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Fish–hook shaped global bifurcation branch of a spatially heterogeneous cooperative system with cross-diffusion

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ABSTRACT

In this paper, we will consider the following strongly coupled cooperative system in a spatially heterogeneous environment with Neumann boundary condition

$$\begin{cases} \Delta u + u\big(\lambda - u + b(x)v\big) = 0, & x \in \Omega, \\ \Delta\big[\big(1 + k\rho(x)u\big)v\big] + v\big(\mu - v + d(x)u\big) = 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$; k is a positive constant, λ and μ are real constants which may be non-positive; $b(x) \geqslant\not\equiv 0$ and $d(x) \geqslant\not\equiv 0$ are continuous functions in $\bar{\Omega}$; $\rho(x)$ is a smooth positive function in $\bar{\Omega}$ with $\partial_\nu \rho(x)|_{\partial\Omega} = 0$; ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. For the case $\mu > 0$, we show that if $|\mu|$ is small and k is large, a spatial segregation of $\rho(x)$ and $b(x)$ can cause the positive solution curve to form an **unbounded** fish–hook ($\subset$) shaped curve with parameter λ. For the case $\mu < 0$, if $|\mu|$ is small and k is large, and the cooperative effect is strong for species u and very weak for species v, then the positive solution set also forms an **unbounded** fish–hook shaped continuum. These results are quite different from those of predator–prey systems and the cooperative system under Dirichlet boundary condition, both of which can form a **bounded** continuum. Our results deduce that the spatial heterogeneity of environments can produce multiple coexistence states.

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Our method of analysis is based on the bifurcation theory and the Lyapunov–Schmidt procedure.

1. Introduction

Besides the interactions between species, the spatial heterogeneity also influences the population dynamics, which can be observed in many scientific experiments. For example, Huffaker [15] found that a predator–prey system consisting of two species of mites could collapse to extinction quickly in small homogeneous environments, but would persist longer in suitable heterogeneous environments. Therefore, it is very important to study the heterogeneous effects of the environments on population dynamics. On the other hand, different concentration levels of species can affect the diffusive direction of another interacting species, which is called cross-diffusion. One can see [26,30] for further ecological background. To study the combined heterogeneous effects of the interactions and cross-diffusion on the set of positive stationary solutions, we study the following Lotka–Volterra cooperative system with cross-diffusion in a spatially heterogeneous environment:

$$\begin{cases} u_t = d_1 \Delta u + u\big(a_1 - b_1 u + c_1(x) v\big), & x \in \Omega,\ t > 0, \\ v_t = \Delta\big[\big(d_2 + \rho(x) u\big) v\big] + v\big(a_2 - b_2 v + c_2(x) u\big), & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \qquad v(x,0) = v_0(x) \geqslant 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with smooth boundary $\partial\Omega$; u and v represent the cooperative species; d_1, d_2, b_1 and b_2 are positive constants, d_1 and d_2 represent the natural dispersive forces of movements of the species, respectively; b_1 and b_2 represent the intra-specific pressures of u and v; a_1 and a_2 are real constants which may be non-positive, representing the birth or death rates of the species; $c_1(x) \geqslant\not\equiv 0$ and $c_2(x) \geqslant\not\equiv 0$ are continuous functions in $\bar{\Omega}$ representing the inter-specific interactions; $\rho(x)$ is a smooth positive function in $\bar{\Omega}$ with $\partial_\nu \rho(x)|_{\partial\Omega} = 0$; ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. The system is known as the Lotka–Volterra cooperative system, which was introduced by Shigesada et al. [36] to model the segregation phenomenon of two species.

It should be emphasized that there is a nonlinear diffusion $\Delta(\rho(x) v u)$ in the diffusion term. One can see

$$\Delta\big[\big(d_2 + \rho(x) u\big) v\big] = \nabla \cdot \big[\big(d_2 + \rho(x) u\big) \nabla v + v \nabla\big(\rho(x) u\big)\big],$$

$\rho(x)$ is known as the cross-diffusion pressure, which means a tendency that v diffuses to the low density region of $\rho(x)u$, and moreover the tendency not only depends on the population pressure of u but also on the heterogeneity of the environments. In particular, when $\rho(x)$ is spatially homogeneous, v moves to low density region of u. Furthermore, we can see that the diffusive flux of v is $J = -[(d_2 + \rho(x)u)\nabla v + v\nabla(\rho(x)u)]$, then the diffusive flux across the boundary $\partial\Omega$ is

$$J \cdot \nu = -\big(d_2 + \rho(x) u\big) \partial_\nu v - \rho v \partial_\nu u - u v \partial_\nu \rho = 0,$$

that is, there is no flux across $\partial\Omega$. $\partial\Omega$ acts as a perfect barrier for both species, the system is self-contained [1].

Since the pioneering work [36], strongly coupled elliptic system has received increasing attention. Many authors have studied population models with cross-diffusion terms from various mathematical viewpoints: the global existence of classical solutions [2,3,28]; the existence of positive solutions [20,22–24,29,34,35]; and the existence of nonconstant positive steady-states [26,27,31,32,37,38]. The main methods to study the existence of positive stationary solutions are the bifurcation theory, sub-supersolution and fixed point index theory. In 1998, Du and Lou [10] applied the bifurcation theory and the Lyapunov–Schmidt reduction to a predator–prey system with Holling–Tanner response and obtained an S-shaped global bifurcation branch. In the above papers, the coefficients are all spatially homogeneous, while more and more interesting papers studying the spatially heterogeneous effects have appeared. For example, Du et al. [8,9,11–13] have mainly studied degenerate effects of intra-specific pressures in some predator–prey or competitive models; Hutson et al. [16–19] have mainly studied the spatial effects of birth rates in some diffusive competitive models. Recently, Kuto [21] studied a Lotka–Volterra predator–prey system with cross-diffusion in a spatially heterogeneous environment. By the methods of the bifurcation theory and Lyapunov–Schmidt reduction, Kuto obtained the global bifurcation branch of positive stationary solutions and found that the spatial segregation of $\rho(x)$ and $d(x)$ could cause the bifurcation branch to form a bounded fish–hook curve. However, little attention has been paid to the cooperative system. As far as we know, only Delgado et al. [7] and Ling and Pedersen [24] studied some cooperative systems with cross-diffusion, where the coefficients are spatially homogeneous. In this paper, we use the bifurcation theory and Lyapunov–Schmidt reduction to investigate the structure of positive stationary solutions of (1.1) and especially study the spatial heterogeneous effects on the set of the positive stationary solutions and obtain rather different results. To focus our attention on the heterogeneous effect, we study a simpler strongly coupled elliptic system

$$\begin{cases} \Delta u + u\big(\lambda - u + b(x)v\big) = 0, & x \in \Omega, \\ \Delta\big[\big(1 + k\rho(x)u\big)v\big] + v\big(\mu - v + d(x)u\big) = 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

which can be obtained by a rescaling. We remark that our all results are true for the following system

$$\begin{cases} \Delta\big[\big(1 + k\rho(x)v\big)u\big] + u\big(\lambda - u + b(x)v\big) = 0, & x \in \Omega, \\ \Delta v + v\big(\mu - v + d(x)u\big) = 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega. \end{cases}$$

From the ecological viewpoint, we are only interested in positive solutions of (1.2). It is said that (u, v) is a positive solution of (1.2) if $u > 0$ and $v > 0$ in $\bar{\Omega}$. From system (1.2), we can see that the presence of v is beneficial to u due to the cooperative character; while in the equation of v, there is a balance between the cooperation (term $+\, d(x)uv$) and the repulsive force in the diffusion (term $+\, k\rho(x)uv$). Thus it is quite interesting to investigate the necessary balance between both terms to obtain the positive solution curve of (1.2).

Throughout the paper, we denote the average of $f(x)$ over Ω by $⨍ f(x) = \frac{1}{|\Omega|}\int_\Omega f\,dx$ and $\|u\|_\infty = \max_{\bar{\Omega}} |u(x)|$. We denote $\lambda_1(q)$ by the least eigenvalue of the problem

$$-\Delta u + q(x)u = \lambda u \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega,$$

where $q(x)$ is continuous in $\bar{\Omega}$. We know that the mapping $q \to \lambda_1(q) : C(\bar{\Omega}) \to \mathbb{R}$ is continuous and monotone increasing.

We now show our main results.

Theorem 1.1. *If $\mu > 0$ is sufficiently small and k is sufficiently large, $\|b\|_\infty \|d\|_\infty < \frac{\min_{\bar{\Omega}} \rho}{\|\rho\|_\infty}$, and*

$$⨍_\Omega b(x)\rho(x) < ⨍_\Omega b(x) ⨍_\Omega \rho(x),$$

then the set of positive solutions of (1.2) *forms an unbounded smooth curve*

$$\Gamma_p = \{(u(x;s), v(x;s), \lambda(s)):\ s > 0\}$$

with

$$\big(u(x;0), v(x;0), \lambda(0)\big) = \big(0, \mu, \lambda^*\big)$$

for a negative number λ^*. Furthermore, there exists a small positive number μ^* such that the following hold:

(i) *if $0 < \mu \leqslant \mu^*/3$, then $\lambda'(0) > 0$, Γ_p supercritically bifurcates from $(0, \mu, \lambda^*)$*;
(ii) *if $2\mu^*/3 \leqslant \mu \leqslant \mu^*$, then $\lambda'(0) < 0$, Γ_p subcritically bifurcates from $(0, \mu, \lambda^*)$. In this case, for $\underline{\lambda} = \min_{0\leqslant s\leqslant C} \lambda(s)$, if $\lambda \in [-\mu \frac{\|b\|_\infty \|\rho\|_\infty}{\min_{\bar{\Omega}} \rho}, \underline{\lambda})$, (1.2) has no positive solutions; if $\lambda \in [\lambda^*, \infty)$ or $\lambda = \underline{\lambda}$, (1.2) has at least one positive solution; if $\lambda \in (\underline{\lambda}, \lambda^*)$, (1.2) has at least two positive solutions. Here C is a sufficiently large number.*

Theorem 1.2. *If $\mu < 0$ is sufficiently close to 0 and k is sufficiently large, $\|b\|_\infty \|d\|_\infty < \frac{\min_{\bar{\Omega}} \rho}{\|\rho\|_\infty}$, then the set of positive solutions of* (1.2) *forms an unbounded smooth curve*

$$\Gamma_p = \{(u(x;s), v(x;s), \lambda(s)):\ s > 0\},$$

with

$$\big(u(x;0), v(x;0), \lambda(0)\big) = \big(\lambda_*, 0, \lambda_*\big)$$

for a positive number λ_*. Furthermore, if $b(x)$ is very large in $\bar{\Omega}$ and $d(x)$ is very small in $\bar{\Omega}$, there exists $\mu_* < 0$ such that if $\mu_* \leqslant \mu < 0$, for $\underline{\lambda} = \min_{0\leqslant s\leqslant C} \lambda(s)$, if $\lambda \in [-\frac{\mu}{\|d\|_\infty}, \underline{\lambda})$, (1.2) *has no positive solutions; if $\lambda \in [\lambda_*, \infty)$ or $\lambda = \underline{\lambda}$,* (1.2) *has at least one positive solution; if $\lambda \in (\underline{\lambda}, \lambda_*)$,* (1.2) *has at least two positive solutions. Here C is a sufficiently large number.*

We should note that if $\rho(x), b(x)$ and $d(x)$ are all spatially homogeneous, then under the weakly cooperative condition ($bd < 1$), we have $\lambda_\xi(\xi, \varepsilon) > 0$. Thus in case $\mu > 0$, (1.2) has a unique positive solution if $\lambda \in (\lambda^*, \infty)$ and no positive solutions if $\lambda \leqslant \lambda^*$; in case $\mu < 0$, (1.2) has a unique positive solution if $\lambda \in (\lambda_*, \infty)$ and no positive solution if $\lambda \leqslant \lambda_*$. More precisely, the positive solution set can be explicitly expressed by

$$\Gamma_p = \left\{\left(\frac{\lambda + b\mu}{1 - bd}, \frac{\mu + \lambda d}{1 - bd}, \lambda\right):\ \lambda > -b\mu \text{ for } \mu > 0\right\}$$

and

$$\Gamma_p = \left\{\left(\frac{\lambda + b\mu}{1 - bd}, \frac{\mu + \lambda d}{1 - bd}, \lambda\right):\ \lambda > -\mu/d \text{ for } \mu < 0\right\}.$$

From Theorems 1.1 and 1.2, we can see that the spatial heterogeneity can cause Γ_p to become an unbounded fish–hook shaped branch with respect to λ, i.e. can produce multiple coexistence states.

Clearly, we know that if $b(x) \equiv \text{const}$ or $\rho(x) \equiv \text{const}$, then $⨍_\Omega b(x) ⨍_\Omega \rho(x) = ⨍_\Omega b(x)\rho(x)$; if both $b(x)$ and $\rho(x)$ are spatially heterogeneous, then either

$$⨍_\Omega b(x) ⨍_\Omega \rho(x) < ⨍_\Omega b(x)\rho(x) \quad \text{or} \quad ⨍_\Omega b(x) ⨍_\Omega \rho(x) > ⨍_\Omega b(x)\rho(x)$$

may hold. In particular, if there is a segregation of $\rho(x)$ and $b(x)$, $\operatorname{suup}(\rho-\varepsilon)_+ \cap \operatorname{suup} b = \emptyset$ with an arbitrary positive number ε satisfying $\varepsilon < \fint_\Omega \rho(x)$, then

$$\fint_\Omega b(x) \fint_\Omega \rho(x) > \varepsilon \fint_\Omega b(x) \geqslant \fint_\Omega b(x)\rho(x);$$

while if $b(x) \equiv \rho(x)$, then

$$\fint_\Omega b(x) \fint_\Omega \rho(x) = \left(\fint_\Omega b(x)\right)^2 \leqslant \fint_\Omega b^2(x) = \fint_\Omega b(x)\rho(x).$$

It is known that when $\lambda < 0$, $\mu < 0$, $k = 0$ and $b(x) \equiv d(x) \equiv 0$, (1.2) has a globally stable trivial solution $(0,0)$. Moreover, $\operatorname{suup}(\rho-\varepsilon)_+$ provides a domain in which the cross-diffusion effect is comparatively strong and $\operatorname{suup} b$ gives a favorable domain for u in which u increases due to the cooperation. Our first result shows that, if u has a death rate, v has a small but not very small birth rate, the inter-specific cooperation is quite small, then a spatial segregation of $\rho(x)$ and $b(x)$ can produce even multiple coexistence steady-states when k is very large. We see that the cross-diffusion changes the stationary patterns of (1.2). In our second result, the cross-diffusion can be spatially homogeneous ($\rho(x) \equiv \text{const}$). Then our second result implies that although v has a small death rate, the cooperation is rather weak for v, and v moves high away from u, a very strong cooperation for u can still produce even multiple coexistence steady-states.

Compared with the results in [21], we find that the results are quite different. For the case $\mu > 0$, a spatial segregation can yield an unbounded fish–hook shaped bifurcation branch; for the case $\mu < 0$, strong cooperation for u and weak for v can also yield an unbounded fish–hook shaped bifurcation branch; while in [21], a spatial segregation asserts a bounded fish–hook shaped bifurcation branch. The shape of the curve is similar, but the curve of the predator–prey is bounded. This is because upper and lower bounds can be obtained for the bifurcation parameter in the predator–prey system, while we can only deduce a lower bound for the bifurcation parameter. In an ecological viewpoint, this is due to the cooperation character and the predator character.

Furthermore, Delgado et al. [7] discussed (1.2) with spatially homogeneous coefficients under Dirichlet boundary condition. As the principal eigenvalue λ_1 of $-\Delta$ under Dirichlet boundary condition is positive, then from the proof of item (3) in Theorem 1.1 we can see that if $\beta\lambda_1 > c$ (i.e. $k\rho\lambda_1 > d$ in our system), for any fixed $\mu > \lambda_1$, the bifurcation branch bifurcating from $(0,\theta_\mu)$ forms a bounded continuum which joins $(\theta_\lambda, 0)$ with bifurcation parameter λ. This result is very different from the result under Neumann boundary condition. As $\lambda_1 = 0$ in this case, the bifurcation branch is always unbounded with bifurcation parameter λ. From an ecological viewpoint [1], we know that the Dirichlet boundary condition corresponds to a lethal boundary, all species who encounter $\partial\Omega$ die, so the positive solution branch is bounded if the cross-diffusion is large; however, the Neumann boundary condition is a no-flux boundary condition, the species encountering $\partial\Omega$ are always "reflected" back into Ω. Then even though the cross-diffusion is large in this case, the upper bound for λ cannot be obtained. All the above results show the interesting and complicated spatio-temporal patterns of the positive stationary solutions due to the spatial heterogeneity.

Remark 1.3. We point out that our present paper only concerns the weak cooperation (i.e., $\|b\|_\infty \|d\|_\infty < \frac{\min_{\bar\Omega}\rho}{\|\rho\|_\infty}$). But for the strong cooperation case $\|b\|_\infty \|d\|_\infty > \frac{\min_{\bar\Omega}\rho}{\|\rho\|_\infty}$, we leave it for the further study as it is much complicated [6,25].

The organization of our paper is as follows: In Section 2, we mainly give some preliminary results, including a priori estimates and non-existence regions of the positive stationary solutions and the local and global bifurcation branch. In Section 3, we firstly introduce a perturbed problem by a suitable rescaling and the Lyapunov–Schmidt reduction, then study the detailed positive solution structure of the limiting system. In Section 4, we show the detailed profile of Γ_p as a perturbation of Γ_∞. Finally, we give the main results of the paper.

2. Coexistence regions

2.1. An equivalent semilinear elliptic system

In this subsection, we reduce (1.2) to an equivalent semilinear elliptic system. To do so, we let

$$V = \big(1 + k\rho(x)u\big)v, \tag{2.1}$$

then by the one-to-one correspondence between (u, v) and (u, V), one sees that the new unknown function (u, V) satisfies the following semilinear elliptic system:

$$\begin{cases} \Delta u + u\left(\lambda - u + \dfrac{b(x)V}{1+k\rho(x)u}\right) = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{V}{1+k\rho(x)u}\left(\mu - \dfrac{V}{1+k\rho(x)u} + d(x)u\right) = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu V = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Obviously, (1.2) and (2.2) have the same semitrivial solution sets

$$\Gamma_u = \big\{(\lambda, 0, \lambda)\colon \lambda > 0\big\} \quad \text{and} \quad \Gamma_V = \big\{(0, \mu, \lambda)\colon \lambda \in \mathbb{R},\ \mu > 0\big\}.$$

Furthermore, (u, v) is a positive solution of (1.2) if and only if (u, V) is a positive solution of (2.2). Once we have found a bifurcation point on Γ_u or Γ_V to the positive solution of (2.2), we immediately obtain the positive solution branch of (1.2) bifurcating from the same point. So in the following, we mainly apply the bifurcation theory to (2.2).

In order to apply the bifurcation theory, we firstly introduce sets

$$S_u = \left\{(\lambda, \mu) \in \mathbb{R}^2\colon \lambda_1\left(-\frac{\mu + \lambda d(x)}{1 + \lambda k\rho(x)}\right) = 0\right\}, \tag{2.3}$$

and

$$S_V = \big\{(\lambda, \mu) \in \mathbb{R}^2\colon \lambda_1\big(-\lambda - \mu b(x)\big) = 0 \text{ for } \mu \geqslant 0\big\}. \tag{2.4}$$

We have the following lemma with respect to S_u and S_V, which will be important to obtain the local positive solution branch.

Lemma 2.1. *For fixed $\mu < 0$, there exists a monotone decreasing function $\lambda = \lambda_*(\mu) > 0$ with $\lambda_*(0) = 0$ and $\lim_{\mu\to-\infty} \lambda_*(\mu) = +\infty$ such that*

$$S_u = \big\{(\lambda, \mu) \in \mathbb{R}^2\colon \lambda = \lambda_*(\mu) \text{ for } \mu < 0\big\}.$$

While if $\mu > 0$, then

$$\lambda_1\left(-\frac{\mu + \lambda d(x)}{1 + \lambda k\rho(x)}\right) < 0$$

for any λ.

Furthermore, for fixed $\mu > 0$, there exists a monotone decreasing function $\lambda = \lambda^*(\mu) = \lambda_1(-\mu b(x)) \leqslant 0$ satisfying $\lambda^*(0) = 0$ such that

$$S_V = \{(\lambda, \mu) \in \mathbb{R}^2\colon\ \lambda = \lambda^*(\mu) \text{ for } \mu > 0\}.$$

A similar argument to that of Lemma A.1 in [23] can deduce the above lemma, so we omit the proof of Lemma 2.1.

At the end of the subsection, we point out the linear stability of the semitrivial solutions.

Lemma 2.2.

(i) *If $\lambda < 0$, $\mu < 0$, the trivial solution $(0, 0)$ of (2.2) is linearly asymptotically stable; if $\lambda > 0$ or $\mu > 0$, $(0, 0)$ is unstable.*
(ii) *If $\mu < 0$, $0 < \lambda < \lambda_*$, the semitrivial positive solution $(\lambda, 0)$ of (2.2) is linearly asymptotically stable; if $\mu < 0$, $\lambda > \lambda_*$ or $\mu > 0$, $(\lambda, 0)$ is unstable.*
(iii) *If $\lambda < \lambda^*$, the semitrivial positive solution $(0, \mu)$ of (2.2) is linearly asymptotically stable; if $\lambda > \lambda^*$, $(0, \mu)$ is unstable.*

Proof. The proof is very similar to that of Proposition 4.1 of [6], but we give the proof of (ii) for the convenience of the readers.

We linearize the corresponding parabolic system of (2.2) at $(\lambda, 0)$ and obtain

$$\begin{cases} -\Delta\phi + \lambda\phi - \dfrac{\lambda b(x)}{1+\lambda k\rho(x)}\psi = \sigma\phi & \text{in } \Omega, \\ -\Delta\psi - \dfrac{\mu + \lambda d(x)}{1+\lambda k\rho(x)}\psi = \sigma\psi & \text{in } \Omega, \\ \partial_\nu\phi = \partial_\nu\psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

If $\psi \equiv 0$, then

$$-\Delta\phi = (\sigma - \lambda)\phi,$$

$\sigma - \lambda$ is an eigenvalue of $-\Delta$ in Ω under Neumann boundary condition, thus $\sigma - \lambda \geqslant 0$, i.e. $\sigma \geqslant \lambda > 0$.

If $\psi \not\equiv 0$, then σ is an eigenvalue of $-\Delta - \frac{\mu+\lambda d(x)}{1+\lambda k\rho(x)}$ in Ω. From the property of the eigenvalue, we know

$$\operatorname{Re}\sigma \geqslant \lambda_1\left(-\frac{\mu + \lambda d(x)}{1+\lambda k\rho(x)}\right).$$

If $\mu < 0$, Lemma 2.1 asserts that there exists a unique λ_* such that

$$\lambda_1\left(-\frac{\mu + \lambda_* d(x)}{1+\lambda_* k\rho(x)}\right) = 0,$$

then if $0 < \lambda < \lambda_*$,

$$\lambda_1\left(-\frac{\mu + \lambda d(x)}{1+\lambda k\rho(x)}\right) > \lambda_1\left(-\frac{\mu + \lambda_* d(x)}{1+\lambda_* k\rho(x)}\right) = 0,$$

so if $\mu < 0$, $0 < \lambda < \lambda_*$, the eigenvalue satisfies $\operatorname{Re}\sigma > 0$, $(\lambda, 0)$ is linearly asymptotically stable.

If $\mu < 0$, $\lambda > \lambda_*$ or $\mu > 0$, we know

$$\lambda_1\left(-\frac{\mu + \lambda\, d(x)}{1 + \lambda k \rho(x)}\right) < 0,$$

then for $\sigma_1 = \lambda_1(-\frac{\mu+\lambda\, d(x)}{1+\lambda k\rho(x)})$, there exists a positive eigenfunction ψ such that

$$-\Delta\psi - \frac{\mu + \lambda\, d(x)}{1 + \lambda k \rho(x)}\psi = \sigma_1 \psi,$$

and a unique

$$\phi = (-\Delta + \lambda - \sigma_1)^{-1}\left[\frac{\lambda b(x)}{1 + \lambda k \rho(x)}\psi\right] > 0$$

exists, i.e. a positive eigenfunction (ϕ, ψ) exists with the corresponding negative eigenvalue σ_1 for (2.5), thus $(\lambda, 0)$ is unstable. □

2.2. A priori estimates

In this subsection, we mainly obtain a priori estimates of positive solutions of (2.2), which can also give the non-existence regions of positive solutions.

Lemma 2.3. *Let $c = \frac{\|\rho\|_\infty}{\min_{\bar\Omega} \rho}$. Suppose $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ and (u, V) is a positive solution of* (2.2)*, then*

$$\max\{0, \lambda\} \leqslant u \leqslant \frac{\lambda + c\mu\|b\|_\infty}{1 - c\|b\|_\infty\|d\|_\infty},$$
$$0 < V \leqslant \frac{(\mu + \lambda\|d\|_\infty)[(1 - c\|b\|_\infty\|d\|_\infty) + k\|\rho\|_\infty(\lambda + c\mu\|b\|_\infty)]}{(1 - c\|b\|_\infty\|d\|_\infty)^2}.$$

Proof. Since

$$-\Delta u = u\bigl(\lambda - u + b(x)v\bigr) \geqslant u(\lambda - u),$$

the inequality $u \geqslant \lambda$ follows easily by a simple comparison argument.

By the maximum principle, we can obtain

$$\begin{cases} \|u\|_\infty \leqslant \lambda + \|b\|_\infty \dfrac{\|V\|_\infty}{1 + k\min_{\bar\Omega}\rho\|u\|_\infty}, \\ \|V\|_\infty \leqslant \bigl(1 + k\|\rho\|_\infty\|u\|_\infty\bigr)\bigl(\mu + \|d\|_\infty\|u\|_\infty\bigr). \end{cases}$$

Furthermore, we can see

$$\begin{aligned} \|u\|_\infty &\leqslant \lambda + \|b\|_\infty\bigl(\mu + \|d\|_\infty\|u\|_\infty\bigr)\frac{1 + k\|\rho\|_\infty\|u\|_\infty}{1 + k\min_{\bar\Omega}\rho\|u\|_\infty}, \\ &\leqslant \lambda + \frac{\|b\|_\infty\|\rho\|_\infty}{\min_{\bar\Omega}\rho}\bigl(\mu + \|d\|_\infty\|u\|_\infty\bigr), \end{aligned}$$

thus

$$\left(1-\frac{\|b\|_\infty\|d\|_\infty\|\rho\|_\infty}{\min_{\bar{\Omega}}\rho}\right)\|u\|_\infty \leqslant \lambda+\frac{\mu\|b\|_\infty\|\rho\|_\infty}{\min_{\bar{\Omega}}\rho}, \tag{2.6}$$

the estimate of u is obtained.

Similarly, we can get

$$\left(1-\frac{\|b\|_\infty\|d\|_\infty\|\rho\|_\infty}{\min_{\bar{\Omega}}\rho}\right)\|V\|_\infty \leqslant \big(\mu+\lambda\|d\|_\infty\big)\big(1+k\|\rho\|_\infty\|u\|_\infty\big), \tag{2.7}$$

the estimate of V is also obtained. □

Remark 2.4. It should be noted that in [21] the spatial dimension $N \leqslant 3$ is required to derive a priori estimates of positive solutions. However, we do not restrict the spatial dimension in our paper.

From (2.6) and (2.7) in the proof of Lemma 2.3, we immediately deduce the following non-existence region of positive solutions of (2.2).

Lemma 2.5. *Suppose $\|b\|_\infty\|d\|_\infty < \frac{1}{c}$ with c defined in Lemma* 2.3. *If λ satisfies the following conditions*

$$\begin{cases} \lambda \leqslant -c\mu\|b\|_\infty, & \text{if } \mu \geqslant 0, \\ \lambda \leqslant -\dfrac{\mu}{\|d\|_\infty}, & \text{if } \mu < 0, \end{cases} \tag{2.8}$$

then (2.2) *does not have any positive solutions.*

Note that if the cross-diffusion is spatially homogeneous ($\rho(x) \equiv \text{const}$), by the same method, we know that if $\|b\|_\infty\|d\|_\infty < 1$, $\lambda \leqslant -\mu\|b\|_\infty$, then there does not exist any positive solution in case $\mu > 0$; the results are the same if $\mu < 0$. The differences imply that we have to impose stronger restrictions on $b(x)$ and $d(x)$, but we can obtain a smaller non-existence region of positive solutions, that is the spatially heterogeneous cross-diffusion may cause a larger existence region of positive solutions. In Section 4, we surely deduce that the spatial heterogeneity can produce a larger coexistence region, in particular, can produce multiple coexistence states.

2.3. Bifurcation from semitrivial solutions

In the subsection, we will regard λ as the bifurcation parameter and apply the local bifurcation theory [4] to obtain positive solutions of (2.2) bifurcating from the semitrivial solution sets.

With regard to Lemma 2.1, we can define the positive functions ψ_* and ϕ^* such that

$$-\Delta\psi_* - \frac{\mu+\lambda_* d(x)}{1+\lambda_* k\rho(x)}\psi_* = 0 \quad \text{in } \Omega, \qquad \partial_\nu\psi_* = 0 \quad \text{on } \partial\Omega, \qquad \|\psi_*\|_2 = 1, \tag{2.9}$$

and

$$-\Delta\phi^* - \big(\lambda^*+\mu b(x)\big)\phi^* = 0 \quad \text{in } \Omega, \qquad \partial_\nu\phi^* = 0 \quad \text{on } \partial\Omega, \qquad \big\|\phi^*\big\|_2 = 1. \tag{2.10}$$

It should be noted that we assume $\mu < 0$ in (2.9) and $\mu > 0$ in (2.10).

Furthermore, we define the following Banach spaces:

$$X = W^{2,p}_\nu(\Omega) \times W^{2,p}_\nu(\Omega), \qquad Y = L^p(\Omega) \times L^p(\Omega), \tag{2.11}$$

where

$$W_\nu^{2,p}(\Omega)=\left\{u\in W^{2,p}(\Omega)\colon\ \partial_\nu u=0 \text{ on } \partial\Omega\right\},\quad p>N.$$

Thus $X\subset C^1(\bar\Omega)\times C^1(\bar\Omega)$ by the Sobolev embedding theorem.

Then, we obtain the following lemma by the local bifurcation theory.

Lemma 2.6. *For any fixed $(\mu,k,\rho(x),b(x),d(x))$, the following local bifurcation properties hold true*:

(i) *If $\mu<0$, then a branch of positive solutions of (2.2) bifurcates from Γ_u if and only if $\lambda=\lambda_*>0$. To be precise, all positive solutions of (2.2) near $(\lambda_*,0,\lambda_*)\in X\times\mathbb{R}$ can be parameterized as*:

$$\left\{\left(\lambda_*+s\left(\phi_*+s\tilde u(s)\right),s\left(\psi_*+s\tilde V(s)\right),\lambda(s)\right)\in X\times\mathbb{R}\colon\ 0<s\leqslant\delta_*\right\}\tag{2.12}$$

for some $(\phi_*,\psi_*)\in X$ and $\delta_*>0$. Here $(\tilde u(s),\tilde V(s),\lambda(s))$ is a smooth function with $\lambda(0)=\lambda_*$ and $\int_\Omega\tilde V\psi_*=0$.

If $\mu>0$, there is no positive solutions bifurcating from Γ_u.

(ii) *For $\mu>0$, a branch of positive solutions of (2.2) bifurcates from Γ_V if and only if $\lambda=\lambda^*<0$. More precisely, all positive solutions of (2.2) near $(0,\mu,\lambda^*)$ can be parameterized as*

$$\left\{\left(s\left(\phi^*+s\bar u(s)\right),\mu+s\left(\psi^*+s\bar V(s)\right),\lambda(s)\right)\colon\ 0<s\leqslant\delta^*\right\}\tag{2.13}$$

for some $(\phi^*,\psi^*)\in X$ and $\delta^*>0$. Here $(\bar u(s),\bar V(s),\lambda(s))$ is a smooth function satisfying $\lambda(0)=\lambda^*$ and $\int_\Omega\bar u\phi^*=0$.

If $\mu<0$, there is no such semitrivial solutions $(0,\mu)$.

Proof. (i) Let $U=u-\lambda$ and define the operator $\Phi: X\times\mathbb{R}\to Y$ by

$$\Phi(U,V,\lambda)=\begin{pmatrix}\Delta U+(U+\lambda)\left(-U+\frac{b(x)V}{1+k\rho(x)(U+\lambda)}\right)\\ \Delta V+\frac{V}{1+k\rho(x)(U+\lambda)}\left(\mu-\frac{V}{1+k\rho(x)(U+\lambda)}+d(x)(U+\lambda)\right)\end{pmatrix},$$

then clearly $\Phi(0,0,\lambda)=0$.

After some simple computations, we can see

$$\Phi_{(U,V)}(0,0,\lambda)[\phi,\psi]=\begin{pmatrix}\Delta\phi-\lambda\phi+\frac{\lambda b(x)}{1+\lambda k\rho(x)}\psi\\ \Delta\psi+\frac{\mu+\lambda d(x)}{1+\lambda k\rho(x)}\psi\end{pmatrix}.$$

Note (2.9) yields that in case $\mu<0$,

$$\mathcal{N}\left(\Phi_{(U,V)}(0,0,\lambda_*)\right)=\operatorname{span}\left\{\left(\phi_*,\psi_*\right)\right\},$$

where $\phi_*=(-\Delta+\lambda_*)^{-1}[\frac{\lambda_* b(x)}{1+\lambda_* k\rho(x)}\psi_*]>0$ in Ω. If $(f,g)\in\mathcal{R}(\Phi_{(U,V)}(0,0,\lambda_*))$, then there exists $(\phi,\psi)\in X$ such that

$$\begin{cases}\Delta\phi-\lambda_*\phi+\dfrac{\lambda_* b(x)}{1+\lambda_* k\rho(x)}\psi=f & \text{in } \Omega,\\ \Delta\psi+\dfrac{\mu+\lambda_* d(x)}{1+\lambda_* k\rho(x)}\psi=g & \text{in } \Omega,\\ \partial_\nu\phi=\partial_\nu\psi=0 & \text{on } \partial\Omega.\end{cases}\tag{2.14}$$

Since $\lambda_1(-\frac{\mu+\lambda_* d(x)}{1+\lambda_* k\rho(x)})=0$, together with the elliptic regularity theory and the compactness of this kind of operator $(-\Delta+I)^{-1}$, we can apply the Fredholm alternative theorem to obtain that the second equation of (2.14) is solvable if and only if $\int_\Omega g\psi_*=0$. Then for the obtained solution ψ, we know

$$\phi=(-\Delta+\lambda_*)^{-1}\left(\frac{\lambda_* b(x)}{1+\lambda_* k\rho(x)}\psi-f\right).$$

So,

$$\mathcal{R}\big(\Phi_{(U,V)}(0,0,\lambda_*)\big)=\left\{(f,g)\in Y\colon \int\limits_\Omega g\psi_*=0\right\}.$$

Thus $\dim\mathcal{N}(\Phi_{(U,V)}(0,0,\lambda_*))=\operatorname{codim}\mathcal{R}(\Phi_{(U,V)}(0,0,\lambda_*))=1$. Furthermore, it can be calculated that

$$\Phi_{(U,V)\lambda}(0,0,\lambda_*)[\phi_*,\psi_*]=\begin{pmatrix}-\phi_*+\frac{b(x)}{(1+\lambda_* k\rho(x))^2}\psi_*\\ \frac{d(x)-\mu k\rho(x)}{(1+\lambda_* k\rho(x))^2}\psi_*\end{pmatrix},$$

as $\int_\Omega \frac{d(x)-\mu k\rho(x)}{(1+\lambda_* k\rho(x))^2}\psi_*^2>0$,

$$\Phi_{(U,V)\lambda}(0,0,\lambda_*)[\phi_*,\psi_*]\notin\mathcal{R}\big(\Phi_{(U,V)}(0,0,\lambda_*)\big).$$

Consequently we can apply the local bifurcation theory to Φ at $(0,0,\lambda_*)$. It should be noted that the possibility of other bifurcation points except $\lambda=\lambda_*$ is excluded by the virtue of the Krein–Rutman theorem. Using $u=U+\lambda$, we immediately obtain the local bifurcation branch (2.12). (ii) The proof is similar to that of (i). We sketch the main procedure. Let $\bar V=V-\mu$ and define $\Psi: X\times\mathbb{R}\to Y$ by

$$\Psi(u,\bar V,\lambda)=\begin{pmatrix}\Delta u+u\big(\lambda-u+\frac{b(x)(\bar V+\mu)}{1+k\rho(x)u}\big)\\ \Delta\bar V+\frac{\bar V+\mu}{1+k\rho(x)u}\big(\mu-\frac{\bar V+\mu}{1+k\rho(x)u}+d(x)u\big)\end{pmatrix},$$

$$\Psi_{(u,\bar V)}(0,0,\lambda)[\phi,\psi]=\begin{pmatrix}\Delta\phi+(\lambda+\mu b(x))\phi\\ \Delta\psi+\mu(d(x)+\mu k\rho(x))\phi-\mu\psi\end{pmatrix},$$

$$\Psi_{(u,\bar V)\lambda}(0,0,\lambda)[\phi,\psi]=\begin{pmatrix}\phi\\0\end{pmatrix}.$$

Then (2.10) yields that

$$\mathcal{N}\big(\Psi_{(u,\bar V)}\big(0,0,\lambda^*\big)\big)=\operatorname{span}\big\{\big(\phi^*,\psi^*\big)\big\},$$

where $\psi^*=(-\Delta+\mu)^{-1}[\mu(d(x)+\mu k\rho(x))\phi^*]$ and

$$\mathcal{R}\big(\Psi_{(u,\bar V)}\big(0,0,\lambda^*\big)\big)=\left\{(f,g)\in Y\colon \int\limits_\Omega f\phi^*=0\right\},$$

then

$$\dim\mathcal{N}\big(\Psi_{(u,\bar V)}\big(0,0,\lambda^*\big)\big)=\operatorname{codim}\mathcal{R}\big(\Psi_{(u,\bar V)}\big(0,0,\lambda^*\big)\big)=1.$$

Moreover,

$$\Psi_{(u,\bar{V})\lambda}\big(0,0,\lambda^*\big)\big[\phi^*,\psi^*\big]=\begin{pmatrix}\phi^*\\0\end{pmatrix},$$

$\Psi_{(u,\bar{V})\lambda}(0,0,\lambda^*)[\phi^*,\psi^*]\notin\mathcal{R}(\Psi_{(u,\bar{V})}(0,0,\lambda^*))$, we obtain the local bifurcation branch (2.13). $\square$

Remark 2.7. If there is no cross-diffusion, system (1.2) becomes

$$\begin{cases}\Delta u+u\big(\lambda-u+b(x)v\big)=0, & x\in\Omega,\\ \Delta v+v\big(\mu-v+d(x)u\big)=0, & x\in\Omega,\\ \partial_\nu u=\partial_\nu v=0, & x\in\partial\Omega.\end{cases}$$

By the local bifurcation theory, we can see that the cross-diffusion does not affect the set S_V, but causes S_u to become $\{(\lambda,\mu)\in\mathbb{R}^2\colon\ \lambda_1(-\mu-\lambda d(x))=0\}$. The result shows that the spatial heterogeneity of the cross-diffusion can only affect the bifurcation point on Γ_u, the bifurcation point on Γ_V is unaffected.

2.4. Global bifurcation branch

In this subsection, we obtain the global bifurcation branch together with the local branch obtained in Lemma 2.6.

Theorem 2.8. *For any fixed $(\mu,k,\rho(x),b(x),d(x))$ with $\|b\|_\infty\|d\|_\infty<\frac{1}{c}$ (c is defined in Lemma 2.3), if $\mu>0$, the positive solution set of (2.2) (with parameter λ) forms an* **unbounded** *continuum $\Gamma_p\subset X\times\mathbb{R}$ bifurcating from $(u,V,\lambda)=(0,\mu,\lambda^*)\in\Gamma_V$; if $\mu<0$, then the positive solution set of (2.2) also forms an* **unbounded** *continuum Γ_p bifurcating from $(u,V,\lambda)=(\lambda_*,0,\lambda_*)\in\Gamma_u$.*

Proof. For any fixed $\mu>0$, the local bifurcation branch (2.13) exists. We let $\hat{\Gamma}$ be any maximum extension in $X\times\mathbb{R}$ as a connected set of solutions of (2.2). According to the global bifurcation theorem [33], $\hat{\Gamma}$ must satisfy one of the following:

(i) $\hat{\Gamma}$ is unbounded in $X\times\mathbb{R}$;

(ii) $\hat{\Gamma}$ meets the trivial or a semilinear solution curve at some point except for $(0,\mu,\lambda^*)$.

Recall that positive solutions of (2.2) bifurcate from $\{(0,\mu,\lambda)\colon\ \lambda\in\mathbb{R},\ \mu>0\}$ if and only if $\lambda=\lambda^*$ and no positive solutions bifurcate from $\{(\lambda,0,\lambda)\colon\ \lambda>0\}$ if $\mu>0$. In addition, the non-degeneracy of the trivial solution can be easily verified.

Thus when $\mu>0$, (ii) is excluded. Due to the a priori estimates of positive solutions, the local bifurcation branch (2.13) can be extended to $\lambda\to\infty$, $\hat{\Gamma}$ is unbounded in $X\times\mathbb{R}$.

If $\mu<0$, for the local bifurcation branch (2.12), the proof is essentially the same as that of the case $\mu>0$. We only need to note that there do not exist such semitrivial solutions $(0,\mu)$ in this case. Thus the positive solution set of (2.2) also forms an unbounded continuum bifurcating from $(u,V,\lambda)=(\lambda_*,0,\lambda_*)$. The theorem is proved. $\square$

From Theorem 2.8 and the non-existence region of positive solutions in Lemma 2.5, we know under weak cooperation, if $\mu>0$, (2.2) has at least one positive solution if $\lambda>\lambda^*$; if $\mu<0$, (2.2) has at least one positive solution if $\lambda>\lambda_*$. Either $\mu>0$ or $\mu<0$ asserts the **unbounded** continuum. In summary, the sufficient condition for the existence of positive solutions can be given by

$$\lambda_1\left(-\frac{\mu+\lambda d(x)}{1+\lambda k\rho(x)}\right)<0\quad\text{and}\quad\lambda_1\big(-\lambda-\mu b(x)\big)<0.$$

3. Limiting system

3.1. Lyapunov–Schmidt reduction

To study the heterogeneous effects on the shape of the positive solution curve, we introduce the following change of variables in (2.2):

$$u = \varepsilon w, \qquad V = \varepsilon z, \qquad \lambda = \varepsilon\alpha, \qquad \mu = \varepsilon\beta, \qquad k = \frac{1}{\varepsilon}, \tag{3.1}$$

where ε is a small positive number and α and β are real numbers. By (3.1), the function (w, z) satisfies the following perturbed problem of semilinear elliptic equations:

$$\begin{cases} \Delta w + \varepsilon w\left(\alpha - w + \dfrac{b(x)z}{1+\rho(x)w}\right) = 0, & x \in \Omega, \\ \Delta z + \dfrac{\varepsilon z}{1+\rho(x)w}\left(\beta - \dfrac{z}{1+\rho(x)w} + d(x)w\right) = 0, & x \in \Omega, \\ \partial_\nu w = \partial_\nu z = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Thus, (3.2) has two semitrivial solutions

$$(w, z) = (\alpha, 0) \quad \text{and} \quad (w, z) = (0, \beta)$$

in addition to the trivial solution $(w, z) = (0, 0)$. In the following, we regard α as the bifurcation parameter. We will give the exact structure of the positive solution set of (3.2) when $\varepsilon > 0$ is sufficiently small.

To apply the Lyapunov–Schmidt reduction, we firstly introduce a linear operator $H : X \to Y$ and a nonlinear operator $B : X \times \mathbb{R} \to Y$ by

$$H(w, z) = (\Delta w, \Delta z), \tag{3.3}$$

$$B(w, z, \alpha) = \left(w\left(\alpha - w + \frac{b(x)z}{1+\rho(x)w}\right), \frac{z}{1+\rho(x)w}\left(\beta - \frac{z}{1+\rho(x)w} + d(x)w\right)\right), \tag{3.4}$$

where X and Y are defined in (2.11). Consequently, (3.2) is equivalent to

$$H(w, z) + \varepsilon B(w, z, \alpha) = \mathbf{0}. \tag{3.5}$$

Denote by X_1 and Y_1 the L^2-orthogonal space of $\mathbb{R}^2$ in X and Y, respectively. Moreover, let $P : X \to X_1$ and $Q : Y \to Y_1$ be the orthogonal projections. So for any $(w, z) \in X$, there exists a unique $(r, s) \in \mathbb{R}^2$ such that

$$(w, z) = (r, s) + \mathbf{u}, \quad \mathbf{u} = P(w, z). \tag{3.6}$$

By virtue of $H((r, s)) = \mathbf{0}$ and $(I - Q)H(X_1) = \mathbf{0}$, (3.5) is then reduced to

$$QH(\mathbf{u}) + \varepsilon QB\big((r, s) + \mathbf{u}, \alpha\big) = \mathbf{0} \tag{3.7}$$

and

$$(I - Q)B\big((r, s) + \mathbf{u}, \alpha\big) = \mathbf{0}.$$

By a similar argument to that of [23, Lemma 3.1], the implicit function theorem and a compactness argument give the following lemma.

Lemma 3.1. *For any $C > 0$, there exist a small positive number ε_0 and a neighborhood N of $\{(w, z, \alpha, \varepsilon) = (r, s, \alpha, 0) \in X \times \mathbb{R}^2\colon |r|, |s|, |\alpha| \leqslant C\}$ such that all positive solutions of (3.7) contained in N can be expressed by*

$$K = \big\{\big((r, s) + \varepsilon \mathbf{U}(r, s, \alpha, \varepsilon), \alpha, \varepsilon\big)\colon |r|, |s|, |\alpha| \leqslant C + \varepsilon_0,\ |\varepsilon| \leqslant \varepsilon_0\big\},$$

where $\mathbf{U}(r, s, \alpha, \varepsilon)$ is an X_1-valued smooth function. Therefore,

$$(w, z, \alpha, \varepsilon) = \big((r, s) + \varepsilon \mathbf{U}(r, s, \alpha, \varepsilon), \alpha, \varepsilon\big) \in K$$

is a solution of (3.5) if and only if

$$\Phi^{\varepsilon}(r, s, \alpha) = (I - Q)B\big((r, s) + \varepsilon \mathbf{U}(r, s, \alpha, \varepsilon), \alpha\big) = \mathbf{0}.$$

3.2. Some properties of the limiting positive solution set

In the subsection, we derive the exact expression of the limiting solution set as $\varepsilon \to 0$ and give some useful properties of the limiting solution set. This will be very important to obtain the positive solution set of (3.2) when $\varepsilon > 0$ is small.

Note that $(I - Q)(w, z) = (⨍_\Omega w, ⨍_\Omega z)$, then

$$\Phi^0(r, s, \alpha) = \begin{pmatrix} r\big(\alpha - r + s ⨍_\Omega \frac{b(x)}{1+r\rho(x)}\big) \\ s\big(⨍_\Omega \frac{1}{1+r\rho(x)}\big(\beta - \frac{s}{1+r\rho(x)} + r\,d(x)\big)\big) \end{pmatrix}. \tag{3.8}$$

Thus, $\mathcal{N}(\Phi^0) = \mathcal{L}_0 \cup \mathcal{L}_w \cup \mathcal{L}_z \cup \mathcal{L}_p$, where

$$\mathcal{L}_0 = \big\{(0, 0, \alpha)\colon \alpha \in \mathbb{R}\big\},$$
$$\mathcal{L}_w = \big\{(\alpha, 0, \alpha)\colon \alpha \in \mathbb{R}\big\},$$
$$\mathcal{L}_z = \big\{(0, \beta, \alpha)\colon \alpha \in \mathbb{R}\big\},$$

and

$$\mathcal{L}_p = \big\{\big(r, f(r), g(r)\big)\colon r \in \mathbb{R}\big\},$$

with

$$f(r) = ⨍_\Omega \frac{\beta + r\,d(x)}{1 + r\rho(x)} \Big/ ⨍_\Omega \frac{1}{(1 + r\rho(x))^2}, \qquad g(r) = r - f(r) ⨍_\Omega \frac{b(x)}{1 + r\rho(x)}. \tag{3.9}$$

Note that $\mathcal{L}_p$ contains the limiting set of positive solutions of (3.2) as $\varepsilon \to 0$.

By virtue of (3.9), if $\beta > 0$, then

$$f(r) > 0 \quad \text{for } r \in [0, \infty).$$

While if $\beta < 0$, we can find a unique positive constant r_0 such that

$$\begin{cases} f(r) < 0, & \text{for } r \in [0, r_0), \\ f(r) > 0, & \text{for } r \in (r_0, \infty). \end{cases} \tag{3.10}$$

In the following, we aim to study the profiles of $g(r)$. Since we need the a priori estimates to obtain our final results, we assume the assumptions in Lemma 2.3 hold true in the following.

Lemma 3.2. *Suppose $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma* 2.3. *Then we have*

$$g(0) = -\beta \fint_\Omega b(x), \qquad \lim_{r\to+\infty} g(r) = +\infty$$

and

$$g'(0) = 1 - \fint_\Omega d(x) \fint_\Omega b(x) - \beta\left(\fint_\Omega b(x) \fint_\Omega \rho(x) - \fint_\Omega b(x)\rho(x)\right). \tag{3.11}$$

Furthermore, the following hold:

(i) *If $\fint_\Omega b(x)\rho(x) \geqslant \fint_\Omega b(x) \fint_\Omega \rho(x)$, then $g'(0) > 0$ for $\beta > 0$.*
(ii) *If $\fint_\Omega b(x)\rho(x) < \fint_\Omega b(x) \fint_\Omega \rho(x)$, then*

$$\begin{cases} g'(0) > 0, & \text{if } \beta < \dfrac{1 - \fint_\Omega d(x) \fint_\Omega b(x)}{\fint_\Omega b(x) \fint_\Omega \rho(x) - \fint_\Omega b(x)\rho(x)}, \\ g'(0) < 0, & \text{if } \beta > \dfrac{1 - \fint_\Omega d(x) \fint_\Omega b(x)}{\fint \Omega b(x) \fint_\Omega \rho(x) - \fint_\Omega b(x)\rho(x)}. \end{cases} \tag{3.12}$$

(iii) *If $\beta < 0$, for the zero point r_0 of f, we have*

$$\begin{cases} g'(r_0) > 0, & \text{if } \fint_\Omega \dfrac{1}{(1+r_0\rho(x))^2} > \fint_\Omega \dfrac{d(x)-\beta\rho(x)}{(1+r_0\rho(x))^2} \fint_\Omega \dfrac{b(x)}{1+r_0\rho(x)}, \\ g'(r_0) < 0, & \text{if } \fint_\Omega \dfrac{1}{(1+r_0\rho(x))^2} < \fint_\Omega \dfrac{d(x)-\beta\rho(x)}{(1+r_0\rho(x))^2} \fint_\Omega \dfrac{b(x)}{1+r_0\rho(x)}. \end{cases} \tag{3.13}$$

Proof. From the expression, $g(0) = -\beta \fint_\Omega b(x)$ follows easily. As

$$\frac{g(r)}{r} = 1 - f(r) \fint_\Omega \frac{b(x)}{r(1+r\rho(x))} = 1 - \frac{\fint_\Omega \frac{\beta + r\,d(x)}{1+r\rho(x)} \fint_\Omega \frac{b(x)}{r(1+r\rho(x))}}{\fint_\Omega \frac{1}{(1+r\rho(x))^2}},$$

we can deduce that

$$\lim_{r\to+\infty} \frac{g(r)}{r} = 1 - \frac{\fint_\Omega \frac{d(x)}{\rho(x)} \fint_\Omega \frac{b(x)}{\rho(x)}}{\fint_\Omega \frac{1}{\rho^2(x)}}.$$

While

$$\fint_\Omega \frac{d(x)}{\rho(x)} \fint_\Omega \frac{b(x)}{\rho(x)} \leqslant \|b\|_\infty \|d\|_\infty \left(\fint_\Omega \frac{1}{\rho(x)}\right)^2 \leqslant \|b\|_\infty \|d\|_\infty \fint_\Omega \frac{1}{\rho^2(x)},$$

thus

$$0 < \frac{\fint_\Omega \frac{d(x)}{\rho(x)} \fint_\Omega \frac{b(x)}{\rho(x)}}{\fint_\Omega \frac{1}{\rho^2(x)}} \leqslant \|b\|_\infty \|d\|_\infty < \frac{1}{c} \leqslant 1,$$

which yields that $\lim_{r\to+\infty} g(r) = +\infty$.

As

$$g'(r) = 1 - f'(r) \fint_\Omega \frac{b(x)}{1+r\rho(x)} + f(r) \fint_\Omega \frac{b(x)\rho(x)}{(1+r\rho(x))^2},$$

$$f'(r) = \frac{\fint_\Omega \frac{d(x)-\beta\rho(x)}{(1+r\rho(x))^2} \fint_\Omega \frac{1}{(1+r\rho(x))^2} + 2\fint_\Omega \frac{\beta+rd(x)}{1+r\rho(x)} \fint_\Omega \frac{\rho(x)}{(1+r\rho(x))^3}}{(\fint_\Omega \frac{1}{(1+r\rho(x))^2})^2},$$

then

$$g'(0) = 1 - \fint_\Omega d(x) \fint_\Omega b(x) - \beta\left(\fint_\Omega b(x) \fint_\Omega \rho(x) - \fint_\Omega b(x)\rho(x)\right).$$

Thus (i) and (ii) follow clearly.

It is easy to see that

$$g'(r_0) = 1 - \frac{\fint_\Omega \frac{d(x)-\beta\rho(x)}{(1+r_0\rho(x))^2} \fint_\Omega \frac{b(x)}{1+r_0\rho(x)}}{\fint_\Omega \frac{1}{(1+r_0\rho(x))^2}},$$

thus (iii) follows. □

Remark 3.3. As

$$f(r) = \fint_\Omega \frac{\beta + r\,d(x)}{1+r\rho(x)} \Big/ \fint_\Omega \frac{1}{(1+r\rho(x))^2},$$

$f(r_0) = 0$ in the case $\beta < 0$, we know that r_0 does not depend on $b(x)$. So if $\|b\|_\infty$ is very small, then

$$\fint_\Omega \frac{1}{(1+r_0\rho(x))^2} - \fint_\Omega \frac{d(x)-\beta\rho(x)}{(1+r_0\rho(x))^2} \fint_\Omega \frac{b(x)}{1+r_0\rho(x)}$$

is positive; however, if $\min_{\bar\Omega} b(x)$ is very large, then

$$\fint_\Omega \frac{1}{(1+r_0\rho(x))^2} - \fint_\Omega \frac{d(x)-\beta\rho(x)}{(1+r_0\rho(x))^2} \fint_\Omega \frac{b(x)}{1+r_0\rho(x)}$$

is negative.

Thus, either $g'(r_0) > 0$ or $g'(r_0) < 0$ can hold.

Note that if the coefficients of (3.2) are all spatially homogeneous, that is $\rho(x), b(x)$ and $d(x)$ are all constants, then

$$f(r) = (\beta + rd)(1 + r\rho), \qquad g(r) = r - b(\beta + rd),$$

we can see

$$g'(r) = 1 - bd > 0 \quad \text{for all } r > 0.$$

4. Construction of the perturbed set of positive solutions

4.1. Case $\beta > 0$

Let $\beta > 0$. Then by Lemma 3.2, we can find sufficiently large numbers A and C such that

$$A = g(C) = \max_{r \in [0,C]} g(r). \tag{4.1}$$

Furthermore, Lemma 2.5 asserts that (3.2) has no positive solutions if $\alpha \leqslant -c\beta\|b\|_\infty$. So in this subsection, we let $\alpha \in [-c\beta\|b\|_\infty, A]$.

In the following, the positive solution set of (3.2) will be constructed when $\varepsilon > 0$ is small. To be more precise, we have the following theorem:

Theorem 4.1. *Assume $\beta > 0$, $\|b\|_\infty\|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. Then there exist a small constant $\varepsilon_0 > 0$ and a family of bounded smooth curves*

$$\{S(\xi,\varepsilon) = (r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)) \in \mathbb{R}^3\colon (\xi,\varepsilon) \in [0, C_\varepsilon] \times [0,\varepsilon_0]\}$$

such that for any $\varepsilon \in (0,\varepsilon_0]$, all positive solutions of (3.2) with $\alpha \in [-c\beta\|b\|_\infty, A]$ can be expressed by

$$\begin{aligned}\Gamma^\varepsilon = \{&(w(\xi,\varepsilon), z(\xi,\varepsilon), \alpha(\xi,\varepsilon)) = ((r,s) + \varepsilon\mathbf{U}(r,s,\alpha,\varepsilon), \alpha)\colon \\ &(r,s,\alpha) = (r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)),\ \xi \in (0, C_\varepsilon)\}\end{aligned} \tag{4.2}$$

where $\mathbf{U}(r,s,\alpha,\varepsilon)$ is the X_1-valued smooth function defined in Lemma 3.1 and $S(\xi,\varepsilon)$ is a certain smooth function satisfying

$$S(\xi,0) = (\xi, f(\xi), g(\xi)) \quad \textit{and} \quad S(0,\varepsilon) = (0,\beta,\alpha^*(\varepsilon)).$$

Here $\alpha^*(\varepsilon)$ is defined by

$$\alpha^*(\varepsilon) = \frac{\lambda^*(\varepsilon\beta)}{\varepsilon}, \tag{4.3}$$

and C_ε is a certain smooth positive function in $\varepsilon \in [0,\varepsilon_0]$ with $C_0 = C$ and $\alpha(C_\varepsilon,\varepsilon) = A$. Furthermore, Γ^ε can be extended to the range of $\alpha \in [A,\infty)$ as a positive solution curve of (3.2).

In order to prove Theorem 4.1, we take several steps. As the first step, we construct local branches of positive solutions of (3.5) near $(0, \beta, -\beta \fint_\Omega b(x))$.

Lemma 4.2. *Assume $\beta > 0$, then there exist a neighborhood U^* of $(0, \beta, -\beta \fint_\Omega b(x))$ and a positive number $\bar{\delta}^*$ such that for each $\varepsilon \in [0, \bar{\delta}^*]$,*

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U^* \cap \big(\bar{\mathbb{R}}_+^2 \times \mathbb{R}\big) = \big\{\big(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)\big)\colon \xi \in \big[0, \bar{\delta}^*\big]\big\} \cup \big\{(0,\beta,\alpha) \in U^*\big\}, \quad (4.4)$$

for a smooth function $(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon))$ with

$$\big(r(\xi,0), s(\xi,0), \alpha(\xi,0)\big) = \big(\xi, f(\xi), g(\xi)\big), \qquad \big(r(0,\varepsilon), s(0,\varepsilon), \alpha(0,\varepsilon)\big) = \big(0, \beta, \alpha^*(\varepsilon)\big).$$

Proof. By Lemma 2.6 and the change of variable (3.1), we know that there exist a neighborhood V_ε of $(w, z, \alpha) = (0, \beta, \alpha^*(\varepsilon))$ and a positive number $\delta = \delta(\varepsilon)$ such that all positive solutions of (3.2) contained in V_ε can be given by

$$\big(w(\xi,\varepsilon), z(\xi,\varepsilon), \alpha(\xi,\varepsilon)\big) = \big(\xi\big(\phi^* + W(\xi,\varepsilon)\big), \beta + \xi\big(\psi^* + Z(\xi,\varepsilon)\big), \alpha(\xi,\varepsilon)\big)$$

for $\xi \in [0, \delta]$, where $(\phi^*, \psi^*) \in X$ is defined in Lemma 2.6, $(W(\xi,\varepsilon), Z(\xi,\varepsilon), \alpha(\xi,\varepsilon))$ is a certain smooth function with $\alpha(0,\varepsilon) = \alpha^*(\varepsilon)$ and $\int_\Omega W(\xi,\varepsilon)\phi^* = 0$.

We define the subset $U_\varepsilon \subset \mathbb{R}^3$ by

$$U_\varepsilon = \left\{(r, s, \alpha) \in \mathbb{R}^3\colon r = \fint_\Omega w,\ s = \fint_\Omega z \text{ for } (w, z, \alpha) \in V_\varepsilon\right\},$$

and put

$$r(\xi,\varepsilon) = \fint_\Omega w(\xi,\varepsilon), \qquad s(\xi,\varepsilon) = \fint_\Omega z(\xi,\varepsilon).$$

As (3.2) is equivalent to $\Phi^\varepsilon(r, s, \alpha) = 0$ for small ε, we obtain

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U_\varepsilon \cap \big(\bar{\mathbb{R}}_+^2 \times \mathbb{R}\big) = \big\{\big(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)\big)\colon \xi \in [0, \delta]\big\} \cup \big\{(0,\beta,\alpha) \in U_\varepsilon\big\}.$$

Since $\lim_{\varepsilon \to 0} \alpha^*(\varepsilon) = -\beta \fint_\Omega b(x)$ (the proof is similar to [21, Lemma 4.6], we omit the proof), $(0, \beta, \alpha(0,\varepsilon)) = (0, \beta, \alpha^*(\varepsilon))$ is the bifurcation point, so if $\varepsilon > 0$ is small, U_ε contains a neighborhood $U^*(\subset \mathbb{R}^3)$ of $(0, \beta, -\beta \fint_\Omega b(x))$. The proof is complete. □

Lemma 4.3. *Assume $\beta > 0$. Then there exist a positive number $\bar{\delta}$ and a neighborhood U of $\{(r, f(r), g(r))\colon 0 \leqslant r \leqslant C\}$ such that for any $\varepsilon \in [0, \bar{\delta}]$, all positive solutions of (3.2) contained in $U \cap (X \times [-c\beta\|b\|_\infty, A])$ can be parameterized as (4.2).*

Proof. To prove the lemma, we mainly use the perturbation theory of Du and Lou [10, Appendix]. For the number $\bar{\delta}^* > 0$ obtained in Lemma 4.2, we define

$$\mathcal{L}_p\big(\big[\bar{\delta}^*/2, C\big]\big) = \big\{\big(r, f(r), g(r)\big)\colon r \in \big[\bar{\delta}^*/2, C\big]\big\}.$$

As some computations can yield that

$$\det \Phi^0_{(r,s)}\big(r, f(r), g(r)\big) = rf(r)g'(r) \fint_\Omega \frac{1}{(1 + r\rho(x))^2},$$

and $f(\bar r) > 0$ for any $(\bar r, f(\bar r), g(\bar r)) \in \mathcal{L}_p([\bar\delta^*/2, C])$, we can see that if $g'(\bar r) \neq 0$, $\Phi^0_{(r,s)}(\bar r, f(\bar r), g(\bar r))$ is invertible. The implicit function theorem then asserts that there exist a positive number $\delta = \delta(\bar r)$ and a neighborhood $W_{\bar r}$ of $(\bar r, f(\bar r))$ such that for each $\varepsilon \in [0, \delta]$,

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U_{\bar r} = \big\{\big(r(\alpha, \varepsilon), s(\alpha, \varepsilon), \alpha\big)\colon \alpha \in \big(g(\bar r) - \delta, g(\bar r) + \delta\big)\big\}, \tag{4.5}$$

where $U_{\bar r} = W_{\bar r} \times (g(\bar r) - \delta, g(\bar r) + \delta)$ and $(r(\alpha, \varepsilon), s(\alpha, \varepsilon))$ is a smooth function satisfying

$$\big(r(g(\bar r), 0), s(g(\bar r), 0)\big) = \big(\bar r, f(\bar r)\big).$$

While if $g'(\bar r) = 0$, then $\operatorname{rank}\Phi^0_{(r,s)}(\bar r, f(\bar r), g(\bar r)) = 1$, which means

$$\dim\mathcal{N}\big(\Phi^0_{(r,s)}\big(\bar r, f(\bar r), g(\bar r)\big)\big) = \operatorname{codim}\mathcal{R}\big(\Phi^0_{(r,s)}\big(\bar r, f(\bar r), g(\bar r)\big)\big) = 1.$$

Furthermore, we can show that

$$\Phi^0_\alpha\big(\bar r, f(\bar r), g(\bar r)\big) = \begin{pmatrix} \bar r \\ 0 \end{pmatrix},$$

$\Phi^0_\alpha(\bar r, f(\bar r), g(\bar r)) \notin \mathcal{R}(\Phi^0_{(r,s)}(\bar r, f(\bar r), g(\bar r)))$ can be verified. Then the bifurcation theory of Crandall and Rabinowitz [5, Theorem 3.2 and Remark 3.3] yields that there exist a positive number $\delta = \delta(\bar r)$ and a neighborhood $U_{\bar r}$ of $(\bar r, f(\bar r), g(\bar r))$ such that for any $\varepsilon \in [0, \delta]$,

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U_{\bar r} = \big\{\big(r(\xi, \varepsilon), s(\xi, \varepsilon), \alpha(\xi, \varepsilon)\big)\colon \xi \in (-\delta, \delta)\big\}, \tag{4.6}$$

with a smooth function $(r(\xi, \varepsilon), s(\xi, \varepsilon), \alpha(\xi, \varepsilon))$ satisfying

$$\big(r(0, 0), s(0, 0), \alpha(0, 0)\big) = \big(\bar r, f(\bar r), g(\bar r)\big).$$

So either (4.5) or (4.6) holds, we know

$$\mathcal{L}_p\big(\big[\bar\delta^*/2, C\big]\big) \subset \bigcup\big\{U_{\bar r}\colon \bar r \in \big[\bar\delta^*/2, C\big]\big\}.$$

As $\mathcal{L}_p([\bar\delta^*/2, C])$ is compact, we can find finitely many points $\{r_j\}_{j=1}^n$ such that

$$\begin{cases} \big(r_j, f(r_j), g(r_j)\big) \in \mathcal{L}_p\big(\big[\bar\delta^*/2, C\big]\big), & \text{for } 1 \leqslant j \leqslant n, \\ \mathcal{L}_p\big(\big[\bar\delta^*/2, C\big]\big) \subset \displaystyle\bigcup_{j=1}^n U_j, & \text{where } U_j = U_{r_j}. \end{cases}$$

In addition, we put $U_0 = U^*$. Without loss of generality, we can assume

$$U_j \cap U_{j+1} \neq \emptyset \quad \text{for all } 0 \leqslant j \leqslant n - 1.$$

Let $\delta_j = \delta(r_j)$. Then (4.5) and (4.6) yield that for each $\varepsilon \in [0, \delta_j]$ $(1 \leqslant j \leqslant n)$, there exists a smooth function $(r_j(\xi, \varepsilon), s_j(\xi, \varepsilon), \alpha_j(\xi, \varepsilon))$ such that

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U_j = \big\{\big(r_j(\xi, \varepsilon), s_j(\xi, \varepsilon), \alpha_j(\xi, \varepsilon)\big)\colon \xi \in (-\delta_j, \delta_j)\big\} = J_j^\varepsilon, \tag{4.7}$$

with $(r_j(0, 0), s_j(0, 0,), \alpha_j(0, 0,)) = (r_j, f(r_j), g(r_j))$.

Furthermore, we set

$$J_0^\varepsilon = \big\{(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)) \colon \xi \in \big[0, \bar{\delta}^*\big]\big\},$$

$$U = \bigcup_{j=0}^{n} U_j.$$

Thus, Lemma 4.2 and (4.7) imply that

$$\mathcal{N}\big(\Phi^\varepsilon\big) \cap U \cap \big(\mathbb{R}_+^2 \times \mathbb{R}\big) = \bigcup_{j=0}^{n} J_j^\varepsilon \quad \text{for } \varepsilon \in \Big[0, \min_{0 \leqslant j \leqslant n} \delta_j\Big], \tag{4.8}$$

with $\delta_0 = \bar{\delta}^*$. So we know that $\mathcal{N}(\Phi^\varepsilon) \cap U \cap (\mathbb{R}_+^2 \times \mathbb{R})$ is a one-dimensional submanifold. Indeed, with the aid of the perturbation theory, we can construct a smooth curve $S(\xi,\varepsilon) = (r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon))$ which satisfies

$$\begin{cases} \displaystyle\bigcup_{j=0}^{n} J_j^\varepsilon = S\big((0, C_\varepsilon), \varepsilon\big), \\ \big(r(\xi,0), s(\xi,0), \alpha(\xi,0)\big) = \big(\xi, f(\xi), g(\xi)\big), \\ \big(r(0,\varepsilon), s(0,\varepsilon), \alpha(0,\varepsilon)\big) = \big(0, \beta, \alpha^*(\varepsilon)\big), \end{cases}$$

for small $\varepsilon > 0$ and $\xi \in [0, C_\varepsilon]$. The lemma is proved. □

If we can show that (3.2) has no positive solution outside U for $\alpha \in [-c\beta\|b\|_\infty, A]$ in the next step, Lemmas 4.2 and 4.3 can yield Theorem 4.1. More precisely, we have the following lemma.

Lemma 4.4. *Assume $\beta > 0$ and $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. For any fixed neighborhood $V (\subset \mathbb{R}^3)$ of $\{(r, f(r), g(r)) \colon 0 \leqslant r \leqslant C\}$, there exists a small number $\varepsilon_1 > 0$ such that if $\varepsilon \in [0, \varepsilon_1]$, any positive solution (w, z) of (3.2) with $\alpha \in [-c\beta\|b\|_\infty, A]$ can be expressed by*

$$(w, z) = (r, s) + \varepsilon \mathbf{U}(r, s, \alpha, \varepsilon).$$

Here $(r, s, \alpha) \in V$ and $\mathbf{U}(r, s, \alpha, \varepsilon)$ is the X_1-valued function in Lemma 3.1.

Proof. We will prove the lemma by a contradiction argument. Suppose that there exists a sequence $\{(\alpha_n, \varepsilon_n)\}$ with $\alpha_n \in [-c\beta\|b\|_\infty, A]$ and $\lim_{n\to\infty} \varepsilon_n = 0$ such that (3.2) has a positive solution (w_n, z_n) and $(w_n, z_n, \alpha_n) \notin V$ for all $n \in \mathbb{N}$.

If we can find a sequence $\{(r_j, s_j)\}$ and a subsequence $\{(w_{n(j)}, z_{n(j)}, \alpha_{n(j)})\}$ such that

$$\begin{cases} (w_{n(j)}, z_{n(j)}) = (r_j, s_j) + \varepsilon_{n(j)} \mathbf{U}(r_j, s_j, \alpha_{n(j)}, \varepsilon_{n(j)}), & \text{for all } j \in \mathbb{N}, \\ \lim_{j\to\infty} (r_j, s_j, \alpha_{n(j)}) = \big(r, f(r), g(r)\big), & \text{for some } r \in \big[-c\beta\|b\|_\infty, A\big], \end{cases} \tag{4.9}$$

then the desired contradiction is obtained. As Lemma 2.3 and (3.1) yield the following:

$$\left(1 - \frac{\|b\|_\infty \|d\|_\infty \|\rho\|_\infty}{\min_{\bar{\Omega}} \rho}\right) w_n \leqslant \alpha_n + \frac{\beta\|b\|_\infty \|\rho\|_\infty}{\min_{\bar{\Omega}} \rho} \leqslant M,$$

$$\left(1 - \frac{\|b\|_\infty \|d\|_\infty \|\rho\|_\infty}{\min_{\bar{\Omega}} \rho}\right) z_n \leqslant \big(\beta + \alpha_n \|d\|_\infty\big)\big(1 + \|\rho\|_\infty \|w_n\|_\infty\big) \leqslant M$$

for a sufficiently large number M independent of n, $\{(w_n, z_n)\}$ is uniformly bounded in $C(\bar{\Omega})$. Let $\bar{w}_n = w_n/\|w_n\|_\infty$, $\bar{z}_n = z_n/\|z_n\|_\infty$, then

$$\begin{cases} \Delta \bar{w}_n + \varepsilon_n \bar{w}_n\left(\alpha_n - w_n + \dfrac{b(x)z_n}{1+\rho(x)w_n}\right) = 0, & x \in \Omega, \\ \Delta \bar{z}_n + \dfrac{\varepsilon_n \bar{z}_n}{1+\rho(x)w_n}\left(\beta - \dfrac{z_n}{1+\rho(x)w_n} + d(x)w_n\right) = 0, & x \in \Omega, \\ \partial_\nu \bar{w}_n = \partial_\nu \bar{z}_n = 0, & x \in \partial\Omega. \end{cases} \tag{4.10}$$

Since $\{(w_n, z_n, \alpha_n)\}$ is uniformly bounded in $C(\bar{\Omega}) \times C(\bar{\Omega}) \times \mathbb{R}$,

$$\left\{\bar{w}_n\left(\alpha_n - w_n + \frac{b(x)z_n}{1+\rho(x)w_n}\right)\right\} \quad \text{and} \quad \left\{\frac{\bar{z}_n}{1+\rho(x)w_n}\left(\beta - \frac{z_n}{1+\rho(x)w_n} + d(x)w_n\right)\right\}$$

are also uniformly bounded. Then the elliptic regularity (see [14]) deduces that there exists a subsequence $\{(w_{n(j)}, z_{n(j)}, \alpha_{n(j)})\}$ such that

$$\lim_{j\to\infty} (\bar{w}_{n(j)}, \bar{z}_{n(j)}, \alpha_{n(j)}) = (\bar{w}, \bar{z}, \alpha_\infty) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \times \mathbb{R}, \tag{4.11}$$

for some $(\bar{w}, \bar{z}, \alpha_\infty)$.

Let $n \to \infty$ in (4.10), we know

$$\Delta \bar{w} = \Delta \bar{z} = 0 \quad \text{in } \Omega, \qquad \partial_\nu \bar{w} = \partial_\nu \bar{z} = 0 \quad \text{on } \partial\Omega. \tag{4.12}$$

Since $\|\bar{w}\|_\infty = \|\bar{z}\|_\infty = 1$, we know that $\bar{w} = \bar{z} = 1$ in $\bar{\Omega}$. Thus we can find nonnegative constants r and s such that

$$\lim_{j\to\infty} (w_{n(j)}, z_{n(j)}) = (r, s) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}), \tag{4.13}$$

as $\{(w_n, z_n)\}$ is bounded and positive. Together with Lemma 3.1, $(w_{n(j)}, z_{n(j)})$ can be parameterized as

$$(w_{n(j)}, z_{n(j)}) = (r_j, s_j) + \varepsilon_{n(j)}\mathbf{U}(r_j, s_j, \alpha_{n(j)}, \varepsilon_{n(j)})$$

for sufficiently large $j \in \mathbb{N}$. Moreover, $\lim_{j\to\infty}(r_j, s_j) = (r, s)$.

Integrating the two equations in (4.10), we obtain

$$\begin{cases} \displaystyle\fint_\Omega \bar{w}_{n(j)}\left(\alpha_{n(j)} - w_{n(j)} + \frac{b(x)z_{n(j)}}{1+\rho(x)w_{n(j)}}\right) = 0, \\ \displaystyle\fint_\Omega \frac{\bar{z}_{n(j)}}{1+\rho(x)w_{n(j)}}\left(\beta - \frac{z_{n(j)}}{1+\rho(x)w_{n(j)}} + d(x)w_{n(j)}\right) = 0. \end{cases} \tag{4.14}$$

We let $j \to \infty$ in (4.14), then

$$\begin{cases} \displaystyle\alpha_\infty - r + s\fint_\Omega \frac{b(x)}{1+r\rho(x)} = 0, \\ \displaystyle\fint_\Omega \frac{1}{1+r\rho(x)}\left(\beta - \frac{s}{1+r\rho(x)} + r\,d(x)\right) = 0. \end{cases}$$

Obviously,

$$s = f(r), \qquad \alpha_\infty = g(r).$$

So (4.9) is proved, which shows the lemma. □

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemmas 4.3 and 4.4, we need only to show that Γ^ε can be extended to $\alpha \in [A, \infty)$ as a positive solution curve of (3.2). Let $\hat{\Gamma}^\varepsilon$ be any maximum extension in the direction $\alpha \geqslant A$. By virtue of the global bifurcation theorem, $\hat{\Gamma}^\varepsilon$ must satisfy one of the following:

(i) $\hat{\Gamma}^\varepsilon$ is unbounded in $X \times \mathbb{R}$;
(ii) $\hat{\Gamma}^\varepsilon$ meets a certain bifurcation point except for $(0, \beta, \alpha^*(\varepsilon))$.

As no positive solutions bifurcate from other semitrivial solution curve $\{(\alpha, 0, \alpha): \alpha > 0\}$, and the trivial solution is non-degenerate, (ii) is excluded. As (w, z) is bounded due to the a priori estimates, Γ^ε can be extended to $\alpha \in [A, \infty)$. The theorem is proved. □

4.2. Unbounded fish–hook shaped branch

Due to Theorem 4.1, we obtain the following unbounded fish–hook shaped bifurcation branch.

Theorem 4.5. *Assume $\beta > 0$, $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3,*

$$\fint_\Omega b(x)\rho(x) < \fint_\Omega b(x) \fint_\Omega \rho(x).$$

Then for any small constant $\eta > 0$, there exists a small positive number ε_2 such that if

$$(\beta, \varepsilon) \in \left[\frac{1 - \fint_\Omega d(x) \fint_\Omega b(x)}{\fint_\Omega b(x) \fint_\Omega \rho(x) - \fint_\Omega b(x)\rho(x)} + \eta, \eta^{-1}\right] \times [0, \varepsilon_2],$$

the positive solution set of (3.2) contains an unbounded fish–hook shaped curve Γ^ε bifurcating from the semitrivial solution curve $(0, \beta, \alpha^*(\varepsilon))$. Furthermore, there exists a number

$$\underline{\alpha}(\varepsilon) = \min_{\xi \in [0, C_\varepsilon]} \alpha(\xi, \varepsilon) < \alpha^*(\varepsilon)$$

such that

(i) *if $\alpha \in [-c\beta\|b\|_\infty, \underline{\alpha}(\varepsilon))$, (3.2) has no positive solutions;*
(ii) *if $\alpha \in [\alpha^*(\varepsilon), \infty)$ or $\alpha = \underline{\alpha}(\varepsilon)$, (3.2) has at least one positive solution;*
(iii) *if $\alpha \in (\underline{\alpha}(\varepsilon), \alpha^*(\varepsilon))$, (3.2) has at least two positive solutions.*

Proof. For the smooth curve $S(\xi, \varepsilon) = (r(\xi, \varepsilon), s(\xi, \varepsilon), \alpha(\xi, \varepsilon))$ defined in Theorem 4.1, $S(\xi, 0) = (\xi, f(\xi), g(\xi))$, we can show that

$$\lim_{\varepsilon \to 0} \big(s(\xi, \varepsilon), \alpha(\xi, \varepsilon)\big) = \big(f(\xi), g(\xi)\big) \quad \text{in } C^1\big([0, C]\big) \times C^1\big([0, C]\big),$$

with C defined in (4.1).

Since when $\beta > \frac{1-⨍_\Omega d(x)⨍_\Omega b(x)}{⨍_\Omega b(x)⨍_\Omega \rho(x) - ⨍_\Omega b(x)\rho(x)}$, $g'(0) < 0$. Then for any fixed small positive number η, we can find a small $\varepsilon_2 > 0$ such that if $(\beta, \varepsilon) \in [\frac{1-⨍_\Omega d(x)⨍_\Omega b(x)}{⨍_\Omega b(x)⨍_\Omega \rho(x) - ⨍_\Omega b(x)\rho(x)} + \eta, \eta^{-1}] \times [0, \varepsilon_2]$, $\alpha_\xi(0, \varepsilon) < 0$.

With regard to $\alpha(\xi, \varepsilon) \leqslant \alpha(C_\varepsilon, \varepsilon)$ for $\xi \in [0, C_\varepsilon]$, we know that there exists

$$\underline{\alpha}(\varepsilon) = \alpha\big(\underline{\xi}(\varepsilon), \varepsilon\big) = \min_{\xi\in[0,C_\varepsilon]} \alpha(\xi, \varepsilon)$$

for some $\underline{\xi}(\varepsilon) \in (0, C_\varepsilon)$.

Define

$$K_\varepsilon(\alpha) = \big\{\xi \in (0, C_\varepsilon)\colon \alpha(\xi, \varepsilon) = \alpha\big\}.$$

Obviously, $K_\varepsilon(\alpha)$ has no element if $\alpha \in [-c\beta\|b\|_\infty, \underline{\alpha}(\varepsilon))$, at least one element if $\alpha \in [\alpha^*(\varepsilon), A)$ or $\alpha = \underline{\alpha}(\varepsilon)$, and at least two elements if $\alpha \in (\underline{\alpha}(\varepsilon), \alpha^*(\varepsilon))$. While (4.2) implies that the number of elements of $K_\varepsilon(\alpha)$ is equal to the number of positive solutions of (3.2) for sufficiently small ε and $\alpha \in [-c\beta\|b\|_\infty, A]$. As Γ^ε can be extended in the direction $\alpha \in [A, \infty)$, we know that (3.2) has at least one positive solution for $\alpha \in [A, \infty)$. The proof is complete. □

Remark 4.6. If $⨍_\Omega b(x)\rho(x) \geqslant ⨍_\Omega b(x)⨍_\Omega \rho(x)$, then for any small $\eta > 0$, there exists ε_2 such that if $(\beta, \varepsilon) \in [\eta, \eta^{-1}] \times [0, \varepsilon_2]$, the bifurcation at $(0, \beta, \alpha^*(\varepsilon))$ is supercritical; or $⨍_\Omega b(x)\rho(x) < ⨍_\Omega b(x)⨍_\Omega \rho(x)$, then for any small $\eta > 0$, there exists ε_2 such that if

$$(\beta, \varepsilon) \in \left[\eta, \frac{1-⨍_\Omega d(x)⨍_\Omega b(x)}{⨍_\Omega b(x)⨍_\Omega \rho(x) - ⨍_\Omega b(x)\rho(x)} - \eta\right] \times [0, \varepsilon_2],$$

the bifurcation at $(0, \beta, \alpha^*(\varepsilon))$ is supercritical. In the two cases (including the case that the cross-diffusion is spatially homogeneous), we can only deduce that if $\alpha \in (\alpha^*(\varepsilon), \infty)$, (3.2) has at least one positive solution by this method.

While if a spatial segregation of $b(x)$ and $\rho(x)$ enables $⨍_\Omega b(x)\rho(x) < ⨍_\Omega b(x)⨍_\Omega \rho(x)$ to hold, then we can deduce that if $\alpha \in (\underline{\alpha}(\varepsilon), \alpha^*(\varepsilon))$, (3.2) has at least two positive solutions for suitable fixed β besides the existence of positive solutions when $\alpha \in (\alpha^*(\varepsilon), \infty)$. We see that the spatial heterogeneity can generate complicated and interesting spatio-temporal patterns of stationary solutions.

On the other hand, as we know if the coefficients of (3.2) are all spatially homogeneous, $g'(r) > 0$ for all r, then if $\varepsilon > 0$ is sufficiently small, $\alpha_\xi(\xi, \varepsilon) > 0$ can be obtained for any $\xi \in (0, C_\xi)$. The positive solution curve is a monotone curve with respect to α in either case $\beta > 0$ or $\beta < 0$. Thus, (3.2) has a unique positive solution if $\alpha > \alpha^*(\varepsilon)$ in case $\beta > 0$ and $\alpha > \alpha_*(\varepsilon)$ in case $\beta < 0$, the latter case will be seen in the next subsection.

4.3. *Case $\beta < 0$*

Since

$$g(r_0) = 0, \qquad g(r) \to +\infty \quad \text{as } r \to +\infty,$$

there exist two sufficiently large numbers A_1 and C_1 such that

$$A_1 = \max_{r\in[r_0,C_1]} g(r).$$

Lemma 2.5 tells us that if $\alpha \leqslant -\frac{\beta}{\|d\|_\infty}$, (3.2) has no positive solutions, then we let $\alpha \in [-\frac{\beta}{\|d\|_\infty}, A_1]$ in the subsection.

Completely similar to the case $\beta > 0$, we only need to replace $r \in [0, C]$ by $r \in [r_0, C_1]$ and can obtain the following conclusions:

Theorem 4.7. *Assume $\beta < 0$, $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. Then there exist a small $\varepsilon_0 > 0$ and a family of bounded curves*

$$\bigl\{S(\xi,\varepsilon) = \bigl(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)\bigr):\ (\xi,\varepsilon) \in [0, C_\varepsilon] \times [0, \varepsilon_0]\bigr\}$$

such that for any fixed $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (3.2) with $\alpha \in [-\frac{\beta}{\|d\|_\infty}, A_1]$ can be expressed by

$$\Gamma_\varepsilon = \bigl\{(w,z,\alpha) = \bigl((r,s) + \varepsilon\mathbf{U}(r,s,\alpha,\varepsilon), \alpha\bigr):\ (r,s,\alpha) = \bigl(r(\xi,\varepsilon), s(\xi,\varepsilon), \alpha(\xi,\varepsilon)\bigr), \xi \in (0, C_\varepsilon)\bigr\} \tag{4.15}$$

where $S(\xi,\varepsilon)$ is a certain smooth function with

$$S(\xi,0) = \bigl(r_0 + \xi, f(r_0+\xi), g(r_0+\xi)\bigr), \qquad S(0,\varepsilon) = \bigl(\alpha_*(\varepsilon), 0, \alpha_*(\varepsilon)\bigr).$$

Here $\alpha_*(\varepsilon) = \frac{\lambda_*(\varepsilon\beta)}{\varepsilon} > 0$ and C_ε is a certain smooth function in $[0, \varepsilon_0]$ such that $C_0 = C_1$ and $\alpha(C_\varepsilon, \varepsilon) = A_1$. Furthermore, Γ_ε can be extended in the direction $\alpha \in [A_1, \infty)$ as a positive solution curve of (3.2).

Lemma 4.8. *Assume $\beta < 0$. There exist a neighborhood $U_*(\subset \mathbb{R}^3)$ of $(r_0, 0, r_0)$ and a positive number $\bar\delta_*$ such that for any $\varepsilon \in [0, \bar\delta_*]$,*

$$\mathcal{N}\bigl(\Phi^\varepsilon\bigr) \cap U_* \cap \bigl(\bar{\mathbb{R}}^2_+ \times \mathbb{R}\bigr) = \bigl\{\bigl(\hat r(\xi,\varepsilon), \hat s(\xi,\varepsilon), \hat\alpha(\xi,\varepsilon)\bigr) \in \mathbb{R}^3:\ \xi \in [0, \bar\delta_*]\bigr\} \cup \bigl\{(\alpha, 0, \alpha) \in U_*\bigr\},$$

with a certain smooth function $\hat S(\xi,\varepsilon) = (\hat r(\xi,\varepsilon), \hat s(\xi,\varepsilon), \hat\alpha(\xi,\varepsilon))$ satisfying

$$\hat S(\xi,0) = \bigl(r_0 + \xi, f(r_0+\xi), g(r_0+\xi)\bigr), \qquad \hat S(0,\varepsilon) = \bigl(\alpha_*(\varepsilon), 0, \alpha_*(\varepsilon)\bigr).$$

It should be noted that we can deduce $\lim_{\varepsilon\to 0} \alpha_*(\varepsilon) = r_0$ by a similar proof to that of [21, Lemma 4.4].

Lemma 4.9. *Assume $\beta < 0$, then there exist a small $\bar\delta > 0$ and a neighborhood $U(\subset X \times \mathbb{R})$ of*

$$\bigl\{\bigl(r, f(r), g(r)\bigr):\ r_0 \leqslant r \leqslant C_1\bigr\}$$

such that if $\varepsilon \in [0, \bar\delta]$, all positive solutions of (3.5) contained in $U \cap (X \times [-\frac{\beta}{\|d\|_\infty}, A_1])$ can be parameterized as (4.15).

Lemma 4.10. *Assume $\beta < 0$, $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. For any neighborhood V of $\{(r, f(r), g(r)):\ r \in [r_0, C_1]\}$, all positive solutions of (3.2) with $\alpha \in [-\frac{\beta}{\|d\|_\infty}, A_1]$ can be expressed by*

$$(w,z,\alpha) = (r,s) + \varepsilon\mathbf{U}(r,s,\alpha,\varepsilon) \quad \text{with } (r,s,\alpha) \in V,$$

when $\varepsilon > 0$ is sufficiently small.

Then Lemmas 4.9 and 4.10 imply Theorem 4.7. Consequently we deduce the following unbounded fish–hook shaped bifurcation curve.

Theorem 4.11. *Assume $\beta < 0$, $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. If $\min_{\bar\Omega} b(x)$ is very large and $\|d\|_\infty$ is very small such that $g'(r_0) < 0$, then for any small number $\eta > 0$, there exists $\varepsilon_1 > 0$ such that if $(\beta, \varepsilon) \in [-\eta^{-1}, -\eta] \times [0, \varepsilon_1]$, the positive solution set of (3.2) forms an unbounded fish–hook shaped smooth curve bifurcating from $(\alpha_*(\varepsilon), 0, \alpha_*(\varepsilon))$. Furthermore, the bifurcation at $(\alpha_*(\varepsilon), 0, \alpha_*(\varepsilon))$ is subcritical. For $\underline{\alpha}(\varepsilon) = \min_{\xi \in [0, C_\varepsilon]} \alpha(\xi, \varepsilon)$,*

(i) *if $\alpha \in [-\frac{\beta}{\|d\|_\infty}, \underline{\alpha}(\varepsilon))$, (3.2) has no positive solutions;*
(ii) *if $\alpha \in [\alpha_*(\varepsilon), \infty)$ or $\alpha = \underline{\alpha}(\varepsilon)$, (3.2) has at least one positive solution;*
(iii) *if $\alpha \in (\underline{\alpha}(\varepsilon), \alpha_*(\varepsilon))$, (3.2) has at least two positive solutions.*

5. Main results

In this section, we give the main results. Assume $\|b\|_\infty \|d\|_\infty < \frac{1}{c}$ with c defined in Lemma 2.3. Let $\mu = \varepsilon\beta$, $k = \frac{1}{\varepsilon}$, then (1.2) becomes

$$\begin{cases} \Delta u + u\big(\lambda - u + b(x)v\big) = 0, & x \in \Omega, \\ \Delta\big[\big(1 + \varepsilon^{-1}\rho(x)u\big)v\big] + v\big(\varepsilon\beta - v + d(x)u\big) = 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Clearly, (w, z) is a positive solution of (3.2) if and only if

$$(u, v, \lambda) = \varepsilon\left(w, \frac{z}{1 + \rho(x)w}, \alpha\right) \tag{5.2}$$

is a positive solution of (5.1).

Then by Theorems 4.1 and 4.7, we know that for any fixed $(\beta, \rho(x), b(x), d(x))$, for small $\varepsilon > 0$, all positive solutions of (5.1) can be expressed by

$$\Gamma_p = \big\{\big(u(\xi, \varepsilon), v(\xi, \varepsilon), \lambda(\xi, \varepsilon)\big) \colon \xi \in [0, C_\varepsilon]\big\},$$

where $(u(\xi, \varepsilon), v(\xi, \varepsilon), \lambda(\xi, \varepsilon)) = \varepsilon(w(\xi, \varepsilon), \frac{z(\xi,\varepsilon)}{1+\rho(x)w(\xi,\varepsilon)}, \alpha(\xi, \varepsilon))$, $(w(\xi, \varepsilon), z(\xi, \varepsilon), \alpha(\xi, \varepsilon))$ is defined in (4.2) or (4.15). In both cases of $\beta > 0$ and $\beta < 0$, by the one-to-one correspondence of (5.2), Theorems 4.5 and 4.11 imply the main results stated in the Introduction.

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References

[1] R.S. Cantrell, C. Cosner, Spatial Ecology via Reaction–Diffusion Equation, Wiley Ser. Math. Comput. Biol., John Wiley and Sons, 2003.
[2] Y.S. Choi, R. Lui, Y. Yamada, Existence of global solutions for the Shigesada–Kawasaki–Teramoto model with strongly coupled cross-diffusion, Discrete Contin. Dyn. Syst. 10 (2004) 719–730.
[3] Y.S. Choi, R. Lui, Y. Yamada, Existence of global solutions for the Shigesada–Kawasaki–Teramoto model with weak cross diffusion, Discrete Contin. Dyn. Syst. 9 (2003) 1193–1200.
[4] M.G. Crandall, P.H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.
[5] M.G. Crandall, P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues, and linearized stability, Arch. Ration. Mech. Anal. 52 (1973) 161–180.
[6] M. Delgado, J. López-Gómez, A. Suárez, On the symbiotic Lotka–Volterra model with diffusion and transport effects, J. Differential Equations 160 (2000) 175–262.
[7] M. Delgado, M. Montenegro, A. Suárez, A Lotka–Volterra symbiotic model with cross-diffusion, J. Differential Equations 246 (2009) 2131–2149.

[8] Y. Du, S.B. Hsu, A diffusive predator–prey model in heterogeneous environment, J. Differential Equations 203 (2004) 331–364.
[9] Y. Du, X. Liang, A diffusive competition model with a protection zone, J. Differential Equations 244 (2008) 61–86.
[10] Y. Du, Y. Lou, S-shaped global bifurcation curve and Hopf bifurcation of positive solutions to a predator–prey model, J. Differential Equations 144 (1998) 390–440.
[11] Y. Du, R. Peng, M. Wang, Effect of a protection zone in the diffusive Leslie predator–prey model, J. Differential Equations 246 (2009) 3932–3956.
[12] Y. Du, J. Shi, A diffusive predator–prey model with a protection zone, J. Differential Equations 229 (2006) 63–91.
[13] Y. Du, J. Shi, Some recent results on diffusive predator–prey models in spatially heterogeneous environment, in: H. Brummer, X. Zhao, X. Zou (Eds.), Nonlinear Dynamics and Evolution Equations, in: Fields Inst. Commun., vol. 48, Amer. Math. Soc., Providence, RI, 2006, pp. 95–135.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin, 1983.
[15] C.B. Huffaker, Experimental studies on predator: Dispersion factors and predator–prey oscillations, Hilgardia 27 (1958) 343–383.
[16] V. Hutson, Y. Lou, K. Mischaikow, Convergence in competition models with small diffusion coefficients, J. Differential Equations 211 (2005) 135–161.
[17] V. Hutson, Y. Lou, K. Mischaikow, Spatial heterogeneity of resources versus Lotka–Volterra dynamics, J. Differential Equations 185 (2002) 97–136.
[18] V. Hutson, Y. Lou, K. Mischaikow, P. Poláčik, Competing species near a degenerate limit, SIAM J. Math. Anal. 34 (2002) 453–491.
[19] V. Hutson, K. Mischaikow, P. Poláčik, The evolution of dispersal rates in a heterogeneous time-periodic environment, J. Math. Biol. 43 (2001) 501–533.
[20] T. Katoda, K. Kuto, Positive steady states for a prey–predator model with some nonlinear diffusion terms, J. Math. Anal. Appl. 323 (2006) 1387–1401.
[21] K. Kuto, Bifurcation branch of stationary solutions for a Lotka–Volterra cross-diffusion system in a spatially heterogeneous environment, Nonlinear Anal. Real World Appl. 10 (2009) 943–965.
[22] K. Kuto, Y. Yamada, Coexistence problem for a prey–predator model with density-dependent diffusion, Nonlinear Anal. 71 (2009) e2223–e2232.
[23] K. Kuto, Y. Yamada, Multiple coexistence states for a predator–prey system with cross-diffusion, J. Differential Equations 197 (2004) 315–348.
[24] Z. Ling, M. Pedersen, Coexistence of two species in a strongly coupled cooperative model, Math. Comput. Modelling 45 (2007) 371–377.
[25] Y. Lou, Necessary and sufficient condition for the existence of positive solutions of certain cooperative system, Nonlinear Anal. 26 (1996) 1079–1095.
[26] Y. Lou, W.M. Ni, Diffusion, self-diffusion and cross-diffusion: an elliptic approach, J. Differential Equations 154 (1999) 157–190.
[27] Y. Lou, W.M. Ni, Diffusion vs cross-diffusion, J. Differential Equations 131 (1996) 79–131.
[28] Y. Lou, W.M. Ni, Y. Wu, On the global existence of a cross-diffusion system, Discrete Contin. Dyn. Syst. 4 (1998) 193–203.
[29] K. Nakashima, Y. Yamada, Positive steady states for prey–predator models with cross-diffusion, Adv. Differential Equations 6 (1996) 1099–1122.
[30] A. Okubo, L.A. Levin, Diffusion and Ecological Problems: Modern Perspective, second ed., Interdiscip. Appl. Math., vol. 14, Springer-Verlag, New York, 2001.
[31] P.Y.H. Pang, M. Wang, Strategy and stationary pattern in a three-species predator–prey model, J. Differential Equations 200 (2004) 245–273.
[32] R. Peng, M. Wang, G. Yang, Stationary patterns of the Holling–Tanner prey–predator model with diffusion and cross-diffusion, Appl. Math. Comput. 196 (2008) 570–577.
[33] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.
[34] W. Ruan, Positive steady-state solutions of a competing reaction–diffusion system with large cross-diffusion coefficients, J. Math. Anal. Appl. 197 (1996) 558–578.
[35] K. Ryu, I. Ahn, Coexistence theorem of steady states for nonlinear self-cross diffusion systems with competitive dynamics, J. Math. Anal. Appl. 283 (2003) 45–65.
[36] N. Shigesada, K. Kawasaki, E. Teramoto, Spatial segregation of interacting species, J. Theoret. Biol. 79 (1979) 83–99.
[37] M. Wang, Stationary patterns of strongly coupled prey–predator models, J. Math. Anal. Appl. 292 (2004) 484–505.
[38] Y.X. Wang, W.T. Li, H.B. Shi, Stationary patterns of a ratio-dependent predator–prey system with cross-diffusion, Math. Model. Anal., in press.