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Existence of solutions to a model for sparse, one-dimensional fluids

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ABSTRACT

We prove the global existence of solutions to a model for a viscous, compressible, barotropic fluid initially occupying a general open subset of a finite, one-dimensional interval. The fluid equations are applied only on the support of the density, understood in the sense of distributions. This support must be tracked and accommodation must be made for the possibly infinite number of collisions of fluid packets occurring on a possibly dense set of collision times. Our approach avoids certain nonphysical properties of solutions which are constructed as limits of solutions in which artificial mass has been inserted.

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1. Introduction

We prove the existence of solutions to a model for a viscous, compressible, barotropic fluid initially occupying a general open subset of a finite, one-dimensional interval. The novel feature is that the equations of viscous, compressible flow are applied only on the support of the density, understood in the sense of distributions. We formulate a notion of weak solution which insures that the momentum of a fluid packet isolated from the spatial boundary on some time interval is conserved on that time interval. We thus avoid the occurrence of the nonphysical solutions which are known to result from the insertion of artificial mass. The analysis requires that we track the fluid set and accommodate the possibly infinite number of collisions of fluid packets occurring on a possibly dense set of collision times. The solutions we construct are minimally regular, but after a certain finite time the fluid fills the entire spatial domain and the solution acquires somewhat greater regularity. The main result is stated in Theorem 1 below.

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The fluid equations considered here are the Navier–Stokes equations of one-dimensional compressible flow:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + P(\rho)\big)_x = \big(\mu(\rho) u_x\big)_x, \end{cases} \tag{1.1}$$

where ρ and u are the unknown functions of x and t representing respectively the fluid density and velocity, $P = P(\rho)$ is the pressure, and $\mu = \mu(\rho)$ is the viscosity coefficient. These equations are to be applied in an open set $V = \{(x,t)\colon\ x \in V_t,\ \ t > 0\}$ in $(0,1) \times (0,\infty)$ where V_t is a maximal open subset of $(0,1)$ on which $\rho(\cdot,t) >$ a.e. Initial data

$$\big(\rho(\cdot,0), u(\cdot,0)\big) = (\rho_0, u_0) \tag{1.2}$$

is given on an open set $V_0 \subset (0,1)$ and the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0 \tag{1.3}$$

are to hold in a reasonable sense at times t for which $0 \in \partial V_t$ or $1 \in \partial V_t$. We shall take ρ to be zero in the complement V^c but we avoid defining a velocity u in V^c.

Before describing our results in detail we recall at a somewhat informal level two motivating examples. In the first of these we are given initial data as above with V_0 an interval (a_0, b_0) whose closure is contained in $(0,1)$. One approach is to insert artificial mass of positive density δ in the complement of V_0, to extend the initial velocity to all of $(0,1)$ in some reasonable way, then to solve the system (1.1) on all of $(0,1)\times(0,\infty)$ with this extended initial data $(\rho_0^\delta, u_0^\delta)$ and with the boundary conditions (1.3) imposed at all times. A solution $(\overline{\rho}, \overline{u})$ with the original initial data is then obtained in the limit as $\delta \to 0$ and the limiting velocity $\overline{u}$ will be in the space $L^2_{loc}((0,\infty); H^1_0((0,1)))$ (these statements are justified by known results and are included in the auxiliary result Theorem 2 below, which we are therefore anticipating here). A completely different approach, perhaps better motivated by physical considerations, is to instead track the fluid initially on V_0 by solving the system (1.1) only in the region $a(t) < x < b(t)$, where a and b are the free boundaries defined by $\dot a(t) = u(a,t)$ and $a(0) = a_0$, and similarly for $b(t)$. The momentum $\int_{a(t)}^{b(t)} (\rho u)(x,t)\,dx$ should be constant in time, at least until the fluid strikes one of the endpoints $x = 0$ or $x = 1$, at which time the mathematical nature of the problem changes. An easy computation shows formally that this momentum conservation will hold if

$$F \equiv \mu(\rho) u_x - P(\rho) = 0 \tag{1.4}$$

at $x = a(t)$ and $x = b(t)$, this being the condition that the stress at either endpoint of the fluid is zero. There results a solution (ρ, u) defined only where there is fluid, a solution which we may regard as physically satisfactory. However, to compare it to the other solution $(\overline{\rho}, \overline{u})$, which is defined on all of $(0,1)\times(0,\infty)$, we ask whether (ρ, u) can be extended as a weak solution to all of $(0,1)\times(0,T)$, where T is the first time that the fluid strikes the boundary. Observe that, unless $\mu(0) = 0$, (1.1) will hold in regions where $\rho = 0$ only if $u_{xx} = 0$, which we therefore stipulate. The remaining condition is that the resulting extension must satisfy the appropriate jump conditions across the free boundaries. An easy computation (see the introduction to [6] for example) shows that the jump condition at a for the mass equation is precisely the free boundary condition $\dot a = u(a,t)$, and the jump condition at a for the momentum equation is that $[F(a(t),t)] \equiv F(a(t)+,t) - F(a(t)-,t) = 0$; similarly for b. In view of (1.4) and the requirement that $u_{xx} = 0$ in vacuum regions, we therefore conclude that we must take $u(x,t) = u(a(t)+,0)$ for $x < a(t)$ and similarly for $x > b(t)$, at least as long as $[a(t), b(t)] \subset (0,1)$. Now observe that the system (1.1) respects Galilean transformations, so we may arrange that $\dot a(t) < 0$ on a time interval $[0,T)$ before the fluid strikes the boundary. Thus for $t < T$ and $x < a(t)$, $u(x,t) = \dot a(t) < 0$. In particular, u is not in the space $L^2_{loc}((0,T); H^1_0((0,1)))$ and therefore (ρ, u) *cannot* be the

restriction to the support of ρ of the solution $(\overline{\rho}, \overline{u})$ obtained as the limit of solutions with artificial mass.

A similar but more striking example was constructed in Hoff and Serre [8], in which two fluids are initially separated but are rapidly approaching. Considered in the whole space $x \in \mathbb{R}$, the "physical" solution is obtained by solving two (unrelated) free-boundary problems, as described above, up until the time that the two fluids collide. The collision time is then a new initial time at which there is a single fluid whose future evolution is described by the solution of a single free-boundary value problem with initial data consisting of the combined states of the merged fluids. This construction clearly guarantees that the *separate* momenta of the individual fluids are conserved before the collision. This solution (ρ, u) is defined only where there is fluid, but it can in fact be extended as a weak solution to the left and to the right of the two-fluid configuration. It cannot, however, be extended to the region between the two fluids before the collision time: exactly the same considerations as in our first example show that, in the region between the two fluids, u would have to equal both the velocity of the leading edge of the fluid on the left as well as the velocity of the trailing edge of the fluid on the right. These two velocities would therefore have to agree, so that no collision could have occurred, contrary to the construction. Of course, there is no physical significance of this non-extendability, and the solution (ρ, u) defined only where there is fluid is completely satisfactory. The rather striking result is that, if artificial mass of density δ is inserted between the fluids at the initial time, then the resulting solution (ρ^δ, u^δ) (of the free-boundary value problem for a single fluid) will converge in the limit as $\delta \to 0$ to a solution $(\overline{\rho}, \overline{u})$ of (1.1) which *is* defined in the whole space, and which therefore cannot be the physical solution (ρ, u), which as we just saw has no extension to the whole space. Moreover, the limiting weak solution $(\overline{\rho}, \overline{u})$ exhibits rather bizarre nonphysical behavior: it consists of two separate fluids, rapidly approaching initially, but which *never collide*. The total momentum of the combined fluids is conserved but the separate momenta of the two isolated fluids evidently are not.

With these examples in mind we formulate the following notion of locally momentum-conserving solution of (1.1) for the interval $0 < x < 1$:

Definition. Let (ρ_0, u_0, V_0) be given with V_0 an open subset of $(0,1)$ and ρ_0 bounded and nonnegative on $(0,1)$, positive a.e. on V_0, and zero a.e. on V_0^c. Assume that u_0 is measurable on V_0 with $\rho_0 u_0^2$ integrable on V_0 and let $T \leqslant \infty$. A locally momentum-conserving solution of (1.1)–(1.3) on $(0,1) \times [0,T)$ with initial data (ρ_0, u_0, V_0) is then a triple (ρ, u, V) with the following properties:

(a) $\rho : [0,T) \to L^\infty((0,1))$ is bounded on $[0,t]$ for all $t < T$;
(b) for every $t \in [0,T)$ there is an open set $V_t \subset (0,1)$ such that $\rho(\cdot, t)$ is positive a.e. on V_t and zero a.e. on V_t^c, and

$$V \equiv \big\{(x,t)\colon\ t \in (0,T) \text{ and } x \in V_t\big\}$$

is open in $(0,1) \times (0,T)$;
(c) $u \in L^1_{loc}(V) \subset \mathcal{D}'(V)$ and the distribution derivative u_x is in $L^2(V \cap \{(0,1) \times (0,t)\})$ for all $t < T$;
(d) there is a locally bounded map $w : [0,T) \to L^1((0,1))$ such that $w = 0$ a.e. in V^c, $w(\cdot, 0) = \rho_0 u_0$, $w = \rho u$ in V, and $w^2/\rho = \rho u^2 \in L^1(V \cap \{(0,1) \times (0,t)\})$ for all $t < T$;
(e) if $0 \in \partial V_t$ for $t \in [t_1, t_2] \subset [0,T)$, where $t_1 < t_2$, then

$$\lim_{M \to 0} \left(\int_{t_1}^{t_2} \int_0^{z(M,t)} (\rho u)(x,t)\, dx \right) \Big/ \left(\int_0^{z(M,t)} \rho(x,t)\, dx \right) dt = 0,$$

where $z(M,t)$ is defined for $0 \leqslant M \leqslant \int_0^1 \rho_0\,dx$ by

$$z(M,t) \equiv \inf\left\{ z \in (0,1):\ \int_0^z \rho(x,t) = M \right\};$$

a similar condition holds at $x = 1$;

(f) for $0 \leqslant t_1 \leqslant t_2 < T$ and for φ Lipschitz on $[0,1] \times [t_1, t_2]$,

$$\int_{V_t} (\rho\varphi)(x,t)\,dx\Big|_{t=t_1}^{t=t_2} = \iint_{V \cap \{(0,1)\times(t_1,t_2)\}} (\rho\varphi_t + \rho u \varphi_x)\,dx\,dt,$$

where $\rho(\cdot,0) = \rho_0$;

(g) for t_1, t_2 and φ as above with $\varphi(0,t) = \varphi(1,t) = 0$ for $t \in [t_1, t_2]$,

$$\int_{V_t} (\rho u\varphi)(x,t)\,dx\Big|_{t=t_1}^{t=t_2} = \iint_{V \cap \{(0,1)\times(t_1,t_2)\}} \bigl(\rho u \varphi_t + \rho u^2 \varphi_x + P(\rho)\varphi_x - \mu(\rho) u_x \varphi_x\bigr)\,dx\,dt,$$

where $(\rho u)(\cdot,0) = \rho_0 u_0$.

Observe that solutions in the sense of this definition are in fact locally momentum conserving: suppose for example that $(a,b) \subset\subset (c,d) \subset\subset (0,1)$ and that $\rho(\cdot,t) = 0$ a.e. on $(c,d) - (a,b)$ for $t \in [t_1, t_2]$. We may then choose a smooth function φ of x with $\operatorname{supp}\varphi \subset (c,d)$ and $\varphi(x) = 1$ for $x \in (a,b)$. Applying (g), we then find that the local momentum $\int_a^b (\rho u)(x,t)\,dx$ is constant over the interval $[t_1, t_2]$, as required. Pursuing this a step further, we suppose additionally that a and b depend on t and that ρ and u are smooth with ρ positive for $t \in [t_1, t_2]$ and $x \in [a(t), b(t)]$. We would then deduce from (g) by simple integration by parts that the second equation in (1.1) holds in a strict pointwise sense for these x and t. Then applying (g) to the same φ as above we would find that the boundary condition (1.4) holds at the endpoints $a(t)$ and $b(t)$. In other words, the boundary condition (1.4) is the *natural* boundary condition for the weak form in (g) of the momentum equation. Although the solutions we construct are not sufficiently regular for this internal boundary condition to hold in a pointwise sense, its essential consequence is the conservation of local momentum, and this does hold for solutions in the sense of our definition.

Note that the weak forms in (f) and (g) imply that ρ and ρu are continuous functions of time into the duals of the spaces in which the test functions lie; this regularity will be noted in the existence theorem below. The weak form in (g) appears to correspond to the momentum equation in (1.1) with the right side replaced by $(\chi_V \mu(\rho) u_x)_x$, where χ_V is the characteristic function of V. We emphasize however, that u need not be defined in V^c: the definition does not require the specification of velocities of nonexistent fluid particles. Finally, concerning the boundary condition (e) at $x = 0$ and $x = 1$, one easily checks that if (ρ, u) is a solution in the above sense with u defined and continuous near $(0, t_0)$, for example, then in fact $u(0, t_0) = 0$. There is a more subtle point, however, concerning whether or not (e) implies that fluid particles which strike the boundary must then adhere to the boundary at later times. A similar, related question is whether particles which collide in the interior of $(0,1)$ must then adhere to one another at later times. This issue of adherence of fluid particles is related to the question of uniqueness and will be discussed further below following the statement of Theorem 1.

We now give a precise formulation of our major result. We make the following assumptions (A_P) concerning the pressure P:

(a) $P \in C^1([0,\infty))$, $P(0) = 0$, and $P(\rho) > 0$ for $\rho > 0$;
(b) $\liminf_{\rho\to\infty} \frac{P(\rho)}{\rho} > 0$;
(c) there is a $\overline{\rho}$ such that P is (nonstrictly) increasing on $[\overline{\rho},\infty)$ and $P(\overline{\rho}) \geqslant P(\rho)$ for $\rho \in [0,\overline{\rho})$;
(d) there is a $\tilde{\rho} \in [0,\infty)$ such that, if

$$G(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2}\, ds,$$

then G is continuous and nonnegative on $[0,\infty)$.

In particular, the pressures $P(\rho) = C\rho^\gamma$ of an ideal isothermal fluid ($\gamma = 1$) or of an ideal polytropic fluid with constant entropy ($\gamma > 1$) are included. No global monotonicity is assumed. Concerning the viscosity μ we make the following assumptions (A_μ):

(a) $\mu \in C([0,\infty)) \cap C^1((0,\infty))$;
(b) there is a positive $\overline{\mu}$ such that $0 < \overline{\mu} \leqslant \mu(\rho)$ for all $\rho \in [0,\infty)$.

We remark that scaling limits of the Boltzmann equation (see [1, p. 176] or the introduction to [11], for example) suggest that, modulo multiplicative constants, μ should be a positive power of temperature. In particular, μ should be a positive constant for isothermal flow, and this case is included. However, for an ideal polytropic fluid with constant entropy, temperature is a positive power of density and therefore so too will μ. In particular, $\mu(0) = 0$ in this case, which is excluded by (b). Whether or not the isentropic assumption is consistent with the presence of viscosity is a matter of some debate; the lower bound in (b) is imposed here simply because the analysis requires it.

Before we state our main result we show that, if ρ_0 and V_0 are as in the Definition above, then we may without loss of generality take V_0 to be an at most countable union of open intervals with disjoint closures. To see this, let ρ_0 and V_0 be as in the Definition and define for each $x \in V_0$ the set $I_x = \bigcup_{I\in\mathcal{S}_x} I$ where

$$\mathcal{S}_x = \big\{I\colon\ I \text{ is an open interval, } x \in I, \text{ and } \big|I \cap V_0^c\big| = 0\big\}.$$

It is easy to check that I_x is an open interval, that $|I_x \cap V_0^c| = 0$, and that if $\bar{I}_x \cap \bar{I}_y \neq \emptyset$, then $I_x = I_y$. There are therefore at most countably many distinct intervals I_x, say $\{I_{x_j}\}$. We then set $\tilde{V}_0 = \bigcup_j I_{x_j}$. It follows easily that $|\tilde{V}_0 \cap V_0^c| = 0$, so that $\rho_0 > 0$ a.e. on $\tilde{V}_0$. To check that $\rho_0 = 0$ a.e. on $\tilde{V}_0^c$ we first show that $V_0 \subset \tilde{V}_0$: if $x \in V_0$ then there is a small interval $I'_x \subset V_0$ centered at x because V_0 is open. It is clear that $I'_x \in \mathcal{S}_x$ and therefore that $I'_x \subset I_x \subset \tilde{V}_0$. Thus $V_0 = \bigcup_{x\in V_0} I'_x \subset \tilde{V}_0$. Then $\tilde{V}_0^c \subset V_0^c$ and so $\rho_0 = 0$ a.e. on $\tilde{V}_0^c$. We may therefore replace V_0 by $\tilde{V}_0$.

The following is the main result of this paper:

Theorem 1. *Assume that the above hypotheses (A_P) and (A_μ) on P and μ are satisfied for some fixed choice of $\tilde{\rho}$ and let initial data (ρ_0, u_0, V_0) be given such that*

$$V_0 = \bigcup_{j=1}^{\infty} \big(a_j(0), b_j(0)\big)$$

where $\{(a_j(0), b_j(0))\}$ *are nonempty open intervals in* $(0,1)$ *with disjoint closures. Assume that* ρ_0 *is bounded and nonnegative with* $\rho_0 > 0$ *a.e. on* V_0 *and* $\rho_0 = 0$ *a.e. on* V_0^c *and that* u_0 *is measurable on* V_0 *with* $\rho_0 u_0^2 \in L^1(V_0)$. *Define*

$$M_0 = \int_{V_0} \rho_0(x)\,dx \tag{1.5}$$

and

$$E_0 = \int_{V_0} \big[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\big]\,dx, \tag{1.6}$$

where G *is as defined in* (A_P), *and assume finally that there is a positive constant* C_1 *such that, for all* j,

$$\tilde{P}\big(b_j(0) - a_j(0)\big) + \int_{a_j(0)}^{b_j(0)} \big[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\big]\,dx \leqslant C_1 \int_{a_j(0)}^{b_j(0)} \rho_0\,dx, \tag{1.7}$$

where $\tilde{P} \equiv P(\tilde{\rho})$. *Then there is a globally defined locally momentum-conserving solution* (ρ, u, V) *of the corresponding initial–boundary value problem* (1.1)–(1.3) *on* $(0,1)\times(0,\infty)$ *in the sense of the above Definition, satisfying*

$$\rho \in C\big([0,\infty); W^{1,2}\big((0,1)\big)^*\big), \tag{1.8}$$

$$\rho u \in C\big([0,\infty); H^{-1}\big((0,1)\big)\big) \cap C\big([0,\infty); L^2_w\big((0,1)\big)\big) \tag{1.9}$$

(*thus for each* $\varphi \in L^2((0,1))$, *the map* $t \to \int_0^1 (\rho u)(x,t)\varphi(x)\,dx$ *is continuous on* $[0,\infty)$), *and for all* $t \geqslant 0$,

$$\int_0^1 \rho(x,t)\,dx = M_0 \tag{1.10}$$

and

$$\sup_t \int_{V_t} \big[\big(\tfrac{1}{2}\rho u^2 + G(\rho)\big)(x,t)\big]\,dx + \int_0^\infty \int_{V_t} \mu(\rho) u_x^2\,dx\,dt \leqslant E_0 + \tilde{P}. \tag{1.11}$$

Also, there is a constant C *depending on the assumed properties of* P *and* μ *and on upper bounds for* $\|\rho_0\|_\infty$, M_0, E_0, *and* C_1 *such that*

$$\sup_t \big\|\rho(\cdot,t)\big\|_\infty \leqslant C \tag{1.12}$$

and

$$\int_0^T \big\|u(\cdot,t)\big\|_\infty^2\,dt \leqslant C(1+T) \tag{1.13}$$

for all T. Next, there is a map $X : V_0 \times [0,\infty) \to [0,1]$ with the following properties:

$$X(y,0) = y, \quad y \in V_0; \tag{1.14}$$

$$\big|X(y,t_2) - X(y,t_1)\big| \leqslant C|t_2 - t_1|^{1/2} \tag{1.15}$$

for $t_2, t_1 \geqslant 0$ and $y \in V_0$;

$$C(t)^{-1} \leqslant \frac{\partial X}{\partial y}(y,t) = \frac{\rho_0(y)}{\rho(X(y,t),t)} \leqslant C(t) \tag{1.16}$$

for $t \geqslant 0$ and a.a. $y \in V_0$;

$$V_t = \bigcup_{j=1}^{\infty} \big(a_j(t), b_j(t)\big) \tag{1.17}$$

where $a_j(t) = X(a_j(0),t)$ and $b_j(t) = X(b_j(0),t)$; and for all y_1, $y_2 \in V_0$ and $t \geqslant 0$,

$$\int_{y_1}^{y_2} \rho_0(y)\,dy = \int_{X(y_1,t)}^{X(y_2,t)} \rho(x,t)\,dx. \tag{1.18}$$

Finally, there is a time T such that $\overline{V}_t = [0,1]$ for all $t \geqslant T$, both u and $F = \mu(\rho)u_x - P(\rho)$ are Hölder continuous on $[0,1] \times [T,\infty)$, and $u(0,t) = u(1,t) = 0$ for $t \geqslant T$.

The hypothesis (1.7) excludes fluid packets of strictly positive energies but vanishingly small masses, and is required for the proof of (1.13). The estimate (1.13) shows in particular that a fixed particle can move only a finite distance in finite time, and this fact is needed for the construction of the functions X described in (1.14)–(1.16). The local conservation of mass (1.18) justifies the interpretation that $X(y,t)$ is the position at time t of the fluid particle initially at y. The solution is not sufficiently regular, however, to allow the conclusion $\dot{X}(t) = u(X,t)$, at least before time T. After time T the fluid has filled the entire interval and the solution acquires significantly greater regularity properties; these are detailed further in Theorem 2 in Section 2 below. We also remark that, in the case that P is superlinear in ρ near $\rho = 0$, we may take $\tilde{\rho} = 0$ in (d) of (A_P) so that the $\tilde{P}$ terms in (1.7) and (1.11) vanish. But if $\tilde{P} > 0$, as is required for the isothermal pressure of an ideal gas, these terms remain and apparently represent a sort of potential energy reflecting the size of the physical domain occupied by the fluid. More detailed expressions of this interpretation appear below in (2.2) for a single fluid and in (3.3) for finitely many fluids.

The proof of Theorem 1 consists of three steps, corresponding to the cases that V_0 is first a single open interval, then a finite union of open intervals, and finally a general open set. In Theorem 2 of Section 2 we give a complete existence and regularity theory for the single fluid case with either or both endpoints fixed or free. Numerous authors have studied this and related problems, among which we cite [2,4,10,13–22]. Our results in Theorem 2 go beyond these in several respects, the most important of which is that quite general discontinuous initial data is allowed, this being necessary to accommodate the new initial-value problems that must be solved in step two when different fluids collide. We obtain very detailed information about the regularity of solutions in the single fluid case, including the precise regularity (1.16) of the particle trajectories and their relation to the Lagrangean evolution of density, as well as the regularization that occurs in the initial layer, specific to this system, such as the Hölder continuity of u and F. We also allow quite general pressures and we do not assume that the density is strictly positive on the fluid set. We also establish an important spreading effect when at least one of the boundaries is free, this being the basis for the assertion in

Theorem 1 that the fluid fills the entire interval in finite time. Of course, the existence theory given here for a single fluid overlaps with the results of the papers cited above in certain respects, these being dictated in a natural way by the system itself. Our Theorem 2 and its proof are most strongly influenced by the analyses in [5–7,17].

In the second step we consider the case that V_0 is the finite union of open intervals in $(0,1)$. The separate fluids on these subintervals can and do collide and merge and eventually fill the entire region. We apply the results of Theorem 2 for a single fluid repeatedly at each collision time to construct a solution in the sense of the above Definition, and we detail those estimates which are independent of the number of fluids. Aside from mass and energy conservation, these are limited to the pointwise bounds (1.12), (1.13), (1.15), and (1.16). In particular, while u and F will be fairly regular locally between collision times, this regularity depends on the number of fluids and therefore plays no further role in the analysis.

Finally in Section 4 we complete the proof of Theorem 1 by obtaining the solution (ρ, u) as a limit point as $N \to \infty$ of the solutions (ρ^N, u^N) of Theorem 3 for N fluids. First, the uniform estimates corresponding to (1.15) and (1.16) enable us to obtain in Lemma 4.1 the strong convergence of the trajectories X^N. Then from this strong convergence we deduce that the sets V^N converge to an open set V in a certain sense. Next we prove that the densities $\rho^N(\cdot,t)$ converge *strongly* in $L^2((0,1))$ for every t. The argument is related to but somewhat more direct than that occurring in the more difficult multidimensional case treated in [12] (see also the exposition in [3]). Briefly, we show that the time-antiderivatives of the functions $F^N(X^N(y,t),t)$ are strongly compact, this being a direct consequence of the uniform conservation of mass and conservation of energy. This strong compactness is then shown to drive the strong compactness of the Lagrangean densities $\rho^N(X^N(\cdot,t),t)$. The convergence of the X^N together with the special relationship in (1.16) that $\rho^N(X^N,t)\,dX^N = \rho_0\,dy$, which is really an expression of the conservation of mass, then enables us to deduce the strong compactness of the Eulerian densities $\rho^N(\cdot,t)$. These arguments are given in Lemma 4.2 in Section 4. This strong convergence of densities together with the weak convergence of u^N derived from the uniform bound (1.13) then implies the weak convergence of $\rho^N u^N$. This is not sufficient, however, to deal with the terms ρu^2 and $X_V \mu(\rho) u_x$ occurring in (g) of the definition of weak solution. For these terms we apply arguments similar to those given in [12] but differing in that we must first cover V by a countable union of open rectangles, decompose the integrals in question by a subordinate partition of unity, apply the aforementioned convergence arguments on each rectangle, then prove that these local convergence results can be reassembled globally. Statements and proofs are given in Lemma 4.3 in Section 4.

Solutions in the sense of our Definition are certainly not uniquely determined by their initial data. Consider for example a single fluid initially occupying a small interval $(0,h)$ with large positive velocity. Theorem 2 guarantees the existence of a solution whose density is supported on an interval which is transported by the velocity, and by Galilean invariance we may arrange that the fluid continues to move rapidly to the right on some time interval. By reflection we can then construct a second fluid initially occupying the interval $(-h,0)$ and moving rapidly to the left. Combining the two fluids, we then have a fluid initially occupying the interval $(-h,h)$ which instantaneously separates at the interior point $x=0$. The corresponding solution is easily seen to be a solution in the sense of our Definition, but Theorem 2 below gives a *different* solution for the composite fluid, again a solution in the sense of our Definition, for which the support of the density is the connected interval whose endpoints are the transports of the points $x=-h$ and $x=h$ and in which no separation has occurred. This simple example raises an important modeling issue: must fluid particles which come into contact adhere for all subsequent times, or can spontaneous cavitation occur? In constructing solutions for *finitely* many interacting fluids in Theorem 3, we choose to enforce adhesion, but this is a choice, and is dictated neither by the form of the system (1.1) nor by our Definition of weak solution. These considerations strongly suggest that the physically correct solution will depend on the constitutive properties of the fluid itself, and should be selected as the limit of a system in which capillarity effects are included; this we propose to address elsewhere. Returning to the above example, we note that in the second solution described above, there is one and only one particle trajectory $X(y,t)$ for each initial point $y \in (-h,h)$, whereas there are two such trajectories corresponding to $x=0$ in the first solution. A reasonable conjecture is that this "complete Lagrangean structure"–that

is, the existence of a unique particle path corresponding to each initial point in the fluid–may in fact be a (purely mathematical) selection principle guaranteeing uniqueness. The above example also raises a related issue concerning continuous dependence on initial data and adhesion: if the two fluids are initially displaced slightly so as to occupy instead the intervals $(-h-\varepsilon,-\varepsilon)$ and $(\varepsilon,h+\varepsilon)$ for some small $\varepsilon>0$, then the corresponding solutions constructed in Theorem 3 will clearly not depend continuously on ε as $\varepsilon\to 0$.

Even the above conjecture that complete Lagrangean structure should be a uniqueness condition may be too simple, however, for the following reason. While we do enforce adhesion and non-cavitation for finitely many fluids, we do not know whether these persist in the limit as the number of fluids goes to infinity. Consider a hypothetical example in which, in the notation of Theorem 1, two intervals $(a_k(0),b_k(0))$ and $(a_{k+1}(0),b_{k+1}(0))$ are adjacent with no interviewing fluid. In our analysis we first construct a solution (ρ^N,u^N) corresponding to finitely many fluids initially on the truncated set $V_0^N=\bigcup_{j=1}^N(a_j(0),b_j(0))$ and then obtain the solution (ρ,u) of Theorem 1 in the limit as $N\to\infty$. In particular, modulo subsequences, $a_j^N(t)\to a(t)$ and $b_j^N(t)\to b(t)$ for every j. It may therefore be possible that $b_k^N(t)<a_{k+1}^N(t)$ for every t in some time interval including t_0, that in the limit, $b_k(t)<a_{k+1}(t)$ for every such $t\neq t_0$, but that $b_k(t_0)=a_{k+1}(t_0)$. Thus in the limiting solution, two adjacent fluids have come into contact at time t_0 but they separate immediately thereafter. While we do not construct an explicit example of such behavior, there is nothing in our results which precludes it. In particular, if there were such a solution, we could then merge the two fluids in question and restart the solution process at the new initial time t_0 to obtain a *different* solution in which $b_k(t)=a_{k+1}(t)$ for all $t\geqslant t_0$. Exactly the same question arises with respect to adhesion to the boundary: it may be possible that for the limiting solution, $0\in\partial V_{t_0}$ for some time t_0 but that $0\notin\partial V_t$ for immediately subsequent times t. These considerations suggest a revised uniqueness condition which includes the requirement that if $X(y_1,t_0)=X(y_2,t_0)$ for some $y_1,y_2\in V_0$ and $t_0\geqslant 0$, then $X(y_1,t)=X(y_2,t)$ for all $t\geqslant t_0$, with similar conditions holding at $x=0$ and $x=1$. Even if correct, this is again a purely mathematical criterion requiring justification on some physical grounds. These questions regarding adhesion, uniqueness, and continuous dependence are well beyond the scope of the present paper, however, and will therefore be pursued elsewhere.

2. Single fluids

In this section we construct solutions of (1.1) in which the initial density is supported on a single finite interval (a_0,b_0). We consider four cases in which either of the endpoints of the fluid region is fixed or free, with no-slip conditions imposed at fixed boundary points and momentum-conservation conditions imposed at free boundary points. Specifically, we refer to cases I–IV in which (1.1) is to be solved on a set $\{(x,t)\colon\ t>0 \text{ and } a(t)<x<b(t)\}$, where

I: $a\equiv 0, b\equiv 1$,
II: a is free and $b\equiv 1$,
III: $a\equiv 0$ and b is free,
IV: a and b are both free.

If a is a fixed boundary point we impose the condition $u(a(t),t)=0$ and if a is a free boundary point we require that $\dot a(t)=u(a(t),t)$ and $\mu(\rho)u_x-P(\rho)=0$ at $(a(t),t)$; similarly for b.

We employ the standard notation for Hölder semi-norms for a given function w mapping a set S in x–t space into $\mathbb{R}$:

$$\langle w\rangle_S^{\alpha,\beta}=\sup\frac{|w(x_2,t_2)-w(x_1,t_1)|}{|x_2-x_1|^\alpha+|t_2-t_1|^\beta},$$

where the sup is taken over pairs of distinct points in S.

The existence of solutions and their regularity properties in each of the cases I–IV are given in the following theorem:

Theorem 2. *Assume that the functions P and μ satisfy (A_P) and (A_μ) and let (a_0, b_0), ρ_0, and u_0 be given with $a_0 < b_0$, $a_0 = 0$ in cases* I *and* III *and $b_0 = 1$ in cases* I *and* II*, and $\rho_0, u_0 : (a_0, b_0) \to \mathbb{R}$ measurable. Assume that ρ_0 is positive a.e. on (a_0, b_0), $\rho_0 \in L^\infty$, and*

$$E_0 \equiv \int_{a_0}^{b_0} \big[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\big]\,dx < \infty.$$

Then there exist continuous curves $a, b : [0,\infty) \to \mathbb{R}$ with $a(0) = a_0$, $b(0) = b_0$ and $a(t) < b(t)$ for all t, and there exist functions $\rho(\cdot, t)$ and $u(\cdot, t)$ defined for $t \geqslant 0$ and $x \in (a(t), b(t))$ such that $(\rho(\cdot, 0), u(\cdot, 0)) = (\rho_0, u_0)$, $\rho(\cdot, t)$ is positive a.e. on $(a(t), b(t))$, and the distribution derivative $u_x \in L^2(V)$, where V is the open set $V \equiv \{(x, t)\colon\ t > 0$ and $a(t) < x < b(t)\}$. The mass equation

$$\int_{a(t)}^{b(t)} \rho(x,t)\,dx = \int_{a_0}^{b_0} \rho_0(x)\,dx \equiv M_0 \tag{2.1}$$

holds for $t \geqslant 0$; and if $H(t)$ is the energy functional

$$H(t) = \int_{a(t)}^{b(t)} \big[\big(\tfrac{1}{2}\rho u^2 + G(\rho)\big)(x,t)\big]\,dx - P(\tilde\rho)\big(b(t) - a(t)\big) + \int_0^t \int_{a(s)}^{b(s)} \mu(\rho) u_x^2\,dx\,ds,$$

then for $t_2 \geqslant t_1 > 0$,

$$H(t_2) = H(t_1) \leqslant H(0). \tag{2.2}$$

If the essential inf of ρ_0 is positive then in fact $H(t) = H(0)$ for all $t \geqslant 0$. Also, the following regularity conditions are satisfied, in which $C(w_1, \ldots)$ denotes a generic positive constant depending on P and μ and on upper bounds for $\|\rho_0\|_\infty$, M_0, E_0, and $w_1, \ldots$:

$$\big\|\rho(\cdot,t)\big\|_\infty \leqslant \begin{cases} C & \text{in case I or in cases II–IV if } \tilde P = 0, \\ C(t) & \text{otherwise}; \end{cases} \tag{2.3}$$

and for $\tau > 0$,

$$\langle u, F\rangle^{1/2,1/4}_{\{\tau \leqslant t \leqslant T\}} \leqslant \begin{cases} C(\tau^{-1}, M_0^{-1}) & \text{if } \tilde P = 0, \\ C(T, \tau^{-1}, M_0^{-1}) & \text{if } \tilde P > 0, \end{cases} \tag{2.4}$$

where $F \equiv \mu(\rho)u_x - P(\rho)$. Also, $u_x(\cdot, t)$, $F_x(\cdot, t) \in L^2((a(t), b(t)))$ for $t > 0$, $(F_x/\rho)_x \in L^2(V \cap \{\tau \leqslant t \leqslant T\})$ for $0 < \tau < T < \infty$, and the following energy estimates hold, in which $\sigma(t) = \min\{1, t\}$:

$$\sup_{0<t\leqslant T} \int_{a(t)}^{b(t)} \big[\sigma(t) u_x(x,t)^2\,dx + \sigma(t)^2\, F_x(x,t)^2\big]\,dx + \int_0^T \int_{a(t)}^{b(t)} \left[\sigma F_x^2 + \sigma^2 \left(\frac{F_x}{\rho}\right)_x^2\right] dx\,dt$$
$$\leqslant \begin{cases} C & \text{if } \tilde P = 0, \\ C(T) & \text{if } \tilde P > 0. \end{cases} \tag{2.5}$$

The boundary conditions $u = 0$ at fixed boundaries and $F = 0$ at free boundaries hold in a strict pointwise sense for $t > 0$ as described at the beginning of this section, and the following weak forms of (1.1) *hold: for $0 \leqslant t_1 \leqslant t_2$ and for φ Lipschitz on $\overline{V} \cap \{t_1 \leqslant t \leqslant t_2\}$,*

$$\int_{a(t)}^{b(t)} (\rho\varphi)(x,t)\,dx\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2}\int_{a(t)}^{b(t)} (\rho\varphi_t + \rho u\varphi_x)\,dx\,dt; \tag{2.6}$$

and for t_1, t_2 and φ as above with $\varphi = 0$ at fixed boundary points,

$$\begin{aligned}&\int_{a(t)}^{b(t)} (\rho u\varphi)(x,t)\,dx\Big|_{t=t_1}^{t=t_2}\\ &\quad= \int_{t_1}^{t_2}\int_{a(t)}^{b(t)} \big(\rho u\varphi_t + \rho u^2\varphi_x + P(\rho)\varphi_x - \mu(\rho)u_x\varphi_x\big)\,dx\,dt.\end{aligned} \tag{2.7}$$

Next, for each pair $t_1, t_2 \geqslant 0$ there is a bi-Lipschitz homeomorphism $X(\cdot; t_1, t_2) : [a(t_1), b(t_1)] \to [a(t_2), b(t_2)]$ such that

$$X(y; t_1, t_2) = y + \int_{t_1}^{t_2} u\big(X(y; t_1, s), s\big)\,ds; \tag{2.8}$$

$$\langle X\rangle^{1,1/2}_{V\cap\{0\leqslant t_1,t_2\leqslant T\}} \leqslant \begin{cases} C & \text{if } \tilde{P} = 0,\\ C(T) & \text{if } \tilde{P} > 0; \end{cases} \tag{2.9}$$

$$a(t) = X(a_0; 0, t) \quad \text{and} \quad b(t) = X(b_0; 0, t); \tag{2.10}$$

and if $y \in (a_0, b_0)$ is a Lebesgue point of ρ_0, then the map $t \to \rho(X(y; 0, t), t)$ is in $C^{1/4}([0,\infty)) \cap C^1((0,\infty))$, the representation

$$\rho\big(X(y;0,t),t\big) = \rho_0(y)\exp\left[-\int_0^t \big(\mu(\rho)^{-1}\big(F + P(\rho)\big)\big)\big(X(y;0,s),s\big)\,ds\right] \tag{2.11}$$

holds,

$$C(t)^{-1}\rho_0(y) \leqslant \rho\big(X(y;0,t),t\big) \leqslant \begin{cases} C\rho_0(y) & \text{in cases II–IV if } \tilde{P} = 0,\\ C(t)\rho_0(y) & \text{otherwise} \end{cases} \tag{2.12}$$

(C also depends on an upper bound for M_0^{-1} in case I*), and for almost all y,*

$$\frac{\partial X(y; t_1, t_2)}{\partial y} = \frac{\rho(y, t_1)}{\rho(X(y; t_1, t_2), t_2)} \tag{2.13}$$

for $t_1, t_2 \geqslant 0$. Finally, in cases II, III, *and* IV,

$$\limsup_{t\to\infty}\big[b(t) - a(t)\big] = \infty. \tag{2.14}$$

Observe that the continuity of a and b insures that the term $b(t) - a(t)$ in the definition of H just before (2.2) is bounded in finite time, so that each of the energy terms in the definition of H is bounded by $C(t)$ as well and by C in case I or when $\tilde{P} = 0$. We remark that the regularity of the mapping $t \to \rho(X(y;0,t),t)$ described just before (2.11) is closely related to a result of Hoff and Tsyganov [9] for the multidimensional case in which this same map is shown to extend to an analytic function of complex time.

Proof of Theorem 2. The proof consists of a sequence of *a priori* bounds for global-in-time approximate solutions (ρ^δ, u^δ) depending on a parameter δ. These bounds must be independent of δ and should be retained in the limit, say as $\delta \to 0$. A standard approach is to take (ρ^δ, u^δ) to be the smooth solution corresponding to mollified initial data $(\rho_0^\delta, u_0^\delta)$, where δ is the width of the mollifying kernel. We are unaware, however, of a global existence theory for smooth solutions as general as is required here. (But see Kawohl [11] for a smooth existence theory for the corresponding Navier–Stokes system for nonbarotropic flow in which temperature effects are included. Kawhol's result does not include the barotropic case, however.) We have therefore chosen instead to generate approximate solutions by the method introduced in [5]: the system (1.1) is first written in Lagrangean coordinates in which the convection terms are absent and the spatial domain is fixed. This Lagrangean system is then approximated by a system of semi-discrete difference equations in which the spatial variable is discretized and time remains continuous. These difference equations comprise a system of ordinary differential equations for which local existence is guaranteed and global existence follows from the discrete version of the energy bound (2.2). Discrete versions of the other bounds in (2.3) and (2.5) are then derived and are shown to be retained in the limit as the discretization goes to zero. The limiting solution is reconverted to Eulerian coordinates and the other bounds and properties in the statement of the theorem are shown to follow as consequences. While completely rigorous, this approach is quite technical and lengthy and may obscure the main analytical ideas. We therefore do not report these details here but instead choose for the sake of clarity and economy to give a more heuristic exposition in which it is assumed that there is a global smooth solution (ρ^δ, u^δ) corresponding to smooth approximate initial data $(\rho_0^\delta, u_0^\delta)$. We give formal derivations of the bounds in (2.1)–(2.5) and (2.12) and then show how the solution of the theorem is obtained in the limit as $\delta \to 0$ under the assumption that $\rho_0^\delta \to \rho_0$ and $\rho_0^\delta u_0^\delta \to \rho_0 u_0$ in $L^1((0,1))$ as $\delta \to 0$. For the time being we drop the superscript δ.

First, the conservation of mass (2.1) is immediate from (1.1), and to derive the energy balance (2.2) we compute from (1.1) that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \left[\tfrac{1}{2}\rho u^2 + G(\rho)\right] dx = \int_{a(t)}^{b(t)} \left[(\mu u_x)_x u + \left((\tilde{P} - P)u\right)_x\right] dx, \tag{2.15}$$

which together with the boundary conditions implies (2.2) with equality: $H(t) = H(0)$ for all t. Next we derive an estimate for the term $b(t) - a(t)$ appearing in (2.2). We compute

$$\dot{b} - \dot{a} = u|_a^b = \int_a^b u_x\,dx \leqslant C\left(\int_a^b \mu u_x^2\,dx\right)^{1/2} (b-a)^{1/2}$$

and then integrate this separable ode in t and apply (2.2) to bound the space–time integral of μu_x^2 in terms of $b - a$. The result is that

$$b(t) - a(t) \leqslant C(t).$$

This together with (2.15) then shows that

$$\sup_{0\leqslant t\leqslant T}\int_{a(t)}^{b(t)}\big[\big(\tfrac{1}{2}\rho u^2+G(\rho)\big)(x,t)\big]\,dx+\int_0^T\int_{a(s)}^{b(s)}\mu(\rho)u_x^2\,dx\,ds$$
$$\leqslant\begin{cases}C & \text{if } \tilde{P}=0,\\ C(T) & \text{if } \tilde{P}>0.\end{cases}\tag{2.16}$$

The next step is to derive the pointwise bounds (2.3) and (2.12) for ρ. We give the details only for case I, which is the most complicated. In this case $b(t)-a(t)\equiv 1$ so that the $\tilde{P}$ terms in the definition of H are constant in t and the remaining terms on the left are separately bounded by E_0. We begin with the following technical fact about mass transport, which we derive from the convexity of G on $[\overline{\rho},\infty)$ (recall the hypotheses (A_P)):

Lemma. *There is a $\rho'>0$ depending on P and on upper bounds for E_0 and M_0^{-1} such that for case* I *and for any $t_2\geqslant t_1\geqslant 0$,*

$$\int_{\{y:\ \rho(X(y;t_1,t_2),t_2)\leqslant\rho'\}}\rho(y,t_1)\,dy\geqslant\tfrac{1}{2}M_0.$$

Proof. Fix $t_2\geqslant t_1$ and define $A\equiv\{x:\ \rho(x,t_2)\geqslant\rho'\}$ for a given $\rho'>0$. Then $|A|\leqslant M_0/\rho'$ by (2.1), and therefore from Jensen's inequality,

$$\begin{aligned}G\bigg(\frac{1}{|A|}\int_A\rho(x,t_2)\vee\overline{\rho}\,dx\bigg)&\leqslant\frac{1}{|A|}\int G\big(\rho(x,t_2)\big)\vee\overline{\rho}\big)\,dx\\&\leqslant\frac{1}{|A|}\int\big[G\big(\rho(x,t_2)\big)+G(\overline{\rho})\big]\,dx\\&\leqslant\frac{1}{|A|}\big(E_0+G(\overline{\rho})\big),\end{aligned}$$

so that

$$\int_A\rho(x,t_2)\,dx\leqslant|A|G^{-1}\bigg(\frac{1}{|A|}\big(E_0+G(\overline{\rho})\big)\bigg).$$

The quantity on the right above will be bounded above by $\frac{1}{2}M_0$ if $E_0+G(\overline{\rho})\leqslant|A|G(M_0/2|A|)$. However, $G(\rho)\geqslant c\rho\log\rho$ by our hypotheses (A_P), so the latter requirement will hold if $2(E_0+G(\overline{\rho}))\leqslant cM_0\log(M_0/2|A|)$, and therefore if ρ' is sufficiently large, because $M_0/|A|\geqslant\rho'$. The integral of $\rho(\cdot,t_2)$ over A^c will then be bounded below by $\frac{1}{2}M_0$, and this integral is the same as the integral in the conclusion of the lemma, by the conservation of mass. □

Next we derive a preliminary upper bound for $\|\rho(\cdot,t)\|_\infty$, again for case I. To do this we let $m(\rho)$ be the antiderivative of $\mu(\rho)/\rho$ satisfying $m(1)=0$. We then have from (1.1) that, for any two trajectories $x(t)=X(y;0,t)$ and $x_0(t)=X(y_0;0,t)$,

$$\frac{d}{dt}\big[m\big(\rho(\cdot,t)\big)\big|_{x_0(t)}^{x(t)}+I(t)\big]+P\big(\rho(\cdot,t)\big)\big|_{x_0(t)}^{x(t)}=0,\tag{2.17}$$

where

$$I(t) = \int_{x_0(t)}^{x(t)} (\rho u)(s,t)\,ds.$$

Note that $|I(t)| \leqslant C$ by (2.1) and (2.2). Now fix $\rho_1 > \max\{\|\rho_0\|_\infty, \overline{\rho}, \rho'\}$ where $\overline{\rho}$ and ρ' are as in (A_P) and in the lemma respectively, and choose $\rho_2 > \rho_1$. Fix y and therefore $x(t)$ and suppose that, at some time $t_2 > 0$, $\rho(x(t_2), t_2) = \rho_2$. There is then a time $t_1 \in (0, t_2)$ such that $\rho(x(t_1), t_1) = \rho_1$ and $\rho(x(t), t) \in [\rho_1, \rho_2]$ for $t \in [t_1, t_2]$. Temporarily fix another trajectory $x_0(t) = X(y_0; 0, t)$. Then by hypothesis (c) in (A_P),

$$\alpha(t) \equiv \frac{P(\rho(x(t),t)) - P(\rho(x_0(t),t))}{m(\rho(x(t),t)) - m(\rho(x_0(t),t))} \geqslant 0$$

for every $t \in [t_1, t_2]$. We replace the pressure term in (2.17) by $\alpha m|_{x_0}^{x}$ and integrate the resulting linear ode in t. An easy estimate for the integrating factor using the nonnegativity of α then shows that there is a $\theta \in (0,1)$ such that

$$m|_{\rho(x_0(t_2),t_2)}^{\rho_2} = \theta m|_{\rho(x_0(t_1),t_1)}^{\rho_1} + O(1).$$

In particular, for choices of y_0 for which $\rho(x_0(t_2), t_2) \leqslant \rho'$,

$$\log \frac{\rho_2}{\rho(x_0(t_2),t_2)} \leqslant C \log \frac{\rho_1}{\rho(x_0(t_1),t_1)} + C$$

by our assumption (b) in (A_μ), so that

$$\rho_2^{1/C} \rho(y_1, t_1) \leqslant C \rho_1 \rho\big(x_0(t_2), t_2\big)^{1/C} \leqslant \rho_1 \big(\rho'\big)^{1/C} \leqslant C,$$

where $y_1 \equiv x_0(t_1) = X(y_0; 0, t_1)$ (recall $x_0(t) = X(y_0; 0, t) = X(y_1; t_1, t)$). We now integrate over the set $\{y_1\colon\ \rho(x_0(t_2), t_2) = \rho(X(y_1; t_1, t_2), t_2) \leqslant \rho'\}$ and apply the lemma to obtain

$$\tfrac{1}{2} M_0 \rho_2^{1/C} \leqslant C.$$

This shows that ρ_2 cannot be arbitrarily large and therefore proves that

$$\big\|\rho(\cdot, t)\big\| \leqslant C\big(M_0^{-1}\big) \tag{2.18}$$

for all t.

To derive the more refined bounds (2.12), again for case I, we define $g(y,t) = m|_{\rho_0(y)}^{\rho(X(y;0,t),t)}$, write the ode for $g(y,t) - g(y_0,t)$, where $y, y_0 \in (0,1)$, integrate as above, and apply (2.18) to bound the terms involving P. The result is that

$$\int_{\rho_0(y)}^{\rho(X(y;0,t),t)} \frac{\mu(s)}{s}\,ds \leqslant \int_{\rho_0(y_0)}^{\rho(X(y_0;0,t),t)} \frac{\mu(s)}{s}\,ds + C(1+t).$$

Recalling that $\mu(\rho) \geqslant \underline{\mu} > 0$, we then obtain that

$$\left[\frac{\rho(X(y;0,t),t)}{\rho_0(y)}\right]^{\underline{\mu}/C} \rho_0(y_0) \leqslant C e^{Ct}.$$

The upper bound in (2.12) then follows by integration with respect to y_0, and the proof of the lower bound is similar. This completes the proofs of (2.3) and (2.12) for case I. For the other cases the argument can be simplified considerably by choosing x_0 in (2.17) to be one of the free boundary points and then observing that the two terms in (2.17) involving $x_0(t)$ cancel by the boundary condition $F(x_0(t), t) = 0$. We omit the details.

Once the L^∞ bound for ρ in (2.3) has been proved, the bounds in (2.5) can be derived in a relatively straightforward but somewhat lengthy sequence of energy estimates. The essential points are that careful attention must be paid to the initial layer effects involving σ and that the bounds are closed with no dependence on regularity properties of ρ. The details are essentially the one-dimensional versions of estimates for the multidimensional case first derived in [6] and [7]. The Hölder continuity estimates in (2.4) then follow in a straightforward way from (2.5), the details again being nearly identical to those in [6] and [7]. We do note here for later reference the bound

$$\bigl\| F(\cdot, t) \bigr\|_\infty \leqslant C\bigl(1 + t^{-1/2}\bigr), \tag{2.19}$$

which follows directly from (2.5).

Next, for the smooth solutions presently under consideration, we may take (2.8) to be the *definition* of X, so that the Hölder continuity (2.9) follows directly from the bounds in (2.4), and the representations (2.11) and (2.13) follow from the differential equations (1.1) by straightforward calculus.

Again, all of the above analysis applies to smooth approximate solutions $(\rho^\delta, u^\delta), a^\delta, b^\delta$ with corresponding particle trajectories $X^\delta(\cdot; t_1, t_2)$ and fluxes $F^\delta \equiv \mu(\rho^\delta)u^\delta_x - P(\rho^\delta)$, and with initial data ρ^δ_0 and u^δ_0 which are standard mollifiers of ρ_0 and u_0. In particular, the bounds in (2.1)–(2.12) apply with constants C independent of δ. Now, it is clear from the uniform Hölder continuity of a^δ, b^δ and X^δ that there is a subsequence $\delta \to 0$ such that a^δ and b^δ converge uniformly on bounded sets in $[0, \infty)$ to functions a and b and that X^δ converges uniformly on bounded sets in $[a_0, b_0] \times [0, \infty) \times [0, \infty)$ to a function X. We then define the open set $V \equiv \{(x, t)\colon\ t > 0 \text{ and } a(t) < x < b(t)\}$ and apply the uniform Hölder continuity (2.4) to obtain a further subsequence $\delta \to 0$ such that u^δ and F^δ converge uniformly on compact sets in V to functions u and F. It is clear that the limiting functions inherit the bounds and properties in (2.4), (2.5) (except for the statements and bounds concerning $(F_x/\rho)_x$), and (2.8)–(2.10). It is also easy to see that for each $t > 0$, $u(\cdot, t)$ extends Hölder continuously to zero at a fixed boundary point and that $F(\cdot, t)$ extends Hölder continuously to zero at a free boundary point.

The convergence of ρ^δ is somewhat more complicated. We take $\rho^\delta(x, t) = 0$ for $x \notin (a^\delta(t), b^\delta(t))$ and claim that for the same sequence $\delta \to 0$ fixed above and for each $t \geqslant 0$, $\rho^\delta(\cdot, t)$ converges strongly to a function $\rho(\cdot, t)$ in $L^2(a(t), b(t))$. To prove this we compute from (1.1) or (2.11) that, at a point $(X(y; 0, t), t)$,

$$\frac{d}{dt}\rho^\delta = -\mu\bigl(\rho^\delta\bigr)^{-1}\rho^\delta\bigl(F^\delta + P\bigl(\rho^\delta\bigr)\bigr).$$

Writing this relation for $\delta = \delta_1$ and δ_2 and subtracting and integrating, we find that, for $t \geqslant 0$,

$$\begin{aligned}
&\bigl|\rho^{\delta_2}\bigl(X^{\delta_2}(y; 0, t), t\bigr) - \rho^{\delta_1}\bigl(X^{\delta_1}(y; 0, t), t\bigr)\bigr| \\
&\quad\leqslant C(t)\biggl[\bigl|\rho_0^{\delta_2}(y) - \rho_0^{\delta_1}(y)\bigr| + \int_0^t \bigl|F^{\delta_2}\bigl(X^{\delta_2}(y; 0, s), s\bigr) - F^{\delta_1}\bigl(X^{\delta_1}(y; 0, s), s\bigr)\bigr|\,ds\biggr].
\end{aligned}$$

This together with the strong convergence of F^δ shows that $\rho^\delta(X^\delta(y; 0, t), t)$ converges, say to $r(y, t)$, for every Lebesgue point $y \in (a_0, b_0)$ of ρ_0. Also, writing (2.13) for the approximate solution with $t_1 = 0$ and noting that $\frac{\partial X^\delta}{\partial y}(\cdot, t) \to \frac{\partial X}{\partial y}(\cdot, t)$ in $\mathcal{D}'((a(0), b(0)))$, we obtain

$$\frac{\partial X}{\partial y}(y; 0, t) = \frac{\rho_0(y)}{r(y, t)} \tag{2.20}$$

for almost all y. We now define $\rho(x,t) = r(X(x;t,0),t)$ and claim that $\rho^\delta(\cdot,t) \to \rho(\cdot,t)$ strongly in L^2 for each t. First, it is easy to show that $\rho^\delta(\cdot,t) \rightharpoonup \rho(\cdot,t)$ weakly in $L^2((a(t),b(t)))$, and to check weak convergence of the squares we take $\varphi \in \mathcal{D}((a(t),b(t)))$ and apply (2.20) and the corresponding statement for the approximate solutions to compute that

$$\begin{aligned}
\int \rho^\delta(x,t)^2\varphi(x)\,dx &= \int \rho^\delta\big(X^\delta(y;0,t),t\big)^2\varphi\big(X^\delta(y;0,t)\big)\frac{\rho_0^\delta(y)}{\rho^\delta(X^\delta(y;0,t),t)}\,dy \\
&= \int \rho^\delta\big(X^\delta(y;0,t),t\big)\varphi\big(X^\delta(y;0,t)\big)\rho_0^\delta(y)\,dy \\
&\to \int r(y,t)\varphi\big(X(y;0,t)\big)\rho_0(y)\,dy = \int r\big(X(x;t,0),t\big)^2\varphi(x)\,dx \\
&= \int \rho(x,t)^2\varphi(x)\,dx.
\end{aligned}$$

The extension to test functions $\varphi \in L^2(a(t),b(t))$ is straightforward. This proves the claim that $\rho^\delta(\cdot,t)$ converges to $\rho(\cdot,t)$ strongly in $L^2(a(t),b(t))$ for every $t \geqslant 0$. It then follows in a straightforward way that (2.1), (2.3), and (2.11)–(2.13) and the weak forms (2.6) and (2.7) hold, that $F = \mu(\rho)u_x - P(\rho)$, and that F_x satisfies the remaining estimates in (2.5).

Next we prove the energy relations (2.2). First, we have that for the approximate solutions (ρ^δ, u^δ) and for times $t \geqslant \tau \geqslant 0$,

$$\begin{aligned}
&\int_{a^\delta(\cdot)}^{b^\delta(\cdot)} \big[\big(\tfrac{1}{2}\rho^\delta\big(u^\delta\big)^2 + G\big(\rho^\delta\big)\big)(x,\cdot)\big]\,dx\Big|_\tau^t + \int_\tau^t \int_{a^\delta(s)}^{b^\delta(s)} \mu\big(\rho^\delta\big)\big(u_x^\delta\big)^2\,dx\,ds \\
&\quad = E_0 + \tilde{P}\big(b^\delta - a^\delta\big)\big|_\tau^t. \qquad (2.21)
\end{aligned}$$

If we set $\tau = 0$ and then let $\delta \to 0$ we obtain from elementary weak convergence considerations that $H(t) \leqslant H(0)$ for all t. On the other hand if we fix $\tau > 0$ in (2.21) and then let $\delta \to 0$ we obtain that $H(t) = H(\tau)$ for $t \geqslant \tau > 0$. Indeed, the single integrals and the term on the right in (2.21) converge to the corresponding terms with a^δ, b^δ, ρ^δ and u^δ replaced by a, b, ρ and u by the Hölder continuity (2.4) and (2.9) and the modes of convergence described above. Convergence of the double integral follows from the uniform convergence of X^δ and $F^\delta = \mu(\rho^\delta)u_x^\delta - P(\rho^\delta)$ away from $t = 0$ and from the strong L^2 convergence of $\rho^\delta(\cdot,t)$. We thus have that $H(t) = H(\tau) \leqslant H(0)$ for $t \geqslant \tau > 0$, and this proves (2.2). To prove equality when the essential inf of ρ_0 is positive, we write

$$H(\tau) = \int_{a(\tau)}^{b(\tau)} \big[\big(\tfrac{1}{2}\rho u^2 + G(\rho)\big)(x,\tau)\big]\,dx + \int_0^\tau \int_{a(s)}^{b(s)} \mu(\rho)u_x^2\,dx\,ds - \tilde{P}\big(b(\tau) - a(\tau)\big) \qquad (2.22)$$

and take the limit as $\tau \to 0$. It is clear that the last term on the right converges to $\tilde{P}(b_0 - a_0)$ and that the middle term goes to zero. Also, we can apply (2.13) to write the term involving G as

$$\int_{a_0}^{b_0} G\big(\rho\big(X(y;0,\tau),\tau\big)\big)\frac{\rho_0(y)}{\rho(X(y;0,\tau),\tau)}\,dy.$$

The integrand here converges pointwise a.e. to $G(\rho_0(y))$ as $\tau \to 0$ by (2.11) and (2.19), so the integral converges to $\int_{a_0}^{b_0} G(\rho_0(y))\,dy$ by (2.12) and the bounded convergence theorem. In fact, this same

argument shows that $\int_{a(\tau)}^{b(\tau)} f(\rho(x,t))\,dx$ converges to $\int_{a_0}^{b_0} f(\rho_0(x))\,dx$ for any continuous $f:[0,\infty)\to[0,\infty)$. We shall apply this fact to obtain the convergence of the remaining term on the right side of (2.22). First observe that, by our assumption that $\operatorname{ess\,inf}\rho_0>0$ and by (2.12), $\rho(\cdot,t)$ is essentially bounded below away from zero for finite times, and therefore $\|u(\cdot,t)\|_{L^2}\leqslant C$, say for $t\in[0,1]$, by (2.2). Temporarily take both $\rho(\cdot,t)$ and $u(\cdot,t)$ to be zero outside $(a(t),b(t))$. We then have from the weak form (2.7) that $(\rho u)(\cdot,\tau)\to\rho_0u_0$ in H^{-1} as $\tau\to0$. On the other hand, the uniform L^2 bound for $u(\cdot,\tau)$ shows that, modulo subsequences, $u(\cdot,t)\rightharpoonup w$ weakly in L^2 for some w. But this together with the strong convergence $\rho(\cdot,\tau)\to\rho_0$ implies that $(\rho u)(\cdot,\tau)\rightharpoonup\rho_0 w$, so that in fact $w=u_0$ and therefore $u(\cdot,\tau)\rightharpoonup u_0$ as $\tau\to0$. The weak convergence $(\sqrt{\rho}u)(\cdot,\tau)\to\sqrt{\rho_0}u_0$ then follows, and as a consequence,

$$\liminf_{\tau\to0}\int_{a(\tau)}^{b(\tau)}\bigl(\rho u^2\bigr)(x,\tau)\,dx\geqslant\int_{a_0}^{b_0}\rho_0u_0^2\,dx.$$

Thus $\liminf_{\tau\to0}H(\tau)\geqslant H(0)$, and this together with (2.2) proves that $H(t)=H(0)$ for all t when $\operatorname{ess\,inf}\rho_0>0$.

Finally we prove the large-time result (2.14) for the case that at least one of the boundary points, say b, is free. We let $L^\delta(t)\equiv b^\delta(t)-a^\delta(t)$, temporarily omit the δ, and compute

$$L_t=u\big|_a^b=\int_a^b u_x\,dx=\int_a^b\mu^{-1}(F+P). \tag{2.23}$$

Straightforward computations show that

$$\int_a^b\mu^{-1}F\,dx=-\frac{d}{dt}\int_a^b g\rho u\,dx+\int_a^b\rho u(g_t+ug_x)\,dx \tag{2.24}$$

where

$$g(x,t)\equiv\int_a^x\frac{dy}{\mu(\rho(y,t))}$$

and

$$g_t+ug_x=\int_a^x\bigl[\mu^{-1}+\rho\mu^{-2}\mu'\bigr]u_x\,dx.$$

Thus

$$|g_t+ug_x|\leqslant C\int_a^b|u_x|\,dx$$

where the constant C depends on an upper bound for $\|\rho(\cdot,t)\|_\infty$. Substituting into (2.23) and replacing the δ, we then obtain

$$L_t^\delta \geqslant -\frac{d}{dt}\int_{a^\delta}^{b^\delta} g^\delta \rho^\delta u^\delta - C\left(\int_{a^\delta}^{b^\delta} \rho^\delta |u^\delta|\right)\left(\int_{a^\delta}^{b^\delta} |u_x^\delta|\right) + \int_{a^\delta}^{b^\delta} (\mu^\delta)^{-1} P^\delta. \tag{2.25}$$

Now suppose that, contrary to (2.14), $L(t) \leqslant \bar{L} < \infty$ for all t. Then given any $T > 0$ there is a positive $\delta_0(T)$ such that $L^\delta(t) \leqslant 2\bar{L}$ for $\delta \leqslant \delta_0$ and $t \leqslant T$. The bounds in (2.2) and (2.3) will then be independent of t for these δ and t and therefore so will be the constant C in (2.25). Integrating (2.25) and noting that, from its definition, $|g^\delta(x,t)| \leqslant CL^\delta \leqslant C\bar{L}$, we then obtain

$$L^\delta(T) \geqslant \int_0^T \int_{a^\delta}^{b^\delta} (\mu^\delta)^{-1} P^\delta \, dx\, dt - C - CT^{1/2}. \tag{2.26}$$

Clearly,

$$\int_{[a^\delta,b^\delta]\cap\{\rho^\delta < \frac{M_0}{4\bar{L}}\}} \rho^\delta \, dx \leqslant \tfrac{1}{2} M_0$$

for any time $t \leqslant T$, since $L^\delta(t) \leqslant 2\bar{L}$, and therefore

$$\int_{[a^\delta,b^\delta]\cap\{\rho^\delta \geqslant \frac{M_0}{4\bar{L}}\}} \rho^\delta \, dx \geqslant \tfrac{1}{2} M_0$$

by the conservation of mass. Also, our hypotheses on P insure that there is a positive C such that $P(\rho) \geqslant C^{-1}\rho$ for $\rho \in [\frac{M_0}{4\bar{L}}, \|\rho\|_\infty]$. Thus for any $t \leqslant T$ and for a new constant C,

$$\begin{aligned}\int_{a^\delta}^{b^\delta} P(\rho^\delta)\, dx &\geqslant \int_{[a^\delta,b^\delta]\cap\{\rho^\delta > \frac{M_0}{4\bar{L}}\}} P(\rho^\delta)\, dx \\ &\geqslant C^{-1} \int_{[a^\delta,b^\delta]\cap\{\rho^\delta > \frac{M_0}{4\bar{L}}\}} \rho^\delta \, dx \geqslant C^{-1}.\end{aligned}$$

Substituting into (2.26) and then taking the limit as $\delta \to 0$, we conclude that

$$L(T) \geqslant C^{-1}T - C - CT^{1/2}$$

for all T. This contradicts the assumption that $L(T) \leqslant \bar{L}$ for all T, however, and therefore proves (2.14). This completes the proof of Theorem 2. □

3. Finitely many fluids

In this section we construct solutions of (1.1) in the sense of the Definition of Section 1 for the case that the initial data consists of finitely many fluids on $(0,1)$.

Theorem 3. *Assume that the functions P and μ satisfy (A_P) and (A_μ) and let ρ_0 and u_0 be given measurable functions defined on $V_0 \equiv \bigcup_{j=1}^N (a_j(0), b_j(0))$, where $\{(a_j(0), b_j(0))\}$ are open intervals contained in $(0,1)$ with disjoint closures. Assume that $\rho_0 \in L^\infty$ is positive a.e. on V_0, that*

$$E_0 \equiv \sum_{j=1}^N \int_{a_j(0)}^{b_j(0)} \left[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\right] dx < \infty,$$

and that there is a constant C_1 such that, for each j,

$$\tilde{P}\big(b_j(0) - a_j(0)\big) + \int_{a_j(0)}^{b_j(0)} \left[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\right] dx \leqslant C_1 \int_{a_j(0)}^{b_j(0)} \rho_0\, dx. \tag{3.1}$$

Then there is a corresponding solution (ρ, u) to (1.1) with initial data (ρ_0, u_0, V_0) in the sense of the Definition of Section 1. The mass equation

$$\int_{V_t} \rho(x,t)\, dx = \int_{V_0} \rho_0(x)\, dx \equiv M_0 \tag{3.2}$$

holds for $t \geqslant 0$, where V_t is as in the Definition; and if $H(t)$ is the energy functional

$$H(t) = \int_{V_t} \left[\big(\tfrac{1}{2}\rho u^2 + G(\rho)\big)(x,t)\right] dx - \tilde{P}|V_t| + \int_0^t \int_{V_s} \mu(\rho) u_x^2\, dx\, ds,$$

then

$$H(t_2) = H(t_1) \leqslant H(0) = E_0 - \tilde{P}|V_0| \tag{3.3}$$

for $t_2 \geqslant t_1 > 0$. If $\operatorname{ess\,inf}_{V_0} \rho_0 > 0$ then $H(t) = H(0)$ for all $t \geqslant 0$. Also, the following regularity conditions are satisfied, in which C is a generic positive constant as described in Theorem 2 and which is independent of N:

$$\big\|\rho(\cdot,t)\big\|_\infty \leqslant C; \tag{3.4}$$

$$\int_0^t \big\|u(\cdot,s)\big\|_\infty^2\, ds \leqslant C(1+t); \tag{3.5}$$

there are times $0 < t_1 < t_2 < \cdots < t_{K+1} = \infty$ such that u is locally Hölder continuous on each of the sets $\{(x,t)\colon\ x \in V_t,\ t_k < t < t_{k+1}\}$; and there is a map X from $V_0 \times [0,\infty)$ onto V_t such that for $y \in V_0$ and $0 \leqslant t_1, t_2 \leqslant T$,

$$\bigl|X(y,t_2)-X(y,t_1)\bigr| \leqslant C(T)|t_2-t_1|^{1/2}, \tag{3.6}$$

$$X(y,t) = y + \int_0^t u\bigl(X(y,s),s\bigr)\,ds, \tag{3.7}$$

and $V_t = \bigcup_{j=1}^N (a_j(t), b_j(t))$, *where* $a_j(t) = X(a_j(0),t)$ *and* $b_j(t) = X(b_j(0),t)$. *Also, for almost all* $y \in V_0$ *the map* $t \to \rho(X(y,t),t)$ *is in* $C^{1/4}([0,\infty))$ *and*

$$C(t)^{-1} \leqslant \frac{\partial X}{\partial y}(y,t) = \frac{\rho_0(y)}{\rho(X(y,t),t)} \leqslant C(t). \tag{3.8}$$

Finally, there is a time $\overline{T} < \infty$ *which may depend on a lower bound for* M_0 *but which is independent of N such that* $\overline{V}_t = [0,1]$ *for all* $t \geqslant \overline{T}$.

Proof. First the fluid intervals may be ordered left to right, so that $b_j(0) < a_{j+1}(0)$ for all j. We apply Theorem 2 to solve (1.1) with initial data (ρ_0, u_0) restricted to $(a_j(0), b_j(0))$ for each j, imposing free boundary conditions at a_j for $j > 1$ and at b_j for $j < N$, and fixed or free boundary conditions at a_1 according as $a_1(0) = 0$ or $a_1(0) > 0$, and similarly for b_N. We say that a collision occurs at time $t > 0$ if $a_1(t) = 0$, $b_N(1) = 1$, or $b_j(t) = a_{j+1}(t)$, with nonequality at earlier times. The continuity of $a_j(t)$ and $b_j(t)$ together with (2.14) guarantee that there is a first time $t_1 \in (0,\infty)$ at which one or more collisions occur. We then solve new initial-value problems at the new initial time t_1, at which either a free boundary point a_1 or b_N has become a fixed boundary point or two adjacent intervals (a_j, b_j) and (a_{j+1}, b_{j+1}) have merged into a single interval (a_j, b_{j+1}). It is clear that the new initial conditions $(\rho(\cdot,t_1), u(\cdot,t_1))$ satisfy the hypotheses of Theorem 2 on the new fluid intervals and that the solution process may therefore be continued until the next collision time t_2. By (2.14) there will be at most finitely many collision times $t_1,\dots,t_K$, and after time t_K the intervals (a_j, b_j) will have merged into the single interval $(0,1)$ (the endpoints comprising a set of measure zero). It is also clear that the integral curves $X(y,t) \equiv X(y;0,t)$ can be matched Lipschitz-continuously across the collision times. If we define V_t as in the statement preceding (3.8) and V as in the Definition of Section 1, then (ρ, u, V) will be a weak solution of (1.1) in the required sense and (3.2), (3.3), (3.6), (3.7), and the equality in (3.8) will hold.

Next we prove the final statement of the theorem that the last collision time t_K is bounded above by a constant $\overline{T}$ which is independent of N. Let $L(t) = |V_t|$ and compute just as in the proof of (2.14) that, except at collision times,

$$\frac{dL}{dt} = \int_{V_t} \mu(\rho)^{-1}(F+P)\,dx. \tag{3.9}$$

Now let $t < t_K$ and suppose that $\{(c_i,d_i)\}$ are the disjoint fluid intervals present at time t (that is, these intervals have disjoint closures and their union is the interior of the union of the closures of the intervals $\{(a_j,b_j)\}$). Since $t < t_K$, each interval (c_i,d_i) has at least one free boundary point. If d_i is free we define

$$g_i(x,t) = \int_{c_i}^x \frac{ds}{\mu(\rho(s,t))}$$

for $x \in (c_i, d_i)$ and if c_i is free we replace the lower limit above by d_i. Then just as in the proof of (2.14), $|g_i(x,t)| \leqslant C(d_i - c_i)$,

$$|g_{i,t} + u g_{i,x}| \leqslant \int_{c_i}^{d_i} |u_x|\,dx,$$

and

$$\int_{c_i}^{d_i} \mu(\rho)^{-1} F = \frac{d}{dt} \int_{c_i}^{d_i} g_i \rho u\,dx - \int_{c_i}^{d_i} \rho(g_{i,t} + u g_{i,x})\,dx.$$

Substituting into (3.9), summing over i, and applying (3.2) and (3.3) we then obtain

$$\frac{dL}{dt} \geqslant \int_{V_t} \mu(\rho)^{-1} P\,dx + \sum_i \frac{d}{dt} \int_{c_i}^{d_i} g_i \rho u\,dx - C\Big(\int_{V_t} u_x\,dx\Big)^{1/2}.$$

The argument given above in the proof of (2.14) shows that the first term on the right here is bounded below by a positive constant depending on a lower bound for M_0, so that by (3.3) again,

$$L(t) \geqslant C^{-1} t - C - C t^{1/2}.$$

This holds as long as $t < t_K$. Since $L(t) \leqslant 1$ for all t, t_K must be bounded above by a constant independent of N.

Next we derive the pointwise bounds in (3.4) and (3.8). The argument is similar to that given above in the proof of Theorem 2, the essential point being that the bounds must be independent of N. First, just as in (2.17) we have that, for almost all $y \in V_0$ and for t different from a collision time,

$$\frac{d}{dt} m\big(\big(\rho\big(X(y,t),t\big)\big)\big) = -(F + P)$$

where $m'(\rho) = \mu(\rho)/\rho$. Then since $\rho(X(y,t),t)$ is continuous at collision times by construction,

$$m\big(\rho\big(X(y,t),t\big)\big) = m\big(\rho_0(y)\big) - \int_0^t (F+P)\big(X(y,s),s\big)\,ds. \tag{3.10}$$

Now, the construction also shows that there must be one boundary point, say a_1, which is free at every time t before the final collision time t_K. The boundary condition $F(a_1(t),t) = 0$ together with (1.1) then implies at least formally that for such t,

$$\frac{d}{dt} \int_{a_1(t)}^{X(y,t)} (\rho u)(x,t)\,dx = F\big(X(y,t),t\big),$$

so that

$$\int_{a_1(\cdot)}^{X(y,\cdot)} (\rho u)(x,\cdot)\,dx\Big|_0^t = \int_0^t F\big(X(y,s),s\big)\,ds.$$

(The rigorous justification is an easy argument based on the weak form (2.7) of the momentum equation.) We substitute the above result into (3.10) and note that $\int_{V_t} \rho|u|\,dx \leqslant C$ by (3.2) and (3.3). Discarding the P term, which is nonnegative, we obtain the upper bound $\rho(X(y,t),t) \leqslant C\rho_0(y)$ for almost all y; then applying this upper bound in (3.10) we obtain

$$\int_{\rho_0(y)}^{\rho(X(y,t),t)} \frac{\mu(s)}{s}\,ds \geqslant -C - Ct,$$

from which the lower bound in (3.8) follows. These estimates prove (3.4) and (3.8) up to the final collision time t_K. For $t > t_K$ $V_t = (0,1)$ up to a set of measure zero and we simply apply the results (2.3) and (2.12) for case I of Theorem 2. Observe that the lower bound in (3.8) is independent of N because there is an upper bound for t_K which is independent of N.

There remains to prove the bound (3.5) for u. We first claim that there is a constant C such that if t is not a collision time and if $(a_j(t), b_k(t))$ is a fluid interval at time t, that is, if $(a_j(t), b_k(t)) \subset V_t$ except for a finite number of points, then there is an $\bar{x} \in (a_j(t), b_k(t))$ such that $|u(\bar{x},t)| \leqslant C$. Granting this for the moment we then have that, if t is not a collision time and if x is in such a fluid interval $(a_j(t), b_k(t))$, then

$$\big|u(x,t)\big| \leqslant C + \int_{a_j(t)}^{b_k(t)} \big|u_x(x,t)\big|\,dx.$$

It follows that

$$\big\|u(\cdot,t)\big\|_\infty \leqslant C + \int_{V_t} \big|u_x(x,t)\big|\,dx$$

for all but finitely many t; (3.5) then follows from (3.3). Now to prove the existence of $\bar{x}$, first for times $t > t_K$, we recall from (2.4) that $u(\cdot,t)$ is Hölder continuous on $(0,1)$, so that there is a point $\bar{x} \in (0,1)$ such that

$$\int_0^1 \big(\rho u^2\big)(x,t)\,dx = u(\bar{x},t)^2 \int_0^1 \rho(x,t)\,dx = u(\bar{x},t)^2 M_0.$$

The left side here is bounded above by $2(E_0 + \tilde{P})$ by (3.3) and so $u(\bar{x},t)^2 \leqslant C$, as required. Finally, if $t < t_K$ is not a collision time and if $(a_j(t), b_k(t))$ is a maximal fluid interval, then the energy relation (3.3) holds for the isolated fluid on (a_j, b_k) for times less than or equal to t. Since $u(\cdot,t)$ is Hölder continuous on $(a_j(t), b_k(t))$, there must be an $\bar{x} \in (a_j(t), b_k(t))$ such that

$$\tfrac{1}{2}u(\bar{x},t)^2 \int_{a_j(t)}^{b_k(t)} \rho(x,t)\,dx = \int_{a_j(t)}^{b_k(t)} \tfrac{1}{2}\big(\rho u^2\big)(x,t)\,dx$$

$$\leqslant \int_{a_j(0)}^{b_k(0)} \left[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\right] dx + \tilde{P}\big(b_k(t) - a_j(t)\big)$$

$$\leqslant C_1 \int_{a_j(0)}^{b_k(0)} \rho_0\, dx + \tilde{P}\big(b_k(t) - a_j(t)\big) \tag{3.11}$$

by (3.1). To complete the argument we need a bound for $b_k(t) - a_j(t)$. Let $(a_i(t), b_i(t))$ be a subset of $(a_j(t), b_k(t))$; then for noncollision times $s < t$,

$$\frac{d}{ds}\big(b_i(s) - a_i(s)\big) = \int_{a_i(s)}^{b_i(s)} u_x\, dx \leqslant C\big(b_i(s) - a_i(s)\big)^{1/2}\left(\int_{a_i(s)}^{b_i(s)} \mu u_x^2\, dx\right)^{1/2}.$$

We integrate this separable ode with respect to s, use the fact that $b_i - a_i$ is continuous at collision times, sum over i, and then apply (3.1) and (3.3) to obtain

$$b_k(t) - a_j(t) \leqslant C\big[\big(b_k(0) - a_j(0)\big) + \int_{a_j(0)}^{b_k(0)} \big[\tfrac{1}{2}\rho_0 u_0^2 + G(\rho_0)\, dx\big]$$

$$\leqslant C \int_{a_j(0)}^{b_k(0)} \rho_0\, dx,$$

again by (3.1). We substitute this into (3.11) and then use the fact that mass is conserved on (a_j, b_k) to conclude that $u(\bar{x}, t) \leqslant C$. This proves the claim and completes the proof of Theorem 3. □

4. Infinitely many fluids–proof of Theorem 1

In this section we complete the proof of Theorem 1 by taking the limit of solutions for finitely many initial fluids as the number of fluids goes to infinity. Specifically, we let (ρ_0, u_0, V_0) be as described in the statement of Theorem 1 and define $V_0^N = \bigcup_{j=1}^N (a_j(0), b_j(0))$ and take ρ_0^N and u_0^N to be the restrictions of ρ_0 and u_0 to V_0^N. Theorem 3 then applies to show that there is a corresponding global weak solution (ρ^N, u^N) with particle trajectories X^N in the sense of (3.7), all satisfying (3.3)–(3.8) with constants independent of N. (Observe that we cannot assume now that the intervals (a_j, b_j) are ordered left to right.)

We divide the overall proof into three separate lemmas, corresponding to the convergence of X^N, ρ^N, and u^N.

Lemma 4.1. *There is a sequence $N_i \to \infty$ such that X^{N_i} converges uniformly on $\overline{V}_0^M \times [0, T]$ for every M and T to a function $X : \bigcup_{j=1}^\infty [a_j(0), b_j(0)] \times [0, \infty) \to [0, 1]$. X is Hölder continuous in t and locally Lipschitz continuous in y: for all $y \in V_0$ and $t_1, t_2 \geqslant 0$,*

$$\big|X(y, t_2) - X(y, t_1)\big| \leqslant C|t_2 - t_1|^{1/2}, \tag{4.1}$$

and for a.a. $y \in V_0$ and for all $t \geqslant 0$,

$$C(t)^{-1} \leqslant \frac{\partial X}{\partial y} \leqslant C(t). \tag{4.2}$$

Also, if $a_j(t) \equiv X(a_j(0),t)$ and $b_j(t) \equiv X(b_j(0),t)$, then the intervals $\{(a_j(t),b_j(t))\}$ are disjoint, $X(\cdot,t)$ restricted to $[a_j(0),b_j(0)]$ is a bi-Lipschitz homeomorphism $[a_j(0),b_j(0)] \leftrightarrow [a_j(t),b_j(t)]$, and if $V_t \equiv \bigcup_{j=1}^{\infty}(a_j(t),b_j(t))$, then the corresponding set V defined in the Definition of Section 1 *is open.*

Proof. The existence of the subsequence and its convergence follow from the Ascoli–Arzela theorem and a diagonal process, and the regularity estimates (4.1) and (4.2) are inherited by X from the corresponding properties (3.6) and (3.8) of X^N. The other statements follow easily. □

Observe that, while $X(\cdot,t)$ is one-to-one on V_0 (an easy consequence of the conservation of mass), it may in general fail to be one-to-one on $\bigcup_{j=1}^{\infty}[a_j(0),b_j(0)]$, and the inverse map $Y(\cdot,t)$ will in general fail to be continuous from V_t to V_0. Indeed, if a collision occurs at time t, say $b_j(t) = a_k(t) \equiv x_0$, then $X(b_j(0),t) = X(a_k(0),t) = x_0$ and $\lim_{x\to x_0-} Y(x,t) = b_j(0) \neq a_k(0) = \lim_{x\to x_0+} Y(x,t)$.

Next we consider the convergence of the approximate densities ρ^{N_i}:

Lemma 4.2. *There is a further subsequence N_i such that for every $t \geqslant 0$, $\rho^{N_i}(\cdot,t)$ converges strongly in $L^2((0,1))$ to a function $\rho(\cdot,t)$ which is positive a.e. on V_t and zero a.e. on V_t^c and which satisfies the conclusions* (1.12) *and* (1.18) *of Theorem* 1. *Also, for almost every $y \in V_0$, the map $t \to \rho(X(y,t),t)$ is continuous and* (1.16) *holds.*

Proof. Recall the definition $F^N = \mu(\rho^N)u_x^N - P(\rho^N)$. The bound (2.19) for a single fluid, applied at each of the (finitely many) collision times associated with the solution (ρ^N, u^N), shows that the map $t \to \|F^N(\cdot,t)\|_{L^\infty(V_t^N)}$ is in $L^1([0,T])$ for each T. We can therefore define $\mathcal{F}^N : V_0^N \times [0,\infty) \to \mathbb{R}$ by

$$\mathcal{F}^N(y,t) = \int_0^t F^N\big(X^N(y,s),s\big)\,ds.$$

We shall show that the sequence $\{\mathcal{F}^N\}$ is strongly compact and then that the convergence of $\mathcal{F}^N$ dominates the convergence of ρ^N. First we compute from the bounds (3.3) and (3.4) for F^N and (4.2) for $\frac{\partial X^N}{\partial y}$ that

$$\int_0^T \int_{V_0^N} \big(\mathcal{F}_t^N\big)^2\,dy\,dt \leqslant C(T). \tag{4.3}$$

Next, a straightforward computation based on (1.1) shows that

$$\mathcal{F}_y^N(y,t) = \rho_0(y)\big[u^N\big(X^N(y,t),t\big) - u^N(y,0)\big],$$

so that, by (3.4) and (3.8),

$$\int_{V_0^N} \big(\mathcal{F}_y^N\big)^2(y,t)\,dy \leqslant C\bigg[\int_{V_t^N} \big(\rho^N\big(u^N\big)^2\big)(x,t)\,dx + \int_{V_0^N} \rho_0 u_0^2\,dx\bigg] \leqslant C. \tag{4.4}$$

It follows easily from (4.3) and (4.4) that $\{\mathcal{F}^N\}_{N\geqslant M}$ is uniformly bounded in $C^{1/2,1/4}(V_0^M \times [0,T])$ for every M and T. There is therefore a subsequence of the sequence of Lemma 4.1, still denoted N_i, such that $\mathcal{F}^{N_i}$ converges uniformly on $\overline{V}_0^M \times [0,T]$ for every M and $T > 0$.

Now again define $m : (0,\infty) \to \mathbb{R}$ by $m(1) = 0$ and $m'(\rho) = \mu(\rho)/\rho$. Then for almost all $y \in V_0$ and at times t other than collision times,

$$\frac{d}{dt} m\big(\rho^N\big(X^N(y,t),t\big)\big) = -\big(F^N + P^N\big)\big(X^N(y,t),t\big).$$

We let $M \leqslant N$ and subtract the corresponding equation for ρ^M and integrate in t (recall that the maps $t \to \rho^N(X^N(y,t),t)$ are continuous at collision times by construction). Applying the fact that for any two densities ρ_1 and ρ_2 occurring here there is a ξ such that

$$\left|\frac{P(\rho_2) - P(\rho_1)}{m(\rho_2) - m(\rho_1)}\right| = \left|\frac{\xi P'(\xi)}{\mu(\xi)}\right| \leqslant C(t),$$

we obtain the bound

$$\big|m\big(\rho^N\big(X^N(y,t),t\big)\big) - m\big(\rho^M\big(X^M(y,t),t\big)\big)\big| \leqslant C(t)\big\|\mathcal{F}^N - \mathcal{F}^M\big\|_{L^\infty(V_0^M\times[0,T])}$$

for almost all $y \in V_0^M$ and for $t \in [0,T]$. It then follows from the uniform bound (3.4) and from our assumption that μ is bounded below away from zero that

$$\big|\rho^N\big(X^N(y,t),t\big) - \rho^M\big(X^M(y,t),t\big)\big| \leqslant C(t)\big\|\mathcal{F}^N - \mathcal{F}^M\big\|_{L^\infty(V_0^M\times[0,T])}.$$

This together with the strong convergence of the $\mathcal{F}^{N_i}$ shows that, for almost all $y \in V_0$, $\rho^{N_i}(X^{N_i}(y,\cdot),\cdot)$ converges uniformly for $t \in [0,T]$ to a continuous function $r(y,\cdot)$. We then define $\rho(\cdot,t)$ on each $(a_j(t), b_j(t))$ by the relation $\rho(X(y,t),t) = r(y,t)$ and we take $\rho(x,t) = 0$ for $x \in V_t^c$. Thus $\rho^{N_i}(X^{N_i}(\cdot,t),t)$ converges in $L^1((0,1))$ to $\rho(X(\cdot,t),t)$ (and therefore in L^p for every $p < \infty$). We can then approximate $\rho(\cdot,t)$ by smooth functions, change variables, and apply (4.2) to conclude that $\rho^{N_i}(\cdot,t) \to \rho(\cdot,t)$ in $L^p((0,1))$ for $p < \infty$. It is clear that ρ inherits the L^∞ bound in (3.4), and the local mass conservation property (1.18) follows from the corresponding property for ρ^{N_i} and the strong convergence of ρ^{N_i} and X^{N_i}.

Finally we recall from (3.8) the relation

$$\rho^N\big(X^N(\cdot,t),t\big)\frac{\partial X^N}{\partial y}(\cdot,t) = \rho_0.$$

We have just shown that for $N = N_i$, $\rho^N(X^N(\cdot,t),t) \to \rho(X(\cdot,t),t)$ strongly in L^2, and from Lemma 4.1, $\frac{\partial X^N}{\partial y}(\cdot,t) \rightharpoonup \frac{\partial X}{\partial y}(\cdot,t)$, first in $\mathcal{D}'((0,1))$, then weakly in L^2 by the uniform bound (4.2). The left side above therefore converges weakly in $L^2((0,1))$ to $\rho(X(\cdot,t),t)\frac{\partial X}{\partial y}(\cdot,t)$. This proves the equality in (1.16), and the bounds in (1.16) then follow from (4.2). □

Before proceeding we recall the following elementary facts and notations (see Feireisl [3, pp. 28–29] for further discussion). Let X be a separable Banach space and denote by $C([0,T];X_w)$ the vector space of functions $f : [0,T] \to X^*$ such that for each $\varphi \in X$ the map $t \to f(t)\cdot\varphi$ is continuous on $[0,T]$. An application of the Ascoli–Arzela theorem shows that if $\{f_N\} \subset C([0,T];X_w)$ with $\sup_{t\in[0,T]}\|f_N(t)\|_{X^*} \leqslant K$ for some K and all N, and if there is a dense set $\mathcal{D} \subset X$ such that the maps $t \to f_N(t)\cdot\varphi$ are equicontinuous on $[0,T]$ for each fixed $\varphi \in \mathcal{D}$, then there is a subsequence N_i and an element $f \in C([0,T];X_w)$ such that $f_N \to f$ in $C([0,T];X_w)$ in the sense that

$$\sup_{t\in[0,T]} \big|f_{N_i}(t)\cdot\varphi - f(t)\cdot\varphi\big| \to 0$$

as $N_i \to \infty$ for each $\varphi \in X$. Moreover, if Y is a Banach space compactly imbedded in X, then $\sup_{t\in[0,T]} \|f_{N_i}(t) - f(t)\|_{Y^*} \to 0$ as $N_i \to \infty$. We apply these facts below to prove the convergence of the momentum and energy densities $\rho^N u^N$ and $\rho^N (u^N)^2$ and the viscous terms $\chi_{V_N} \mu^N u_x^N$:

Lemma 4.3. *There is a further subsequence $N_i \to \infty$ such that*:

(a) u^{N_i} *converges to a function u weakly in $L^2((0,1)\times(0,T))$ for all $T > 0$ (here we take $u^{N_i} = 0$ in the complement of V^{N_i})*;
(b) $u_x^{N_i}$ *converges to u_x weakly in $L^2(K)$ for every compact set $K \subset V$*;
(c) $\rho^{N_i} u^{N_i}$ *converges to ρu in $C([0,T]; L^2_w((0,1))) \cap C([0,T]; H^{-1}((0,1)))$ for all $T > 0$*;
(d) *for $T > 0$ and φ Lipschitz on $[0,1]\times[0,T]$,*

$$\iint\limits_{V^{N_i}\cap\{0\leqslant t\leqslant T\}} \rho^{N_i}\big(u^{N_i}\big)^2 \varphi_x \,dx\,dt \to \iint\limits_{V\cap\{0\leqslant t\leqslant T\}} \rho u^2 \varphi_x \,dx\,dt;$$

(e) *and for φ as in* (d)*,*

$$\iint\limits_{V^{N_i}\cap\{0\leqslant t\leqslant T\}} \mu\big(\rho^{N_i}\big) u_x^{N_i} \varphi_x \,dx\,dt \to \iint\limits_{V\cap\{0\leqslant t\leqslant T\}} \mu(\rho) u_x \varphi_x \,dx\,dt.$$

The limiting functions u and ρu satisfy

$$\int\limits_0^T \big\|u(\cdot,t)\big\|_\infty^2 \,dt \leqslant C(1+T), \quad T > 0, \tag{4.5}$$

and

$$\sup_{t\geqslant 0} \int\limits_{V_t} \big[\tfrac{1}{2}\rho u^2 + G(\rho)\big](x,t)\,dx + \iint\limits_V \mu(\rho) u_x^2 \,dx\,dt \leqslant E_0 + \tilde{P}. \tag{4.6}$$

Proof. The convergence of u^{N_i} and $u_x^{N_i}$ in (a) and (b) follows immediately from the uniform bounds (3.3) and (3.5). To prove (c) we apply the weak form of the second equation in (1.1) to a test function $\varphi \in \mathcal{D}((0,1))$ to obtain that

$$\begin{aligned}
&\left| \int \big(\rho^N u^N\big)(x,t_2)\varphi(x)\,dx - \int \big(\rho^N u^N\big)(x,t_1)\varphi(x)\,dx \right| \\
&\quad = \left| \int\limits_{t_1}^{t_2} \int\limits_{V_t^N} \big[\rho^N \big(u^N\big)^2 + P\big(\rho^N\big) - \mu\big(\rho^N\big) u_x^N\big]\varphi_x \,dx\,dt \right| \\
&\quad \leqslant C(\varphi)\big(|t_2 - t_1| + |t_2 - t_1|^{1/2}\big),
\end{aligned}$$

again by the bounds in (3.3) and (3.5). Thus for each such φ, the maps $t \to \int (\rho^N u^N)(x,t)\varphi(x)\,dx$ are equicontinuous on $[0,T]$. Noting also that

$$\big\|\big(\rho^N u^N\big)(\cdot,t)\big\|_{L^2} \leqslant C$$

for all N and t by these same bounds, we conclude that there is a further subsequence $\rho^{N_i}u^{N_i}$ which converges in $C([0,T];L^2_w((0,1)))$ as well as in $C([0,T];H^{-1}((0,1)))$ for all $T>0$. An elementary argument based on the strong convergence of ρ^{N_i} in Lemma 4.2 and the weak convergence of u^{N_i} in (a) then shows that the limiting function is ρu in both cases. This proves (c).

Before proving (d) and (e) we show that if Δ denotes symmetric difference, then

$$\big|V_t^{N_i}\Delta V_t\big|\to 0 \quad \text{as } N_i\to\infty \tag{4.7}$$

for every t. To see this we surpress the t and define $I_j=(a_j,b_j)$ and $I_j^N=(a_j^N,b_j^N)$ for $N=N_i$. Then

$$V_t^N\Delta V_t\subset\Big[\bigcup_j\big(I_j^N-I_j\big)\Big]\cup\Big[\bigcup_j\big(I_j-I_j^N\big)\Big]. \tag{4.8}$$

It is clear from the uniform convergence in Lemma 4.1 that $|I_j^N-I_j|\to 0$ for each j, and (3.8) shows that

$$\big|I_j^N\big|=\big|X^N\big(b_j^N,t\big)-X^N\big(a_j^N,t\big)\big|\leqslant C(t)\big(b_j(0)-a_j(0)\big).$$

Thus

$$\begin{aligned}\sum_j\big|I_j^N-I_j\big|&\leqslant\sum_1^J\big|I_j^N-I_j\big|+\sum_{J+1}^\infty\big|I_j^N\big|\\&\leqslant\sum_1^J\big|I_j^N-I_j\big|+C(t)\sum_{J+1}^\infty\big(b_j(0)-a_j(0)\big),\end{aligned}$$

which can be made arbitrarily small by taking first J large then $N=N_i$ large. The argument for the other term on the right side of (4.8) is similar. This proves the claim (4.7).

Now to prove (d) we first let φ be smooth on $[0,1]\times[0,T]$ and derive a preliminary bound:

$$\begin{aligned}&\left|\int_0^T\int_{V_t^N}\rho^N\big(u^N\big)^2\varphi_x\,dx\,dt-\int_0^T\int_{V_t^N\cap V_t}\rho^N\big(u^N\big)^2\varphi_x\,dx\,dt\right|\\&\quad\leqslant C(\varphi)\int_0^T\big\|u^N(\cdot,t)\big\|_\infty\bigg(\int_{V_t^N-V_t}\big(\rho^Nu^N\big)^2dx\bigg)^{1/2}\big|V_t^N-V_t\big|^{1/2}dt\\&\quad\leqslant C(\varphi)\bigg(\int_0^T\big\|u^N(\cdot,t)\big\|_\infty^2\bigg)^{1/2}\bigg(\int_0^T\big|V_t^N-V_t\big|\,dt\bigg)^{1/2},\end{aligned}$$

which goes to zero as $N=N_i\to\infty$ by (3.5) and (4.7). Then

$$\int_0^T\int_0^1\big(\rho^N\big(u^N\big)^2-\rho u^2\big)\varphi_x\,dx\,dt$$

$$= \iint\limits_{V\cap\{0\leqslant t\leqslant T\}} \bigl(\rho^N (u^N)^2 - \rho u^2\bigr)\varphi_x\,dx\,dt + o_N(1)$$

$$= \iint\limits_{V\cap\{0\leqslant t\leqslant T\}} \bigl(\rho^N u^N - \rho u\bigr)u^N\varphi_x\,dx\,dt$$

$$+ \iint\limits_{V\cap\{0\leqslant t\leqslant T\}} \rho u\varphi_x\bigl(u^N - u\bigr)\,dx\,dt + o_N(1), \tag{4.9}$$

where $o_N(1)$ denotes a term which goes to zero as $N = N_i \to \infty$. The second term on the right here goes to zero as $N_i \to \infty$ by (a), since $\rho u\varphi_x \in L^2$. To estimate the first term we cover V by a countable collection of open rectangles $\{R_k\}$ such that for each k there is a j with $\overline{R}_k \subset \{(x,t)\colon\ t > 0$ and $x \in (a_j(t), b_j(t))\}$, and then construct a partition of unity $\{\psi_k\}$ on V such that for each k, $\operatorname{supp}\psi_k \subset R_k$. The integrand in the first term on the right in (4.9) may then be written $\sum_k(\rho^N u^N - \rho u)u^N\varphi_x\psi_k$, and since each corresponding partial sum is bounded by the integrable function $|\rho^N u^N - \rho u||u^N\varphi_x|$, we can write the first integral on the right side of (4.9) as $\sum_k \alpha_k^N$ where

$$\alpha_k^N = \int_0^T \int_0^1 \bigl(\rho^N u^N - \rho u\bigr)u^N\varphi_x\psi_k\,dx\,dt.$$

Now for k fixed and $N = N_i$ large,

$$\operatorname{supp}\psi_k \subset \bigl\{(x,t)\colon\ t > 0 \text{ and } x \in \bigl(a_j^N(t), b_j^N(t)\bigr)\bigr\}$$

for some j, and therefore by (3.3) and (3.5),

$$\begin{aligned}
\bigl|\alpha_k^N\bigr| &\leqslant \int_0^T \bigl\|\bigl(\rho^N u^N - \rho u\bigr)(\cdot,t)\bigr\|_{H^{-1}((0,1))}\bigl\|\bigl(u^N\varphi\psi_k\bigr)(\cdot,t)\bigr\|_{H_0^1(V_t^N)}\,dt \\
&\leqslant C(k,\varphi)\int_0^T \bigl\|\bigl(\rho^N u^N - \rho u\bigr)(\cdot,t)\bigr\|_{H^{-1}((0,1))}\bigl(\bigl\|u^N(\cdot,t)\bigr\|_\infty + \bigl\|u_x^N(\cdot,t)\bigr\|_{L^2(V_t^N)}\bigr)\,dt \\
&\leqslant C(k,\varphi,T)\sup_{0\leqslant t\leqslant T}\bigl\|\bigl(\rho^N u^N - \rho u\bigr)(\cdot,t)\bigr\|_{H^{-1}((0,1))},
\end{aligned}$$

which goes to zero as $N = N_i \to \infty$, by (c). A cruder bound gives

$$\begin{aligned}
\sum_{K+1}^\infty \bigl|\alpha_k^N\bigr| &\leqslant C(\varphi)\sum_{K+1}^\infty \biggl(\iint \bigl(\rho^N u^N - \rho u\bigr)^2\psi_k\,dx\,dt\biggr)^{1/2}\biggl(\iint \psi_k\,dx\,dt\biggr)^{1/2} \\
&\leqslant C(\varphi)\biggl(\sum_{K+1}^\infty \iint \psi_k\,dx\,dt\biggr)^{1/2},
\end{aligned}$$

which goes to zero as $K \to \infty$, uniformly in N. These estimates show that

$$\sum_k \alpha_k^N = \int_0^T \int_0^1 \bigl(\rho^N u^N - \rho u\bigr) u^N \varphi_x \, dx \, dt \to 0$$

as $N = N_i \to \infty$ and together with (4.9) complete the proof of (d) for smooth functions φ. An easy argument based on the uniform bounds (3.5) and (4.5) then shows that (d) holds for Lipschitz test functions as well.

The proof of (e) is similar to that of (d). We again fix a smooth function φ and abbreviate $\chi^N = \chi_{V^N}$ and $\chi = \chi_V$. An easy estimate based on (4.7) similar to that given above preceding (4.8) then shows that

$$\int\int \bigl(\chi^N - \chi^N \chi\bigr) \mu^N u_x^N \varphi_x \, dx \, dt = o_N(1).$$

Then

$$\begin{aligned}
&\int\int \chi^N \mu^N u_x^N \varphi_x - \int\int \chi \mu u_x \varphi_x \\
&\quad = \int\int \chi^N \chi \bigl(\mu^N - \mu\bigr) u_x^N \varphi_x + \int\int \chi^N \chi \mu \bigl(u_x^N - u_x\bigr) \varphi_x \\
&\qquad + \int\int \bigl(\chi^N \chi - \chi\bigr) \mu u_x \varphi_x + o_N(1).
\end{aligned}$$

The first term on the right goes to zero as $N = N_i \to \infty$ by (3.3) and the strong convergence of ρ^{N_i} in Lemma 4.2, and the third term goes to zero by (3.3) and (4.7). The middle term may be written $\sum_k \beta_k^N$, where

$$\beta_k^N = \int\int_{R_k} \chi^N \chi \mu \bigl(u_x^N - u_x\bigr) \varphi_x \psi_k$$

and $\{\psi_k\}$ and $\{R_k\}$ are as in the proof of (d). We apply (b) and (3.3) to show that each $\beta_k^N \to 0$ as $N = N_i \to \infty$ and that $\sum_{K+1}^{\infty} |\beta_k^N| \to 0$ as $K \to \infty$, uniformly in N, as in the proof of (d). These estimates prove (e) for smooth functions φ, and the extension to Lipschitz test functions is elementary, as before.

The bound in (4.5) follows easily from the corresponding bound (3.5) for u^N and the convergence in (a). To prove (4.6) we write the corresponding equality with ρ, u, and V replaced by ρ^N, u^N, and V^N, and then take the limit as $N = N_i \to \infty$: First, the strong convergence $\rho^{N_i}(\cdot, t) \to \rho(\cdot, t)$ shows that

$$\int_{V_t^{N_i}} G\bigl(\rho^{N_i}(x,t)\bigr) \, dx \to \int_{V_t} G\bigl(\rho(x,t)\bigr) \, dx$$

as $N_i \to \infty$ (but some care must be taken with the domains of integration since $G(0)$ need not be zero). Next, this same strong convergence together with (b) shows that for compact $K \subset V$ and for N_i large enough that $K \subset V^{N_i}$,

$$\iint_K \mu u_x^2\,dx\,dt \leqslant \liminf_{N_i\to\infty} \iint_K \mu^{N_i}\bigl(u_x^{N_i}\bigr)^2\,dx\,dt \leqslant \liminf_{N_i\to\infty} \iint_{V^{N_i}} \mu^{N_i}\bigl(u_x^{N_i}\bigr)^2\,dx\,dt,$$

so that

$$\iint_V \mu u_x^2\,dx\,dt \leqslant \liminf_{N_i\to\infty} \iint_{V^{N_i}} \mu^{N_i}\bigl(u_x^{N_i}\bigr)^2\,dx\,dt.$$

For the kinetic energy term we observe that, for each fixed t, the sequence $\{(\sqrt{\rho^N}u^N)(\cdot,t)\}$ is uniformly bounded in $L^2((0,1))$ and therefore, modulo subsequences, is weakly convergent, say to w. But then Lemma 4.2 applies to show that $(\rho^N u^N)(\cdot,t) \rightharpoonup \sqrt{\rho(\cdot,t)}w$, so that by (c), $w = \sqrt{\rho}u$. We conclude that $(\sqrt{\rho^N}u^N)(\cdot,t) \rightharpoonup (\sqrt{\rho}u)(\cdot,t)$ as $N = N_i \to \infty$, and thus finally that

$$\int \bigl(\rho u^2\bigr)(x,t)\,dx \leqslant \liminf_{N\to\infty} \int \bigl(\rho^N\bigl(u^N\bigr)^2\bigr)(x,t)\,dx.$$

These estimates prove (4.6) and complete the proof of the lemma. □

It is now clear from Lemmas 4.1–4.3 that, with the exception of (e), the limits ρ, u, and V satisfy the conditions of the Definition of Section 1. In particular, the weak forms (f) and (g) follow by writing these weak forms for the approximate solutions (ρ^{N_i}, u^{N_i}) and noting the modes of convergence described above and the strong convergence of the approximate initial data. The properties (1.8)–(1.10) then follow immediately from the weak forms, and the other properties (1.11)–(1.18) have already been established in these same lemmas. The final statement of Theorem 1 follows from the fact $\overline{V}_t^N = [0,1]$ for all N and for $t \geqslant \overline{T}$, where $\overline{T}$ is as in the final statement of Theorem 3, and by the Hölder continuity (2.4) of u and F for case I of Theorem 2.

There remains to prove that the constructed solution (ρ, u) satisfies the weak form of the boundary condition in (e) of the Definition of Section 1. Suppose therefore that $0 \in \partial V_t$ for $t \in [t_1, t_2]$, as in the Definition, and let $\varepsilon > 0$ be given. Then for each $t \in [t_1, t_2]$ there is a boundary point $a_K(t) \leqslant \varepsilon/2$. By (4.1) we can choose a single K so that $a_K(t) \leqslant \varepsilon$ for all such t and we can choose a point $y_0 > a_K(0)$ such that $X(y_0, t) \leqslant \varepsilon$ for all such t as well. Now define a positive number δ by

$$\delta = \int_0^{y_0} \rho_0(y)\,dy = \int_0^{X(y_0,t)} \rho(x,t)\,dx$$

and suppose that $0 < M \leqslant \delta$. Then from the definition of $z(M,t)$ there are points $y_j \in V_0$ such that $y_j \uparrow z(M,0)$ and $X(y_j,t) \uparrow z(M,t)$ for each t. It follows that $X(y_j,t) \leqslant \varepsilon$ for $t \in [t_1,t_2]$ and for all j because

$$\int_0^{X(y_j,t)} \rho(x,t)\,dx \leqslant \int_0^{z(M,t)} \rho(x,t)\,dx = M \leqslant \delta = \int_0^{X(y_0,t)} \rho(x,t)\,dx,$$

so that $X(y_j,t) \leqslant X(y_0,t) \leqslant \varepsilon$. We then have that

$$\left|\int_{t_1}^{t_2}\int_0^{z(M,t)}(\rho u)(x,t)\,dx\,dt\right| = \left|\lim_j\lim_{N_i}\int_{t_1}^{t_2}\int_0^{X^{N_i}(y_j,t)}\bigl(\rho^{N_i}u^{N_i}\bigr)(x,t)\,dx\,dt\right|$$

$$= \left|\lim_j\lim_{N_i}\int_0^{y_j}\int_{t_1}^{t_2}\rho_0(y)u^{N_i}\bigl(X^{N_i}(y,t),t\bigr)\,dt\,dy\right|$$

$$= \left|\lim_j\int_0^{y_j}\rho_0(y)\bigl[X(y,t_2)-X(y,t_1)\bigr]dy\right|$$

$$\leqslant \varepsilon\lim_j\int_0^{y_j}\rho_0(y)\,dy = \varepsilon M = \varepsilon\int_0^{z(M,t)}\rho(x,t)\,dx.$$

Thus given $\varepsilon > 0$ there is a $\delta > 0$ such that the ratio in (e) is less than ε when $0 < M \leqslant \delta$. This proves the required boundary property and completes the proof of Theorem 1. □

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