jussieu.fr (J.-M. Coron), guerrero@ann.jussieu.fr (S. Guerrero).

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and

$$H = \{ y \in L^2(\Omega)^N \colon \ \nabla \cdot y = 0 \text{ in } \Omega, \ \ y \cdot n = 0 \text{ on } \partial\Omega \}.$$

In this paper, we deal with the following Stokes control system:

$$\begin{cases} y_t - \Delta y + \nabla p = v 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y|_{t=0} = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y_0 \in H$ is the initial condition and $v = (v_j)_{j=1}^N$ is the control function.

It is well known that the null controllability for this system holds, that is to say, for every $y^0 \in H$ and every $T > 0$, there exists $v \in L^2(Q)^N$ such that the solution $y \in L^2(0, T; V) \cap C^0([0, T]; H)$ of (1) satisfies

$$y|_{t=T} = 0 \quad \text{in } \Omega.$$

For a proof of this result, see, for instance, [9] or [4].

The main objective of this paper is to prove that system (1) is null controllable by means of $N - 1$ scalar controls, that is to say, when $v_i = 0$ for some given $i \in \{1, \ldots, N\}$. This result has been proved in [5] when $\omega$ "touches" the boundary $\partial\Omega$, that is to say, when $\overline{\omega} \cap \partial\Omega \neq \emptyset$. The major novelty of this paper is to remove this geometric property and prove the null controllability result for every open set $\omega \subset \Omega$.

Our main result is given in the following theorem.

**Theorem 1.** *There exists a constant $C > 0$ depending only on $\Omega$ and $\omega$ such that, for every $y_0 \in H$ and every $i \in \{1, \ldots, N\}$, there exists a control $v \in L^2(Q)^N$ with $v_i \equiv 0$ in $Q$ satisfying*

$$\|v\|_{L^2(Q)^N} \leqslant e^{C(1+1/T^9)} \|y_0\|_{L^2(\Omega)^N}$$

*and such that the solution $y$ of (1) satisfies*

$$y|_{t=T} = 0 \quad \text{in } \Omega.$$

**Remark 1.** As proved in [11], there are nonempty Lipschitz bounded connected open subset $\Omega$ of $\mathbf{R}^3$ such that, even with $\omega := \Omega$, the null controllability of the control system (1) does not hold with two vanishing components for the control (i.e. if one imposes, for example, $v_1 = v_2 = 0$). See also [2] for a torus.

In order to prove Theorem 1, we introduce the *adjoint system*:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0 & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi_T & \text{in } \Omega, \end{cases}$$

with $\varphi_T \in H$. Then, it is well known (see, e.g. [1, Theorem 2.44, pp. 56–57]) that the result stated in Theorem 1 is equivalent to the following *observability inequality*:

$$\int_\Omega |\varphi|_{t=0}|^2 \, dx \leqslant e^{C(1+1/T^9)} \sum_{j=1,\, j \neq i}^{N} \iint_{\omega \times (0,T)} |\varphi_j|^2 \, dt \, dx, \tag{2}$$

for some $C$ depending only on $\Omega$ and $\omega$.

The proof of this inequality is based on *Carleman inequalities*. The general idea is to get profit of the fact that $\Delta\pi = 0$ in order to have equations in $\varphi_j$ $(j \neq i)$ which do not depend neither on $\varphi_i$ nor on $\pi$ (see Eq. (18) below). The only problem is that these equations are heat equations which are satisfied by some derivatives of $\varphi_j$ $(j \neq i)$ and so no boundary conditions are prescribed. Therefore, for the moment, we have only

$$\sum_{j=1,\, j\neq i}^{N} \iint\limits_{Q} \rho_1(t,x)|\varphi_j|^2\, dt\, dx \leqslant C \sum_{j=1,\, j\neq i}^{N} \left( \iint\limits_{\omega\times(0,T)} \rho_1(t,x)|\varphi_j|^2\, dt\, dx + \iint\limits_{\Sigma} \rho_2(t,x) \left| \frac{\partial \nabla \Delta \varphi_j}{\partial n} \right|^2 dt\, d\sigma \right),$$

where the boundary terms on the right-hand side have to be estimated. For that, the idea is to use a priori estimates relying on the regularizing effect of the Stokes system (see Lemma 1 below). This will provide an estimate of the boundary terms but with an additional integral depending on $\varphi_i$:

$$\sum_{j=1,\, j\neq i}^{N} \iint\limits_{Q} \rho_1(t,x)|\varphi_j|^2\, dt\, dx \leqslant C \left( \sum_{j=1,\, j\neq i}^{N} \iint\limits_{\omega\times(0,T)} \rho_1(t,x)|\varphi_j|^2\, dt\, dx + \iint\limits_{Q} \rho_3(t,x)|\varphi_i|^2\, dt\, d\sigma \right).$$

Finally, using the divergence-free condition on $\varphi$ and the properties of the weight functions, we can absorb the term depending on $\varphi_i$ with the help of the left-hand side.

Let us remark that, even in the case of $N$ scalar controls, our proof of (2) is simpler than the ones given in [9] and [4]: in these papers, a local estimate of the pressure had to be performed. Indeed, the main advantage of our estimate is that we do not have to deal with the pressure all along the proof. These ideas were already developed in [8].

This paper is organized as follows. In Section 2, we present some technical results, most of them known, which will be used in the proof of Theorem 1. Finally, in Section 3 we prove the observability inequality (2).

## 2. Some previous results

For the proof of the observability inequality needed to establish Theorem 1, we follow a classical approach, consisting of obtaining a suitable weighted-like estimate (so-called *Carleman estimate*) for the associated adjoint system. For a systematic use of this kind of estimates see, for instance, [7] or [9].

In order to establish these Carleman inequalities, we need to define some weight functions:

$$\alpha(t,x) = \frac{\exp\{20\lambda\|\eta^0\|_\infty\} - \exp\{\lambda(18\|\eta^0\|_\infty + \eta^0(x))\}}{t^9(T-t)^9},$$

$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(t,x), \qquad \xi(t,x) = \frac{e^{\lambda(18\|\eta^0\|_\infty + \eta^0(x))}}{t^9(T-t)^9}, \qquad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(t,x). \tag{3}$$

Here, $\eta^0 \in C^2(\overline{\Omega})$ satisfies

$$\left|\nabla\eta^0\right| > 0 \quad \text{in } \overline{\Omega}\setminus\omega_0, \qquad \eta^0 > 0 \quad \text{in } \Omega \quad \text{and} \quad \eta^0 \equiv 0 \quad \text{on } \partial\Omega, \tag{4}$$

where $\omega_0$ is a nonempty open subset of $\mathbf{R}^N$ such that $\overline{\omega}_0 \subset \omega$. The existence of such a function $\eta^0$ is given in [7, Lemma 1.1, Chapter 1] (see also [1, Lemma 2.68, Chapter 1]). Weights of the kind (3) were first considered in [7].

Accordingly, we define $I_0(s,\lambda;\cdot)$ as follows:

$$I_0(s,\lambda,g) := s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla g|^2\,dt\,dx + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|g|^2\,dt\,dx,$$

for $g : \Omega \to \mathbf{R}$ and

$$I_0(s,\lambda,g) := \sum_{i=1}^{N} I_0(s,\lambda,g_i),$$

for $g = (g_1,\dots,g_N) : \Omega \to \mathbf{R}^N$. From this expression, we also introduce

$$I(s,\lambda;g) := s^{-1} \iint_Q e^{-2s\alpha}\xi^{-1}\big(|g_t|^2 + |\Delta g|^2\big)\,dt\,dx + I_0(s,\lambda,g), \tag{5}$$

for $g = \Omega \to \mathbf{R}^N$.

Now, we state all the technical results we need. The first one is a regularity result for the solutions of Stokes system:

**Lemma 1.** *For every $T > 0$ and every $f \in L^2(0,T;H)$, there exists a unique*

$$u \in L^2\big(0,T;H^2(\Omega)^N\big) \cap H^1(0,T;H)$$

*such that, for some $p \in L^2(0,T;H^1(\Omega))$,*

$$\begin{cases} u_t - \Delta u + \nabla p = f & \text{in } Q, \\ \nabla\cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

*Furthermore, there exists a constant $C > 0$ depending only on $\Omega$ such that*

$$\|u\|_{L^2(0,T;H^2(\Omega)^N)} + \|u\|_{H^1(0,T;L^2(\Omega)^N)} \leqslant C\|f\|_{L^2(0,T;L^2(\Omega)^N)}. \tag{7}$$

In order to deal with more regular solutions, let us introduce some compatibility conditions. We will say that $f$ satisfies the compatibility condition of order $r$ if, for any nonnegative integer $k \leqslant r-1$, we have

$$\nabla p^k(x) = \sum_{i=0}^{k}\big(\partial_t^i\Delta^{k-i}f\big)(0,x), \quad x \in \partial\Omega,$$

where $p^0 \equiv 0$ and, for $k > 0$, $p^k$ is the solution of the Neumann boundary-value problem

$$\begin{cases} \Delta p^k = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n}p^k = \displaystyle\sum_{i=0}^{k-1}\big(\partial_t^i\Delta^{k-i}f\big)(0,x) & \text{on } \partial\Omega. \end{cases}$$

One has the following lemma (see, for instance, [12, Section IV], [10, Theorem 6, pp. 100–101], [14]):

**Lemma 2.** *Let $T > 0$ and let $r$ be a positive integer. There exists $C > 0$ depending only on $r$ and $\Omega$ such that, for every $f \in L^2(0,T;H^{2r}(\Omega)^N \cap H) \cap H^r(0,T;H)$ satisfying the compatibility condition of order $r$, the solution $u$ of (6) satisfies*

$$u \in X_r := L^2\big(0,T;H^{2r+2}(\Omega)^N\big) \cap H^{r+1}\big(0,T;L^2(\Omega)^N\big),$$
$$\|u\|_{X_r} \leqslant C\big(\|f\|_{L^2(0,T;H^{2r}(\Omega)^N)} + \|f\|_{H^r(0,T;L^2(\Omega)^N)}\big). \tag{8}$$

The second result is a nice property coming from the definition of the previous weights:

**Lemma 3.** *Let $r \in \mathbf{R}$. There exists $C > 0$ depending only on $r$, $\Omega$, $\omega_0$ and $\eta^0$ such that, for every $T > 0$ and every $u \in L^2(0,T;H^1(\Omega))$,*

$$s^2\lambda^2 \iint_Q e^{-2s\alpha}\xi^{r+2}|u|^2\,dt\,dx$$
$$\leqslant C\bigg(\iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2\,dt\,dx + s^2\lambda^2 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^{r+2}|u|^2\,dt\,dx\bigg), \tag{9}$$

*for every $\lambda \geqslant C$ and every $s \geqslant CT^{18}$.*

**Proof.** In this proof, we denote by $C$ various positive constants depending only on $r$, $\Omega$ $\omega_0$ and $\eta^0$. By a density argument, we can suppose that $u \in C^0([0,T];C^1(\overline{\Omega}))$. Then, we consider the following integral

$$s\lambda \iint_Q e^{-2s\alpha}\xi^{r+1}\big(\nabla\eta^0\cdot\nabla u\big)u\,dt\,dx,$$

where the weight functions $\xi$, $\alpha$ and $\eta^0$ were introduced in (3)–(4). We integrate by parts with respect to $x$; this gives

$$s\lambda \iint_Q e^{-2s\alpha}\xi^{r+1}\big(\nabla\eta^0\cdot\nabla u\big)u\,dt\,dx$$
$$= \frac{s\lambda}{2}\iint_\Sigma e^{-2s\alpha}\xi^{r+1}\frac{\partial\eta^0}{\partial n}|u|^2\,dt\,d\sigma - \frac{s\lambda}{2}\iint_Q e^{-2s\alpha}\nabla\cdot\big(\xi^{r+1}\nabla\eta^0\big)|u|^2\,dt\,dx$$
$$- s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^{r+2}\big|\nabla\eta^0\big|^2|u|^2\,dt\,dx.$$

We observe that from (4), one has $\frac{\partial\eta^0}{\partial n} \leqslant 0$, so that we deduce

$$s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^{r+2}\big|\nabla\eta^0\big|^2|u|^2\,dt\,dx$$
$$\leqslant s\lambda\iint_Q e^{-2s\alpha}\xi^{r+1}\big|\nabla\eta^0\big||\nabla u||u|\,dt\,dx + \frac{s\lambda}{2}\iint_Q e^{-2s\alpha}\big|\nabla\cdot\big(\xi^{r+1}\nabla\eta^0\big)\big||u|^2\,dt\,dx. \tag{10}$$

From (3), we obtain

$$\left|\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)\right| \leqslant C\lambda\xi^{r+1},$$

for $\lambda \geqslant C$. Using Cauchy–Schwarz's inequality for the first term in the right-hand side of (10), we obtain, for every $\epsilon > 0$,

$$\begin{aligned} s^2\lambda^2 &\iint\limits_Q e^{-2s\alpha}\xi^{r+2}\left|\nabla\eta^0\right|^2|u|^2\,dt\,dx \\ &\leqslant \frac{C}{\epsilon}\iint\limits_Q e^{-2s\alpha}\xi^r|\nabla u|^2\,dt\,dx + \epsilon s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^{r+2}|u|^2\,dt\,dx, \end{aligned}$$

for $s \geqslant CT^{18}/\epsilon$ and $\lambda \geqslant C$. Taking $\epsilon$ small enough, we get the desired estimate (9). □

The third technical result concerns the Laplace operator:

**Lemma 4.** *Let $\gamma(x) = \exp\{\lambda\eta^0(x)\}$ for $x \in \Omega$ and let $r \in \mathbf{R}$. Then, there exists $C > 0$ depending only on $r$, $\Omega$, $\omega_0$ and $\eta^0$ such that, for every $T > 0$ and every $u \in H_0^1(\Omega)$,*

$$\begin{aligned} &\tau^{r+3}\lambda^{r+5}\int\limits_\Omega e^{2\tau\gamma}\gamma^{r+3}|u|^2\,dx + \tau^{r+1}\lambda^{r+3}\int\limits_\Omega e^{2\tau\gamma}\gamma^{r+1}|\nabla u|^2\,dx \\ &\quad\leqslant C\left(\tau^r\lambda^{r+1}\int\limits_\Omega e^{2\tau\gamma}\gamma^r|\Delta u|^2\,dx + \tau^{r+3}\lambda^{r+5}\int\limits_{\omega_0} e^{2\tau\gamma}\gamma^{r+3}|u|^2\,dx\right), \end{aligned} \tag{11}$$

*for every $\lambda \geqslant C$ and every $\tau \geqslant C$.*

The proof of this lemma can be readily deduced from the corresponding result for parabolic equations included in [7, Remark 1.2, Chapter 1]. The original result was stated for $r = 0$; then, using this result for the function $\gamma^{r/2}u \in H_0^1(\Omega)$, we obtain (11).

The fourth and last technical result is an estimate which holds for energy solutions of heat equations with non-homogeneous Neumann boundary conditions:

**Lemma 5.** *There exists $C > 0$ depending only on $\Omega$, $\omega_0$ and $\eta^0$ such that, for every $T > 0$, every $u_0 \in L^2(\Omega)$, every $f_1 \in L^2(Q)$, every $f_2 \in L^2(Q)^N$ and every $f_3 \in L^2(\Sigma)$, the weak solution $u$ of*

$$\begin{cases} u_t - \Delta u = f_1 + \nabla\cdot f_2 & \text{in } Q, \\ \dfrac{\partial u}{\partial n} + f_2\cdot n = f_3 & \text{on } \Sigma, \\ u|_{t=0} = u_0 & \text{in } \Omega \end{cases} \tag{12}$$

*satisfies*

$$\begin{aligned} I_0(s,\lambda;u) \leqslant C\Bigg(&s^3\lambda^4\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|u|^2\,dt\,dx + \iint\limits_Q e^{-2s\alpha}|f_1|^2\,dt\,dx \\ &+ s^2\lambda^2\iint\limits_Q e^{-2s\alpha}\xi^2|f_2|^2\,dt\,dx + s\lambda\iint\limits_\Sigma e^{-2s\alpha^*}\xi^*|f_3|^2\,dt\,d\sigma\Bigg), \end{aligned} \tag{13}$$

*for every $\lambda \geqslant C$ and every $s \geqslant C(T^9 + T^{18})$.*

Let us recall the definition of a weak solution: we say that $u$ is a *weak solution* to (12) if it satisfies

$$\begin{cases} u \in L^2\big(0,T;H^1(\Omega)\big) \cap C^0\big([0,T];L^2(\Omega)\big), \\ \dfrac{d}{dt}\int_\Omega uv\,dx + \displaystyle\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f_1(t,x)v\,dx - \int_\Omega f_2(t,x)\cdot\nabla v\,dx + \int_{\partial\Omega} f_3(t,x)v\,d\sigma \\ \quad \text{in } \mathcal{D}'(0,T),\ \forall v\in H^1(\Omega), \\ u(x,0) = u^0(x) \quad \text{in } \Omega. \end{cases} \tag{14}$$

It is well known that, for $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$, $f_3 \in L^2(\Sigma)$ and $u_0 \in L^2(\Omega)$, (12) possesses exactly one weak solution $u$.

Lemma 5 is essentially proved in [3]. In fact, the inequality proved there concerns the same weight functions as in (13) but with $t(T-t)$ instead of $t^9(T-t)^9$. Then, one can follow the steps of the proof in [3] (see Theorem 1 in that reference) and adapt the arguments just taking into account that

$$\partial_t\alpha := \alpha_t \leqslant CT\xi^{10/9} \quad \text{and} \quad \partial_{tt}\alpha := \alpha_{tt} \leqslant CT^2\xi^{11/9}, \tag{15}$$

with $C>0$ independent of $s$, $\lambda$ and $T$.

## 3. Proof of the observability inequality (2)

In this section we denote by $C$ various positive constants which depend only on $\Omega$ and $\omega$ (they depend also in general on the choice of $\eta^0$ and $\omega_0$ but one can consider that $\eta^0$ as well as $\omega_0$ depend on $\Omega$ and $\omega$). Without any lack of generality, we treat the case of dimension 2. The same proof can be performed in dimension 3. We introduce the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0 & \text{in } Q, \\ \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi_T & \text{in } \Omega, \end{cases} \tag{16}$$

and define $\varphi_1 : \Omega \to \mathbf{R}$ and $\varphi_2 : \Omega \to \mathbf{R}$ by $(\varphi_1,\varphi_2) = \varphi$.

We are going to establish estimate (2) for $i=2$. Of course, the same can be done for $i=1$. One has the following proposition.

**Proposition 1.** *There exists a positive constant $C$ depending only on $\Omega$ and $\omega$ such that*

$$\begin{aligned} &s^8\lambda^{10}\iint_Q e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8\iint_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx \\ &\quad\leqslant Cs^9\lambda^{10}\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^9|\varphi_1|^2\,dt\,dx, \end{aligned} \tag{17}$$

*for every $s \geqslant C(T^9+T^{18})$ and every $\lambda \geqslant C$.*

**Remark 2.** From the Carleman inequality (17), one can follow the same steps as in [6] in order to prove the observability inequality (2) for $N=2$ and $i=2$ and so once Proposition 1 is established, the proof of Theorem 1 is finished.

**Proof of Proposition 1.** Note that, by a simple density argument, we may assume, without loss of generality, that $\varphi$ is of class $C^6$ on $[0,T]\times\overline{\Omega}$. We also observe that, using the divergence-free condition, we have that

$$\Delta\pi = 0 \quad \text{in } \Omega\times(0,T).$$

Then, we apply the operator $\nabla\Delta = (\partial_1\Delta, \partial_2\Delta)$ to the equation satisfied by $\varphi_1$. Denoting $\psi_1 := \nabla\Delta\varphi_1 \in \mathbf{R}^2$, we have

$$\psi_{1,t} + \Delta\psi_1 = 0 \quad \text{in } Q. \tag{18}$$

We apply Lemma 5 to $\psi_1$ and we have

$$I_0(s,\lambda;\psi_1) \leqslant C\left(s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\psi_1|^2\,dt\,dx + s\lambda \iint\limits_{\Sigma} e^{-2s\alpha^*}\xi^*\left|\frac{\partial\psi_1}{\partial n}\right|^2 d\sigma\,dt\right), \tag{19}$$

for every $\lambda \geqslant C$ and $s \geqslant C(T^9 + T^{18})$.

The rest of the proof is divided in three steps.

- In Step 1, we will prove that $I_0(s,\lambda;\psi_1)$ can be estimated from below by the left-hand side of inequality (17).
- In Step 2, we will estimate the normal derivative appearing in the right-hand side of (19).
- Finally, in Step 3, we will estimate all the local terms by the local term of $\varphi_1$ appearing in the right-hand side of (17).

**Step 1.** (1.1) *Estimate of* $\varphi_1$. We use Lemma 3 for $u := \Delta\varphi_1$ and $r = 3$. We get the existence of a positive constant $C$ such that

$$\begin{aligned} &s^5\lambda^6 \iint\limits_{Q} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx - Cs^5\lambda^6 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx \\ &\quad \leqslant Cs^3\lambda^4 \iint\limits_{Q} e^{-2s\alpha}\xi^3|\psi_1|^2\,dt\,dx, \end{aligned} \tag{20}$$

for every $\lambda \geqslant C$ and every $s \geqslant CT^{18}$.

Next, we would like to recover a term in $\varphi_1$ and a term in $\nabla\varphi_1$ using $\Delta\varphi_1$. This is done by applying Lemma 4 for $u := \varphi_1 \in H_0^1(\Omega)$ and $r = 5$:

$$\begin{aligned} &\tau^8\lambda^{10}\int\limits_{\Omega} e^{2\tau\gamma}\gamma^8|\varphi_1|^2\,dx + \tau^6\lambda^8\int\limits_{\Omega} e^{2\tau\gamma}\gamma^6|\nabla\varphi_1|^2\,dx \\ &\quad \leqslant C\left(\tau^5\lambda^6\int\limits_{\Omega} e^{2\tau\gamma}\gamma^5|\Delta\varphi_1|^2\,dx + \tau^8\lambda^{10}\int\limits_{\omega_0} e^{2\tau\gamma}\gamma^8|\varphi_1|^2\,dx\right), \end{aligned}$$

for every $\lambda, \tau \geqslant C$. Now, we take

$$\tau := \frac{s\exp\{18\lambda\|\eta^0\|_\infty\}}{t^9(T-t)^9}$$

in this inequality. Observe that this can be done whenever $s \geqslant CT^{18}$ and $\lambda \geqslant C$ (recall that we must have $\tau \geqslant C$):

$$\begin{cases} s^8\lambda^{10} \int\limits_\Omega e^{2s\xi}\xi^8|\varphi_1|^2\,dx + s^6\lambda^8 \int\limits_\Omega e^{2s\xi}\xi^6|\nabla\varphi_1|^2\,dx \\ \quad \leqslant C\left(s^5\lambda^6 \int\limits_\Omega e^{2s\xi}\xi^5|\Delta\varphi_1|^2\,dx + s^8\lambda^{10} \int\limits_{\omega_0} e^{2s\xi}\xi^8|\varphi_1|^2\,dx\right), \quad t \in (0,T) \end{cases}$$

(the definition of $\xi$ is given in (3)). Then, we multiply this inequality by

$$\exp\left\{-2s\frac{e^{20\lambda\|\eta^0\|_\infty}}{t^9(T-t)^9}\right\},$$

we integrate in $(0,T)$ and we obtain (recall the definition of $\alpha$ also given in (3)):

$$s^8\lambda^{10} \iint\limits_Q e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8 \iint\limits_Q e^{-2\alpha}\xi^6|\nabla\varphi_1|^2\,dt\,dx$$
$$\leqslant C\left(s^5\lambda^6 \iint\limits_Q e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx + s^8\lambda^{10} \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx\right),$$

for every $s \geqslant CT^{18}$ and every $\lambda \geqslant C$. Combining this with (20), we get the following estimate for $\varphi_1$:

$$s^8\lambda^{10} \iint\limits_Q e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^6|\nabla\varphi_1|^2\,dt\,dx + s^5\lambda^6 \iint\limits_Q e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx$$
$$\leqslant C\left(s^3\lambda^4 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi_1|^2\,dt\,dx + s^8\lambda^{10} \int_0^T \int\limits_{\omega_0} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx\right.$$
$$\left. + s^5\lambda^6 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx\right), \tag{21}$$

for every $s \geqslant C(T^9 + T^{18})$ and every $\lambda \geqslant C$.

(1.2) *Estimate of* $\varphi_2$. We recall that the minimum of the weights $e^{-2s\alpha}$ and $\xi$ is reached at the boundary $\partial\Omega$, where $\alpha = \alpha^*$ and $\xi = \xi^*$ do not depend on $x$; see (3) for more details. From the divergence-free condition $\partial_2\varphi_2 = -\partial_1\varphi_1$, we find

$$s^6\lambda^8 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^6|\partial_2\varphi_2|^2\,dt\,dx = s^6\lambda^8 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^6|\partial_1\varphi_1|^2\,dt\,dx$$
$$\leqslant s^6\lambda^8 \iint\limits_Q e^{-2s\alpha}\xi^6|\nabla\varphi_1|^2\,dt\,dx. \tag{22}$$

Using $\varphi_2|_{\partial\Omega} = 0$ and $\Omega$ bounded we have that

$$\int\limits_\Omega |\varphi_2|^2\,dx \leqslant C \int\limits_\Omega |\partial_2\varphi_2|^2\,dx,$$

where $C$ only depends on $\Omega$. Since $\alpha^*$ and $\xi^*$ do not depend on $x$, we also have that

$$s^6\lambda^8 \iint_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx \leqslant Cs^6\lambda^8 \iint_Q e^{-2s\alpha^*}(\xi^*)^6|\partial_2\varphi_2|^2\,dt\,dx.$$

Combining this with (22), we obtain

$$s^6\lambda^8 \iint_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx \leqslant s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|\nabla\varphi_1|^2\,dt\,dx. \tag{23}$$

**Step 2.** In this step, we estimate the boundary term in the right-hand side of (19):

$$\left\| s^{1/2}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\frac{\partial\nabla\Delta\varphi_1}{\partial n}\right\|^2_{L^2(\Sigma)}.$$

Using integrations by parts, we readily have

$$\begin{aligned}&\left\| s^{1/2}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\frac{\partial\nabla\Delta\varphi_1}{\partial n}\right\|^2_{L^2(\Sigma)}\\ &\quad\leqslant \left\|(s\xi^*)^{2/9}\lambda^{1/2}e^{-s\alpha^*}\varphi_1\right\|_{L^2(H^5(\Omega))}\left\|(s\xi^*)^{7/9}\lambda^{1/2}e^{-s\alpha^*}\varphi_1\right\|_{L^2(H^4(\Omega))}\end{aligned} \tag{24}$$

(recall that $\alpha^*$ and $\xi^*$ do not depend on $x$). Our goal is to estimate these two terms.

In order to do this, we first consider the function

$$\widetilde{\varphi} := s^{17/9}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{17/9}\varphi := \theta_1(t)\varphi.$$

Let us point out that $\widetilde{\varphi}$, together with $\widetilde{\pi} := \theta_1(t)\pi$, fulfills the following problem (see (16)):

$$\begin{cases} -\widetilde{\varphi}_t - \Delta\widetilde{\varphi} + \nabla\widetilde{\pi} = -\theta_{1,t}\varphi & \text{in } Q,\\ \nabla\cdot\widetilde{\varphi} = 0 & \text{in } Q,\\ \widetilde{\varphi} = 0 & \text{on } \Sigma,\\ \widetilde{\varphi}|_{t=T} = 0 & \text{in } \Omega.\end{cases} \tag{25}$$

From (7), we get

$$\|\widetilde{\varphi}\|_{L^2(0,T,H^2(\Omega)^2)} + \|\widetilde{\varphi}\|_{H^1(0,T,L^2(\Omega)^2)} \leqslant C\|\theta_{1,t}\varphi\|_{L^2(Q)^2}.$$

From the definition of the weight functions (see (3)), we see that

$$|\theta_{1,t}| \leqslant CTs^{26/9}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3 \leqslant Cs^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3, \tag{26}$$

for every $s \geqslant CT^9$ and every $\lambda \geqslant C$, so

$$\|\widetilde{\varphi}\|_{L^2(0,T,H^2(\Omega)^2)} + \|\widetilde{\varphi}\|_{H^1(0,T,L^2(\Omega)^2)} \leqslant C\left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(Q)^2}. \tag{27}$$

Let now

$$\widehat{\varphi} := s^{7/9}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{7/9}\varphi := \theta_2(t)\varphi.$$

It is clear that $\widehat{\varphi}$, together with $\widehat{\pi} := \theta_2(t)\pi$, fulfills system (25) with $\theta_1$ replaced by $\theta_2$. Using (8) with $r = 1$, we find

$$\|\widehat{\varphi}\|_{L^2(0,T,H^4(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)} \leqslant C\left(\|\theta_{2,t}\varphi\|_{L^2(0,T;H^2(\Omega)^2)} + \left\|(\theta_{2,t}\varphi)_t\right\|_{L^2(0,T;L^2(\Omega)^2)}\right). \tag{28}$$

Estimating the weight functions as in (26), we have

$$|\theta_{2,t}| \leqslant Cs^{17/9}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{17/9} = C\theta_1$$

and

$$|\theta_{2,tt}| \leqslant Cs^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3,$$

for every $s \geqslant CT^9$, so

$$\begin{aligned}
&\|\widehat{\varphi}\|_{L^2(0,T,H^4(\Omega)^2)} + \|\widehat{\varphi}\|_{H^2(0,T;L^2(\Omega)^2)} \\
&\quad \leqslant C\left(\|\widetilde{\varphi}\|_{L^2(0,T;H^2(\Omega)^2)} + \|\widetilde{\varphi}\|_{H^1(0,T;L^2(\Omega)^2)} + \left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(0,T;L^2(\Omega)^2)}\right).
\end{aligned}$$

Using (27), we get

$$\|\widehat{\varphi}\|_{L^2(0,T,H^4(\Omega)^2)} + \|\widehat{\varphi}\|_{H^2(0,T;L^2(\Omega)^2)} \leqslant C\left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(0,T;L^2(\Omega)^2)}. \tag{29}$$

Finally, we define the function

$$\varphi^* =: s^{-1/3}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{-1/3}\varphi.$$

The same computations performed with $\widetilde{\varphi}$ and $\widehat{\varphi}$ and an application of (8) with $r = 2$ lead to

$$\begin{aligned}
\|\varphi^*\|_{L^2(0,T;H^6(\Omega)^2)} \leqslant C\Big(&\|\widehat{\varphi}\|_{L^2(0,T,H^4(\Omega)^2)} + \|\widehat{\varphi}\|_{H^2(0,T;L^2(\Omega)^2)} \\
&+ \left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(0,T;L^2(\Omega)^2)}\Big).
\end{aligned}$$

Combining this with (29), we get

$$\|\varphi^*\|_{L^2(0,T;H^6(\Omega)^2)} \leqslant C\left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(Q)^2}. \tag{30}$$

Thanks to an interpolation argument between the spaces $L^2(H^6)$ and $L^2(H^4)$, estimates (29) and (30) provide

$$\left\|s^{2/9}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{2/9}\varphi\right\|_{L^2l(0,T;H^5(\Omega)^2)} \leqslant C\left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|_{L^2(Q)^2}, \tag{31}$$

for $\lambda \geqslant C$ and $s \geqslant CT^9$. Coming back to (24) and using (28) and (31), we find that

$$\left\|s^{1/2}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\frac{\partial\nabla\Delta\varphi_1}{\partial n}\right\|^2_{L^2(\Sigma)} \leqslant C\left\|s^3\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^3\varphi\right\|^2_{L^2(Q)^2}, \tag{32}$$

for $\lambda \geqslant C$ and $s \geqslant CT^9$.

This ends Step 2.

Putting together (19), (21) and (32), we have for the moment the following inequality:

$$\begin{aligned}
&s^8\lambda^{10}\iint\limits_Q e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8\iint\limits_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx\\
&\quad + s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi\big(\big|\Delta^2\varphi_1\big|^2 + s^2\lambda^2\xi^2|\nabla\Delta\varphi_1|^2 + s^4\lambda^4\xi^4|\Delta\varphi_1|^2\big)\,dt\,dx\\
&\quad\leqslant C\Bigg(s^6\lambda\iint\limits_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi|^2\,dt\,dx + s^8\lambda^{10}\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx\\
&\quad + s^5\lambda^6\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx + s^3\lambda^4\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx\Bigg),
\end{aligned}\tag{33}$$

for every $s \geqslant C(T^9 + T^{18})$ and every $\lambda \geqslant C$.

Now, we see that the first term in the right-hand side can be absorbed by the left-hand side as long as $\lambda \geqslant C$. For the moment, we have

$$\begin{aligned}
&s^8\lambda^{10}\iint\limits_Q e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8\iint\limits_Q e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx\\
&\quad + s\lambda^2\iint\limits_Q e^{-2s\alpha}\xi\big(\big|\Delta^2\varphi_1\big|^2 + s^2\lambda^2\xi^2|\nabla\Delta\varphi_1|^2 + s^4\lambda^4\xi^4|\Delta\varphi_1|^2\big)\,dt\,dx\\
&\quad\leqslant C\Bigg(s^8\lambda^{10}\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^5\lambda^6\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx\\
&\quad + s^3\lambda^4\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx\Bigg),
\end{aligned}\tag{34}$$

for every $s \geqslant C(T^9 + T^{18})$ and every $\lambda \geqslant C$.

**Step 3.** In this final step, we estimate the two last local terms in the right-hand side of (34) in terms of $|\varphi_1|^2$ and small constants multiplied by the left-hand side of (34).

We start by estimating the term on $\nabla\Delta\varphi_1$. Let $\omega_1$ be an open subset satisfying $\omega_0 \Subset \omega_1 \Subset \omega$ and let $\rho_1 \in C_c^2(\omega_1)$ with $\rho_1 \equiv 1$ in $\omega_0$ and $0 \leqslant \rho_1$. Then, an integration by parts gives

$$\begin{aligned}
&s^3\lambda^4\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx\\
&\quad\leqslant s^3\lambda^4\iint\limits_{\omega_1\times(0,T)} \rho_1 e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx\\
&\quad= -s^3\lambda^4\iint\limits_{\omega_1\times(0,T)} \rho_1 e^{-2s\alpha}\xi^3\Delta^2\varphi_1\,\Delta\varphi_1\,dt\,dx + \frac{s^3\lambda^4}{2}\iint\limits_{\omega_1\times(0,T)} \Delta\big(\rho_1 e^{-2s\alpha}\xi^3\big)|\Delta\varphi_1|^2\,dt\,dx.
\end{aligned}$$

Using the Cauchy–Schwarz's inequality for the first term and estimate

$$\left|\Delta\left(\rho_1 e^{-2s\alpha}\xi^3\right)\right| \leqslant Cs^2\lambda^2\xi^5 e^{-2s\alpha}, \quad s \geqslant CT^{18},\ \lambda \geqslant C,$$

for the second one, we obtain for every $\epsilon > 0$

$$s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx$$
$$\leqslant Cs^5\lambda^6 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx + \epsilon s\lambda^2 \iint\limits_{Q} e^{-2s\alpha}\xi\left|\Delta^2\varphi_1\right|^2\,dt\,dx,$$

for every $s \geqslant CT^{18}$ and every $\lambda \geqslant C$ ($C$ depending also on $\varepsilon$). Using this in (34), we obtain

$$s^8\lambda^{10}\iint\limits_{Q} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^6\lambda^8 \iint\limits_{Q} e^{-2s\alpha^*}(\xi^*)^6|\varphi_2|^2\,dt\,dx$$
$$+ s\lambda^2 \iint\limits_{Q} e^{-2s\alpha}\xi\left(\left|\Delta^2\varphi_1\right|^2 + s^2\lambda^2\xi^2|\nabla\Delta\varphi_1|^2 + s^4\lambda^4\xi^4|\Delta\varphi_1|^2\right)dt\,dx$$
$$\leqslant C\left(s^8\lambda^{10}\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^8|\varphi_1|^2\,dt\,dx + s^5\lambda^6 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx\right), \tag{35}$$

for every $s \geqslant C(T^9 + T^{18})$ and every $\lambda \geqslant C$.

Let us now estimate $\Delta\varphi_1$. Let $\omega_2$ be an open subset satisfying $\omega_1 \Subset \omega_2 \Subset \omega$ and let $\rho_2 \in C_c^2(\omega_2)$ with $\rho_2 \equiv 1$ in $\omega_1$ and $\rho_2 \geqslant 0$. Then, an integration by parts gives

$$s^5\lambda^6 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx \leqslant s^5\lambda^6 \iint\limits_{\omega_2\times(0,T)} \rho_2 e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx$$
$$= -s^5\lambda^6 \iint\limits_{\omega_2\times(0,T)} \rho_2 e^{-2s\alpha}\xi^5(\nabla\Delta\varphi_1\cdot\nabla\varphi_1)\,dt\,dx$$
$$- s^5\lambda^6 \iint\limits_{\omega_2\times(0,T)} \nabla\left(\rho_2 e^{-2s\alpha}\xi^5\right)\cdot\nabla\varphi_1\Delta\varphi_1\,dt\,dx.$$

Using again the Cauchy–Schwarz's inequality for the first term and estimate

$$\left|\nabla\left(\rho_2 e^{-2s\alpha}\xi^5\right)\right| \leqslant Cs\lambda\xi^6 e^{-2s\alpha}, \quad s \geqslant CT^{18},\ \lambda \geqslant C,$$

for the second one, we obtain for every $\epsilon > 0$

$$s^5\lambda^6 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx$$
$$\leqslant Cs^7\lambda^8 \iint\limits_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^7|\nabla\varphi_1|^2\,dt\,dx + \epsilon s^3\lambda^4 \iint\limits_{Q} e^{-2s\alpha}\xi^3|\nabla\Delta\varphi_1|^2\,dt\,dx, \tag{36}$$

for every $s \geqslant CT^{18}$ and every $\lambda \geqslant C$ ($C$ depending also on $\varepsilon$). Finally, we locally estimate $\nabla\varphi_1$ in terms of $\varphi_1$ by a completely analogous argument:

$$s^7\lambda^8 \iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7|\nabla\varphi_1|^2\,dt\,dx$$

$$\leqslant Cs^9\lambda^{10} \iint\limits_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^9|\varphi_1|^2\,dt\,dx + \epsilon s^5\lambda^6 \iint\limits_{Q} e^{-2s\alpha}\xi^5|\Delta\varphi_1|^2\,dt\,dx.$$

This estimate, together with (35) and (36), readily gives the desired Carleman inequality (17). This concludes the proof of Proposition 1. □

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