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Conservative solutions to a system of variational wave equations ☆

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ABSTRACT

We investigate a system of variational wave equations which is the Euler–Lagrange equations of a variational principle arising in the theory of nematic liquid crystals and a few other physical contexts. We establish the global existence of an energy-conservative weak solution to its Cauchy problem for initial data of finite energy. The main difficulty arises from the possible concentration of energy. We construct the solution by introducing a new set of variables depending on the energy, whereby all singularities are resolved.

1. Introduction

In this paper, we are concerned with the following system of variational wave equations

$$u_{ktt} - c_k(u)\big[c_k(u)u_{kx}\big]_x = \sum_{i=1}^{n}(c_k c_{ku_i} u_{kx} - c_i c_{iu_k} u_{ix})u_{ix} \quad (k = 1, 2, \dots, n), \tag{1.1}$$

where $u(t,x) = (u_1(t,x), \dots, u_n(t,x))$ is the unknown vector function, c_i $(i = 1, 2, \dots, n): R^n \to R$ are smooth and positive functions, that is, there are positive numbers $\dot{c} < \ddot{c}$ such that

$$0 < \dot{c} \leqslant c_i(z) \leqslant \ddot{c} < \infty \quad \text{and} \quad \sup_z \big|\nabla c_i(z)\big| < \infty, \quad z \in R^n \ (i = 1, 2, \dots, n). \tag{1.2}$$

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System (1.1) is derived from a variational principle of the form

$$\delta \int A^{ij}_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\,\mathrm{d}x = 0, \tag{1.3}$$

where we use the summation convention, see [1,12] for details. Here, $x \in R^{d+1}$ are the space–time independent variables and $u : R^{d+1} \to R^n$ are the dependent variables. The coefficients $A^{ij}_{\alpha\beta} : R^n \to R$ are smooth functions and $A^{ij}_{\alpha\beta} = A^{ij}_{\beta\alpha} = A^{j\,i}_{\alpha\beta}$. Consider the special case $d = 1$ and

$$A_{\alpha\beta} = \left(A^{ij}_{\alpha\beta}\right)_{2\times 2} = \begin{cases} \mathrm{diag}(-c^2_\alpha(u), 1), & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases} \tag{1.4}$$

then the Lagrangian density of (1.3) is

$$\mathcal{L}(u, \nabla u) = \sum_{i=1}^{n}\left[u^2_{it} - c^2_i(u)u^2_{ix}\right], \tag{1.5}$$

for which the Euler–Lagrange equations are (1.1).

For the case $n = 1$, system (1.1) reduces to the variational wave equation

$$u_{tt} - c(u)\left[c(u)u_x\right]_x = 0. \tag{1.6}$$

This equation arises in a number of various physical contexts, for example, it describes, to the first order, the motion of long waves on a neutral dipole chain in the continuum limit [10,29]. For another important example, Eq. (1.6) is a simplified model for the director field of a nematic liquid crystal [12,17]. Even for smooth initial data, there is the well-known fact that the solution of this equation can develop cusp-type singularities in finite time, which is attributed to the term $c(u)c'(u)u^2_x$, see Glassey et al. [8,9]. It is therefore necessary to study the global existence of weak solutions. There are at least two natural distinct classes of admissible weak solutions, which are called dissipative and conservative solutions. The dissipative solution loses all the energy at the blowup time, while the conservative solution will preserve its energy in time. The existence of a dissipative weak solution to its initial value problem, as well as for related asymptotic models, has been extensively studied by Zhang and Zheng [19–26] and Hunter and Zheng [13]. The more relevant results of the first-order asymptotic equation (which is also called Hunter–Saxton equation) and its geometric interpretation are presented in [3,4,7,14–16,18] and the references therein. In [6], Bressan and Zheng established an energy-conservative weak solution to the Cauchy problem of Eq. (1.6). Recently, Holden and Raynaud [11] have shown that it possesses a global semigroup for conservative weak solutions. Moreover, the global conservative weak solutions to its asymptotic equations have been obtained in [5,21].

Another simplified model arising from the theory of nematic liquid crystals is a system of equations

$$\begin{cases} u_{tt} - c_1[c_1u_x]_x = aa'\left(v^2_t - c^2_2v^2_x\right) - a^2c_2c'_2v^2_x, \\ \left(a^2v_t\right)_t - \left(c^2_2a^2v_x\right)_x = 0, \end{cases} \tag{1.7}$$

where $c_1 = c_1(u)$, $c_2 = c_2(u)$ and $a = a(u)$ are functions of u alone. We refer the reader to Ref. [2] of Ali and Hunter for the detailed derivation and more background information of the system. Recently, Zhang and Zheng [27,28] have established the global existence of an energy-conservative weak solution to its initial value problem for initial data of finite energy. We notice that system (1.7) is a particular case of (1.1) for $n = 2$.

In the present paper, we only consider the special case $c_i = c_j$ $(i, j = 1, \dots, n)$ of (1.1). In this case, system (1.1) reduces to

$$u_{ktt} - c(u)\big[c(u)u_{kx}\big]_x = c(u)\sum_{i=1}^{n}(c_{u_i}u_{kx} - c_{u_k}u_{ix})u_{ix} \quad (k = 1, 2, \dots, n), \tag{1.8}$$

where $c : R^n \to R$ is a smooth and positive function satisfying

$$0 < c_1 \leqslant c(z) \leqslant c_2 < \infty \quad \text{and} \quad \sup_z \big|\nabla c(z)\big| < \infty, \quad z \in R^n, \tag{1.9}$$

for positive numbers $c_1 < c_2$. The general case of (1.1) will be considered in a forthcoming paper.

The purpose of this paper is to establish the global well-posedness of the initial value problem for the system of variational wave equations (1.8) for conservative weak solutions. We shall use the method of energy-dependent coordinates used in papers [5,6,27] to construct an energy-conservative solution of (1.8). This method allows us to resolve all singularities by introducing a new set of variables related to the energy. The global smooth solution of the system in the new variables can be obtained by a priori estimates. Going back to the original variables, we thus recover a global weak solution to system (1.8).

We consider (1.8) and (1.9) with the following initial data

$$u_k(0, x) = u_{k0}(x) \in H^1, \qquad u_{kt}(0, x) = u_{k1}(x) \in L^2 \quad (k = 1, 2, \dots, n). \tag{1.10}$$

To deal with the potential breakdown of regularity of solutions, we need to consider weak solutions instead of classical solutions. It can easily be derived that every smooth solution satisfies the conservation of energy

$$\left[\frac{1}{2}\sum_{i=1}^{n}\big(u_{it}^2 + c^2(u)u_{ix}^2\big)\right]_t - \left[c^2(u)\sum_{i=1}^{n}u_{it}u_{ix}\right]_x = 0, \tag{1.11}$$

which implies the existence of finite-energy weak solutions is possible for all time, even after singularities have formed. Before we state our results, let us first recall the definition of weak solutions introduced by Bressan and Zheng [6] (also see Zhang and Zheng [27,28]).

Definition 1 *(Weak solution).* A vector function $u(t, x)$, defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, is a *weak solution* to the Cauchy problem (1.8), (1.10) if the following hold:

(i) In the t–x plane, the functions u_k $(k = 1, 2, \dots, n)$ are locally Hölder continuous with exponent $1/2$. The vector function $t \mapsto u(t, \cdot)$ is continuously differentiable as a map with values in L^θ_{loc}, for all $1 \leqslant \theta < 2$. Moreover, it is Lipschitz continuous with respect to the L^2 distance, that is

$$\sum_{i=1}^{n}\big\|u_i(t, \cdot) - u_i(s, \cdot)\big\|_{L^2} \leqslant L|t - s| \tag{1.12}$$

for all $t, s \in \mathbb{R}$.

(ii) The functions $u_k(t, x)$ $(k = 1, 2, \dots, n)$ take on the initial conditions in (1.10) pointwise, while their temporal derivatives hold in L^θ_{loc} for $\theta \in [1, 2)$.

(iii) Eqs. (1.8) are satisfied in the distributional sense, that is

$$\iint \left[\phi_t u_{kt} - \big(c(u)\phi\big)_x c(u) u_{kx} + \phi c \sum_{i=1}^{n} (c_{u_i} u_{kx} - c_{u_k} u_{ix}) u_{ix} \right] \mathrm{d}x\,\mathrm{d}t = 0 \quad (k = 1, 2, \ldots, n) \tag{1.13}$$

for all test functions $\phi \in C_c^1(\mathbb{R} \times \mathbb{R})$.

Our conclusions can be stated as follows.

Theorem 1.1 *(Existence). Let condition* (1.9) *hold. Then problem* (1.8), (1.10) *has a global weak solution defined for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$.

The continuous dependence of the solution upon the initial data follows directly from the constructive procedure (see Section 3). Moreover, the total energy

$$\mathcal{E}(t) := \frac{1}{2} \int \sum_{i=1}^{n} \big[u_{it}^2(t, x) + c^2\big(u(t, x)\big) u_{ix}^2(t, x) \big] \, \mathrm{d}x \tag{1.14}$$

remains uniformly bounded by its initial level $\mathcal{E}_0 := \mathcal{E}(0)$. More precisely, we have

Theorem 1.2 *(Continuous dependence). A family of weak solutions to the Cauchy problem* (1.8), (1.10) *can be constructed with the additional properties: For every* $t \in \mathbb{R}$ *one has*

$$\mathcal{E}(t) \leqslant \mathcal{E}_0. \tag{1.15}$$

Moreover, let a sequence of initial conditions satisfy

$$\sum_{i=1}^{n} \big\| \big(u_{i0}^{\nu}\big)_x - (u_{i0})_x \big\|_{L^2} \to 0, \qquad \sum_{i=1}^{n} \big\| u_{i1}^{\nu} - u_{i1} \big\|_{L^2} \to 0$$

and $u_{k0}^{\nu} \to u_{k0}$ $(k = 1, 2, \ldots, n)$ *uniformly on compact sets, as* $\nu \to \infty$*. Then one has the convergence of the corresponding solutions* $u^{\nu} \to u$*, uniformly on bounded subsets of the* t–x *plane.*

It seems from (1.15) that the total energy of our solutions may decrease in time. We emphasize, however, that our solutions are conservative, in the following sense.

Theorem 1.3 *(Conservation of energy). There exists a continuous family* $\{\mu_t;\ t \in \mathbb{R}\}$ *of positive Radon measures on the real line with the following properties.*

(i) *At every time* t*, one has* $\mu_t(\mathbb{R}) = \mathcal{E}_0$.
(ii) *For each* t*, the absolutely continuous part of* μ_t *has density* $\frac{1}{2} \sum_{i=1}^{n} (u_{it}^2 + c^2(u) u_{ix}^2)$ *with respect to the Lebesgue measure.*
(iii) *For almost every* $t \in \mathbb{R}$*, the singular part of* μ_t *is concentrated on the set where* $|\nabla c(u)| = 0$.

In other words, the solutions that we obtain are conservative, in the sense that the total energy represented by the measure μ equals a constant, for almost every time. This energy may only be concentrated on a set of times of zero measure or at points where all $c_{u_k}(u)$ $(k = 1, 2, \ldots, n)$ vanish. If $|\nabla c(u)| \neq 0$ for any vector u, then assertion (iii) states that the set $\{t;\ \mathcal{E}(t) < \mathcal{E}_0\}$ has measure zero.

The rest of the paper is organized as follows. In Section 2, we introduce a new set of dependent and independent variables for smooth solutions, and then derive an equivalent semilinear system of (1.8). Section 3 is devoted to establishing the existence and continuous dependence of solutions for the equivalent semilinear system. In Section 4, we show the Hölder continuity of these solutions u in the original independent variables t, x, and verify that the equations in (1.8) are satisfied in the distributional sense. Moreover, the energy inequality (1.15) is demonstrated in Section 5 and the Lipschitz continuity of the map $t \mapsto u(t,\cdot)$, the continuity of the maps $t \mapsto u_t(t,\cdot)$, $t \mapsto u_x(t,\cdot)$ are established in Section 6. These results complete the proofs of Theorems 1.1 and 1.2. Finally, Section 7 is devoted to the proof of Theorem 1.3.

2. An equivalent system

This section is devoted to deriving an equivalent system of (1.8) for smooth solutions by introducing a new set of dependent and independent variables to replace the original variables.

Let

$$\begin{cases} R_k = u_{kt} + c(u)u_{kx}, \\ S_k = u_{kt} - c(u)u_{kx} \end{cases} \quad (k = 1, 2, \ldots, n), \tag{2.1}$$

and denote $R := (R_1, \ldots, R_n)$, $S := (S_1, \ldots, S_n)$. Then (1.8) can be rewritten as

$$\begin{cases} R_{kt} - cR_{kx} = \dfrac{\nabla c}{2c} \cdot R(R_k - S_k) - \dfrac{c_{u_k}}{4c}|R - S|^2, \\ S_{kt} + cS_{kx} = \dfrac{\nabla c}{2c} \cdot S(S_k - R_k) - \dfrac{c_{u_k}}{4c}|S - R|^2, \\ u_{kt} - c(u)u_{kx} = S_k \end{cases} \quad (k = 1, 2, \ldots, n). \tag{2.2}$$

This system is equivalent to (1.8) for smooth solutions if we supplement (2.2) with initial restriction

$$\text{at } t = 0: \quad u_{kx} = \frac{R_k - S_k}{2c(u)} \quad (k = 1, 2, \ldots, n). \tag{2.3}$$

In fact, for

$$F_k := R_k - S_k - 2c(u)u_{kx} \quad (k = 1, 2, \ldots, n), \tag{2.4}$$

we directly compute

$$\begin{aligned} \partial_t F_k - c(u)\partial_x F_k &= \partial_t(R_k - S_k - 2cu_{kx}) - c(u)\partial_x(R_k - S_k - 2cu_{kx}) \\ &= \sum_{i=1}^n \frac{c_{u_i}}{2c}\big[(R_i + S_i)(R_k - S_k) - 4cu_{it}u_{kx}\big] \\ &= \sum_{i=1}^n \frac{c_{u_i}}{2c}\big[(R_i + S_i + 2cu_{ix})F_k + 2cu_{kx}(R_i - S_i) - 2cu_{ix}(R_k - S_k)\big] \\ &= \sum_{i=1}^n \frac{c_{u_i}}{2c}\big[(R_i + S_i)F_k + 2cu_{kx}F_i\big] \quad (k = 1, 2, \ldots, n) \end{aligned} \tag{2.5}$$

which imply, if all F_k $(k = 1, 2, \ldots, n)$ vanish at time zero, that $F_k \equiv 0$ $(k = 1, 2, \ldots, n)$ for all $t > 0$.

Proposition 1. *Any smooth solution of* (1.8) *is a solution to* (2.2), (2.3). *The converse also holds.*

We notice that the equations for u_k $(k=1,2,\dots,n)$ in (2.2) may be replaced by $u_{kt}+c(u)u_{kx}=R_k$ $(k=1,2,\dots,n)$ together with initial compatibility conditions $2u_{kt}=R_k+S_k$ $(k=1,2,\dots,n)$ (at $t=0$). This follows since we have

$$G_{kt}+c(u)G_{kx}=\sum_{i=1}^{n}\frac{c_{u_i}}{2c}\big[(R_i+S_i)G_k-2cu_{kx}G_i\big]\quad (k=1,2,\dots,n) \tag{2.6}$$

for $G_k:=R_k+S_k-2u_{kt}$ $(k=1,2,\dots,n)$.

For convenience to deal with possibly unbounded values of R_k and S_k $(k=1,2,\dots,n)$, we introduce a new set of dependent variables:

$$\begin{aligned}\ell_k&:=\frac{R_k}{1+|R|^2}, & h_1&:=\frac{1}{1+|R|^2},\\ m_k&:=\frac{S_k}{1+|S|^2}, & h_2&:=\frac{1}{1+|S|^2}\end{aligned}\qquad (k=1,2,\dots,n). \tag{2.7}$$

We denote $\ell:=(\ell_1,\dots,\ell_n)$ and $m:=(m_1,\dots,m_n)$ for notational convenience. It is easy to see that there hold

$$h_1^2+|\ell|^2=h_1, \tag{2.8}$$

$$h_2^2+|m|^2=h_2. \tag{2.9}$$

For $k=1,2,\dots,n$, we now compute

$$\begin{aligned}\ell_{kt}-c(u)\ell_{kx}&=h_1^2\left[\big(1+|R|^2\big)(R_{kt}-cR_{kx})-R_k\sum_{j=1}^{n}2R_j(R_{jt}-cR_{jx})\right]\\&=h_1^2\left\{-\frac{c_{u_k}}{4c}\big(1+|R|^2\big)|R-S|^2+\frac{\nabla c}{2c}\cdot R\big[R_k\big(1+|S|^2\big)-S_k\big(1+|R|^2\big)\big]\right\},\end{aligned} \tag{2.10}$$

which, combined with (2.7)–(2.9), leads to

$$\ell_{kt}-c(u)\ell_{kx}=\frac{1}{h_2}\left[\frac{c_{u_k}}{4c}(2h_1h_2-h_1-h_2)+\frac{c_{u_k}}{2c}\ell\cdot m+\frac{\nabla c}{2c}\cdot\ell(\ell_k-m_k)\right]. \tag{2.11}$$

Similarly, we have

$$\begin{aligned}m_{kt}+c(u)m_{kx}&=h_2^2\left[\big(1+|S|^2\big)(S_{kt}+cS_{kx})-S_k\sum_{j=1}^{n}2S_j(S_{jt}+cS_{jx})\right]\\&=h_2^2\left\{-\frac{c_{u_k}}{4c}\big(1+|S|^2\big)|S-R|^2+\frac{\nabla c}{2c}\cdot S\big[S_k\big(1+|R|^2\big)-R_k\big(1+|S|^2\big)\big]\right\},\end{aligned} \tag{2.12}$$

and combining this with (2.7)–(2.9) yields

$$m_{kt}+c(u)m_{kx}=\frac{1}{h_1}\left[\frac{c_{u_k}}{4c}(2h_1h_2-h_1-h_2)+\frac{c_{u_k}}{2c}\ell\cdot m+\frac{\nabla c}{2c}\cdot m(m_k-\ell_k)\right]. \tag{2.13}$$

Furthermore, by a simple calculation we get

$$\begin{cases} h_{1t} - c(u)h_{1x} = \dfrac{\nabla c}{2ch_2} \cdot \ell(h_1 - h_2), \\ h_{2t} + c(u)h_{2x} = \dfrac{\nabla c}{2ch_1} \cdot m(h_2 - h_1). \end{cases} \tag{2.14}$$

We define the forward and backward characteristics as follows:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} x^{\pm}(s;t,x) = \pm c\big(u\big(s; x^{\pm}(s;t,x)\big)\big), \\ x^{\pm}|_{s=t} = x. \end{cases} \tag{2.15}$$

Then, for a point (t,x), we define the energy-dependent coordinates (X,Y):

$$\begin{cases} X := \displaystyle\int_0^{x^-(0;t,x)} \big[1 + \big|R(0,\xi)\big|^2\big]\,\mathrm{d}\xi, \\ Y := \displaystyle\int_{x^+(0;t,x)}^{0} \big[1 + \big|S(0,\xi)\big|^2\big]\,\mathrm{d}\xi, \end{cases} \tag{2.16}$$

which imply that

$$X_t - c(u)X_x = 0, \qquad Y_t + c(u)Y_x = 0. \tag{2.17}$$

Moreover, for any smooth function f, we obtain by making use of (2.17) that

$$\begin{aligned} f_t + c(u)f_x &= f_X X_t + f_Y Y_t + cf_X X_x + cf_Y Y_x = (X_t + cX_x) f_X = 2cX_x f_X, \\ f_t - c(u)f_x &= f_X X_t + f_Y Y_t - cf_X X_x - cf_Y Y_x = (Y_t - cY_x) f_X = -2cY_x f_Y. \end{aligned} \tag{2.18}$$

We now introduce

$$p := \frac{1 + |R|^2}{X_x}, \qquad q := \frac{1 + |S|^2}{-Y_x}. \tag{2.19}$$

From (2.7), we see that

$$\frac{1}{X_x} = \frac{p}{1 + |R|^2} = ph_1, \qquad \frac{1}{-Y_x} = \frac{q}{1 + |S|^2} = qh_2. \tag{2.20}$$

Noting

$$c_x = \sum_{i=1}^{n} c_{u_i} u_{ix} = \frac{\nabla c}{2c} \cdot (R - S), \tag{2.21}$$

and combining (2.7), (2.17) and (2.20), we compute

$$\begin{aligned}
p_t - c(u)p_x &= \frac{1}{X_x}\sum_{j=1}^{n} 2R_j(R_{jt} - cR_{jx}) - \frac{1}{X_x^2}\big(1+|R|^2\big)\big[(X_x)_t - c(X_x)_x\big] \\
&= \frac{1}{X_x}\left[\sum_{j=1}^{n} 2R_j(R_{jt} - cR_{jx}) - \big(1+|R|^2\big)c_x\right] \\
&= ph_1\left\{\frac{\nabla c}{2c}\cdot R\big[2R\cdot(R-S) - |R-S|^2\big] - \frac{\nabla c}{2c}\cdot(R-S)\big(1+|R|^2\big)\right\} \\
&= ph_1\left[\frac{\nabla c}{2c}\cdot S\big(1+|R|^2\big) - \frac{\nabla c}{2c}\cdot R\big(1+|S|^2\big)\right] \\
&= \frac{\nabla c}{2ch_2}\cdot(m-\ell)p. \qquad (2.22)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
q_t + c(u)q_x &= \frac{1}{-Y_x}\sum_{j=1}^{n} 2S_j(S_{jt} + cS_{jx}) - \frac{1}{-Y_x^2}\big(1+|S|^2\big)\big[(-Y_x)_t + c(-Y_x)_x\big] \\
&= \frac{1}{-Y_x}\left[\sum_{j=1}^{n} 2S_j(S_{jt} + cS_{jx}) + \big(1+|S|^2\big)c_x\right] \\
&= qh_2\left\{\frac{\nabla c}{2c}\cdot S\big[2S\cdot(S-R) - |S-R|^2\big] + \frac{\nabla c}{2c}\cdot(R-S)\big(1+|S|^2\big)\right\} \\
&= qh_2\left[\frac{\nabla c}{2c}\cdot R\big(1+|S|^2\big) - \frac{\nabla c}{2c}\cdot S\big(1+|R|^2\big)\right] \\
&= \frac{\nabla c}{2ch_1}\cdot(\ell-m)q. \qquad (2.23)
\end{aligned}$$

Summing up (2.11), (2.13)–(2.14), (2.18) and (2.22)–(2.23), we obtain a semilinear hyperbolic system with smooth coefficients for the variables h_1, h_2, p, q, ℓ_k, m_k, u_k $(k=1,2,\dots,n)$ in (X,Y) coordinates as follows:

$$\begin{cases}
\ell_{kY} = q\left[\dfrac{c_{u_k}}{8c^2}(2h_1h_2 - h_1 - h_2) + \dfrac{c_{u_k}}{4c^2}\ell\cdot m + \dfrac{\nabla c}{4c^2}\cdot\ell(\ell_k - m_k)\right], \\
m_{kX} = p\left[\dfrac{c_{u_k}}{8c^2}(2h_1h_2 - h_1 - h_2) + \dfrac{c_{u_k}}{4c^2}\ell\cdot m + \dfrac{\nabla c}{4c^2}\cdot m(m_k - \ell_k)\right], \\
u_{kX} = \dfrac{p}{2c}\ell_k \quad \left(\text{or } u_{kY} = \dfrac{q}{2c}m_k\right) \quad (k=1,2,\dots,n),
\end{cases} \qquad (2.24)$$

$$\begin{cases}
h_{1Y} = \dfrac{\nabla c}{4c^2}\cdot\ell(h_1 - h_2)q, \\
h_{2X} = \dfrac{\nabla c}{4c^2}\cdot m(h_2 - h_1)p,
\end{cases} \qquad (2.25)$$

$$\begin{cases}
p_Y = \dfrac{\nabla c}{4c^2}\cdot(m-\ell)pq, \\
q_X = \dfrac{\nabla c}{4c^2}\cdot(\ell-m)pq.
\end{cases} \qquad (2.26)$$

We comment that we have $u_{kXY} = u_{kYX}$ $(k = 1, 2, \dots, n)$, so we may use either u_{kX} or u_{kY} $(k = 1, 2, \dots, n)$ in (2.24). In addition one also has

$$p_Y + q_X = 0, \qquad \left(\frac{q}{c}\right)_X - \left(\frac{p}{c}\right)_Y = 0. \tag{2.27}$$

Indeed, thanks to (2.18), we deduce

$$\begin{cases} c_X = \dfrac{c_t + cc_x}{2cX_x} = \dfrac{\nabla c}{2c} \cdot Rph_1 = \dfrac{\nabla c}{2c} \cdot \ell p, \\ c_Y = \dfrac{c_t - cc_x}{-2cY_x} = \dfrac{\nabla c}{2c} \cdot Sqh_2 = \dfrac{\nabla c}{2c} \cdot mq. \end{cases} \tag{2.28}$$

Then we compute

$$\begin{aligned} \left(\frac{q}{c}\right)_X - \left(\frac{p}{c}\right)_Y &= \frac{1}{c^2}(2cq_X + pc_Y - qc_X) \\ &= \frac{1}{c^2}\left[2c\frac{\nabla c}{4c^2} \cdot (\ell - m)pq + p\frac{\nabla c}{2c} \cdot mq - q\frac{\nabla c}{2c} \cdot \ell p\right] = 0. \end{aligned}$$

We notice that the two functions for $\partial_Y h_1$ and $\partial_X h_2$ in (2.25) may seem to be replaceable by the conserved quantities (2.8) and (2.9), but (2.8) and (2.9) do not yield single-valued functions h_1 and h_2 of $|\ell|^2$ and $|m|^2$, respectively. For these reasons, we keep more equations supplemented by conserved quantities rather than fewer equations involving complicated functions.

We next consider the boundary conditions of system (2.24)–(2.26), corresponding to (1.10). The initial line $t = 0$ in the (t, x) plane is transformed to a curve Γ: $Y = \varphi(X)$ defined through a parametric $x \in \mathbb{R}$

$$\begin{cases} X = \displaystyle\int_0^x \left[1 + \left|R(0,\xi)\right|^2\right] \mathrm{d}\xi, \\ Y = \displaystyle\int_x^0 \left[1 + \left|S(0,\xi)\right|^2\right] \mathrm{d}\xi \end{cases} \tag{2.29}$$

which is non-characteristic. The assumptions $u_{k0} \in H^1$, $u_{k1} \in L^2$ $(k = 1, 2, \dots, n)$ imply that $R_k(0, x), S_k(0, x) \in L^2$ $(k = 1, 2, \dots, n)$. We introduce

$$\mathcal{E}_0 := \frac{1}{4}\int \left[\left|R(0,\xi)\right|^2 + \left|S(0,\xi)\right|^2\right] \mathrm{d}\xi < \infty, \tag{2.30}$$

which equals to the number $\mathcal{E}(0)$ in (1.14). Then the two functions $X = X(x)$, $Y = Y(x)$ are well defined and absolutely continuous. Moreover, X is strictly increasing while Y is strictly decreasing. Therefore, the map $X \mapsto \varphi(X)$ is continuous and strictly decreasing. In addition, we also have $|X + \varphi(X)| \leqslant 4\mathcal{E}_0$ from (2.30). The coordinate transformation maps the domain $[0, \infty) \times \mathbb{R}$ in the (t, x) plane into the set

$$\Omega^+ := \left\{(X, Y);\ Y \geqslant \varphi(X)\right\} \tag{2.31}$$

in the (X, Y) plane. Since the curve Γ is parametrized by the parameter x, then we can assign the boundary data $(\bar{\ell}, \bar{m}, \bar{h}_1, \bar{h}_2, \bar{p}, \bar{q}, \bar{u}) \in L^\infty$ defined by

$$\begin{cases} \bar{\ell} = R(0,x)\bar{h}_1, \\ \bar{m} = S(0,x)\bar{h}_2, \end{cases} \qquad \begin{cases} \bar{h}_1 = \dfrac{1}{1+|R(0,x)|^2}, \\ \bar{h}_2 = \dfrac{1}{1+|S(0,x)|^2}, \end{cases} \qquad \begin{cases} \bar{p} = 1, \\ \bar{q} = 1, \end{cases} \qquad \bar{u} = u_0(x), \tag{2.32}$$

where

$$\begin{cases} R(0,x) = u_1(x) + c\big(u_0(x)\big)u_0'(x), \\ S(0,x) = u_1(x) - c\big(u_0(x)\big)u_0'(x), \end{cases} \tag{2.33}$$

along Γ. Furthermore, (2.8) and (2.9) are identically satisfied along Γ. In fact, thanks to (2.24)–(2.26), we deduce

$$\begin{aligned} \partial_Y\big(|\ell|^2 + h_1^2 - h_1\big) &= \sum_{j=1}^n 2\ell_j \ell_{jY} + (2h_1 - 1)h_{1Y} \\ &= 2q\sum_{j=1}^n \ell_j\left[\frac{c_{u_j}}{8c^2}(2h_1h_2 - h_1 - h_2) + \frac{c_{u_j}}{4c^2}\ell\cdot m + \frac{\nabla c}{4c^2}\cdot\ell(\ell_j - m_j)\right] \\ &\quad + \frac{\nabla c}{4c^2}\cdot\ell(2h_1-1)(h_1-h_2)q \\ &= \frac{\nabla c}{2c^2}\cdot\ell q\big[\big(h_1^2 - h_1\big) + \ell\cdot m + \ell\cdot(\ell - m)\big] = \frac{\nabla c}{2c^2}\cdot\ell q\big(|\ell|^2 + h_1^2 - h_1\big) \end{aligned}$$

and

$$\begin{aligned} \partial_X\big(|m|^2 + h_2^2 - h_2\big) &= \sum_{j=1}^n 2m_j m_{jX} + (2h_2 - 1)h_{2X} \\ &= 2p\sum_{j=1}^n m_j\left[\frac{c_{u_j}}{8c^2}(2h_1h_2 - h_1 - h_2) + \frac{c_{u_j}}{4c^2}\ell\cdot m + \frac{\nabla c}{4c^2}\cdot m(m_j - \ell_j)\right] \\ &\quad + \frac{\nabla c}{4c^2}\cdot m(2h_2-1)(h_2-h_1)p \\ &= \frac{\nabla c}{2c^2}\cdot mp\big[\big(h_2^2 - h_2\big) + \ell\cdot m + m\cdot(m - \ell)\big] = \frac{\nabla c}{2c^2}\cdot mp\big[|m|^2 + h_2^2 - h_2\big], \end{aligned}$$

which imply, by the initial data (2.32), that the identities (2.8) and (2.9) hold for the solutions.

3. Solutions of the equivalent system

In this section, we establish the existence of a unique global solution for system (2.24)–(2.26) with boundary data (2.32) in the energy coordinates (X, Y).

Theorem 3.1. *Let conditions* (1.9), (1.10) *hold. Then problem* (2.24)–(2.26) *with boundary data* (2.32) *has a unique global solution defined for all* $(X, Y) \in \mathbb{R}^2$.

We construct below the solution on the domain Ω^+ where $Y \geqslant \varphi(X)$. On the complementary set $R^2 \setminus \Omega^+$, the solution can be constructed in a completely similar way.

We first note that all right-hand side functions in system (2.24)–(2.26) are locally Lipschitz continuous, thus the construction of a local solution is straightforward by the fixed point method. In order to extend this local solution to the entire domain Ω^+, it suffices to establish a priori estimates.

We observe that the conserved quantities (2.8) and (2.9) hold for the solutions, namely,

$$\sum_{i=1}^{n} \ell_i^2 = h_1(1-h_1), \qquad \sum_{i=1}^{n} m_i^2 = h_2(1-h_2). \tag{3.1}$$

Thus h_1, h_2 are bounded between zero and one and the functions ℓ_k, m_k $(k = 1, 2, \dots, n)$ are uniformly bounded. Integrating the first equation of (2.27) over the characteristic triangle with vertex (X, Y),

$$\begin{aligned}\int_{\varphi^{-1}(Y)}^{X} p\big(X', Y\big)\,\mathrm{d}X' + \int_{\varphi(X)}^{Y} q\big(X, Y'\big)\,\mathrm{d}Y' &= X - \varphi^{-1}(Y) + Y - \varphi(X) \\ &\leqslant 2\big(|X| + |Y| + 4\mathcal{E}_0\big),\end{aligned} \tag{3.2}$$

where φ^{-1} denotes the inverse of φ. The second inequality holds by the energy assumption (2.30). Integrating the first equation of (2.26) vertically and making use of (3.2), since $p, q > 0$ from (2.19) and (2.26), we obtain

$$\begin{aligned}p(X, Y) &= \exp\left\{ \int_{\varphi(X)}^{Y} \frac{\nabla c}{4c^2} \cdot (m - \ell) q \,\mathrm{d}Y' \right\} \\ &\leqslant \exp\left\{ C_0 \int_{\varphi(X)}^{Y} q\big(X, Y'\big)\,\mathrm{d}Y' \right\} \\ &\leqslant \exp\big\{2C_0\big(|X| + |Y| + 4\mathcal{E}_0\big)\big\},\end{aligned} \tag{3.3}$$

where C_0 is a finite number depending only on c_1, c_2 and $\sup|\nabla c|$. Similarly, integrating the second equation of (2.26) horizontally, we get

$$\begin{aligned}q(X, Y) &= \exp\left\{ \int_{\varphi^{-1}(Y)}^{X} \frac{\nabla c}{4c^2} \cdot (\ell - m) p \,\mathrm{d}X' \right\} \\ &\leqslant \exp\left\{ C_0 \int_{\varphi^{-1}(Y)}^{X} p\big(X', Y\big)\,\mathrm{d}X' \right\} \\ &\leqslant \exp\big\{2C_0\big(|X| + |Y| + 4\mathcal{E}_0\big)\big\}.\end{aligned} \tag{3.4}$$

The proof of the global existence of system (2.24)–(2.26) with boundary data (2.32) follows from the local bounds (3.3) and (3.4). We omit the details since they are very similar to those in Bressan and Zheng [6].

For future reference, we here state a useful consequence of the above construction.

Corollary 1. *Let* (1.9) *hold. If the initial data* (u_{k0}, u_{k1}) $(k = 1, 2, \dots, n)$ *are smooth, then the solution of* (2.24)–(2.26), (2.32) *is a smooth function of the variables* (X, Y)*. Moreover, assume that a sequence of smooth functions* $(u^\nu_{k0}, u^\nu_{k1})_{\nu\geqslant 1}$ $(k = 1, 2, \dots, n)$ *satisfies*

$$u^\nu_{k0} \to u_{k0}, \qquad \left(u^\nu_{k0}\right)_x \to (u_{k0})_x, \qquad u^\nu_{k1} \to u_{k1} \quad (k = 1, 2, \dots, n)$$

uniformly on compact subsets of $\mathbb{R}$*. Then one has the convergence of the corresponding solutions*:

$$\left(u^\nu, \ell^\nu, m^\nu, h^\nu_1, h^\nu_2, p^\nu, q^\nu\right) \to (u, \ell, m, h_1, h_2, p, q),$$

uniformly on bounded subsets of the X–Y *plane.*

4. Solutions in the original variables

In the present section, we prove that the solution in the X–Y plane can be expressed by the original variables (t, x). Moreover, we also prove that the solution in the original variables is Hölder continuous and satisfies (1.8) in the distributional sense.

Since the initial data $(u_{k0})_x$ and u_{k1} $(k = 1, 2, \dots, n)$ are assumed only to be in L^2, we first examine from system (2.24)–(2.26) that the regularity of the solution is as follows:

- The functions ℓ_k $(k = 1, 2, \dots, n)$, h_1, p are Lipschitz continuous w.r.t. Y, measurable w.r.t. X.
- The functions m_k $(k = 1, 2, \dots, n)$, h_2, q are Lipschitz continuous w.r.t. X, measurable w.r.t. Y.
- The functions u_k $(k = 1, 2, \dots, n)$ are Lipschitz continuous w.r.t. both X and Y, on bounded subsets of the X–Y plane.

In order to define u_k $(k = 1, 2, \dots, n)$ as functions of the original variables t, x, we need the inverse functions $X = X(t, x)$, $Y = Y(t, x)$. The map $(X, Y) \mapsto (t, x)$ can be constructed as follows. Due to (2.18) and (2.20), we obtain

$$\begin{cases} x_X = \dfrac{1}{2X_x} = \dfrac{1}{2} p h_1, \\ x_Y = \dfrac{1}{2Y_x} = -\dfrac{1}{2} q h_2 \end{cases} \tag{4.1}$$

and

$$\begin{cases} t_X = \dfrac{1}{2cX_x} = \dfrac{1}{2c} p h_1, \\ t_Y = \dfrac{1}{-2cY_x} = \dfrac{1}{2c} q h_2. \end{cases} \tag{4.2}$$

For future reference, we write here the partial derivatives of the inverse mapping, valid at points where $h_1, h_2 \neq 0$,

$$X_x = \frac{1}{p h_1}, \qquad Y_x = -\frac{1}{q h_2}, \qquad X_t = \frac{c}{p h_1}, \qquad Y_t = \frac{c}{q h_2}. \tag{4.3}$$

We use (2.25)–(2.26) and (2.28) to compute

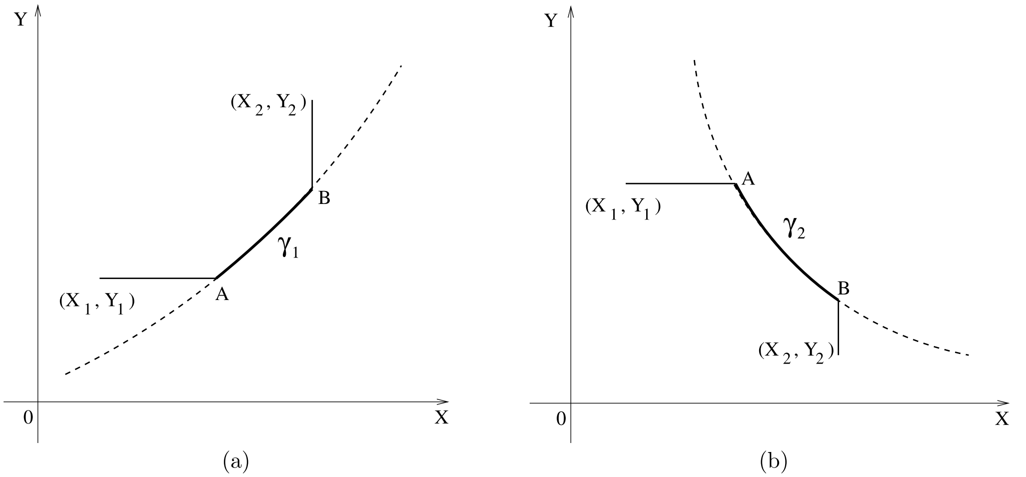

Fig. 1. Paths of integration γ_1, γ_2. The left figure (a) corresponds to the case $X_1 \leqslant X_2$, $Y_1 \leqslant Y_2$, and the right figure (b) to the case $X_1 \leqslant X_2$, $Y_1 \geqslant Y_2$. The dashed curves in figures (a) and (b) are, respectively, the boundaries ∂D_{x^*} and ∂D_{t^*}.

$$\begin{aligned} x_{XY} &= \frac{1}{2}(h_1 p_Y + p h_{1Y}) \\ &= \frac{\nabla c}{8c^2}\cdot \ell(h_1-h_2)pq + \frac{\nabla c}{8c^2}\cdot (m-\ell)h_1 pq = \frac{\nabla c}{8c^2}\cdot (h_1 m - h_2\ell)pq \\ &= \frac{\nabla c}{8c^2}\cdot (m-\ell)h_2 pq - \frac{\nabla c}{8c^2}\cdot m(h_2-h_1)pq = \frac{1}{2}(-h_2 q_X - q h_{2}X) = x_{YX} \end{aligned}$$

and

$$\begin{aligned} t_{XY} - t_{YX} &= \left(\frac{x_X}{c}\right)_Y + \left(\frac{x_Y}{c}\right)_X = \frac{2}{c}x_{XY} - \frac{1}{c^2}(x_X c_Y + x_Y c_X) \\ &= \frac{\nabla c}{4c^3}\cdot (h_1 m - h_2 \ell)pq - \frac{1}{c^2}\left[\frac{\nabla c}{4c^2}\cdot m h_1 pq - \frac{\nabla c}{4c^2}\cdot \ell h_2 qp\right] = 0. \end{aligned}$$

Therefore, the functions $x = x(X,Y)$ and $t = t(X,Y)$ can be obtained by integrating one of the equations in (4.1) and (4.2), respectively. We note that the map constructed above may not be one-to-one mapping. This fact, however, does not cause any real difficulty. Indeed, we need only prove the claim that the values of u do not depend on the choice of (X,Y). That is because if it holds then, for each given point (t^*, x^*), we can choose an arbitrary point (X^*, Y^*) such that $t(X^*, Y^*) = t^*$, $x(X^*, Y^*) = x^*$, and define $u(t^*, x^*) := u(X^*, Y^*)$. To prove this claim, assume (X_1, Y_1) and (X_2, Y_2) are two distinct points such that $t(X_1, Y_1) = t(X_2, Y_2) = t^*$ and $x(X_1, Y_1) = x(X_2, Y_2) = x^*$, we need to show $u(X_1, Y_1) = u(X_2, Y_2)$. Consider two cases:

Case 1. $X_1 \leqslant X_2, Y_1 \leqslant Y_2$. Consider the set

$$D_{x^*} := \{(X,Y);\ x(X,Y) \leqslant x^*\}$$

and denote by ∂D_{x^*} its boundary. Since x is increasing with X and decreasing with Y, this boundary can be represented as the graph of a Lipschitz continuous function: $X - Y = \phi(X+Y)$. We now construct a Lipschitz continuous curve γ_1 (Fig. 1(a)) consisting of the following:

– a horizontal segment joining (X_1, Y_1) with a point $A = (X_A, Y_A)$ on ∂D_{x^*} with $Y_A = Y_1$,
– a portion of the boundary ∂D_{x^*},
– a vertical segment joining (X_2, Y_2) with a point $B = (X_B, Y_B)$ on ∂D_{x^*} with $X_B = X_2$.

From (4.1)–(4.2) and the assumptions $t(X_1, Y_1) = t(X_2, Y_2) = t^*$, $x(X_1, Y_1) = x(X_2, Y_2) = x^*$, we obtain that $t \equiv t^*$ and $x \equiv x^*$ on the curve γ_1. Then, along γ, we have

$$0 = \mathrm{d}x = x_X\,\mathrm{d}X + x_Y\,\mathrm{d}Y = \frac{1}{2}ph_1\,\mathrm{d}X - \frac{1}{2}qh_2\,\mathrm{d}Y,$$
$$0 = \mathrm{d}t = t_X\,\mathrm{d}X + t_Y\,\mathrm{d}Y = \frac{1}{2c}ph_1\,\mathrm{d}X + \frac{1}{2c}qh_2\,\mathrm{d}Y$$

hold almost everywhere. Hence, $h_1\,\mathrm{d}X = h_2\,\mathrm{d}Y = 0$, which means that $\ell_k\,\mathrm{d}X = m_k\,\mathrm{d}Y = 0$ ($k = 1, 2, \ldots, n$). For $k = 1, 2, \ldots, n$, we now compute

$$\begin{aligned} u_k(X_2, Y_2) - u_k(X_1, Y_1) &= \int_\gamma (u_{kX}\,\mathrm{d}X + u_{kY}\,\mathrm{d}Y) \\ &= \int_\gamma \frac{p}{2c}\ell_k\,\mathrm{d}X + \frac{q}{2c}m_k\,\mathrm{d}Y = 0, \end{aligned}$$

which concludes our claim.

Case 2. $X_1 \leqslant X_2$, $Y_1 \geqslant Y_2$. In this case, we consider the set

$$D_{t^*} := \big\{(X, Y);\ t(X, Y) \leqslant t^*\big\},$$

and construct a curve γ_2 connecting (X_1, Y_1) with (X_2, Y_2) as in Fig. 1(b). Details are entirely similar to Case 1.

We next prove that the vector function $u(t, x) = u(X(t, x), Y(t, x))$ thus obtained is Hölder continuous on bounded sets. For $k = 1, 2, \ldots, n$, integrating along any forward characteristic $t \mapsto x^+(t)$ and noticing $Y = const.$ on this kind of characteristics, we get

$$\begin{aligned} \int_0^\tau \big[u_{kt} + c(u)u_{kx}\big]^2\,\mathrm{d}t &= \int_{X_0}^{X_\tau} (2cX_x u_{kX})^2 (2cX_x)^{-1}\,\mathrm{d}X = \int_{X_0}^{X_\tau} \frac{p}{2ch_1}\ell_k^2\,\mathrm{d}X \\ &\leqslant \int_{X_0}^{X_\tau} \frac{p}{2ch_1}|\ell|^2\,\mathrm{d}X = \int_{X_0}^{X_\tau} \frac{1}{2c}p(1-h_1)\,\mathrm{d}X \\ &\leqslant \int_{X_0}^{X_\tau} \frac{1}{2c}p\,\mathrm{d}X \leqslant C_\tau, \end{aligned} \tag{4.4}$$

for some constant C_τ depending only on τ. Similarly, integrating along any backward characteristic $t \mapsto x^-(t)$ and noticing $X = const.$ on this kind of characteristics, we obtain

$$\int_0^\tau \left[u_{kt} - c(u)u_{kx}\right]^2 \mathrm{d}t \leqslant C_\tau \quad (k = 1, 2, \dots, n). \tag{4.5}$$

Thanks to (1.9), the bounds (4.4) and (4.5) imply that the vector function $u = u(t, x)$ is Hölder continuous with exponent $1/2$. Due to (2.15) we obtain all characteristic curves are C^1 with Hölder continuous derivative. Moreover, the functions R_k, S_k $(k = 1, 2, \dots, n)$ at (2.1) are square integrable on bounded subsets of the t–x plane. In addition, notice that

$$u_{kt} + c(u)u_{kx} = 2cX_x u_{kX} = 2c\frac{1}{ph_1} \cdot \frac{p}{2c}\ell_k = \frac{\ell_k}{h_1} = R_k$$

and

$$u_{kt} - c(u)u_{kx} = -2cY_x u_{kY} = -2c\left(-\frac{1}{qh_2}\right) \cdot \frac{q}{2c}m_k = \frac{m_k}{h_2} = S_k$$

for $k = 1, 2, \dots, n$, which indicate that the functions R_k, S_k $(k = 1, 2, \dots, n)$ at (2.1) are indeed the same as recovered from (2.7).

Finally, we prove that the vector function $u = u(t, x)$ satisfies (1.8) in the distributional sense. According to (1.13), we need to show that for $k = 1, 2, \dots, n$,

$$\begin{aligned}
0 &= \iint \Bigg\{\phi_t\left[(u_{kt} + cu_{kx}) + (u_{kt} - cu_{kx})\right] - (c\phi)_x\left[(u_{kt} + cu_{kx}) - (u_{kt} - cu_{kx})\right] \\
&\quad + 2\phi c\sum_{i=1}^n (c_{u_i}u_{kx} - c_{u_k}u_{ix})u_{ix}\Bigg\}\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint \left\{R_k\left[\phi_t - (c\phi)_x\right] + S_k\left[\phi_t + (c\phi)_x\right] + 2\phi c\sum_{i=1}^n (c_{u_i}u_{kx} - c_{u_k}u_{ix})u_{ix}\right\}\mathrm{d}x\,\mathrm{d}t \\
&= \iint \left\{-2cR_kY_x\phi_Y + 2cS_kX_x\phi_X + \phi\left[(S_k - R_k)c_x + 2c\sum_{i=1}^n (c_{u_i}u_{kx} - c_{u_k}u_{ix})u_{ix}\right]\right\}\mathrm{d}x\,\mathrm{d}t \\
&= \iint \left[-2cR_kY_x\phi_Y + 2cS_kX_x\phi_X - \phi\frac{c_{u_k}}{2c}|R - S|^2\right]\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.6}$$

The third identity holds by (2.18). We notice that, from (4.1) and (4.2),

$$\mathrm{d}x\,\mathrm{d}t = \frac{pq}{2c}h_1h_2\,\mathrm{d}X\,\mathrm{d}Y.$$

Therefore, one can rewrite the double integral in (4.6) as

$$\begin{aligned}
&\iint \left[2c\frac{\ell_k}{qh_1h_2}\phi_Y + 2c\frac{m_k}{ph_1h_2}\phi_X - \phi\frac{c_{u_k}}{2c}\sum_{i=1}^n\left(\frac{\ell_i}{h_1} - \frac{m_i}{h_2}\right)^2\right] \cdot \frac{pq}{2c}h_1h_2\,\mathrm{d}X\,\mathrm{d}Y \\
&\quad = \iint \left\{p\ell_k\phi_Y + qm_k\phi_X + \phi pq\left[\frac{c_{u_k}}{4c^2}(2h_1h_2 - h_1 - h_2) + \frac{c_{u_k}}{2c^2}\ell \cdot m\right]\right\}\mathrm{d}X\,\mathrm{d}Y.
\end{aligned} \tag{4.7}$$

By direct calculation, we find that

$$\begin{aligned}(p\ell_k)_Y + (qm_k)_X &= p\ell_{kY} + \ell_k p_Y + qm_{kX} + m_k q_X \\ &= pq\left[\frac{c_{u_k}}{4c^2}(2h_1h_2 - h_1 - h_2) + \frac{c_{u_k}}{2c^2}\ell\cdot m\right] \\ &\quad + pq\sum_{i=1}^{n}\frac{c_{u_i}}{4c^2}\big[\ell_i(\ell_k - m_k) + \ell_k(m_i - \ell_i) + m_i(m_k - \ell_k) + m_k(\ell_i - m_i)\big] \\ &= pq\left[\frac{c_{u_k}}{4c^2}(2h_1h_2 - h_1 - h_2) + \frac{c_{u_k}}{2c^2}\ell\cdot m\right],\end{aligned} \tag{4.8}$$

from which and (4.7) we obtain (4.6) holds. Thus, the integral equations (1.13) hold for every test function $\phi \in C_c^1$.

5. Upper bound on energy

This section is devoted to completing the proof of Theorem 1.2. We establish the energy inequality (1.15) by converting the energy conservation (1.11) formally to the (X, Y) plane.

Using the variables R_k and S_k ($k = 1, 2, \ldots, n$), one can rewrite (1.11) as

$$\left[\frac{1}{4}\big(|R|^2 + |S|^2\big)\right]_t + \left[\frac{1}{4}c(u)\big(|S|^2 - |R|^2\big)\right]_x = 0, \tag{5.1}$$

which, combined with (2.7)–(2.9), gives

$$\left(\frac{1}{4h_1} + \frac{1}{4h_2} - \frac{1}{2}\right)_t + \left[\frac{c}{4}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\right]_x = 0,$$

which means that the 1-form

$$\left(\frac{1}{4h_1} + \frac{1}{4h_2} - \frac{1}{2}\right)\mathrm{d}x - \left[\frac{c}{4}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\right]\mathrm{d}t \tag{5.2}$$

is closed. Making use of the formula

$$\begin{cases} \mathrm{d}t = t_X\,\mathrm{d}X + t_Y\,\mathrm{d}Y = \dfrac{1}{2c}ph_1\,\mathrm{d}X + \dfrac{1}{2c}qh_2\,\mathrm{d}Y, \\ \mathrm{d}x = x_X\,\mathrm{d}X + x_Y\,\mathrm{d}Y = \dfrac{1}{2}ph_1\,\mathrm{d}X - \dfrac{1}{2}qh_2\,\mathrm{d}Y, \end{cases} \tag{5.3}$$

the expression (5.2) can be reduced to

$$\frac{1-h_1}{4}p\,\mathrm{d}X - \frac{1-h_2}{4}q\,\mathrm{d}Y, \tag{5.4}$$

which is also closed, in the X–Y plane. It follows from a direct calculation

$$\begin{aligned}\left(\frac{1-h_1}{4}p\right)_Y &= \frac{\nabla c}{16c^2}\cdot\big[(1-h_1)m - (1-h_2)\ell\big]pq \\ &= -\left(\frac{1-h_2}{4}q\right)_X.\end{aligned} \tag{5.5}$$

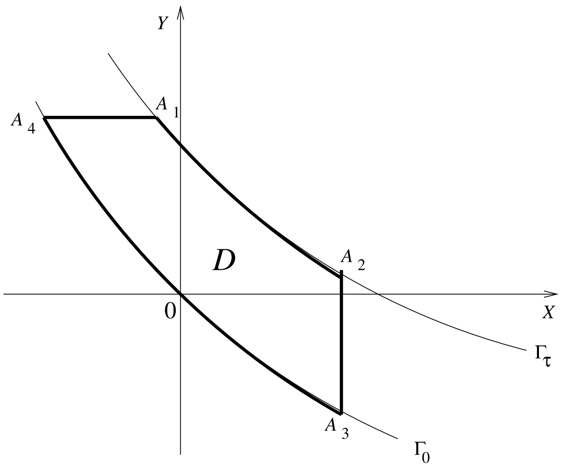

Fig. 2. The region D and its four vertices A_1, A_2, A_3 and A_4.

The solution $u = u(X, Y)$ constructed in Section 3 is conservative, in the sense that the integral of the form (5.4) along every Lipschitz continuous, closed curve in the X–Y plane is zero.

We now use the above fact to establish the energy inequality (1.15). Fix any $\tau > 0$, the case $\tau < 0$ is analogous. For a given $r \gg 1$, define the set

$$D := \{(X, Y);\ 0 \leqslant t(X, Y) \leqslant \tau,\ X \leqslant r,\ Y \leqslant r\}. \tag{5.6}$$

See Fig. 2. By construction, the map $(X, Y) \mapsto (t, x)$ will act as follows:

$$A_1 \mapsto (\tau, a_1), \qquad A_2 \mapsto (\tau, a_2), \qquad A_3 \mapsto (0, a_3), \qquad A_4 \mapsto (0, a_4),$$

for some $a_1 < a_2$ and $a_4 < a_3$. Integrating the 1-form (5.4) along the boundary of D, we find that

$$\begin{aligned}
&\int_{A_1A_2} \frac{1-h_1}{4} p\,\mathrm{d}X - \frac{1-h_2}{4} q\,\mathrm{d}Y \\
&\quad = \int_{A_4A_3} \frac{1-h_1}{4} p\,\mathrm{d}X - \frac{1-h_2}{4} q\,\mathrm{d}Y - \int_{A_4A_1} \frac{1-h_1}{4} p\,\mathrm{d}X - \int_{A_3A_2} \frac{1-h_2}{4} q\,\mathrm{d}Y \\
&\quad \leqslant \int_{A_4A_3} \frac{1-h_1}{4} p\,\mathrm{d}X - \frac{1-h_2}{4} q\,\mathrm{d}Y \\
&\quad = \int_{a_4}^{a_3} \frac{1}{2} \sum_{i=1}^{n} \left[u_{it}^2(0, x) + c^2\big(u(0, x)\big) u_{ix}^2(0, x)\right] \mathrm{d}x,
\end{aligned} \tag{5.7}$$

where the last relation holds by using the fact that the variables h_1, h_2 never assume the value zero at the initial time. On the other hand, we compute by using (5.3) to obtain

$$\int_{a_1}^{a_2} \frac{1}{2} \sum_{i=1}^{n} \left[u_{it}^2(\tau, x) + c^2\big(u(\tau, x)\big) u_{ix}^2(\tau, x)\right] \mathrm{d}x$$

$$= \int_{A_1A_2\cap\{h_1\neq 0\}} \frac{1-h_1}{4} p\,\mathrm{d}X - \int_{A_1A_2\cap\{h_2\neq 0\}} \frac{1-h_2}{4} q\,\mathrm{d}Y$$
$$\leqslant \mathcal{E}_0. \tag{5.8}$$

Letting $r \to +\infty$ in (5.6), one has $a_1 \to -\infty$, $a_2 \to +\infty$. Therefore, combining (5.7) and (5.8), we get $\mathcal{E}(t) \leqslant \mathcal{E}_0$, this proves (1.15).

6. Regularity of trajectories

In this section, we shall show that the vector function $t \mapsto u(t,\cdot)$ is Lipschitz continuous in the L^2 distance and is continuously differentiable as a map with values in L^θ_{loc}, for all $1 \leqslant \theta < 2$. These results will complete the proof of Theorem 1.1.

We now establish the Lipschitz continuity of the vector function $t \mapsto u(t,\cdot)$ in the L^2 distance, that is, (1.12) holds. For any $t, s \in \mathbb{R}$, we have

$$u_k(t,x) - u_k(s,x) = (t-s)\int_0^1 u_{kt}\big(s+\xi(t-s), x\big)\,\mathrm{d}\xi$$

for $k = 1, 2, \ldots, n$. Thus

$$\sum_{i=1}^n \big\|u_i(t,x) - u_i(s,x)\big\|_{L^2} \leqslant |t-s| \int_0^1 \sum_{i=1}^n \big\|u_{it}\big(s+\tau(t-s),\cdot\big)\big\|_{L^2}\,\mathrm{d}\tau$$
$$\leqslant \sqrt{2\mathcal{E}_0}|t-s|. \tag{6.1}$$

Next we prove that the functions $t \mapsto u_{kt}(t,\cdot)$ and $t \mapsto u_{kx}(t,\cdot)$ $(k = 1, 2, \ldots, n)$ are continuous with values in L^θ, which imply the vector function $t \mapsto u(t,\cdot)$ is continuously differentiable as a map with values in L^θ_{loc}, for all $1 \leqslant \theta < 2$.

Let us first establish the argument for smooth initial data. In this case, the solution $u = u(X,Y)$ is a smooth vector function on the entire X–Y plane. Fix a time τ. We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_k(t,\cdot)|_{t=\tau} = u_{kt}(\tau,\cdot), \tag{6.2}$$

where

$$u_{kt}(\tau,\cdot) := u_{kX}X_t + u_{kY}Y_t = \frac{p}{2c}\ell_k \cdot \frac{c}{ph_1} + \frac{q}{2c} m_k \cdot \frac{c}{qh_2} = \frac{\ell_k}{2h_1} + \frac{m_k}{2h_2} \tag{6.3}$$

for $k = 1, 2, \ldots, n$. Notice that (6.3) defines the value of $u_{kt}(\tau,\cdot)$ at almost every point $x \in \mathbb{R}$. By the energy inequality (1.15), we have

$$\int_{\mathbb{R}} \big|u_{kt}(\tau,x)\big|^2\,\mathrm{d}x \leqslant 2\mathcal{E}(t) \leqslant 2\mathcal{E}_0. \tag{6.4}$$

In order to establish the relations (6.2), we consider the set

$$D_\tau := \big\{(X,Y) \,\big|\, t(X,Y) \leqslant \tau\big\}, \tag{6.5}$$

and denote its boundary by Γ_τ. Given $\varepsilon > 0$, due to the energy inequality (1.15), there then exist finitely many disjoint intervals $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2, \dots, N$, such that

$$\min\{h_1(P), h_2(P)\} < 2\varepsilon \tag{6.6}$$

for every point $P = (X(x_P, \tau), Y(x_P, \tau))$ ($x_P \in J := \bigcup_{i=1}^{N} [a_i, b_i]$) and

$$h_1(Q) > \varepsilon, \qquad h_2(Q) > \varepsilon \tag{6.7}$$

for every point $Q = (X(x_Q, \tau), Y(x_Q, \tau))$ ($x_Q \in J' := \mathbb{R} \setminus J$). Noticing the function $u_k = u_k(t, x)$ is smooth in a neighborhood of the set $\{\tau\} \times J'$ and using Minkowski's inequality, we get

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}\Bigg[\int_{\mathbb{R}} \big|u_k(\tau+h, x) - u_k(\tau, x) - h u_{kt}(\tau, x)\big|^\theta \,\mathrm{d}x\Bigg]^{\frac{1}{\theta}} \\
&\quad\leqslant \lim_{h\to 0} \frac{1}{h}\Bigg[\int_J \big|u_k(\tau+h, x) - u_k(\tau, x)\big|^\theta \,\mathrm{d}x\Bigg]^{\frac{1}{\theta}} + \Bigg[\int_J \big|u_{kt}(\tau, x)\big|^\theta \,\mathrm{d}x\Bigg]^{\frac{1}{\theta}}.
\end{aligned} \tag{6.8}$$

Making use of (4.1) and (6.6), we estimate the measure of the "bad" set J

$$\begin{aligned}
\operatorname{meas}(J) &= \int_J \mathrm{d}x = \sum_{i=1}^{N} \int_{(X_{a_i}, Y_{a_i})}^{(X_{b_i}, Y_{b_i})} \frac{1}{2} p h_1 \,\mathrm{d}X - \frac{1}{2} q h_2 \,\mathrm{d}Y \\
&\leqslant \frac{2\varepsilon}{1-2\varepsilon} \sum_{i=1}^{N} \int_{(X_{a_i}, Y_{a_i})}^{(X_{b_i}, Y_{b_i})} \frac{1-h_1}{2} p \,\mathrm{d}X - \frac{1-h_2}{2} q \,\mathrm{d}Y \\
&\leqslant \frac{4\varepsilon}{1-2\varepsilon} \int_{\Gamma_\tau} \frac{1-h_1}{4} p \,\mathrm{d}X - \frac{1-h_2}{4} q \,\mathrm{d}Y \leqslant \frac{4\varepsilon}{1-2\varepsilon} \mathcal{E}_0,
\end{aligned} \tag{6.9}$$

where $(X_{a_i}, Y_{a_i}) = (X(a_i, \tau), Y(a_i, \tau))$ and $(X_{b_i}, Y_{b_i}) = (X(b_i, \tau), Y(b_i, \tau))$. Using Hölder's inequality with conjugate exponents $2/\theta$ and $\kappa := 2/(2-\theta)$, and recalling (6.1), we obtain

$$\begin{aligned}
\int_J \big|u_k(\tau+h, x) - u_k(\tau, x)\big|^\theta \,\mathrm{d}x &\leqslant \operatorname{meas}(J)^{\frac{1}{\kappa}} \Bigg(\int_J \big|u_k(\tau+h, x) - u_k(\tau, x)\big|^2 \,\mathrm{d}x\Bigg)^{\frac{\theta}{2}} \\
&\leqslant \bigg(\frac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\bigg)^{\frac{1}{\kappa}} \big\|u_k(\tau+h, \cdot) - u_k(\tau, \cdot)\big\|_{L^2}^\theta \\
&\leqslant \bigg(\frac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\bigg)^{\frac{1}{\kappa}} \big(h\sqrt{2\mathcal{E}_0}\big)^\theta = 2\bigg(\frac{2\varepsilon}{1-2\varepsilon}\bigg)^{\frac{1}{\kappa}} h^\theta \mathcal{E}_0.
\end{aligned}$$

Thus,

$$\limsup_{h\to 0}\frac{1}{h}\left(\int_J \left|u_k(\tau+h,x)-u_k(\tau,x)\right|^\theta \,\mathrm{d}x\right)^{\frac{1}{\theta}} \leqslant \left(\frac{2\varepsilon}{1-2\varepsilon}\right)^{\frac{1}{\kappa\theta}}(2\mathcal{E}_0)^{\frac{1}{\theta}}. \tag{6.10}$$

Similar argument leads to

$$\begin{aligned}\left(\int_J \left|u_{kt}(\tau,x)\right|^\theta \,\mathrm{d}x\right)^{\frac{1}{\theta}} &\leqslant \operatorname{meas}(J)^{\frac{1}{\kappa\theta}}\left(\int_J \left|u_{kt}(\tau,x)\right|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant \left(\frac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\right)^{\frac{1}{\kappa\theta}}(2\mathcal{E}_0)^{\frac{1}{2}} = \left(\frac{2\varepsilon}{1-2\varepsilon}\right)^{\frac{1}{\kappa\theta}}(2\mathcal{E}_0)^{\frac{1}{\theta}}.\end{aligned} \tag{6.11}$$

Combining (6.8), (6.10) and (6.11), and noticing $\varepsilon > 0$ is arbitrary, we conclude

$$\lim_{h\to 0}\frac{1}{h}\left(\int_{\mathbb{R}} \left|u_k(\tau+h,x)-u_k(\tau,x)-hu_{kt}(\tau,x)\right|^\theta \,\mathrm{d}x\right)^{\frac{1}{\kappa}} = 0 \tag{6.12}$$

for $k = 1,2,\dots,n$. The proofs of continuity of the functions $t \mapsto u_{kt}(t,\cdot)$ $(k = 1,2,\dots,n)$ are similar. Fix $\varepsilon > 0$. Consider the intervals $[a_i, b_i]$ as before. For $k = 1,2,\dots,n$, noticing that the function $u_k = u_k(t,x)$ is smooth on a neighborhood of $\{\tau\}\times J'$, we have

$$\begin{aligned}&\limsup_{h\to 0}\int \left|u_{kt}(\tau+h,x)-u_{kt}(\tau,x)\right|^\theta \,\mathrm{d}x \\ &\quad\leqslant \limsup_{h\to 0}\int_J \left|u_{kt}(\tau+h,x)-u_{kt}(\tau,x)\right|^\theta \,\mathrm{d}x \\ &\quad\leqslant \limsup_{h\to 0}\operatorname{meas}(J)^{\frac{1}{\kappa}}\left(\int_J \left|u_{kt}(\tau+h,x)-u_{kt}(\tau,x)\right|^2\right)^{\frac{\theta}{2}} \mathrm{d}x \\ &\quad\leqslant \limsup_{h\to 0}\left(\frac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\right)^{\frac{1}{\kappa}}\left(\left\|u_{kt}(\tau+h,x)\right\|_{L^2}+\left\|u_{kt}(\tau,x)\right\|_{L^2}\right)^\theta \\ &\quad\leqslant \left(\frac{4\varepsilon}{1-2\varepsilon}\mathcal{E}_0\right)^{\frac{1}{\kappa}}(4\mathcal{E}_0)^\theta,\end{aligned}$$

which completes the proof by the arbitrariness of ε.

For general initial data $(u_{k0}), u_{k1} \in L^2$ $(k = 1,2,\dots,n)$, we let $\{(u^\nu_{k0})_x\}, \{u^\nu_{k1}\} \in C^\infty_c$ $(k = 1,2,\dots,n)$ be a sequence of smooth initial data such that $u^\nu_{k0} \to u_{k0}$ $(k = 1,2,\dots,n)$ uniformly, $(u^\nu_{k0})_x \to (u_{k0})_x$ $(k = 1,2,\dots,n)$ almost everywhere and in L^2, $u^\nu_{k1} \to u_{k1}$ $(k = 1,2,\dots,n)$ almost everywhere and in L^2, and finish the proof by Corollary 1.

The continuity of the functions $t \to u_{kx}(t,\cdot)$ $(k = 1,2,\dots,n)$ as maps with values in L^θ, $1 \leqslant \theta < 2$, can be established by the same method.

7. Energy conservation

This section is devoted to the proof of Theorem 1.3, that is, we show that the total energy of the solution remains constant in time in some sense.

We complete our analysis by using the tool of the *wave interaction potential*. For any fixed time τ, we let $\mu_\tau = \mu_\tau^- + \mu_\tau^+$ be the positive measure on the real line defined as follows. Given any open interval (a,b), let $A = (X_A, Y_A)$ and $B = (X_B, Y_B)$ be the points on Γ_τ (the boundary of D_τ) such that

$$x(A) = a, \qquad X_P - Y_P \leqslant X_A - Y_A \quad \text{for every point } P \in \Gamma_\tau \text{ with } x(P) \leqslant a,$$
$$x(B) = b, \qquad X_P - Y_P \geqslant X_B - Y_B \quad \text{for every point } P \in \Gamma_\tau \text{ with } x(P) \geqslant b.$$

Then we have

$$\mu_\tau\big((a,b)\big) = \mu_\tau^-\big((a,b)\big) + \mu_\tau^+\big((a,b)\big), \tag{7.1}$$

where

$$\mu_\tau^-\big((a,b)\big) := \int_{AB} \frac{1-h_1}{4} p \,\mathrm{d}X, \qquad \mu_\tau^+\big((a,b)\big) := -\int_{AB} \frac{1-h_2}{4} q \,\mathrm{d}Y. \tag{7.2}$$

For all τ, it is easily seen that μ_τ^-, μ_τ^+ are bounded, positive measures, and $\mu_\tau(\mathbb{R}) = \mathcal{E}_0$. We define the wave interaction potential $\Lambda(t)$ by

$$\Lambda(t) := \big(\mu_t^- \otimes \mu_t^+\big)\big\{(x,y);\ x > y\big\}. \tag{7.3}$$

Notice that in the smooth case, (7.2) and (7.3) are, respectively, equivalent to

$$\mu_\tau^-\big((a,b)\big) := \frac{1}{4}\int_a^b \big|R(\tau,x)\big|^2 \,\mathrm{d}x, \qquad \mu_\tau^+\big((a,b)\big) := \frac{1}{4}\int_a^b \big|S(\tau,x)\big|^2 \,\mathrm{d}x,$$

and

$$\Lambda(t) := \frac{1}{16} \iint_{x>y} \big|R(t,x)\big|^2 \big|S(t,y)\big|^2 \,\mathrm{d}x \,\mathrm{d}y.$$

Lemma 1 *(Bounded variation). The map $t \to \Lambda(t)$ has locally bounded variation; that is, there exists a one-sided Lipschitz constant L_0 such that*

$$\Lambda(t) - \Lambda(s) \leqslant L_0 \cdot (t-s), \quad t > s > 0.$$

We first consider the case that the solution is smooth. From (2.2) we obtain

$$\begin{cases} \big(|R|^2\big)_t - \big(c|R|^2\big)_x = \dfrac{\nabla c}{2c} \cdot \big(|R|^2 S - |S|^2 R\big), \\ \big(|S|^2\big)_t + \big(c|S|^2\big)_x = \dfrac{\nabla c}{2c} \cdot \big(|S|^2 R - |R|^2 S\big). \end{cases}$$

Differentiating $\Lambda(t)$ with respect to time we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[16\Lambda(t)\big] &\leqslant -\int 2c|R|^2|S|^2\,\mathrm{d}x + \int\big(|R|^2+|S|^2\big)\,\mathrm{d}x \int\left|\frac{\nabla c}{2c}\cdot\big(|R|^2S-|S|^2R\big)\right|\mathrm{d}x \\ &\leqslant -2c_1\int|R|^2|S|^2\,\mathrm{d}x + 4\mathcal{E}_0\max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{2c}\right\|_{L^\infty}\int\sum_{i=1}^n\big||R|^2S_i-|S|^2R_i\big|\,\mathrm{d}x. \qquad (7.4)\end{aligned}$$

For each $\varepsilon>0$ we have $|R_k|\leqslant\varepsilon^{-\frac12}+\varepsilon^{\frac12}|R|^2$, $|S_k|\leqslant\varepsilon^{-\frac12}+\varepsilon^{\frac12}|S|^2$ $(k=1,2,\dots,n)$. Choosing $\varepsilon>0$ such that

$$4\mathcal{E}_0\max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{2c}\right\|_{L^\infty}\cdot 2n\sqrt{\varepsilon}<c_1,$$

we thus have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[16\Lambda(t)\big] &\leqslant -c_1\int|R|^2|S|^2\,\mathrm{d}x + \frac{16n\mathcal{E}_0^2}{\sqrt{\varepsilon}}\max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{2c}\right\|_{L^\infty} \\ &\leqslant \frac{16n\mathcal{E}_0^2}{\sqrt{\varepsilon}}\max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{2c}\right\|_{L^\infty}.\end{aligned}$$

Hence, the map $t\to\Lambda(t)$ has bounded variation on any bounded interval in the smooth case.

In order to prove Lemma 1 in general case, we consider the above argument in terms of the variables X, Y. For this purpose, we fix $0\leqslant s<t$ and denote $D_{st}:=D_t\setminus D_s$, and then get by using (5.5) and (7.2)

$$\begin{aligned}\Lambda(t)-\Lambda(s) &\leqslant -\iint\limits_{D_{st}}\frac{1-h_1}{4}p\cdot\frac{1-h_2}{4}q\,\mathrm{d}X\,\mathrm{d}Y \\ &\quad + 4\mathcal{E}_0\cdot\iint\limits_{D_{st}}pq\sum_{i=1}^n\frac{|c_{u_i}|}{16c^2}\big|(1-h_1)m_i-(1-h_2)\ell_i\big|\,\mathrm{d}X\,\mathrm{d}Y. \qquad (7.5)\end{aligned}$$

According to (2.8) and (2.9) we find that

$$\ell_k^2\leqslant h_1(1-h_1),\qquad m_k^2\leqslant h_2(1-h_2).$$

Using the interpolation inequality, we observe that for every $\varepsilon>0$ there exists a constant K_ε such that

$$\ell_k\leqslant\varepsilon(1-h_1)+K_\varepsilon h_1,\qquad m_k\leqslant\varepsilon(1-h_2)+K_\varepsilon h_2.$$

Thus we have

$$\big|(1-h_1)m_k-(1-h_2)\ell_k\big|\leqslant\varepsilon(1-h_1)(1-h_2)+K_\varepsilon\big[(1-h_1)h_2+(1-h_2)h_1\big] \qquad (7.6)$$

for $k=1,2,\dots,n$. Combining (7.5) and (7.6), we obtain

$$\begin{aligned}\Lambda(t)-\Lambda(s) &\leqslant \left[n\varepsilon\mathcal{E}_0 \max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{4c^2}\right\|_{L^\infty}-\frac{1}{16}\right]\iint\limits_{D_{st}} pq(1-h_1)(1-h_2)\,\mathrm{d}X\,\mathrm{d}Y \\ &\quad + n\mathcal{E}_0 K_\varepsilon \max_{k=1,\dots,n}\left\|\frac{c_{u_k}}{4c^2}\right\|_{L^\infty}\iint\limits_{D_{st}} pq\big[(1-h_1)h_2+(1-h_2)h_1\big]\,\mathrm{d}X\,\mathrm{d}Y \\ &\leqslant K(t-s), \end{aligned} \tag{7.7}$$

for a suitable large constant K, which reaches the desired conclusion Lemma 1. Here the second inequality holds by the fact that

$$\iint\limits_{D_{st}} \frac{1}{4}\left(\frac{1-h_1}{h_1}+\frac{1-h_2}{h_2}\right)\frac{pq}{2c}h_1h_2\,\mathrm{d}X\,\mathrm{d}Y=(t-s)\mathcal{E}_0, \tag{7.8}$$

which is always valid.

The proof of Theorem 1.3 is similar to that of Theorem 3 in [6] and Theorem 1.3 in [27], but we reproduce it here for completeness.

Consider the three sets

$$\begin{aligned}\Omega_1 &:= \big\{(X,Y);\ h_1(X,Y)=0,\ h_2(X,Y)\neq 0,\ \big|\nabla c\big(u(X,Y)\big)\big|\neq 0\big\},\\ \Omega_2 &:= \big\{(X,Y);\ h_2(X,Y)=0,\ h_1(X,Y)\neq 0,\ \big|\nabla c\big(u(X,Y)\big)\big|\neq 0\big\},\\ \Omega_3 &:= \big\{(X,Y);\ h_1(X,Y)=0,\ h_2(X,Y)=0,\ \big|\nabla c\big(u(X,Y)\big)\big|\neq 0\big\}.\end{aligned}$$

From (2.24), we find there at least exist two integers $\hat{k},\tilde{k}\in\{1,2,\dots,n\}$ such that $\ell_{\hat{k}Y}\neq 0$ on Ω_1 and $m_{\tilde{k}X}\neq 0$ on Ω_2, thus

$$\operatorname{meas}(\Omega_1)=\operatorname{meas}(\Omega_2)=0. \tag{7.9}$$

Let Ω_3^* be the set of Lebesgue points of Ω_3. We assert that

$$\operatorname{meas}\big(\big\{t(X,Y);\ (X,Y)\in\Omega_3^*\big\}\big)=0. \tag{7.10}$$

To prove (7.10), fix any $P^*=(X^*,Y^*)\in\Omega_3^*$ and let $\tau=t(P^*)$; we first prove the following claim:

$$\limsup_{h,k\to 0^+}\frac{\Lambda(\tau-h)-\Lambda(\tau+k)}{h+k}=+\infty. \tag{7.11}$$

By assumption, for any $\varepsilon>0$ arbitrarily small we can find $\delta>0$ with the following property. For any square Q centered at P^* with side of length $l<\delta$, there exist a vertical segment σ and a horizontal segment σ', as in Fig. 3, such that

$$\operatorname{meas}(\Omega_3\cap\sigma)\geqslant(1-\varepsilon)l,\qquad \operatorname{meas}\big(\Omega_3\cap\sigma'\big)\geqslant(1-\varepsilon)l. \tag{7.12}$$

Since $h_1=h_2=0$ at nearly all points close to p^*, we can assume that the endpoints of the two segments σ, σ' are all in Ω_3. Moreover, we assume without loss of generality that there exists an integer $\bar{k}\in\{1,2,\dots,n\}$ such that $c_{u_{\bar{k}}}>\bar{c}>0$ ($\bar{c}$ is a constant) at the point P^*. By integrating the equation for $\ell_{\bar{k}}$ from (2.24) along σ and doing a simple rearrangement, we obtain

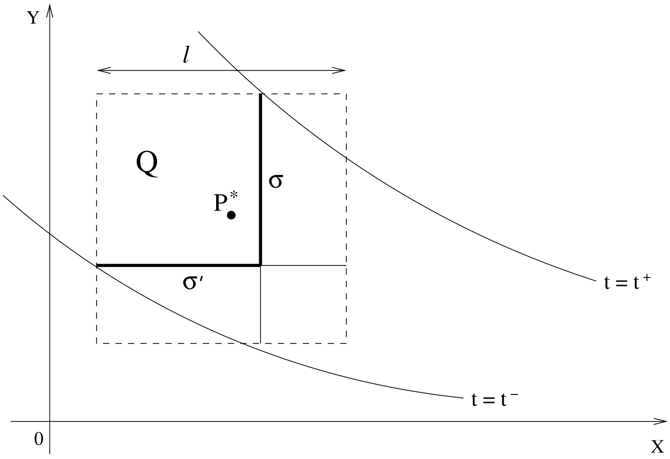

Fig. 3. Lebesgue point of Ω_3. Q is a square centered at the Lebesgue point $P^* \in \Omega_3$, σ and σ' are two segments such that (7.12) holds.

$$\int_\sigma \frac{c_{u_{\bar{k}}}}{4c} \cdot \frac{q}{2c} h_2 = \int_\sigma q \left\{ \frac{c_{u_{\bar{k}}}}{8c^2}(2h_2 - 1)h_1 + \frac{c_{u_{\bar{k}}}}{4c^2} \ell \cdot m + \frac{\nabla c}{4c^2} \cdot \ell(\ell_{\bar{k}} - m_{\bar{k}}) \right\} \mathrm{d}Y. \tag{7.13}$$

Notice that h_1, ℓ_k $(k = 1, 2, \ldots, n)$ are Lipschitz in Y and $h_1 = 0$ means $\ell_k = 0$ $(k = 1, 2, \ldots, n)$, and they are zero on σ on a set with measure greater than $(1 - \varepsilon)l$, then (7.13) leads to

$$\int_\sigma \frac{q}{2c} h_2 \, \mathrm{d}Y = O(1)(\varepsilon l)^2. \tag{7.14}$$

Similarly we have

$$\int_{\sigma'} \frac{p}{2c} h_1 \, \mathrm{d}X = O(1)(\varepsilon l)^2. \tag{7.15}$$

Denote

$$t^+ := \max\bigl\{t(X, Y);\ (X, Y) \in \sigma \cup \sigma'\bigr\}, \qquad t^- := \min\bigl\{t(X, Y);\ (X, Y) \in \sigma \cup \sigma'\bigr\}.$$

Combining (7.14) and (7.15) and noticing (4.2), we get

$$t^+ - t^- \leqslant \int_{\sigma'} t_X \, \mathrm{d}X + \int_\sigma t_Y \, \mathrm{d}Y = \int_{\sigma'} \frac{p}{2c} h_1 \, \mathrm{d}X + \int_\sigma \frac{q}{2c} h_2 \, \mathrm{d}Y = O(1)(\varepsilon l)^2. \tag{7.16}$$

On the other hand, by (7.7) we have

$$\Lambda\bigl(t^-\bigr) - \Lambda\bigl(t^+\bigr) \geqslant \hat{c}(1 - \varepsilon)^2 \ell^2 - \tilde{c}\bigl(t^+ - t^-\bigr)$$

for some constants $\hat{c} > 0$, $\tilde{c} > 0$. Since $\varepsilon > 0$ is arbitrary, this implies (7.11).

The assertion (7.10) follows directly from (7.11) and the fact that the map $t \mapsto \Lambda(t)$ has bounded variation.

We now observe that the singular part of the Radon measure μ_τ is nontrivial only if the set

$$\{P \in \Gamma_\tau;\ h_1(P) = 0 \text{ or } h_2(P) = 0\}$$

has positive 1-dimensional measure. The previous analysis shows that, provided $|\nabla c| \neq 0$, this can occur only on a set of times of measure zero.

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References

[1] G. Ali, J.K. Hunter, Diffractive nonlinear geometrical optics for variational wave equations and the Einstein equations, Comm. Pure Appl. Math. 60 (2007) 1522–1557.
[2] G. Ali, J.K. Hunter, Orientation waves in a director field with rotational inertia, Kinet. Relat. Models 2 (2009) 1–37.
[3] A. Bressan, A. Constantin, Global solutions of the Hunter–Saxton equation, SIAM J. Math. Anal. 37 (2005) 996–1026.
[4] A. Bressan, H. Holden, X. Raynaud, Lipschitz metric for the Hunter–Saxton equation, J. Math. Pures Appl. 94 (2010) 68–92.
[5] A. Bressan, P. Zhang, Y. Zheng, Asymptotic variational wave equations, Arch. Ration. Mech. Anal. 183 (2007) 163–185.
[6] A. Bressan, Y. Zheng, Conservative solutions to a nonlinear variational wave equation, Comm. Math. Phys. 266 (2006) 471–497.
[7] C.M. Dafermos, Generalized characteristics and the Hunter–Saxton equation, J. Hyperbolic Differ. Equ. 8 (2011) 159–168.
[8] R.T. Glassey, J.K. Hunter, Y. Zheng, Singularities of a variational wave equation, J. Differential Equations 129 (1996) 49–78.
[9] R.T. Glassey, J.K. Hunter, Y. Zheng, Singularities and oscillations in a nonlinear variational wave equation, in: J. Rauch, M.E. Taylor (Eds.), Singularities and Oscillations, in: IMA Vol. Math. Appl., vol. 91, Springer, 1997, pp. 37–60.
[10] A. Grundland, E. Infeld, A family of nonlinear Klein–Gordon equations and their solutions, J. Math. Phys. 33 (1992) 2498–2503.
[11] H. Holden, X. Raynaud, Global semigroup of conservative solutions of the nonlinear variational wave equation, Arch. Ration. Mech. Anal. 201 (2011) 871–964.
[12] J.K. Hunter, R.A. Saxton, Dynamics of director fields, SIAM J. Appl. Math. 51 (1991) 1498–1521.
[13] J.K. Hunter, Y. Zheng, On a nonlinear hyperbolic variational equation I, II, Arch. Ration. Mech. Anal. 129 (1995) 305–353, 355–383.
[14] J. Lenells, The Hunter–Saxton equation describes the geodesic flow on a sphere, J. Geom. Phys. 57 (2007) 2049–2064.
[15] J. Lenells, The Hunter–Saxton equation: A geometric approach, SIAM J. Math. Anal. 40 (2008) 266–277.
[16] J. Li, K. Zhang, Global existence of dissipative solutions to the Hunter–Saxton equation via vanishing viscosity, J. Differential Equations 250 (2011) 1427–1447.
[17] R.A. Saxton, Dynamic instability of the liquid crystal director, in: W.B. Lindquist (Ed.), Current Progress in Hyperbolic Systems, in: Contemp. Math., vol. 100, Amer. Math. Soc., 1989, pp. 325–330.
[18] Z. Yin, On the structure of solutions to the periodic Hunter–Saxton equation, SIAM J. Math. Anal. 36 (2004) 272–283.
[19] P. Zhang, Weak solutions to a nonlinear variational wave equation and some related problems, Appl. Math. 51 (2006) 427–466.
[20] P. Zhang, Y. Zheng, On oscillations of an asymptotic equation of a nonlinear variational wave equation, Asymptot. Anal. 18 (1998) 307–327.
[21] P. Zhang, Y. Zheng, Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data, Arch. Ration. Mech. Anal. 155 (2000) 49–83.
[22] P. Zhang, Y. Zheng, Rarefactive solutions to a nonlinear variational wave equation of liquid crystals, Comm. Partial Differential Equations 26 (2001) 381–419.
[23] P. Zhang, Y. Zheng, Singular and rarefactive solutions to a nonlinear variational wave equation, Chin. Ann. Math. Ser. B 22 (2001) 159–170.
[24] P. Zhang, Y. Zheng, On the second-order asymptotic equation of a variational wave equation, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002) 483–509.
[25] P. Zhang, Y. Zheng, Weak solutions to a nonlinear variational wave equation, Arch. Ration. Mech. Anal. 166 (2003) 303–319.
[26] P. Zhang, Y. Zheng, Weak solutions to a nonlinear variational wave equation with general data, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005) 207–226.
[27] P. Zhang, Y. Zheng, Conservative solutions to a system of variational wave equations of nematic liquid crystals, Arch. Ration. Mech. Anal. 195 (2010) 701–727.
[28] P. Zhang, Y. Zheng, Energy conservative solutions to a one-dimensional full variational wave system, Comm. Pure Appl. Math. (2011), doi:10.1002/cpa.20380.
[29] H. Zorski, E. Infeld, New soliton equations for dipole chains, Phys. Rev. Lett. 68 (1992) 1180–1183.