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Blow-up for a semilinear parabolic equation with large diffusion on $\mathbf{R}^N$. II

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ABSTRACT

We are concerned with the Cauchy problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = D\Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = \lambda + \varphi(x), & x \in \mathbf{R}^N, \end{cases} \tag{P}$$

where $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and $\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. In the paper of Fujishima and Ishige (2011) [8] the authors of this paper studied the behavior of the blow-up time and the blow-up set of the solution of (P) as $D \to \infty$ for the case $\int_{\mathbf{R}^N} \varphi(x)\,dx > 0$. In this paper, as a continuation of Fujishima and Ishige (2011) [8], we consider the case

$$\int_{\mathbf{R}^N} \varphi(x)\,dx \leqslant 0,$$

and study the behavior of the blow-up time and the blow-up set of the solution of (P) as $D \to \infty$. The behavior in the case $\int_{\mathbf{R}^N} \varphi(x)\,dx \leqslant 0$ is completely different from the one in the case $\int_{\mathbf{R}^N} \varphi(x)\,dx > 0$.

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1. Introduction

In this paper we are concerned with the Cauchy problem for a semilinear heat equation,

$$\partial_t u = D\Delta u + |u|^{p-1}u, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = \lambda + \varphi(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

where $\partial_t = \partial/\partial t$, $D > 0$, $p > 1$, $N \geqslant 3$, $\lambda > 0$, and

$$\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|)^2\,dx). \tag{1.3}$$

Let T_D be the maximal existence time of the unique classical solution u of (1.1) and (1.2). If $T_D < \infty$, then

$$\limsup_{t \to T_D} \sup_{x \in \mathbf{R}^N} |u(x,t)| = \infty,$$

and we call T_D the blow-up time of the solution u. Furthermore we denote by B_D the blow-up set of the solution u, that is,

$$B_D = \Big\{ x \in \mathbf{R}^N\text{: there exists a sequence } \{(x_n,t_n)\} \subset \mathbf{R}^N \times (0,T_D)$$
$$\text{such that } \lim_{n\to\infty}(x_n,t_n) = (x,T_D),\ \lim_{n\to\infty}|u(x_n,t_n)| = +\infty \Big\}.$$

The blow-up set is an interesting subject for the study of the blow-up problem for the semilinear heat equation (1.1), and has been studied intensively by many mathematicians (see for example [2–12,14–16,18–20,22–27], and a survey [21], which includes a considerable list of references for this topic). Generally speaking, the location of the blow-up set is decided by given data such as the initial function and the boundary conditions and by the balance between the diffusion and the nonlinear term. In particular, if D is sufficiently large, then the behavior of the solution heavily depends on the diffusion driven from Laplacian Δ, and we can expect that the location of the blow-up set is decided by the diffusion term Δ. Indeed, as pointed out in [12,15], and [16], for the Cauchy–Neumann problem for the semilinear heat equation (1.1) in a bounded domain, the limit of the blow-up set B_D as $D \to \infty$ coincides with the limit of the hot spots of the solution of the heat equation as $t \to \infty$ and it is characterized as the set of the maximum points of the projection of the initial function onto the second Neumann eigenspace. In this paper we consider problem (1.1) and (1.2), and study the relationship between the behavior of blow-up time and the blow-up set as $D \to \infty$ and the large time behavior of the solution of the heat equation.

Let φ be a function satisfying (1.3). Then the function

$$(e^{t\Delta}\varphi)(x) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\,dy \tag{1.4}$$

is a unique bounded classical solution of the heat equation with the initial function φ. We denote by $M(\varphi)$ the mass of the function φ, that is,

$$M(\varphi) := \int_{\mathbf{R}^N} \varphi(x)\,dx,$$

and by $C(\varphi)$ the center of the mass of the function φ, that is,

$$C(\varphi) := \int_{\mathbf{R}^N} x\varphi(x)\,dx \Big/ \int_{\mathbf{R}^N} \varphi(x)\,dx \quad \text{if } M(\varphi) > 0.$$

Here we remark that $M(e^{t\Delta}\varphi) = M(\varphi)$ for all $t > 0$ and that $C(e^{t\Delta}\varphi) = C(\varphi)$ for all $t > 0$ if $M(\varphi) > 0$. We denote by $H(t)$ the hot spots of the function $e^{t\Delta}\varphi$, that is,

$$H\big(e^{t\Delta}\varphi\big) := \Big\{x \in \mathbf{R}^N\colon \big(e^{t\Delta}\varphi\big)(x) = \sup_{y\in\mathbf{R}^N} \big(e^{t\Delta}\varphi\big)(y)\Big\}.$$

It is known that, if $M(\varphi) > 0$, then

$$\lim_{t\to\infty} \sup\big\{\big|x - C(\varphi)\big|\colon x \in H\big(e^{t\Delta}\varphi\big)\big\} = 0 \tag{1.5}$$

(see [1] and [8, Section 2.1]).

In [8], under the assumption $M(\varphi) > 0$, the authors of this paper considered blow-up problem (1.1) and (1.2), and studied the behavior of the blow-up time and the blow-up set of the solution as $D \to \infty$. In particular, they propounded the following problem:

" if D is sufficiently large, is the location of the blow-up set for problem (1.1) *and* (1.2) *determined mainly by the large time behavior of the hot spots for the heat equation?",* (Q)

and proved the following theorem, which gave an affirmative answer to (Q) for the case $M(\varphi) > 0$ (see also (1.5)). Let $\zeta_\lambda = \zeta_\lambda(t)$ be a solution of the ordinary differential equation $\zeta' = \zeta^p$ with $\zeta(0) = \lambda$, that is,

$$\zeta_\lambda(t) := \kappa(S_\lambda - t)^{-\frac{1}{p-1}}, \quad \kappa := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \; S_\lambda := \frac{\lambda^{-(p-1)}}{p-1}. \tag{1.6}$$

Theorem 1.1. *Let $N \geqslant 3$ and u be the solution of* (1.1) *and* (1.2) *under condition* (1.3)*. Assume $M(\varphi) > 0$. Then $T_D \leqslant S_\lambda$ for any $D > 0$ and*

$$S_\lambda - T_D = (4\pi S_\lambda)^{-\frac{N}{2}}\lambda^{-p}D^{-\frac{N}{2}}\big[M(\varphi) + O\big(D^{-1}\big)\big] \quad \text{as } D \to \infty.$$

Furthermore

$$\lim_{D\to\infty} \sup\big\{\big|x - C(\varphi)\big|\colon x \in B_D\big\} = 0.$$

In this paper, as a continuation of [8], we consider the following two cases:

$$\text{(A)} \quad M(\varphi) = 0; \qquad \text{(B)} \quad M(\varphi) < 0,$$

and study the behavior of the blow-up time and the blow-up set of the solution u of (1.1) and (1.2) as $D \to \infty$, and give an affirmative answer to problem (Q). For cases (A) and (B), the large time behavior of hot spots for the heat equation is different from the one in the case $M(\varphi) > 0$, while we can prove that the behavior of the blow-up time and the blow-up set as $D \to \infty$ is different from the one in the case $M(\varphi) > 0$.

We introduce some notation. Put $B(x,r) := \{y \in \mathbf{R}^N\colon\ |x-y| < r\}$ for $x \in \mathbf{R}^N$ and $r > 0$. For any $f \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $\eta > 0$, we set

$$H(f) := \Big\{x \in \mathbf{R}^N\colon\ f(x) = \sup_{y\in\mathbf{R}^N} f(y)\Big\},$$

$$H(f,\eta) := \Big\{x \in \mathbf{R}^N\colon\ f(x) \geqslant \sup_{y\in\mathbf{R}^N} f(y) - \eta\Big\}.$$

Now we are ready to state the main results of this paper, which give the behavior of the blow-up time and the blow-up set as $D \to \infty$ for problem (1.1) and (1.2) with $M(\varphi) \leqslant 0$. We first give a result for the case $M(\varphi) = 0$. Put

$$\Xi(\varphi) := \int_{\mathbf{R}^N} x\varphi(x)\,dx.$$

Theorem 1.2. *Let $N \geqslant 3$ and u be the solution of* (1.1) *and* (1.2) *under condition* (1.3)*. Assume $M(\varphi) = 0$. Then $T_D \leqslant S_\lambda$ for any $D > 0$ and*

$$S_\lambda - T_D = \frac{(4\pi S_\lambda)^{-\frac{N}{2}}}{\lambda^p\sqrt{2eS_\lambda}} D^{-\frac{N}{2}-\frac{1}{2}}\big[\big|\Xi(\varphi)\big| + O\big(D^{-\frac{1}{2}}\big)\big] \tag{1.7}$$

as $D \to \infty$. Furthermore, if $\Xi(\varphi) \neq 0$, then

$$\lim_{D\to\infty} \sup_{x\in B_D} \left|\frac{x}{\sqrt{2S_\lambda D}} - \frac{\Xi(\varphi)}{|\Xi(\varphi)|}\right| = 0. \tag{1.8}$$

Remark 1.1. Assume the same conditions as in Theorem 1.2. Then, by Theorem 1.2 we see that the blow-up time T_D converges to S_λ as $D \to \infty$ and the blow-up set B_D moves to the space infinity at the rate $\sqrt{2S_\lambda D}$ in the direction $\Xi(\varphi)$ as $D \to \infty$. On the other hand, the hot spots of $e^{t\Delta}\varphi$ move to the space infinity at the rate $\sqrt{2t}$ in the direction $\Xi(\varphi)$ as $t \to \infty$ (see Section 2.1). Therefore the behavior of the blow-up set B_D as $D \to \infty$ is similar to that of the hot spots $H(e^{DT_D\Delta}\varphi)$ as $D \to \infty$.

Next we give a result for the case $M(\varphi) < 0$.

Theorem 1.3. *Let $N \geqslant 3$ and u be the solution of* (1.1) *and* (1.2) *under condition* (1.3)*. Assume $M(\varphi) < 0$. Then $T_D \leqslant S_\lambda$ for any $D > 0$ and*

$$S_\lambda - T_D = O\big(D^{-\frac{N}{2}-1}\big) \tag{1.9}$$

as $D \to \infty$. Furthermore there exist positive constants C and D_* such that

$$B_D \cap B\big(0, C(D\log D)^{1/2}\big) = \emptyset \tag{1.10}$$

for all $D > D_*$.

Remark 1.2. We give two remarks on Theorem 1.3.

(i) Under the same conditions as in Theorem 1.3, there exists a positive constant C such that

$$H\big(e^{t\Delta}\varphi\big) \cap B\big(0, C(t\log t)^{1/2}\big) = \emptyset$$

for all sufficiently large t (see Section 2.1). Then, by Theorem 1.3 we can see a close relationship between the hot spots $H(e^{D\Delta}\varphi)$ and the blow-up set B_D.

(ii) Let $\varphi \in C_0(\mathbf{R}^N)$ be such that $\varphi(x) \leqslant (\not\equiv)\, 0$ in $\mathbf{R}^N$. Then we have $H(e^{t\Delta}\varphi) = \emptyset$ for all $t > 0$. On the other hand, for any $D > 0$, it is known that the solution of (1.1) and (1.2) blows up in a finite time, however its blow-up set B_D is empty and the solution blows up at the space infinity (see for example [11]).

In this paper, following the strategy in [8], we study the profile of the solution u of (1.1) and (1.2) just before the blow-up time T_D, and prove Theorems 1.2 and 1.3. We remark that the arguments in [8] heavily depend on the behavior of the solution of the heat equation and its hot spots. Since the behavior in our cases $M(\varphi) \leqslant 0$ is different from the one in the case $M(\varphi) > 0$, we cannot apply directly the arguments in [8] to our case $M(\varphi) \leqslant 0$, and the proof of our theorems requires another analysis, in particular, for the behavior of the blow-up set as $D \to \infty$.

The rest of this paper is organized as follows. In Section 2 we give preliminary results on the behavior of $e^{t\Delta}\varphi$ and the solution u of (1.1) and (1.2). Furthermore we recall three propositions on the blow-up problem for semilinear heat equations. In Section 3 we study the behavior of the blow-up time T_D and the blow-up set B_D of (1.1) and (1.2) as $D \to \infty$, and prove Theorems 1.2 and 1.3.

2. Preliminary results

In this section we introduce some notation and recall some properties of the solutions of the heat equation and the solutions of the semilinear heat equation (1.1). Almost all claims in this section have been already proved in [8], and we state them without the details of the proof, except for the large time behavior of the solution of the heat equation for the case $M(\varphi) \leqslant 0$.

We first introduce some notation. For any $q \in [1, \infty]$, we denote by $\|\cdot\|_q$ the usual norm of $L^q(\mathbf{R}^N)$. Let $\mathbf{N} = \{1, 2, \ldots\}$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbf{N} \cup \{0\})^N$, let

$$|\alpha| := \sum_{n=1}^{N} \alpha_n, \qquad \alpha! := \alpha_1! \cdots \alpha_N!, \qquad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

Put

$$G(x,t) := (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}.$$

Let $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ for $i, j \in \{1, \ldots, N\}$. For any sets Λ and Σ, let $f = f(\lambda, \sigma)$ and $h = h(\lambda, \sigma)$ be maps from $\Lambda \times \Sigma$ to $(0, \infty)$. Then we say

$$f(\lambda,\sigma) \preccurlyeq h(\lambda,\sigma)$$

for all $\lambda \in \Lambda$ if, for any $\sigma \in \Sigma$, there exists a positive constant C such that $f(\lambda,\sigma) \leqslant Ch(\lambda,\sigma)$ for all $\lambda \in \Lambda$.

2.1. Behavior of the solutions of the heat equation

In this subsection we recall some properties of $e^{t\Delta}\varphi$, and give a lemma on the hot spots for the heat equation. We first recall the following properties of $e^{t\Delta}\varphi$:

(P1) For any $1 \leqslant r \leqslant q \leqslant \infty$, $m \in \mathbf{N} \cup \{0\}$, and $\phi \in L^r(\mathbf{R}^N)$,

$$\left\| \nabla^m e^{t\Delta}\phi \right\|_q \preccurlyeq t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})-\frac{m}{2}} \|\phi\|_r$$

for $t > 0$. In particular, if $r = q$, then $\|e^{t\Delta}\phi\|_q \leqslant \|\phi\|_q$ for $t > 0$;

(P2) If $M(\varphi) = 0$, then, for any $m \in \mathbf{N} \cup \{0\}$ and $\phi \in L^1(\mathbf{R}^N, (1+|x|)\,dx)$,

$$\left\| \nabla^m e^{t\Delta}\phi \right\|_\infty \preccurlyeq t^{-\frac{N}{2}-\frac{m+1}{2}} \|\phi\|_{L^1(\mathbf{R}^N,(1+|x|)\,dx)} \quad \text{for } t > 0;$$

(P3) Let $\phi \in L^1(\mathbf{R}^N, (1+|x|)^2\,dx)$. Then

$$\lim_{t\to\infty} t^{\frac{N}{2}+1} \left\| e^{t\Delta}\phi - \sum_{|\alpha|\leqslant 2} (-1)^{|\alpha|} \frac{c_\alpha}{\alpha!} \partial_x^\alpha G(t) \right\|_\infty = 0, \quad \text{where } c_\alpha = \int_{\mathbf{R}^N} y^\alpha \phi(y)\,dy.$$

Property (P1) easily follows from (1.4). For properties (P2) and (P3), see Lemma 2.5 in [13] (see also Lemma 2.1 in [8]).

By using properties (P1)–(P3) we study the large time behavior of the hot spots for the heat equation for the case $M(\varphi) \leqslant 0$. The behavior is completely different from the one in the case $M(\varphi) > 0$ (see also [8, Lemma 2.2]).

Lemma 2.1. *Assume condition* (1.3)*. Then there holds the following*:

(i) *If $M(\varphi) = 0$ and $\Xi(\varphi) \neq 0$, then, for any $\delta > 0$, there exist positive constants C_1 and T_1 such that*

$$\bigl(e^{t\Delta}\varphi\bigr)(x) \leqslant \bigl(e^{t\Delta}\varphi\bigr)\bigl(x(t)\bigr) - C_1 t^{-N/2-1/2} \quad \text{if } \bigl|x - x(t)\bigr| \geqslant \delta t^{1/2}, \tag{2.1}$$

$$\bigl\|e^{t\Delta}\varphi\bigr\|_\infty = \bigl(e^{t\Delta}\varphi\bigr)\bigl(x(t)\bigr) + O\bigl(t^{-\frac{N}{2}-1}\bigr) = (4\pi)^{-\frac{N}{2}} t^{-\frac{N}{2}-\frac{1}{2}} \frac{|\Xi(\varphi)|}{\sqrt{2e}} + O\bigl(t^{-\frac{N}{2}-1}\bigr), \tag{2.2}$$

for all $t > T_1$, where $x(t) = \sqrt{2t}\,\Xi(\varphi)/|\Xi(\varphi)|$. In particular, $H(e^{t\Delta}\varphi) \neq \emptyset$ for all $t > T_1$ and

$$\lim_{t\to\infty} t^{-1/2} \sup\bigl\{\bigl|x - x(t)\bigr|:\ x \in H\bigl(e^{t\Delta}\varphi\bigr)\bigr\} = 0;$$

(ii) *If $M(\varphi) < 0$, then, for any $c \in (0, 1/2)$, there exist positive constants C_2 and T_2 such that*

$$\sup_{|x|\leqslant(4ct\log t)^{1/2}} \bigl(e^{t\Delta}\varphi\bigr)(x) \leqslant -C_2 t^{-\frac{N}{2}-c}, \tag{2.3}$$

$$\sup_{|x|\geqslant(4ct\log t)^{1/2}-1} \bigl|\bigl(e^{t\Delta}\varphi\bigr)(x)\bigr| = O\bigl(t^{-\frac{N}{2}-c}\bigr), \qquad \sup_{|x|\geqslant t} \bigl|\bigl(e^{t\Delta}\varphi\bigr)(x)\bigr| = o\bigl(t^{-\frac{N}{2}-1}\bigr), \tag{2.4}$$

$$0 \leqslant \sup_{x\in\mathbf{R}^N} \bigl(e^{t\Delta}\varphi\bigr)(x) = o\bigl(t^{-\frac{N}{2}-1}\bigr), \tag{2.5}$$

for all $t > T_2$. In particular, for any $t > T_2$,

$$H\big(e^{t\Delta}\varphi\big) \cap B\big(0, (4ct\log t)^{1/2}\big) = \emptyset$$

even if $H(e^{t\Delta}\varphi) \neq \emptyset$.

Proof. We first prove assertion (i). We can assume, without loss of generality, that $\Xi(\varphi)/|\Xi(\varphi)| = e_1$, where $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^N$. Then, since $M(\varphi) = 0$ and $\|\partial_t \partial_x^\alpha G\|_\infty = O(t^{-(N+|\alpha|+2)/2})$ as $t \to \infty$ for any $\alpha \in (\mathbf{N} \cup \{0\})^N$, by (P1) and (P3) we have

$$\begin{aligned}
\big(e^{t\Delta}\varphi\big)(x) &= \sum_{|\alpha|\leqslant 2} \frac{(-1)^{|\alpha|}}{\alpha!}\bigg(\int_{\mathbf{R}^N} y^\alpha \varphi(y)\,dy\bigg)\big(\partial_x^\alpha G\big)(x,t) + o\big(t^{-\frac{N}{2}-1}\big) \\
&= \sum_{1\leqslant|\alpha|\leqslant 2} \frac{(-1)^{|\alpha|}}{\alpha!}\bigg(\int_{\mathbf{R}^N} y^\alpha \varphi(y)\,dy\bigg)\big(\partial_x^\alpha G\big)(x,t) + o\big(t^{-\frac{N}{2}-1}\big) \\
&= (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}\bigg[\frac{x_1}{2t}\big|\Xi(\varphi)\big| - \frac{1}{4t}\int_{\mathbf{R}^N} |y|^2\varphi(y)\,dy + O\bigg(\frac{|x|^2}{t^2}\bigg)\bigg] + o\big(t^{-\frac{N}{2}-1}\big) \\
&= (4\pi)^{-\frac{N}{2}} t^{-\frac{N}{2}-\frac{1}{2}} h\bigg(\frac{x}{2t^{1/2}}\bigg) + O\big(t^{-\frac{N}{2}-1}\big)
\end{aligned} \tag{2.6}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large t, where $h(\xi) := |\Xi(\varphi)|e^{-|\xi|^2}\xi_1$ for $\xi \in \mathbf{R}^N$. Let $\delta > 0$. Since

$$(\partial_{\xi_i} h)(\xi) = \big|\Xi(\varphi)\big|e^{-|\xi|^2}(\delta_{1i} - 2\xi_1\xi_i), \quad i = 1, \ldots, N,$$

there exists a positive constant d_δ such that

$$h(\xi) \leqslant h\bigg(\frac{e_1}{\sqrt{2}}\bigg) - d_\delta \quad \text{for all } \xi \in \mathbf{R}^N \text{ with } \bigg|\xi - \frac{e_1}{\sqrt{2}}\bigg| \geqslant \delta.$$

Then, since $x(t)/2t^{1/2} = e_1/\sqrt{2}$, by (2.6) we can find a positive constant C_δ satisfying

$$\big(e^{t\Delta}\varphi\big)(x) \leqslant \big(e^{t\Delta}\varphi\big)\big(x(t)\big) - C_\delta t^{-\frac{N}{2}-\frac{1}{2}}$$

for all $x \in \mathbf{R}^N$ with $|x - x(t)| \geqslant \delta t^{1/2}$ and all sufficiently large t. Thus we obtain (2.1), and see that $H(e^{t\Delta}\varphi) \neq \emptyset$ for all sufficiently large t. Furthermore, by (2.6) we apply the Taylor theorem to obtain

$$\begin{aligned}
&(4\pi)^{\frac{N}{2}} t^{\frac{N+1}{2}}\big[\big(e^{t\Delta}\varphi\big)(x) - \big(e^{t\Delta}\varphi\big)\big(x(t)\big)\big] \\
&= h\bigg(\frac{x}{2t^{1/2}}\bigg) - h\bigg(\frac{x(t)}{2t^{1/2}}\bigg) + O\big(t^{-\frac{1}{2}}\big) \\
&= \bigg(\frac{1}{2}(\nabla_\xi^2 h)\bigg(\frac{e_1}{\sqrt{2}}\bigg)\frac{x - x(t)}{2t^{1/2}}\bigg) \cdot \frac{x - x(t)}{2t^{1/2}} + o\bigg(\bigg|\frac{x - x(t)}{t^{1/2}}\bigg|^2\bigg) + O\big(t^{-\frac{1}{2}}\big)
\end{aligned} \tag{2.7}$$

for all $x \in \mathbf{R}^N$ with $|x - x(t)| \leqslant \delta t^{1/2}$ and all sufficiently large t. Since

$$(\partial_{\xi_i}\partial_{\xi_j}h)\left(\frac{e_1}{\sqrt{2}}\right)=\left|\Xi(\varphi)\right|e^{-|\xi|^2}[-2\delta_{1j}\xi_i-2\xi_1\delta_{ij}-2\delta_{1i}\xi_j+4\xi_1\xi_i\xi_j]|_{\xi=e_1/\sqrt{2}}$$
$$=\begin{cases}-2\sqrt{2}|\Xi(\varphi)|/\sqrt{e} & \text{for } i=j=1,\\ -\sqrt{2}|\Xi(\varphi)|/\sqrt{e} & \text{for } i=j\neq 1,\\ 0 & \text{otherwise,}\end{cases}$$

by (2.7) we have

$$\sup\left\{\left|\frac{x-x(t)}{t^{1/2}}\right|:\ x\in H\big(e^{t\Delta}\varphi\big)\right\}=O\big(t^{-1/4}\big),$$

and obtain

$$0\leqslant(4\pi)^{\frac{N}{2}}t^{\frac{N+1}{2}}\big[\big\|e^{t\Delta}\varphi\big\|_\infty-\big(e^{t\Delta}\varphi\big)\big(x(t)\big)\big]$$
$$\leqslant\frac{1}{2}\left|\big(\nabla^2h\big)\left(\frac{e_1}{\sqrt{2}}\right)\right|\sup\left\{\left|\frac{x-x(t)}{t^{1/2}}\right|^2:\ x\in H\big(e^{t\Delta}\varphi\big)\right\}+O\big(t^{-\frac{1}{2}}\big)=O\big(t^{-\frac{1}{2}}\big)$$

for all sufficiently large t. This together with (2.6) implies (2.2), and assertion (i) follows.

Next we prove assertion (ii). Similarly to (2.6), by (P3) we have

$$\big(e^{t\Delta}\varphi\big)(x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}\left[M(\varphi)+O\left(\frac{1}{t}+\frac{|x|}{t}+\frac{|x|^2}{t^2}\right)\right]+o\big(t^{-\frac{N}{2}-1}\big)$$
$$=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}M(\varphi)+O\big(t^{-\frac{N+1}{2}}e^{-\frac{|x|^2}{8t}}\big)+o\big(t^{-\frac{N}{2}-1}\big)\tag{2.8}$$

for all $x\in\mathbf{R}^N$ and all sufficiently large t. Let $\epsilon\in(0,1)$. Then, by (2.8) we have

$$\sup_{|x|\geqslant\epsilon t}\big|\big(e^{t\Delta}\varphi\big)(x)\big|=o\big(t^{-\frac{N}{2}-1}\big)\tag{2.9}$$

for all sufficiently large t. Furthermore, since $M(\varphi)<0$, by (2.8) we have

$$\big(e^{t\Delta}\varphi\big)(x)=-(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}\big[\big|M(\varphi)\big|+O(\epsilon)+O\big(t^{-1}\big)\big]+o\big(t^{-\frac{N}{2}-1}\big)\tag{2.10}$$

for all $x\in\mathbf{R}^N$ with $|x|\leqslant\epsilon t$ and all sufficiently large t. By (2.10) and $c\in(0,1/2)$, taking a sufficiently small ϵ if necessary, we have

$$\sup_{|x|\leqslant\epsilon t}\big(e^{t\Delta}\varphi\big)(x)\leqslant o\big(t^{-\frac{N}{2}-1}\big),\tag{2.11}$$

$$\sup_{|x|\leqslant(4ct\log t)^{1/2}}\big(e^{t\Delta}\varphi\big)(x)\leqslant-\frac{1}{2}(4\pi t)^{-\frac{N}{2}}t^{-c}\big|M(\varphi)\big|<0,\tag{2.12}$$

$$\sup_{(4ct\log t)^{1/2}-1\leqslant|x|\leqslant\epsilon t}\big|\big(e^{t\Delta}\varphi\big)(x)\big|\leqslant 2(4\pi t)^{-\frac{N}{2}}t^{-c}\big|M(\varphi)\big|,\tag{2.13}$$

for all sufficiently large t. Then, by (2.12) we have (2.3), and by (2.9) and (2.13) we obtain (2.4). On the other hand, by (1.4) we have

$$\sup_{x\in\mathbf{R}^N}\big(e^{t\Delta}\varphi\big)(x) \geqslant \lim_{|x|\to\infty}\big(e^{t\Delta}\varphi\big)(x) = 0.$$

This together with (2.9) and (2.11) implies

$$0 \leqslant \sup_{x\in\mathbf{R}^N}\big(e^{t\Delta}\varphi\big)(x) = o\big(t^{-\frac{N}{2}-1}\big)$$

for all sufficiently large t, and we obtain (2.5). Thus assertion (ii) follows, and the proof of Lemma 2.1 is complete. □

2.2. *Preliminaries for blow-up problem* (1.1) *and* (1.2)

In this subsection we give preliminary results for blow-up problem (1.1) and (1.2). In particular, we study the short time behavior of the solution u of (1.1) and (1.2) and give one lemma on the blow-up estimates of the solution u and its gradient.

Let u be a solution of (1.1) and (1.2) and $T := S_{\lambda+\|\varphi\|_\infty}/2$. Following the argument in Section 3 of [8], we put

$$v(x,t) := \zeta_\lambda(t)^{-p}\big(u(x,t) - \zeta_\lambda(t)\big),$$

$$F(s) := |1+s|^{p-1}(1+s), \qquad f(x,t) := \big[F(s) - F(0) - F'(0)s\big]\big|_{s=\zeta_\lambda(t)^{p-1}v(x,t)}.$$

Then v satisfies

$$\begin{cases} \partial_t v = D\Delta v + f(x,t), & x\in\mathbf{R}^N,\ t>0, \\ v(x,0) = \lambda^{-p}\varphi(x), & x\in\mathbf{R}^N, \end{cases} \tag{2.14}$$

and

$$v(t) = e^{D(t-t')\Delta}v\big(t'\big) + \int_{t'}^{t} e^{D(t-s)\Delta}f(s)\,ds, \quad t > t' \geqslant 0. \tag{2.15}$$

Furthermore, putting

$$z(t) := e^{D(t-T)\Delta}v(T), \qquad g(x) := \int_0^T \big(e^{D(T-s)\Delta}f(s)\big)(x)\,ds,$$

by (2.14) and (2.15) we obtain

$$z(t) = e^{Dt\Delta}v(0) + e^{D(t-T)\Delta}g = \lambda^{-p}e^{Dt\Delta}\varphi + e^{D(t-T)\Delta}g, \quad t \geqslant T. \tag{2.16}$$

Then, by [8, Lemma 3.1] we have:

Lemma 2.2. *Assume condition* (1.3)*. Then, for any* $l\in\{0,1,2\}$ *and* $m\in[0,2]$*, there exist positive constants* C *and* D_1 *such that*

$$\sup_{t\geqslant T}\left\|\nabla^l z(t)\right\|_\infty \leqslant \left\|\nabla^l v(T)\right\|_\infty \leqslant CD^{-\frac{N}{2}-\frac{l}{2}}, \tag{2.17}$$

$$\left\|z(t)-z(s)\right\|_\infty \leqslant C|t-s|D^{-\frac{N}{2}}, \quad t,s\in[T,\infty), \tag{2.18}$$

$$\|g\|_\infty \leqslant CD^{-\frac{N}{2}-1}, \tag{2.19}$$

$$\int_{\mathbf{R}^N} |x|^m\left|g(x)\right| dx \leqslant CD^{\frac{m}{2}-1}, \tag{2.20}$$

for all $D > D_1$.

Furthermore, for the case $M(\varphi)=0$, by (P2) we can refine on the decay estimates of $\|z(t)\|_\infty$ and $\|\nabla z(t)\|_\infty$ in Lemma 2.2.

Lemma 2.3. *Assume the same conditions as in Lemma* 2.2 *and $M(\varphi)=0$. Let $T'>T$. Then there exist positive constants C and D_1 such that*

$$\left\|z(t)\right\|_\infty \leqslant CD^{-\frac{N}{2}-\frac{1}{2}}, \tag{2.21}$$

$$\left\|\nabla z(t)\right\|_\infty \leqslant CD^{-\frac{N}{2}-1}, \tag{2.22}$$

for all $t\geqslant T'$ and all $D>D_1$.

Proof. By (P1), (P2), (2.16), and (2.19) we have

$$\begin{aligned}\sup_{t\geqslant T'}\left\|\nabla^l z(t)\right\|_\infty &\leqslant \lambda^{-p}\sup_{t\geqslant T'}\left\|\nabla^l e^{Dt\Delta}\varphi\right\|_\infty + \sup_{t\geqslant T'}\left\|\nabla^l e^{D(t-T)\Delta} g\right\|_\infty \\ &\preccurlyeq D^{-\frac{N}{2}-\frac{l+1}{2}}\|\varphi\|_{L^1(\mathbf{R}^N,(1+|x|)\,dx)} + D^{-\frac{l}{2}}\|g\|_\infty \preccurlyeq D^{-\frac{N}{2}-\frac{l+1}{2}}\end{aligned}$$

for all sufficiently large D, where $l=0,1$. This implies (2.21) and (2.22), and Lemma 2.3 follows. $\square$

Combining Lemma 2.1 with Lemma 2.2, we obtain the large time behavior of the hot spots of $z(t)$ for all sufficiently large D.

Lemma 2.4. *Assume condition* (1.3)*. Then there hold the following*:

(i) *If $M(\varphi)=0$ and $\Xi(\varphi)\neq 0$, then, for any $\delta>0$, there exist positive constants C_1, C_2, and D_1 such that*

$$z(x,t)\leqslant z\bigl(x(Dt),t\bigr)-C_1D^{-\frac{N}{2}-\frac{1}{2}} \quad \text{if } \left|x-x(Dt)\right|\geqslant \delta(Dt)^{1/2}, \tag{2.23}$$

$$\left|\left\|z(t)\right\|_\infty - z\bigl(x(Dt),t\bigr)\right| \leqslant C_2D^{-\frac{N}{2}-1}, \tag{2.24}$$

for all $t\geqslant T$ and all $D>D_1$. In particular, for any $t\geqslant T$,

$$\lim_{D\to\infty} D^{-1/2}\sup\bigl\{\left|x-x(Dt)\right|:\ x\in H\bigl(z(t)\bigr)\bigr\}=0;$$

(ii) *If $M(\varphi) < 0$, then, for any $c \in (0, 1/2)$, there exist positive constants C_3 and D_2 such that*

$$\sup_{|x| \leqslant (4cDt\log(Dt))^{1/2}} z(x,t) \leqslant -C_3 D^{-\frac{N}{2}-c}, \tag{2.25}$$

$$\sup_{|x| \geqslant (4cDt\log(Dt))^{1/2}-1} \bigl|z(x,t)\bigr| \leqslant C_3 D^{-\frac{N}{2}-c}, \qquad \sup_{|x| \geqslant Dt} \bigl|z(x,t)\bigr| \leqslant C_3 D^{-\frac{N}{2}-1}, \tag{2.26}$$

$$\Bigl|\sup_{x \in \mathbf{R}^N} z(x,t)\Bigr| \leqslant C_3 D^{-\frac{N}{2}-1}, \tag{2.27}$$

for all $t \geqslant T$ and all $D > D_2$. In particular, for any $t \geqslant T$,

$$H\bigl(z(t)\bigr) \cap B\bigl(0, \bigl(4cDt\log(Dt)\bigr)^{1/2}\bigr) = \emptyset, \quad D > D_2.$$

Proof. By (P1), (2.16), and (2.19) we have

$$z(x,t) - \lambda^{-p}\bigl(e^{Dt\Delta}\varphi\bigr)(x) = O\bigl(D^{-\frac{N}{2}-1}\bigr)$$

for all $(x,t) \in \mathbf{R}^N \times [T, \infty)$ and all sufficiently large D. Then Lemma 2.4 follows from Lemma 2.1. □

On the other hand, applying the same argument as in the proof of [8, Proposition 3.1] with Lemma 2.2, we have the following lemma on the blow-up estimates of the solution.

Lemma 2.5. *Assume the same conditions as in Theorem* 1.2*. Then there exist positive constants C_1, C_2, C_3 and D_1 such that*

$$\bigl\|u(t)\bigr\|_\infty \leqslant C_1 (T_D - t)^{-\frac{1}{p-1}}, \tag{2.28}$$

$$\begin{aligned} \bigl\|\nabla u(t)\bigr\|_\infty &\leqslant C_1 \bigl\|\nabla u(T)\bigr\|_\infty (T_D - t)^{-\frac{p}{p-1} - C_2 D^{-\frac{N}{2}}} \\ &\leqslant C_3 \bigl\|\nabla z(T)\bigr\|_\infty (T_D - t)^{-\frac{p}{p-1} - C_2 D^{-\frac{N}{2}}} \end{aligned} \tag{2.29}$$

for all $T \leqslant t < T_D$ and all $D > D_1$, where $T = S_{\lambda + \|\varphi\|_\infty}/2$.

2.3. Some propositions for semilinear heat equations

In this subsection we give two propositions on the profiles of the solutions of semilinear heat equations and one proposition on the location of the blow-up set.

We first give one proposition on the profile of the solution of (1.1) and (1.2).

Proposition 2.1. *Assume condition* (1.3)*. Let u be the solution of* (1.1) *and* (1.2)*. Then there exists a positive constant D_* such that $S_\lambda - D^{-1} < T_D$ and*

$$u(x,t) = \zeta_\lambda(t)\bigl[1 - (p-1)\zeta_\lambda(t)^{p-1} z(x,t) + O\bigl(D^{-N+\frac{4}{3}}\bigr)\bigr]^{-\frac{1}{p-1}} \tag{2.30}$$

for all $(x,t) \in \mathbf{R}^N \times [S_\lambda - D^{-2/3}, S_\lambda - D^{-1}]$ and all $D > D_*$.

This proposition is obtained by the same argument as in the proof of Proposition 4.1 in [8]. Proposition 4.1 in [8] is proved under the assumption $M(\varphi) > 0$, while we used the assumption $M(\varphi) > 0$ in order to obtain

$$\sup_{t \geqslant T} \|z(t)\|_\infty = O\big(D^{-\frac{N}{2}}\big), \qquad \sup_{t \geqslant T} \|\nabla z(t)\|_\infty = O\big(D^{-\frac{N}{2}-\frac{1}{2}}\big),$$

for all sufficiently large D. These estimates actually follow from (2.17) for our case, and the proof of Proposition 4.1 in [8] is applicable to the case $M(\varphi) \leqslant 0$; thus Proposition 2.1 holds true.

Next we give one proposition, which is useful to study the profile of the solution of the semilinear heat equation just before the blow-up time. See [8, Proposition 2.2].

Proposition 2.2. *Let $N \geqslant 1$, $p > 1$, $\epsilon_0 > 0$, and $\{M_\epsilon\}_{0<\epsilon<\epsilon_0} \subset (0, \infty)$ such that*

$$0 < \inf_{0<\epsilon<\epsilon_0} M_\epsilon \leqslant \sup_{0<\epsilon<\epsilon_0} M_\epsilon < \infty.$$

Let $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^1(\mathbf{R}^N)$ and C be a constant such that

$$0 \leqslant M_\epsilon - C\epsilon \leqslant \varphi_\epsilon(x) \leqslant M_\epsilon, \qquad \big|\nabla\varphi_\epsilon(x)\big| \leqslant C\epsilon,$$

for all $x \in \mathbf{R}^N$ and all $\epsilon \in (0, \epsilon_0)$. Assume that there exist constants $t_* \in [0, \liminf_{\epsilon\to+0} S_{M_\epsilon})$, $C_* > 0$, and $\epsilon_* > 0$ such that

$$\sup_{x\in\mathbf{R}^N} \big(e^{t_*\Delta}\varphi_\epsilon\big)(x) \leqslant M_\epsilon - C_*\epsilon, \quad 0 < \epsilon < \epsilon_*.$$

Let u_ϵ be the solution of the problem

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbf{R}^N,\ t > 0, \qquad u(x, 0) = \varphi_\epsilon(x), \quad x \in \mathbf{R}^N,$$

and T_ϵ the blow-up time of u_ϵ. Then $S_{M_\epsilon} < T_\epsilon$ for $\epsilon \in (0, \epsilon_*)$ and

$$\lim_{\epsilon\to 0} \big\| \epsilon^{\frac{1}{p-1}} u_\epsilon(S_{M_\epsilon}) - \kappa M_\epsilon^{\frac{p}{p-1}} \big[\epsilon^{-1}\big(M_\epsilon - e^{S_{M_\epsilon}\Delta}\varphi_\epsilon\big)\big]^{-\frac{1}{p-1}} \big\|_\infty = 0,$$

where κ is the constant given in (1.6)*.*

At the end of this section we recall one proposition, which implies that the location of the blow-up set B_D is determined by the maximum points of the solution just before the blow-up time. See [7, Proposition 4.1].

Proposition 2.3. *Let $N \geqslant 1$, $p > 1$, $\epsilon_0 > 0$, and $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^1(\mathbf{R}^N)$ be nonnegative functions. Assume that*

$$\inf_{0<\epsilon<\epsilon_0} \|\varphi_\epsilon\|_\infty > 0, \qquad \sup_{0<\epsilon<\epsilon_0} \|\varphi_\epsilon\|_\infty < \infty,$$

$$\sup_{0<\epsilon<\epsilon_0} \epsilon^{1/2-\alpha} \|\nabla\varphi_\epsilon\|_\infty < \infty \quad \text{for some } \alpha > 0.$$

Let u_ϵ be the solution of

$$\partial_t u = \epsilon \Delta u + u^p, \quad x \in \mathbf{R}^N,\ t > 0, \qquad u(x,0) = \varphi_\epsilon(x) \geqslant 0, \quad x \in \mathbf{R}^N,$$

and T_ϵ and B_ϵ be the blow-up time and the blow-up set, respectively. Assume that

$$\sup_{0<\epsilon<\epsilon_0} \sup_{0<t<T_\epsilon} (T_\epsilon - t)^{\frac{1}{p-1}} \left\| u_\epsilon(t) \right\|_\infty < \infty.$$

Then, for any $\eta > 0$, there exists a positive constant ϵ_* such that

$$B_\epsilon \subset \left\{ x \in \mathbf{R}^N \colon \varphi_\epsilon(x) \geqslant \|\varphi_\epsilon\|_\infty - \eta \right\}, \quad \epsilon \in (0, \epsilon_*).$$

3. Proof of Theorem 1.2 and Theorem 1.3

In this section we first obtain the behavior of the blow-up time of the solution u of (1.1) and (1.2) as $D \to \infty$. Next, by modifying the arguments in our previous paper [8] we study the profile of the solution u just before the blow-up time, and prove Theorems 1.2 and 1.3.

We first apply the same argument as in the proof of Proposition 3.1 in [8], and obtain

$$S_\lambda \leqslant T_D \quad \text{for all } D > 0.$$

Next we study the behavior of the blow-up time T_D as $D \to \infty$, and prove (1.7) and (1.9).

Proof of (1.7) and (1.9). By the same argument as in the proof of (1.9) in [8] we see that there exists a positive constant D_1 such that

$$S_\lambda - T_D = \lambda^{-p} \sup_{x \in \mathbf{R}^N} \left(e^{D S_\lambda \Delta} \varphi \right)(x) + O\left(D^{-\frac{N}{2}-1} \right)$$

for all $D > D_1$. Then (1.7) and (1.9) follow from (2.2) and (2.5), respectively. $\Box$

Next we study the location of the blow-up set of the solution of (1.1) and (1.2). Let $s_D := S_\lambda - D^{-1}$, and put

$$w(x,\tau) := D^{-\frac{1}{p-1}} u\left(x, s_D + D^{-1}\tau \right). \tag{3.1}$$

Then w satisfies

$$\partial_\tau w = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (-1, \tau_D), \qquad w(x,0) = D^{-\frac{1}{p-1}} u(x, s_D) \quad \text{in } \mathbf{R}^N. \tag{3.2}$$

Here τ_D is the blow-up time of w and

$$T_D = s_D + D^{-1}\tau_D. \tag{3.3}$$

Let ϵ_D be a sufficiently small positive constant to be chosen later, and put

$$\kappa_D := \left\| w(0) \right\|_\infty + \epsilon_D, \qquad \tau_* := S_{\kappa_D} \in (0, \tau_D). \tag{3.4}$$

The choice of ϵ_D depends on whether $M(\varphi) = 0$ or $M(\varphi) < 0$ and is crucial in our analysis. Put

$$\psi_D(x) := \epsilon_D^{-1}\bigl(\|w(0)\|_\infty - w(x,0)\bigr) + 1 = \epsilon_D^{-1} D^{-\frac{1}{p-1}}\bigl(\|u(s_D)\|_\infty - u(x,s_D)\bigr) + 1 \geqslant 1. \tag{3.5}$$

Then we have

$$w(x,0) = \kappa_D - \epsilon_D \psi_D(x). \tag{3.6}$$

Furthermore, since

$$\zeta_\lambda\bigl(s_D + D^{-1}\tau\bigr) = \kappa\bigl(D^{-1}(1-\tau)\bigr)^{-\frac{1}{p-1}} = D^{\frac{1}{p-1}}\zeta_\kappa(\tau), \qquad \kappa^{p-1} = 1/(p-1), \tag{3.7}$$

by (3.1) we apply Proposition 2.1 to obtain

$$w(x,\tau) = \zeta_\kappa(\tau)\left[1 - \frac{D}{1-\tau} z\bigl(x, s_D + D^{-1}\tau\bigr) + O\bigl(D^{-N+\frac{4}{3}}\bigr)\right]^{-\frac{1}{p-1}} \tag{3.8}$$

for all $(x,\tau) \in \mathbf{R}^N \times (-1, 0]$ and all sufficiently large D. In what follows, by using the results on the behavior of the function z given in Section 2.2 we study the location of the maximum points of $w(\cdot, \tau_*)$, and complete the proof of Theorems 1.2 and 1.3.

3.1. Blow-up set for the case $M(\varphi) = 0$

In this subsection we prove (1.8) under the assumption $M(\varphi) = 0$, and complete the proof of Theorem 1.2. Put

$$\epsilon_D := D^{-\frac{N}{2}+\frac{1}{2}}, \qquad x_D := \sqrt{2Ds_D} \int_{\mathbf{R}^N} x\varphi(x)\,dx \Big/ \left|\int_{\mathbf{R}^N} x\varphi(x)\,dx\right| = x(Ds_D). \tag{3.9}$$

By (2.21) and (3.8) we have

$$w(x,0) = \kappa\bigl[1 - Dz(x,s_D)\bigr]^{-\frac{1}{p-1}} + O\bigl(D^{-N+\frac{4}{3}}\bigr), \quad x \in \mathbf{R}^N, \tag{3.10}$$

$$\|w(0)\|_\infty = \kappa\bigl[1 - D\|z(s_D)\|_\infty\bigr]^{-\frac{1}{p-1}} + O\bigl(D^{-N+\frac{4}{3}}\bigr) = \kappa + O\bigl(D^{-\frac{N}{2}+\frac{1}{2}}\bigr), \tag{3.11}$$

for all sufficiently large D. In particular, by (3.4) and (3.11) we have

$$\kappa_D = \kappa + O(\epsilon_D), \qquad \tau_* = S_{\kappa_D} = S_\kappa + O(\epsilon_D) = 1 + O(\epsilon_D), \tag{3.12}$$

as $D \to \infty$. By (2.21), (3.5), (3.10), and (3.11) we apply the mean value theorem to obtain

$$\begin{aligned}\psi_D(x) &= \frac{\kappa\epsilon_D^{-1}(1-\theta_D(x))^{-\frac{p}{p-1}}}{p-1}\bigl[D\|z(s_D)\|_\infty - Dz(x,s_D)\bigr] + O\bigl(\epsilon_D^{-1}D^{-N+\frac{4}{3}}\bigr) + 1 \\ &= \frac{\kappa D^{\frac{N}{2}+\frac{1}{2}}(1-\theta_D(x))^{-\frac{p}{p-1}}}{p-1}\bigl[\|z(s_D)\|_\infty - z(x,s_D)\bigr] + O\bigl(D^{-\frac{N}{2}+\frac{5}{6}}\bigr) + 1\end{aligned}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D, where θ_D is a function in $\mathbf{R}^N$ satisfying $\|\theta_D\|_\infty = O(D^{-\frac{N}{2}+\frac{1}{2}})$ as $D \to \infty$. This together with (2.24) and (3.9) implies that

$$\psi_D(x) = \frac{\kappa D^{\frac{N}{2}+\frac{1}{2}}(1+o(1))}{p-1}\big[z(x_D, s_D) - z(x, s_D)\big] + O\big(D^{-\frac{1}{2}}\big) + O\big(D^{-\frac{N}{2}+\frac{5}{6}}\big) + 1 \tag{3.13}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D, and we obtain

$$\lim_{D\to\infty} \psi_D(x_D) = 1. \tag{3.14}$$

Let $\delta > 0$ and fix it. Then, by (2.23) and (3.13) we can find a constant $c_\delta > 0$ so that

$$\inf_{|x-x_D|\geqslant \delta D^{1/2}} \psi_D(x) \geqslant 1 + c_\delta \tag{3.15}$$

for all sufficiently large D.

Put

$$\psi_D^*(x) := \min\big\{\psi_D(x), 1 + c_\delta\big\} \geqslant 1 \tag{3.16}$$

(see (3.5)), and let w^* be the solution of

$$\begin{cases} \partial_\tau w = \Delta w + w^p, & x \in \mathbf{R}^N,\ \tau > 0, \\ w(x,0) = \kappa_D - \epsilon_D \psi_D^*(x), & x \in \mathbf{R}^N. \end{cases} \tag{3.17}$$

Then, by (3.2), (3.6), and (3.16) we apply the comparison principle to obtain

$$w(x,\tau) \leqslant w^*(x,\tau) \quad \text{in } \mathbf{R}^N \times [0, \tau_*]. \tag{3.18}$$

We apply Proposition 2.2 to study the location of the maximum points of $w(\cdot, \tau_*)$. For this purpose, we prepare the following two lemmas, and verify the assumptions of Proposition 2.2.

Lemma 3.1. *Assume the same conditions as in Theorem* 1.2. *Then there exists a positive constant C such that*

$$\big\|\nabla w^*(0)\big\|_\infty \leqslant C\epsilon_D$$

for all sufficiently large D.

Proof. By (2.18), (2.22), and (2.24) we have

$$\begin{aligned} & z\big(x, s_D + D^{-1}\tau\big) - \big\|z(s_D)\big\|_\infty \\ &\quad = \big[z\big(x, s_D + D^{-1}\tau\big) - z(x, s_D)\big] + \big[z(x, s_D) - z(x_D, s_D)\big] + \big[z(x_D, s_D) - \big\|z(s_D)\big\|_\infty\big] \\ &\quad = O\big(D^{-\frac{N}{2}}\big)\cdot D^{-1} + O\big(D^{-\frac{N}{2}-1}\big)\cdot 2\delta D^{1/2} + O\big(D^{-\frac{N}{2}-1}\big) = O\big(D^{-\frac{N}{2}-\frac{1}{2}}\big) \end{aligned} \tag{3.19}$$

for all $(x,\tau) \in B(x_D, 2\delta D^{1/2}) \times [-1, 0]$ and all sufficiently large D. By (2.21) we have

$$a_D := D\big\|z(s_D)\big\|_\infty \to 0 \quad \text{as } D \to \infty. \tag{3.20}$$

Then, since

$$\zeta_\kappa(\tau)\left[1-\frac{D}{1-\tau}\left\|z(s_D)\right\|_\infty\right]^{-\frac{1}{p-1}}=\zeta_\kappa(\tau+a_D),$$

by (3.8), (3.19), and (3.20) we have

$$\begin{aligned}
w(x,\tau)&=\zeta_\kappa(\tau)\left[1-\frac{D}{1-\tau}\left\|z(s_D)\right\|_\infty+O\left(D^{-\frac{N}{2}+\frac{1}{2}}\right)\right]^{-\frac{1}{p-1}}\\
&=\zeta_\kappa(\tau)\left[1-\frac{D}{1-\tau}\left\|z(s_D)\right\|_\infty\right]^{-\frac{1}{p-1}}\left(1+O\left(D^{-\frac{N}{2}+\frac{1}{2}}\right)\right)^{-\frac{1}{p-1}}\\
&=\zeta_\kappa(\tau+a_D)\left(1+O\left(D^{-\frac{N}{2}+\frac{1}{2}}\right)\right)=\zeta_\kappa(\tau+a_D)+O\left(D^{-\frac{N}{2}+\frac{1}{2}}\right)
\end{aligned}\tag{3.21}$$

for all $(x,\tau)\in B(x_D,2\delta D^{1/2})\times[-1,0]$ and all sufficiently large D. Let

$$\begin{aligned}
W(x,\tau)&:=D^{\frac{N}{2}-\frac{1}{2}}\left[w(x,\tau)-\zeta_\kappa(\tau+a_D)\right],\\
K(x,\tau)&:=D^{\frac{N}{2}-\frac{1}{2}}\left[w(x,\tau)^p-\zeta_\kappa(\tau+a_D)^p\right].
\end{aligned}\tag{3.22}$$

Then, by (3.20), (3.21), and (3.22) we can find a constant $C_1>0$ satisfying

$$\sup_{-1<\tau\leqslant 0}\left\|W(\tau)\right\|_{L^\infty(B(x_D,2\delta D^{1/2}))}\leqslant C_1,\tag{3.23}$$

$$\begin{aligned}
&\sup_{-1<\tau\leqslant 0}\left\|K(\tau)\right\|_{L^\infty(B(x_D,2\delta D^{1/2}))}\\
&=D^{\frac{N}{2}-\frac{1}{2}}\sup_{-1<\tau\leqslant 0}\left|\zeta_\kappa(\tau+a_D)^p\left(1+O\left(D^{-\frac{N}{2}+\frac{1}{2}}\right)\right)-\zeta_\kappa(\tau+a_D)^p\right|\leqslant C_1,
\end{aligned}\tag{3.24}$$

for all sufficiently large D. On the other hand, by (3.2) and (3.22) we have

$$\partial_\tau W-\Delta W=D^{\frac{N}{2}-\frac{1}{2}}\left[w^p-\zeta_\kappa(\tau+a_D)^p\right]=K(x,\tau)\tag{3.25}$$

for all $(x,\tau)\in B(x_D,2\delta D^{1/2})\times[-1,0]$ and all sufficiently large D. Then, by (3.23) and (3.24) we apply the parabolic regularity theorems (see for example [17, Chapter 3, Theorem 11.1]) to (3.25), and see that there exists a positive constant C_2 such that

$$\left|\nabla W(x,\tau)\right|\leqslant C_2\quad\text{in } B\left(x_D,\delta D^{1/2}\right)\times(-1/2,0]\tag{3.26}$$

for all sufficiently large D. Therefore, since $\psi_D^*(x)=1+c_\delta$ in $\mathbf{R}^N\setminus B(x_D,\delta D^{1/2})$ by (3.15) and (3.16), it follows from (3.6), (3.16), (3.17), (3.22), and (3.26) that

$$\begin{aligned}
&\left\|\nabla w^*(0)\right\|_\infty\leqslant\epsilon_D\|\nabla\psi_D\|_{L^\infty(B(x_D,\delta D^{1/2}))}\\
&\quad=\left\|\nabla w(0)\right\|_{L^\infty(B(x_D,\delta D^{1/2}))}=D^{-\frac{N}{2}+\frac{1}{2}}\left\|\nabla W(0)\right\|_{L^\infty(B(x_D,\delta D^{1/2}))}\leqslant C_2D^{-\frac{N}{2}+\frac{1}{2}}=C_2\epsilon_D
\end{aligned}$$

for all sufficiently large D. Thus Lemma 3.1 follows. $\square$

Lemma 3.2. *Assume the same conditions as in Theorem* 1.2. *Then there hold*

$$\sup_{0\leqslant\tau\leqslant\tau_*}\big(e^{\tau\Delta}\psi_D\big)(x_D)=1+o(1), \tag{3.27}$$

$$\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\ \inf_{|x-x_D|\geqslant\delta D^{1/2}}\big(e^{\tau\Delta}\psi_D^*\big)(x)\geqslant 1+c_\delta/2, \tag{3.28}$$

for all sufficiently large D.

Proof. We first prove (3.27). Put

$$Z(x,\tau):=\kappa\big[1-Dz\big(x,s_D+D^{-1}\tau\big)\big]^{-\frac{1}{p-1}}. \tag{3.29}$$

Then we have

$$\partial_\tau Z-\Delta Z=-pD^2Z(x,\tau)^{2p-1}\big|\nabla z\big(x,s_D+D^{-1}\tau\big)\big|^2 \tag{3.30}$$

in $\mathbf{R}^N\times[0,\tau_*]$. On the other hand, by (2.18), (2.21), and (3.12) we have

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau_*}\big\|Z(\tau)-Z(0)\big\|_\infty\\
&\quad\leqslant\frac{\kappa}{p-1}\big[1-\hat\theta_D\big]^{-\frac{p}{p-1}}\sup_{0\leqslant\tau\leqslant\tau_*}\big\|Dz\big(s_D+D^{-1}\tau\big)-Dz(s_D)\big\|_\infty\\
&\quad\preccurlyeq D\cdot D^{-\frac{N}{2}}\cdot D^{-1}\tau_*\leqslant 2D^{-\frac{N}{2}}
\end{aligned} \tag{3.31}$$

for all sufficiently large D, where $\hat\theta_D\in(0,1)$ is a function in $\mathbf{R}^N$ satisfying $\|\hat\theta_D\|_\infty=O(D^{-\frac{N}{2}+\frac{1}{2}})$ as $D\to\infty$. Furthermore, since

$$\lim_{D\to\infty}\sup_{0\leqslant\tau\leqslant\tau_*}\big\|Z(\tau)-\kappa\big\|_\infty=0,$$

by (P1), (2.22), (3.12), (3.29), and (3.30) we obtain

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau_*}\big\|Z(\tau)-e^{\tau\Delta}Z(0)\big\|_\infty\\
&\quad=\sup_{0\leqslant\tau\leqslant\tau_*}\left\|-pD^2\int_0^\tau e^{(\tau-s)\Delta}Z(s)^{2p-1}\big|\nabla z\big(s_D+D^{-1}s\big)\big|^2\,ds\right\|_\infty\\
&\quad\preccurlyeq D^2\int_0^{\tau_*}\big\|Z(s)\big\|_\infty^{2p-1}\big\|\nabla z\big(s_D+D^{-1}s\big)\big\|_\infty^2\,ds=O\big(D^{-N}\big)
\end{aligned} \tag{3.32}$$

for all sufficiently large D. Therefore, by (P1), (3.5), (3.10), (3.31), and (3.32) we obtain

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau_*}\left\|e^{\tau\Delta}\psi_D-\psi_D\right\|_\infty\\
&=\sup_{0\leqslant\tau\leqslant\tau_*}D^{\frac{N}{2}-\frac{1}{2}}\left\|e^{\tau\Delta}w(0)-w(0)\right\|_\infty\\
&=\sup_{0\leqslant\tau\leqslant\tau_*}D^{\frac{N}{2}-\frac{1}{2}}\left\|e^{\tau\Delta}Z(0)-Z(0)\right\|_\infty+O\big(D^{-\frac{N}{2}+\frac{5}{6}}\big)\\
&\leqslant\sup_{0\leqslant\tau\leqslant\tau_*}D^{\frac{N}{2}-\frac{1}{2}}\big[\left\|e^{\tau\Delta}Z(0)-Z(\tau)\right\|_\infty+\left\|Z(\tau)-Z(0)\right\|_\infty\big]+O\big(D^{-\frac{N}{2}+\frac{5}{6}}\big)\\
&=O\big(D^{-\frac{N}{2}-\frac{1}{2}}\big)+O\big(D^{-\frac{1}{2}}\big)+O\big(D^{-\frac{N}{2}+\frac{5}{6}}\big)
\end{aligned}$$

for all sufficiently large D. This together with (3.14) implies

$$\lim_{D\to\infty}\sup_{0\leqslant\tau\leqslant\tau_*}\big(e^{\tau\Delta}\psi_D\big)(x_D)=\lim_{D\to\infty}\psi_D(x_D)=1,$$

and we obtain (3.27).

Next we prove inequality (3.28). By (3.15) and (3.16) we have

$$\psi_D^*(x)\geqslant 1\quad\text{in }\mathbf{R}^N,\qquad \psi_D^*(x)=1+c_\delta\quad\text{in }\mathbf{R}^N\setminus B\big(x_D,\delta D^{1/2}\big).$$

These imply that

$$\big(e^{\tau\Delta}\psi_D^*\big)(x)-1=\int_{\mathbf{R}^N}G(x-y,\tau)\big(\psi_D^*(y)-1\big)\,dy\geqslant c_\delta\int_{|y-x_D|\geqslant\delta D^{1/2}}G(x-y,\tau)\,dy \tag{3.33}$$

for all $x\in\mathbf{R}^N$ and $\tau>0$. Let

$$\Pi_x:=\big\{x+y\colon\ y\cdot(x-x_D)\geqslant 0,\ y\in\mathbf{R}^N\big\}.$$

Then, since $\Pi_x\subset\{y\colon\ |y-x_D|\geqslant\delta D^{1/2}\}$ for $x\in\mathbf{R}^N\setminus B(x_D,\delta D^{1/2})$, by (3.33) we obtain

$$\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x-x_D|\geqslant\delta D^{1/2}}\big(e^{\tau\Delta}\psi_D^*\big)(x)-1\geqslant c_\delta\inf_{\tau_*/2\leqslant\tau\leqslant\tau_*}\inf_{|x-x_D|\geqslant\delta D^{1/2}}\int_{\Pi_x}G(x-y,\tau)\,dy=\frac{c_\delta}{2}.$$

Therefore we obtain inequality (3.28), and the proof of Lemma 3.2 is complete. □

Now we are ready to study the location of the maximum points of $w(\cdot,\tau_*)$. By (3.16) and (3.17) we have

$$\kappa_D-(1+c_\delta)\epsilon_D\leqslant w^*(x,0)\leqslant\kappa_D-\epsilon_D,\quad x\in\mathbf{R}^N. \tag{3.34}$$

Then, by (3.4), (3.11), (3.34), and Lemma 3.1 we can apply Proposition 2.2 to problem (3.17) with $M_\epsilon=\kappa_D$, $C_*=1$, and $t_*=0$, and obtain

$$\lim_{D\to\infty}\left\|\epsilon_D^{\frac{1}{p-1}}w^*(\cdot,\tau_*)-\kappa\kappa_D^{\frac{p}{p-1}}\big[\big(e^{\tau_*\Delta}\psi_D^*\big)(\cdot)\big]^{-\frac{1}{p-1}}\right\|_\infty=0. \tag{3.35}$$

In particular, by (3.12), (3.16), (3.18), and (3.35) we have

$$\epsilon_D^{\frac{1}{p-1}} \|w(\tau_*)\|_\infty \leqslant \epsilon_D^{\frac{1}{p-1}} \|w^*(\tau_*)\|_\infty \preccurlyeq 1 \tag{3.36}$$

for all sufficiently large D.

Let η be a positive constant such that

$$\kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_\delta}{2}\right)^{-\frac{1}{p-1}} + 2\eta < \kappa^{1+\frac{p}{p-1}}(1+\eta)^{-\frac{1}{p-1}} - 2\eta. \tag{3.37}$$

By (3.12), (3.28), and (3.35) we have

$$\epsilon_D^{\frac{1}{p-1}} w^*(x,\tau_*) \leqslant \kappa\kappa_D^{\frac{p}{p-1}}\left(e^{\tau_*\Delta}\psi_D^*\right)(x)^{-\frac{1}{p-1}} + \eta \leqslant \kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_\delta}{2}\right)^{-\frac{1}{p-1}} + 2\eta \tag{3.38}$$

for all $x \in \mathbf{R}^N$ with $|x-x_D| \geqslant \delta D^{1/2}$ and all sufficiently large D. On the other hand, since the function

$$\underline{w}(x,\tau) := \left(\left(e^{\tau\Delta}w(0)\right)(x)^{-(p-1)} - (p-1)\tau\right)^{-\frac{1}{p-1}}$$

is a subsolution of (3.2), we apply the comparison principle to obtain

$$\underline{w}(x,\tau) \leqslant w(x,\tau) \quad \text{in } \mathbf{R}^N \times [0,\tau_*]. \tag{3.39}$$

Since it follows from (3.6), (3.12), and (3.27) that

$$\begin{aligned}\left(e^{\tau_*\Delta}w(0)\right)(x_D)^{-(p-1)} &= \kappa_D^{-(p-1)}\left[1-\kappa_D^{-1}\epsilon_D\left(e^{\tau_*\Delta}\psi_D\right)(x_D)\right]^{-(p-1)} \\ &= \kappa_D^{-(p-1)}\left[1+(p-1)\kappa_D^{-1}\epsilon_D\left(e^{\tau_*\Delta}\psi_D\right)(x_D) + O\left(\epsilon_D^2\right)\right]\end{aligned}$$

for all sufficiently large D, by (1.6) and (3.12) we have

$$\begin{aligned}\underline{w}(x_D,\tau_*) &= \left[\left(e^{\tau_*\Delta}w(0)\right)(x_D)^{-(p-1)} - (p-1)S_{\kappa_D}\right]^{-\frac{1}{p-1}} \\ &= \epsilon_D^{-\frac{1}{p-1}}\kappa\kappa_D^{\frac{p}{p-1}}\left[\left(e^{\tau_*\Delta}\psi_D\right)(x_D) + O(\epsilon_D)\right]^{-\frac{1}{p-1}}\end{aligned} \tag{3.40}$$

for all sufficiently large D. Then, by (3.12), (3.27), (3.37), (3.39), and (3.40) we have

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}} w(x_D,\tau_*) &\geqslant \epsilon_D^{\frac{1}{p-1}} \underline{w}(x_D,\tau_*) = \kappa\kappa_D^{\frac{p}{p-1}}\left[\left(e^{\tau_*\Delta}\psi_D\right)(x_D) + O(\epsilon_D)\right]^{-\frac{1}{p-1}} \\ &\geqslant \kappa^{1+\frac{p}{p-1}}(1+\eta)^{-\frac{1}{p-1}} - \eta > \kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_\delta}{2}\right)^{-\frac{1}{p-1}} + 3\eta\end{aligned} \tag{3.41}$$

for all sufficiently large D. Therefore, by (3.18), (3.38), and (3.41) we obtain

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}} \sup_{|x-x_D|\geqslant\delta D^{1/2}} w(x,\tau_*) &\leqslant \epsilon_D^{\frac{1}{p-1}} \sup_{|x-x_D|\geqslant\delta D^{1/2}} w^*(x,\tau_*) \\ &\leqslant \kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_\delta}{2}\right)^{-\frac{1}{p-1}} + 2\eta < \epsilon_D^{\frac{1}{p-1}} w(x_D,\tau_*) - \eta\end{aligned}$$

for all sufficiently large D. This implies that

$$H\big(\epsilon_D^{\frac{1}{p-1}} w(\cdot,\tau_*),\eta\big) \subset B\big(x_D,\delta D^{1/2}\big) \tag{3.42}$$

for all sufficiently large D.

We complete the proof of (1.8) by using Proposition 2.3. Put

$$\tilde{w}(x,\tau) := \epsilon_D^{\frac{1}{p-1}} w(x,\tau_* + \epsilon_D\tau). \tag{3.43}$$

Then, by (3.2) we have

$$\begin{cases} \partial_\tau \tilde{w} = \epsilon_D \Delta\tilde{w} + \tilde{w}^p & \text{in } \mathbf{R}^N \times (0,\tilde{\tau}_D), \\ \tilde{w}(x,0) = \epsilon_D^{\frac{1}{p-1}} w(x,\tau_*) & \text{in } \mathbf{R}^N, \end{cases} \tag{3.44}$$

where $\tilde{\tau}_D = \epsilon_D^{-1}(\tau_D - \tau_*)$ is the blow-up time of $\tilde{w}$. By (2.28), (3.1), (3.3), and (3.43) we have

$$\begin{aligned} \big|\tilde{w}(x,\tau)\big| &= \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \big|u\big(x, s_D + D^{-1}(\tau_* + \epsilon_D\tau)\big)\big| \\ &\preccurlyeq \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \big(T_D - s_D - D^{-1}\tau_* - D^{-1}\epsilon_D\tau\big)^{-\frac{1}{p-1}} \\ &= \epsilon_D^{\frac{1}{p-1}} (\tau_D - \tau_* - \epsilon_D\tau)^{-\frac{1}{p-1}} = (\tilde{\tau}_D - \tau)^{-\frac{1}{p-1}} \end{aligned} \tag{3.45}$$

for all $(x,\tau) \in \mathbf{R}^N \times [0,\tilde{\tau}_D)$ and all sufficiently large D. Furthermore, by (3.36) and (3.41) we have

$$\big\|\tilde{w}(0)\big\|_\infty = \epsilon_D^{\frac{1}{p-1}} \big\|w(\tau_*)\big\|_\infty \preccurlyeq 1, \qquad \big\|\tilde{w}(0)\big\|_\infty \geqslant \epsilon_D^{\frac{1}{p-1}} w(x_D,\tau_*) \succcurlyeq 1, \tag{3.46}$$

for all sufficiently large D. In particular, we apply the comparison principle to obtain

$$\tilde{\tau}_D \geqslant S_{\|\tilde{w}(0)\|_\infty} \succcurlyeq 1 \tag{3.47}$$

for all sufficiently large D. In addition, since

$$T_D = s_D + D^{-1}\tau_* + D^{-1}\epsilon_D\tilde{\tau}_D, \quad \epsilon_D = D^{-\frac{N}{2}+\frac{1}{2}}, \qquad \lim_{D\to\infty} \big(D^{-1}\epsilon_D\big)^{D^{-\frac{N}{2}}} = 1,$$

by (2.22), (2.29), (3.1), (3.43), and (3.47) we have

$$\begin{aligned} \big\|\nabla\tilde{w}(0)\big\|_\infty &= \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \big\|\nabla u\big(s_D + D^{-1}\tau_*\big)\big\|_\infty \\ &\preccurlyeq \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \big(T_D - s_D - D^{-1}\tau_*\big)^{-\frac{p}{p-1} - C' D^{-\frac{N}{2}}} D^{-\frac{N}{2}-1} \\ &\preccurlyeq D\epsilon_D^{-1} \tilde{\tau}_D^{-\frac{p}{p-1} - C' D^{-\frac{N}{2}}} D^{-\frac{N}{2}-1} \asymp D^{-\frac{1}{2}} \leqslant 1 \end{aligned} \tag{3.48}$$

for all sufficiently large D, where C' is a positive constant. Therefore, by (3.45), (3.46), and (3.48) we can apply Proposition 2.3 with $\varphi_\epsilon = \tilde{w}(0)$ to the solution $\tilde{w}$ of (3.44), and by (3.42) we see that

$$B_D \subset H\big(\tilde{w}(\cdot,0),\eta\big) = H\big(\epsilon_D^{\frac{1}{p-1}} w(\cdot,\tau_*),\eta\big) \subset B\big(x_D, \delta D^{1/2}\big)$$

for all sufficiently large D. This implies

$$\limsup_{D\to\infty} D^{-\frac{1}{2}} \sup\big\{|x-x_D|:\ x\in B_D\big\} \leqslant \delta. \tag{3.49}$$

On the other hand, recalling that $x_D = \sqrt{2Ds_D}\,\Xi(\varphi)/|\Xi(\varphi)|$ and $s_D = S_\lambda - D^{-1}$, we have

$$\left|x_D - \sqrt{2DS_\lambda}\frac{\Xi(\varphi)}{|\Xi(\varphi)|}\right| = \big|\sqrt{2(DS_\lambda-1)} - \sqrt{2DS_\lambda}\big| = O\big(D^{-\frac{1}{2}}\big) \tag{3.50}$$

as $D\to\infty$. Therefore, since $\delta>0$ is arbitrary, by (3.49) and (3.50) we obtain

$$\lim_{D\to\infty} D^{-\frac{1}{2}} \sup_{x\in B_D}\left|x - \sqrt{2DS_\lambda}\frac{\Xi(\varphi)}{|\Xi(\varphi)|}\right| = 0.$$

This implies (1.8), and the proof of Theorem 1.2 is complete. □

3.2. *Blow-up set for the case $M(\varphi)<0$*

In this subsection we prove (1.10) under the assumption $M(\varphi)<0$, and complete the proof of Theorem 1.3. Let $c\in(0,1/2)$ and fix it. In this case we put

$$\epsilon_D := D^{-\frac{N}{2}-c+1},$$

and study the location of the blow-up set B_D. Since $N\geqslant 3$, by (2.17), (2.27), and (3.8) we have

$$\begin{aligned} w(x,0) &= \kappa\big[1 - Dz(x,s_D) + O\big(D^{-N+\frac{4}{3}}\big)\big]^{-\frac{1}{p-1}} \\ &= \kappa\big[1 - Dz(x,s_D)\big]^{-\frac{1}{p-1}} + O\big(D^{-N+\frac{4}{3}}\big), \quad x\in\mathbf{R}^N, \end{aligned} \tag{3.51}$$

$$\big\|w(0)\big\|_\infty = \kappa\Big[1 - D\sup_{x\in\mathbf{R}^N} z(x,s_D) + O\big(D^{-N+\frac{4}{3}}\big)\Big]^{-\frac{1}{p-1}} = \kappa + O\big(D^{-\frac{N}{2}}\big), \tag{3.52}$$

for all sufficiently large D. In particular, by (3.4) and (3.52) we have

$$\kappa_D = \kappa + o(1), \qquad \tau_* = S_{\kappa_D} = S_\kappa + o(1) = 1 + o(1), \tag{3.53}$$

as $D\to\infty$. Furthermore, since $N\geqslant 3$ and $c\in(0,1/2)$, by (2.17), (3.5), (3.51), and (3.52) we have

$$\begin{aligned} \psi_D(x) &= \kappa\epsilon_D^{-1}\big[1 - \big(1 - Dz(x,s_D)\big)^{-\frac{1}{p-1}} + O\big(D^{-N+\frac{4}{3}}\big)\big] + 1 + o(1) \\ &= -\frac{\kappa\epsilon_D^{-1}}{p-1} Dz(x,s_D) + O\big(\epsilon_D^{-1}D^{-N+2}\big) + O\big(\epsilon_D^{-1}D^{-N+\frac{4}{3}}\big) + 1 + o(1) \\ &= -\frac{\kappa\epsilon_D^{-1}}{p-1} Dz(x,s_D) + 1 + o(1) \end{aligned} \tag{3.54}$$

for all $x \in \mathbf{R}^N$ and all sufficiently large D. Then, by (2.26) and (3.54) we have

$$\lim_{D\to\infty} \sup_{|x|\geqslant Ds_D} \psi_D(x) = 1. \tag{3.55}$$

In addition, by (2.25) and (3.54) we can find a positive constant c_* satisfying

$$\inf_{|x|\leqslant(4cDs_D\log(Ds_D))^{1/2}} \psi_D(x) \geqslant 1 + c_* \tag{3.56}$$

for all sufficiently large D.

Put

$$\psi_D^*(x) := \min\{\psi_D(x), 1 + c_*\} \geqslant 1 \tag{3.57}$$

(see (3.5)), and let w^* be the solution of

$$\begin{cases} \partial_\tau w = \Delta w + w^p, & x \in \mathbf{R}^N,\ \tau > 0, \\ w(x,0) = \kappa_D - \epsilon_D \psi_D^*(x), & x \in \mathbf{R}^N. \end{cases} \tag{3.58}$$

Then, by (3.6), (3.57), and (3.58) we apply the comparison principle to obtain

$$w(x,\tau) \leqslant w^*(x,\tau) \quad \text{in } \mathbf{R}^N \times [0,\tau_*] \tag{3.59}$$

for all sufficiently large D.

Next, similarly as in the previous subsection, we give the estimates of $\|\nabla w^*(0)\|_\infty$, $e^{\tau\Delta}\psi_D$, and $e^{\tau\Delta}\psi_D^*$ in the following two lemmas.

Lemma 3.3. *Assume the same conditions as in Theorem* 1.3. *Then there exists a positive constant C such that*

$$\|\nabla w^*(0)\|_\infty \leqslant C\epsilon_D$$

for all sufficiently large D.

Proof. By (2.26) and (3.8) we have

$$w(x,\tau) = \zeta_\kappa(\tau)\bigl(1 + O\bigl(D^{-\frac{N}{2}-c+1}\bigr)\bigr) = \zeta_\kappa(\tau) + O\bigl(D^{-\frac{N}{2}-c+1}\bigr) \tag{3.60}$$

for all $(x,\tau) \in [\mathbf{R}^N \setminus B(0,(4cs_D\log(Ds_D))^{1/2}-1)] \times [-1,0]$ and all sufficiently large D. Put

$$W(x,\tau) := D^{\frac{N}{2}+c-1}\bigl[w(x,\tau) - \zeta_\kappa(\tau)\bigr], \qquad K(x,\tau) := D^{\frac{N}{2}+c-1}\bigl[w(x,\tau)^p - \zeta_\kappa(\tau)^p\bigr]. \tag{3.61}$$

Then, by (3.60) and (3.61) we can find a positive constant C_1 so that

$$\sup_{-1<\tau\leqslant 0} \|W(\tau)\|_{L^\infty(\mathbf{R}^N\setminus B(0,(4cs_D\log(Ds_D))^{1/2}-1))} \leqslant C_1, \tag{3.62}$$

$$\sup_{-1<\tau\leqslant 0} \|K(\tau)\|_{L^\infty(\mathbf{R}^N\setminus B(0,(4cs_D\log(Ds_D))^{1/2}-1))} \leqslant C_1, \tag{3.63}$$

for all sufficiently large D. Furthermore, by (3.2) and (3.61) we have

$$\partial_\tau W - \Delta W = D^{\frac{N}{2}+c-1}\big[w^p - \zeta_\kappa^p\big] = K(x,\tau) \tag{3.64}$$

for all $(x,\tau) \in [\mathbf{R}^N \setminus B(0,(4cs_D \log(Ds_D))^{1/2} - 1)] \times [-1,0]$. Then, by (3.62) and (3.63) we apply the parabolic regularity theorems (see for example [17, Chapter 3, Theorem 11.1]) to (3.64), and see that there exists a positive constant C_2 such that

$$\big|\nabla W(x,\tau)\big| \leqslant C_2 \quad \text{in } \big[\mathbf{R}^N \setminus B\big(0, \big(4cs_D \log(Ds_D)\big)^{1/2}\big)\big] \times (-1/2, 0] \tag{3.65}$$

for all sufficiently large D. Therefore, since $\psi_D^*(x) = 1 + c_*$ in $B(0,(4cs_D \log(Ds_D))^{1/2})$ by (3.56) and (3.57), it follows from (3.6), (3.57), (3.58), (3.61), and (3.65) that

$$\begin{aligned}
\big\|\nabla w^*(0)\big\|_\infty &\leqslant \epsilon_D \|\nabla\psi_D\|_{L^\infty(\mathbf{R}^N \setminus B(0,(4cDs_D \log(Ds_D))^{1/2}))} \\
&= \big\|\nabla w(0)\big\|_{L^\infty(\mathbf{R}^N \setminus B(0,(4cDs_D \log(Ds_D))^{1/2}))} \\
&= D^{-\frac{N}{2}-c+1}\big\|\nabla W(0)\big\|_{L^\infty(\mathbf{R}^N \setminus B(0,(4cDs_D \log(Ds_D))^{1/2}))} \leqslant C_2\epsilon_D
\end{aligned}$$

for all sufficiently large D. Thus we obtain the desired inequality, and the proof of Lemma 3.3 is complete. □

Lemma 3.4. *Assume the same conditions as in Theorem* 1.3. *Then there hold*

$$\sup_{0 \leqslant \tau \leqslant \tau_*} \sup_{|x| \geqslant Ds_D} \big(e^{\tau\Delta}\psi_D\big)(x) = 1 + o(1), \tag{3.66}$$

$$\inf_{\tau_*/2 \leqslant \tau \leqslant \tau_*} \inf_{|x| \leqslant (cDs_D \log(Ds_D))^{1/2}} \big(e^{\tau\Delta}\psi_D^*\big)(x) \geqslant 1 + c_*/2, \tag{3.67}$$

for all sufficiently large D.

Proof. Put

$$Z(x,\tau) := \kappa\big[1 - Dz\big(x, s_D + D^{-1}\tau\big)\big]^{-\frac{1}{p-1}}.$$

Then, by the same argument as in [8, Lemma 5.2] with $A = 1$ we see that

$$\sup_{0 \leqslant \tau \leqslant \tau_*} \big\|Z(\tau) - Z(0)\big\|_\infty = O\big(D^{-\frac{N}{2}}\big),$$

$$\sup_{0 \leqslant \tau \leqslant \tau_*} \big\|Z(\tau) - e^{\tau\Delta}Z(0)\big\|_\infty = O\big(D^{-N+1}\big),$$

for all sufficiently large D. These inequalities together with (3.5) and (3.51) yield

$$\begin{aligned}
&\sup_{0 \leqslant \tau \leqslant \tau_*} \big\|e^{\tau\Delta}\psi_D - \psi_D\big\|_\infty \\
&= \sup_{0 \leqslant \tau \leqslant \tau_*} D^{\frac{N}{2}+c-1}\big\|e^{\tau\Delta}w(0) - w(0)\big\|_\infty \\
&= \sup_{0 \leqslant \tau \leqslant \tau_*} D^{\frac{N}{2}+c-1}\big\|e^{\tau\Delta}Z(0) - Z(0)\big\|_\infty + O\big(D^{-\frac{N}{2}+c+\frac{1}{3}}\big)
\end{aligned}$$

$$\leqslant \sup_{0\leqslant\tau\leqslant\tau_*} D^{\frac{N}{2}+c-1}\big[\big\|e^{\tau\Delta}Z(0)-Z(\tau)\big\|_\infty+\big\|Z(\tau)-Z(0)\big\|_\infty\big]+O\big(D^{-\frac{N}{2}+c+\frac{1}{3}}\big)$$
$$=O\big(D^{-\frac{N}{2}+c}\big)+O\big(D^{c-1}\big)+O\big(D^{-\frac{N}{2}+c+\frac{1}{3}}\big) \tag{3.68}$$

for all sufficiently large D. Then, since $N\geqslant 3$ and $c\in(0,1/2)$, by (3.55) and (3.68) we have

$$\lim_{D\to\infty}\sup_{0\leqslant\tau\leqslant\tau_*}\sup_{|x|\geqslant Ds_D}\big(e^{\tau\Delta}\psi_D\big)(x)=\lim_{D\to\infty}\sup_{|x|\geqslant Ds_D}\psi_D(x)=1.$$

This gives (3.66). Furthermore, since

$$B\big(x,\big(cDs_D\log(Ds_D)\big)^{1/2}\big)\subset B\big(0,\big(4cDs_D\log(Ds_D)\big)^{1/2}\big)\quad\text{if } |x|\leqslant\big(cDs_D\log(Ds_D)\big)^{1/2},$$

by (1.4), (3.56), and (3.57) we have

$$\big(e^{\tau\Delta}\psi_D^*\big)(x)-1\geqslant c_*\int\limits_{|y|\leqslant(4cDs_D\log(Ds_D))^{1/2}}G(x-y,\tau)\,dy$$
$$\geqslant c_*\int\limits_{|z|\leqslant(cDs_D\log(Ds_D))^{1/2}}G(z,\tau)\,dz=c_*\big(1+o(1)\big)\geqslant c_*/2$$

for all $(x,\tau)\in B(0,(cDs_D\log(Ds_D))^{1/2})\times[\tau_*/2,\tau_*]$ and all sufficiently large D. Therefore we obtain (3.67), and the proof of Lemma 3.4 is complete. $\square$

Next we study the location of the maximum points of $w(\cdot,\tau_*)$. Since it follows from (3.57) and (3.58) that

$$\kappa_D-(1+c_*)\epsilon_D\leqslant w^*(x,0)\leqslant\kappa_D-\epsilon_D,\quad x\in\mathbf{R}^N,$$

by (3.53) and Lemma 3.3 we can apply Proposition 2.2 to problem (3.58) with $M_\epsilon=\kappa_D$, $C_*=1$, and $t_*=0$, and obtain

$$\lim_{D\to\infty}\big\|\epsilon_D^{\frac{1}{p-1}}w^*(\cdot,\tau_*)-\kappa\kappa_D^{\frac{p}{p-1}}\big[\big(e^{\tau_*\Delta}\psi_D^*\big)(\cdot)\big]^{-\frac{1}{p-1}}\big\|_\infty=0. \tag{3.69}$$

Then, by (3.53), (3.57), (3.59), and (3.69) we have

$$\epsilon_D^{\frac{1}{p-1}}\big\|w(\tau_*)\big\|_\infty\leqslant\epsilon_D^{\frac{1}{p-1}}\big\|w^*(\tau_*)\big\|_\infty\preccurlyeq 1 \tag{3.70}$$

for all sufficiently large D.

Let η be a positive constant such that

$$\kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_*}{2}\right)^{-\frac{1}{p-1}}+2\eta<\kappa^{1+\frac{p}{p-1}}(1+\eta)^{-\frac{1}{p-1}}-2\eta. \tag{3.71}$$

By (3.53), (3.67), and (3.69) we have

$$\epsilon_D^{\frac{1}{p-1}}w^*(x,\tau_*)\leqslant\kappa\kappa_D^{\frac{p}{p-1}}\big(e^{\tau_*\Delta}\psi_D^*\big)(x)^{-\frac{1}{p-1}}+\eta\leqslant\kappa^{1+\frac{p}{p-1}}\left(1+\frac{c_*}{2}\right)^{-\frac{1}{p-1}}+2\eta \tag{3.72}$$

for all $x \in B(0, (cDs_D \log(Ds_D))^{1/2})$ and all sufficiently large D. On the other hand, since the function

$$\underline{w}(x,\tau) := \left(\left(e^{\tau\Delta} w(0)\right)(x)^{-(p-1)} - (p-1)\tau\right)^{-\frac{1}{p-1}}$$

is a subsolution of (3.2), by the comparison principle we obtain

$$\underline{w}(x,\tau) \leqslant w(x,\tau) \quad \text{in } \mathbf{R}^N \times [0,\tau_*]. \tag{3.73}$$

Then, applying the similar argument as in (3.40) with the aid of (3.66), we obtain

$$\underline{w}(x,\tau) = \epsilon_D^{-\frac{1}{p-1}} \kappa \kappa_D^{\frac{p}{p-1}} \left[\left(e^{\tau_*\Delta}\psi_D\right)(x) + O(\epsilon_D)\right]^{-\frac{1}{p-1}}$$

for all $|x| \geqslant Ds_D$ and all sufficiently large D. This together with (3.66), (3.71), and (3.73) we have

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}} w(x,\tau_*) &\geqslant \epsilon_D^{\frac{1}{p-1}} \underline{w}(x,\tau_*) = \kappa \kappa_D^{\frac{p}{p-1}} \left[\left(e^{\tau_*\Delta}\psi_D\right)(x) + O(\epsilon_D)\right]^{-\frac{1}{p-1}} \\ &\geqslant \kappa^{1+\frac{p}{p-1}} (1+\eta)^{-\frac{1}{p-1}} - \eta > \kappa^{1+\frac{p}{p-1}} \left(1 + \frac{c_*}{2}\right)^{-\frac{1}{p-1}} + 3\eta \end{aligned} \tag{3.74}$$

for all $|x| \geqslant Ds_D$ and all sufficiently large D. Therefore, by (3.59), (3.72), and (3.74) we have

$$\begin{aligned}\epsilon_D^{\frac{1}{p-1}} \sup_{B(0,(cDs_D \log(Ds_D))^{1/2})} w(x,\tau_*) &\leqslant \epsilon_D^{\frac{1}{p-1}} \sup_{B(0,(cDs_D \log(Ds_D))^{1/2})} w^*(x,\tau_*) \\ &\leqslant \kappa^{1+\frac{p}{p-1}} \left(1 + \frac{c_*}{2}\right)^{-\frac{1}{p-1}} + 2\eta < \epsilon_D^{\frac{1}{p-1}} \left\| w(\tau_*) \right\|_\infty - \eta\end{aligned}$$

for all sufficiently large D. Thus we obtain

$$H\left(\epsilon_D^{\frac{1}{p-1}} w(\cdot,\tau_*), \eta\right) \subset \mathbf{R}^N \setminus B\left(0, \left(cDs_D \log(Ds_D)\right)^{1/2}\right) \tag{3.75}$$

for all sufficiently large D.

Next we study the location of the blow-up set of w, and complete the proof of Theorem 1.3. Put

$$\tilde{w}(x,\tau) = \epsilon_D^{\frac{1}{p-1}} w(x, \tau_* + \epsilon_D \tau). \tag{3.76}$$

Then we see that

$$\begin{cases} \partial_\tau \tilde{w} = \epsilon_D \Delta \tilde{w} + \tilde{w}^p & \text{in } \mathbf{R}^N \times (0, \tilde{\tau}_D), \\ \tilde{w}(x,0) = \epsilon_D^{\frac{1}{p-1}} w(x,\tau_*) & \text{in } \mathbf{R}^N, \end{cases} \tag{3.77}$$

where $\tilde{\tau}_D = \epsilon_D^{-1}(\tau_D - \tau_*)$ is the blow-up time of $\tilde{w}$. Similarly to (3.45), we have

$$\left\| \tilde{w}(\tau) \right\|_\infty \preccurlyeq (\tilde{\tau}_D - \tau)^{-\frac{1}{p-1}} \tag{3.78}$$

for all $\tau \in [0, \tilde{\tau}_D)$ and all sufficiently large D. Moreover, by (3.70) and (3.74) we have

$$\|\tilde{w}(0)\|_\infty = \epsilon_D^{\frac{1}{p-1}} \|w(\tau_*)\|_\infty \preccurlyeq 1, \qquad \|\tilde{w}(0)\|_\infty \geqslant \epsilon_D^{\frac{1}{p-1}} \sup_{|x| \geqslant Ds_D} w(x, \tau_*) \succcurlyeq 1, \tag{3.79}$$

for all sufficiently large D. This together with the comparison principle implies that

$$\tilde{\tau}_D \geqslant S_{\|\tilde{w}(0)\|_\infty} \succcurlyeq 1 \tag{3.80}$$

for all sufficiently large D. Furthermore, since $c \in (0, 1/2)$ and

$$T_D = s_D + D^{-1}\tau_* + D^{-1}\epsilon_D \tilde{\tau}_D, \quad \epsilon_D = D^{-\frac{N}{2}-c+1}, \qquad \lim_{D \to \infty} \left(D^{-1}\epsilon_D\right)^{D^{-\frac{N}{2}}} = 1,$$

by (2.17), (2.29), (3.1), (3.76), and (3.80) we have

$$\begin{aligned} \|\nabla \tilde{w}(0)\|_\infty &= \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \left\|\nabla u\left(s_D + D^{-1}\tau_*\right)\right\|_\infty \\ &\preccurlyeq \epsilon_D^{\frac{1}{p-1}} D^{-\frac{1}{p-1}} \left(T_D - s_D - D^{-1}\tau_*\right)^{-\frac{p}{p-1} - C' D^{-\frac{N}{2}}} D^{-\frac{N}{2}-\frac{1}{2}} \\ &\preccurlyeq D\epsilon_D^{-1} \tilde{\tau}_D^{-\frac{p}{p-1} - C' D^{-\frac{N}{2}}} D^{-\frac{N}{2}-\frac{1}{2}} \preccurlyeq D^{c-\frac{1}{2}} \leqslant 1 \end{aligned} \tag{3.81}$$

for all sufficiently large D, where C' is a positive constant. Therefore, by (3.78), (3.79), and (3.81) we can apply Proposition 2.3 with $\varphi_\epsilon = \tilde{w}(0)$ to the solution $\tilde{w}$ of (3.77), and by (3.75) we obtain

$$B_D \subset H\left(\tilde{w}(\cdot, 0), \eta\right) = H\left(\epsilon_D^{\frac{1}{p-1}} w(\cdot, \tau_*), \eta\right) \subset \mathbf{R}^N \setminus B\left(0, \left(cDs_D \log(Ds_D)\right)^{1/2}\right)$$

for all sufficiently large D. Then, since $s_D = S_\lambda - D^{-1}$, there exists a positive constant C such that

$$B_D \cap B\left(0, C(D \log D)^{1/2}\right) = \emptyset$$

for all sufficiently large D, and we obtain (1.10). Thus the proof of Theorem 1.3 is complete. $\square$

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