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# Uniform persistence and Hopf bifurcations in $\mathbb{R}^n_+$

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**Abstract**

We consider parameterized families of flows in locally compact metrizable spaces and give a characterization of those parameterized families of flows for which uniform persistence continues. On the other hand, we study the generalized Poincaré–Andronov–Hopf bifurcations of parameterized families of flows at boundary points of $\mathbb{R}^n_+$ or, more generally, of an $n$-dimensional manifold, and show that this kind of bifurcations produce a whole family of attractors evolving from the bifurcation point and having interesting topological properties. In particular, in some cases the bifurcation transforms a system with extreme non-permanence properties into a uniformly persistent one. We study in the paper when this phenomenon happens and provide an example constructed by combining a Holling-type interaction with a pitchfork bifurcation.
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## 1. Preliminaries

This paper is devoted to the study of some questions related to persistence of flows. This is a topic classically connected to population dynamics, the central issue being to determine whether some components of the population are over the long term driven to extinction or, on the contrary, they will survive and evolve towards some stable states where coexistence between all the components is achieved. The term persistence is given to systems in which strictly positive solutions do not approach the boundary of the nonnegative orthant $\mathbb{R}^n_+$ as $t \to \infty$. The question that arises is that of determining conditions which prevent solutions from approaching the boundary. As G. Butler, H.I. Freedman and P. Waltman remark in [3], this is of great importance in the modeling of biological populations where such conditions rule out the possibility of one of the populations becoming arbitrarily close to zero in a deterministic model and therefore risking extinction in a more realistic interpretation of the model.

Several forms of persistence have been studied. The so called *uniform persistence* or *permanence* is perhaps the most robust concept, that is, more likely to be maintained under suitable small variations of the system of equations. This is always a desirable consideration from the point of view of applications. However, strictly speaking, uniform persistence is not fully robust, although in [43] it has been proved that some forms of weak robustness always hold. These conditions are expressed using the notion of *continuation*, which is one of the forms that robustness adopts in the context of the Conley index theory [5,6]. Also, in the papers [11,19,23,47], some sufficient conditions for robustness are given. One of the aims of the present paper is to give a characterization of those parameterized families of flows for which uniform persistence continues. Another aim is to study the generalized Poincaré–Andronov–Hopf bifurcations of parameterized families of flows at boundary points of $\mathbb{R}^n_+$ or, more generally, of an $n$-dimensional manifold. We see that this kind of bifurcations produce a whole family of attractors evolving from the bifurcation point and having interesting topological properties. A possible consequence of the bifurcation is a qualitative change in the persistence properties of the system. In some cases the bifurcation transforms a system with extreme non-permanence properties into a uniformly persistent one. We study in the paper when this phenomenon happens and provide an example constructed by combining a Holling-type interaction with a pitchfork bifurcation.

In the sequel we fix some terminology and state a few results that will be used along the paper. An attractor of a flow $\varphi : E \times \mathbb{R} \to E$, where $E$ is a locally compact metrizable space, is in this paper an asymptotically stable invariant compactum. A repeller is a negatively asymptotically stable invariant compactum, i.e. an attractor for the reverse flow. The following characterization of repellers is useful (see [36]): An invariant compactum $K$ is a repeller if and only if there is a neighborhood $U$ of $K$ in $E$ such that for every $x \in U - K$ there is $t > 0$ such that $\varphi(x,t) \notin U$. This characterization can be dualized for attractors.

The flow $\varphi$ is said to be dissipative if $\omega(x) \neq \emptyset$ for every $x \in E$ and $\bigcup_{x\in E} \omega(x)$ has compact closure. If $E$ is not compact we shall often consider the Alexandrov compactification $\hat{E} = E \cup \{\infty\}$ and the extended flow

$$\hat{\varphi} : \hat{E} \times \mathbb{R} \to \hat{E}$$

leaving fixed $\infty$. Then dissipativeness is equivalent to $\{\infty\}$ being a repeller (see [10] and [16]). Notice that the dual attractor of $\{\infty\}$ is a global attractor for the flow $\varphi$.

A stronger form of dissipativeness can be given for families of flows. If $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I$, is a (continuous) parameterized family of flows then $\varphi_\lambda$ is said to be uniformly dissipative if $\omega_\lambda(x) \neq \emptyset$ for every $x \in E$ and every $\lambda \in I$ and the set $\Omega = \bigcup_{x\in E,\lambda\in I} \omega_\lambda(x)$ has compact closure.

In this paper $E$ will often be a closed subset of $X$, where $X$ is a locally compact metric space, and we shall denote by $\partial E$ the boundary of $E$ in $X$. We shall say that a flow $\varphi : E \times \mathbb{R} \to E$ is uniformly persistent if there exists $\beta > 0$ such that for every $x \in \mathring{E}$

$$\liminf_{t\to\infty} d\big(\varphi(x,t), \partial E\big) \geqslant \beta.$$

In all the paper, we will suppose that $\partial E$ is invariant for the flow $\varphi$. If $E$ is compact then $\varphi$ is uniformly persistent if and only if $\partial E$ is a repeller of $\varphi$. If $\varphi$ is dissipative and $E$ is not compact then $\varphi$ is uniformly persistent if and only if $\partial E \cup \{\infty\}$ is a repeller for the flow extended to $\hat{E} = E \cup \{\infty\}$ (see [10] for proofs of these results). In this case, there exists a dual attractor $K$ whose region of attraction is the interior $\mathring{E}$. We shall call $K$ the *internal global attractor*. $K$ should not be confused with the global attractor of $\varphi$, which is a larger set.

The Butler–Waltman theorem [4] is one of the most relevant results in the theory of persistent flows. It provides a criterion for uniform persistence which in the more elementary applications may be reduced to readily testable hypotheses. This result shows that some questions of persistence may be addressed by appealing to suitable conditions on the boundary flow. Butler and Waltman stated their result in terms of isolated acyclic coverings but later Garay presented in [10] a reformulation in terms of Morse decompositions. Garay's results are written in the spirit of the Conley index theory and, in particular, he makes use of notions related to chain recurrence. We state the theorem in the form given by Garay [10]. We denote by $W^+(M)$ the stable manifold of $M$, i.e. the set $W^+(M) = \{x \in E \mid \omega(x) \subset M\}$.

**Theorem 1** *(Butler–Waltman & Garay). Let $X$ be a locally compact metric space and let $E$ be a closed subset of $X$. Suppose we are given a dissipative dynamical system $\varphi$ on $E$ for which $\partial E$ is invariant.*

*Let $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ be a Morse decomposition for $\varphi|_M$, where $M$ is the maximal compact invariant set in $\partial E$. Further assume that for each $i \in \{1, 2, \dots, n\}$*

a) *there exists a $\gamma > 0$ such that the set $\{x \in \mathring{E} \mid d(x, M_i) < \gamma\}$ contains no entire trajectories, and*
b) *$\mathring{E} \cap W^+(M_i) = \emptyset$.*

*Then $\varphi$ is uniformly persistent.*

In the paper [43, Theorem 6], a sufficient condition was given for (uniform) continuation of uniform persistence of flows to hold. This was a regularity condition required for all points $x \in \mathring{E}$. We present in this paper an example showing that this result does not hold when we require such condition just for points close to the Morse sets of a decomposition of the maximal invariant compactum of $\partial E$, as in the Butler–Waltman & Garay Theorem (see Remark 9). One of the motivations of the present paper is to provide a version of this result in the spirit of the Butler–Waltman & Garay Theorem. In Theorem 7 and Corollary 8 in this paper we find necessary and sufficient conditions under which this formulation is possible. We also identify weaker conditions which ensure continuation of uniform persistence for bounded trajectories. Another motivation of the paper is to study uniform persistence in the context of bifurcations, in particular those bifurcations arising from a loss of stability at rest points of the flows, such as the generalized Poincaré–Andronov–Hopf bifurcations. We present an example and prove some results where the transition of a system with extreme non-permanence properties into a uniformly persistent one is studied.

We use in the paper some classical notions from algebraic topology, in particular homology, cohomology and duality theory together with some rudiments of shape theory. The notion of homotopy type is too rigid for the study of the topological objects which appear in dynamics. For this reason many authors have used instead Borsuk's shape theory as an essential tool which provides a geometric insight on the global structure of compacta, mainly on those with complicated topological structure as many attractors and invariant sets. For the benefit of the reader we present here a very short introduction, essentially based on the presentation of this subject given by Kapitanski and Rodnianski in [24].

A metrizable space $M$ is said to be an absolute neighborhood retract (notation $M \in \mathrm{ANR}$) if for every homeomorphism $h$ mapping $M$ onto a closed subset $h(M)$ of a metrizable space $X$ there is a neighborhood $U$ of $h(M)$ in $X$ such that $h(M)$ is a retract of $U$.

**Theorem 2.** *A metrizable space $M$ is an* ANR *if and only if for every map $f : Y \to M$ of a closed subset $Y$ of any metrizable space $Y'$ there is a neighborhood $U$ of $Y$ in $Y'$ and a map $f' : U \to M$ being an extension of $f$.*

**Theorem 3.** *A metrizable space $M$ is an ANR if and only if it is homeomorphic to a retract of an open subset of a convex set lying in a Banach space.*

In particular, open subsets of Euclidean spaces are ANRs.

All metric spaces can be viewed as subsets of ANRs. In fact by the Kuratowski–Wojdyslawski theorem [22] every metric space can be embedded into an ANR as a closed subspace.

Let $X$ be a closed subset of an ANR $M$ and $Y$ a closed subset of an ANR $N$. Denote by $\mathbb{U}(X; M)$ (resp. $\mathbb{U}(Y; N)$) the set of all open neighborhoods of $X$ in $M$ (resp. $Y$ in $N$).

Let $\mathbf{f} = \{f : U \to V\}$ be a collection of continuous maps from the neighborhoods $U \in U(X; M)$ to $V \in U(Y; N)$. We call $\mathbf{f}$ a mutation if the following conditions are fulfilled:

1) For every $V \in U(Y; N)$ there exists (at least) a map $f : U \to V$ in $\mathbf{f}$.
2) If $f : U \to V$ is in $\mathbf{f}$ then the restriction $f|_{U_1} : U_1 \to V_1$ is also in $\mathbf{f}$ for every neighborhood $U_1 \subset U$ and every neighborhood $V_1 \supset V$.
3) If the two maps $f, f' : U \to V$ are in $\mathbf{f}$ then there exists a neighborhood $U_1 \subset U$ such that the restrictions $f|_{U_1}$ and $f'|_{U_1}$ are homotopic.

An example of mutation is the identity mutation $\mathbf{id}_{\mathbb{U}(X;M)}$ consisting of the identity maps $i : U \to U$.

The notions of composition of mutations and homotopy of mutations can be defined in a straightforward way that the reader can easily guess (see [24] for details).

Two metric spaces $X$ and $Y$ have the same shape if they can be embedded as closed sets in ANRs $M$ and $N$ in such a way that there exist mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{g} = \{g : V \to U\}$ such that the compositions $\mathbf{gf}$ and $\mathbf{fg}$ are homotopic to the identity mutations $\mathbf{id}_{\mathbb{U}(X;M)}$ and $\mathbf{id}_{\mathbb{U}(Y;N)}$ respectively.

The notion of shape of sets depends neither on the ANRs they are embedded in nor on the embeddings.

Spaces belonging to the same homotopy type have the same shape.

ANRs have the same shape if and only if they have the same homotopy type. A consequence of the two former statements is that the notion of shape may be seen as a generalization of the notion of homotopy type.

For a complete treatment of shape theory we refer the reader to [2,7,9,28,29]. The use of shape in dynamics is illustrated by the papers [12–15,17,24,32–34,36–43]. For information about basic aspects of dynamical systems we recommend [1,35,50]. See also [20,21,48,51] for information about some aspects of permanence. In particular, in [51] Wójcik established interesting relations between permanence and Conley index theory. Concerning Morse decompositions see [6,24–26,41]. Finally, the main references used for algebraic topology have been the books by Hatcher [18] and Spanier [49].

## 2. Uniform continuations of uniform persistence

We start this section by introducing some notions which play an important role in the study on the properties related to robustness of uniform persistence.

**Definition 4.** We say that a compactum $M \subset \partial E$ is externally repelling for a flow $\varphi : E \times \mathbb{R} \to E$ if there is a neighborhood $U$ of $M$ in $E$ such that for every $x \in U - \partial E$ there is a $t > 0$ with $\varphi(x,t) \notin U$.

If $\varphi_\lambda : E \times \mathbb{R} \to E$ with $\lambda \in I = [0,1]$ is a parameterized family of flows we say that $M \subset \partial E$ is uniformly externally repelling if there is a neighborhood $U$ of $M$ in $E$ and a $\lambda_0 > 0$ such that for every $x \in U - \partial E$ and every $\lambda \in [0,\lambda_0]$ there is a $t > 0$ with $\varphi_\lambda(x,t) \notin U$.

**Definition 5.** Let $\varphi_\lambda : E \times \mathbb{R} \to E$ with $\lambda \in I = [0,1]$ be a parameterized family of flows. Suppose $M \subset \partial E$ is an isolated invariant set for $\varphi_0$. We say that $M$ is strongly isolated if there exist a neighborhood $U$ of $M$ in $E$, a $\lambda_0 > 0$ and a compact set $K \subset \mathring{E}$, such that for every $x \in U - \partial E$ and every $\lambda \in [0,\lambda_0]$ the trajectory $\varphi_\lambda(x,\cdot)$ visits $K$ for some $t \in \mathbb{R}$ (not necessarily positive).

We shall discuss in this section some matters related to robustness using the point of view of continuation, a central notion in the Conley index theory. Roughly speaking, we say that a certain property *continues* if whenever we have a (continuous) parameterized family of flows $\varphi_\lambda$, $\lambda \in I = [0,1]$, and $\varphi_0$ has this property then $\varphi_\lambda$ also has this property for small values of $\lambda$. The notion of continuation is applied, in particular, to uniform persistence. We can, also, introduce the stronger notion of uniform continuation.

**Definition 6.** Let $\varphi_\lambda : E \times \mathbb{R} \to E$ with $\lambda \in I = [0,1]$ be a parameterized family of flows. We say that uniform persistence at $\lambda = 0$ continues uniformly if there exist $\lambda_0 > 0$ and $\beta > 0$ such that for every $x \in \mathring{E}$ and every $\lambda \in [0,\lambda_0]$

$$\liminf_{t\to\infty} d\big(\varphi_\lambda(x,t), \partial E\big) \geqslant \beta.$$

It is easy to see that, in a general context, uniform persistence is not a robust property. Fig. 1 was already used in [43] to show that small perturbations of a uniformly persistent flow may destruct this property.

The figure on the left shows the flow for $\lambda = 0$. The figure on the right shows the flow for $\lambda > 0$. For $\lambda = 0$ all orbits are attracted by the point in the center, with the exception of the lower fixed point and the two outer orbits, attracted by it. For $\lambda > 0$ there are two sets of “parallel”

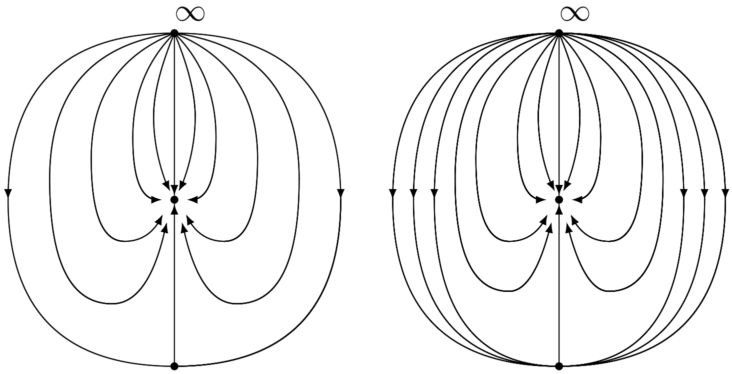


Fig. 1. A uniformly persistent flow $\varphi_0$ and a small perturbation of it.

orbits attracted by the lower fixed point. These sets of orbits shrink as $\lambda$ approaches 0. This family of flows satisfies:

i) It is uniformly dissipative.
ii) $\varphi_0$ is uniformly persistent but uniform persistence at $\lambda = 0$ does not continue uniformly. In fact, for $\lambda > 0$, $\varphi_\lambda$ is not uniformly persistent.
iii) The maximal compact invariant set $M$ of $\varphi_\lambda$ in $\partial E$ is just the lower fixed point; $M$ is externally repelling for $\varphi_0$, but is neither uniformly externally repelling nor strongly isolated.

The following result provides necessary and sufficient conditions for the uniform continuation of uniform persistence.

**Theorem 7.** *Let $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I = [0, 1]$, be a (continuous) uniformly dissipative parameterized family of flows. Suppose that $\varphi_0$ is uniformly persistent, and let $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ be a Morse decomposition for $\varphi_0|_M$ where $M$ is the maximal compact invariant set of $\varphi_0$ in $\partial E$. Then the following conditions are equivalent*:

a) *Uniform persistence at $\lambda = 0$ continues uniformly.*
b) *$M_1, M_2, \dots, M_n$ are strongly isolated.*

**Proof.** We first see that a) $\Rightarrow$ b). Since $\Omega = \bigcup_{x \in E, \lambda \in I} \omega_\lambda(x)$ has compact closure there exists a compact neighborhood $C$ of $\Omega$ in $E$. Moreover, since $\omega_\lambda(x) \subset \mathring{C}$ then for every $x \in E$ and every $\lambda \in I$ there exists $T \geqslant 0$ such that $\varphi_\lambda(x, t) \in \mathring{C}$ for every $t \geqslant T$. Now, by the uniform continuation of uniform persistence there exists $\beta > 0$ such that $\liminf_{t\to\infty} d(\varphi_\lambda(x, t), \partial E) > \beta$ for every $x \in \mathring{E}$ and every $\lambda \leqslant \lambda_0$. We define

$$K = C \cap \bigl\{x \in E \bigm| d(x, \partial E) \geqslant \beta\bigr\},$$

then $K$ is a compact set contained in $\mathring{E}$ and it is easy to see that every trajectory $\varphi_\lambda(x, \cdot)$ visits $K$ for every $\lambda \leqslant \lambda_0$ and $x \in \mathring{E}$. In particular, $M_1, M_2, \dots, M_n$ are strongly isolated.

We see now that b) $\Rightarrow$ a). Suppose that $M_1, M_2, \dots, M_n$ are strongly isolated. Since $\partial E \cup \{\infty\}$ is a repeller for $\hat{\varphi}_0$, it continues to a repeller $\mathcal{R}_\lambda$ for $\hat{\varphi}_\lambda$ with $\lambda$ sufficiently small. This repeller is the maximal invariant set for $\hat{\varphi}_\lambda$ in $W$, where $W$ is an isolating neighborhood of $\partial E \cup \{\infty\}$ for $\hat{\varphi}_0$. However, $\mathcal{R}_\lambda$ could be, at least in principle, different from $\partial E \cup \{\infty\}$. Thus we must prove that under the hypotheses of the theorem $\mathcal{R}_\lambda = \partial E \cup \{\infty\}$.

Since $\partial E \cup \{\infty\}$ is invariant for every $\hat{\varphi}_\lambda$, and $\mathcal{R}_\lambda$ is the maximal invariant set in $W$ then $\partial E \cup \{\infty\}$ must necessarily be contained in $\mathcal{R}_\lambda$ for every $\lambda$ sufficiently small. Moreover, the attractor–repeller decomposition $(M, \{\infty\})$ for $\hat{\varphi}_0|_{\partial E \cup \{\infty\}}$ continues to an attractor–repeller decomposition for $\hat{\varphi}_\lambda|_{\mathcal{R}_\lambda}$. From the condition of uniform dissipativeness we have that $\{\infty\}$ is the maximal compact invariant set in $\hat{E} - \bar{\Omega}$ for every flow $\hat{\varphi}_\lambda$. It follows from this that the repeller $\{\infty\}$ continues to itself which, in turn, implies that $M^\lambda$, the continuation of $M$ for $\hat{\varphi}_\lambda|_{\mathcal{R}_\lambda}$, is the dual attractor of $\{\infty\}$. Moreover, the Morse decomposition $\mathcal{M}$ continues to a Morse decomposition $\mathcal{M}^\lambda = \{M_1^\lambda, M_2^\lambda, \ldots, M_n^\lambda\}$ of $M^\lambda$. Thus $M_i^\lambda \subset U_i$ for every $i \in \{1, \ldots, n\}$ and $\lambda$ sufficiently small, where $U_i$ is the neighborhood given by the property of $M_i$ being strongly isolated. Since $M_i^\lambda$ is invariant for $\varphi_\lambda$ and there are no complete orbits contained in $U_i - \partial E$ it follows that $M_i^\lambda \subset \partial E$. Let $C = C_1 \cup \cdots \cup C_n$, where $C_i$ is the compactum in the definition of strong isolation of $M_i$. Then $C \subset \mathring{E}$ and the trajectory $\varphi_\lambda(x, \cdot)$ visits $C$ for every $x \in \bigcup U_i$. Suppose now that $\lambda_0$ is chosen in such a way that $\mathcal{R}_\lambda \subset \hat{E} - C$ for $\lambda \leqslant \lambda_0$. Then, if $\partial E \cup \{\infty\} \neq \mathcal{R}_\lambda$, the flow $\varphi_\lambda$ has a trajectory $\gamma(x) \subset \mathcal{R}_\lambda - (\partial E \cup \{\infty\})$. This implies that there is an $i \in \{1, \ldots, n\}$ such that $\omega_\lambda(x) \subset M_i^\lambda$ and thus $\gamma(x) \cap U_i \neq \emptyset$. We must then have that $\gamma(x) \cap C \neq \emptyset$, which is in contradiction with the fact that $\mathcal{R}_\lambda \subset \hat{E} - C$. This proves that $\mathcal{R}_\lambda = \partial E \cup \{\infty\}$ and thus, $\varphi_\lambda$ is uniformly persistent. Moreover, the internal global attractor for $\hat{\varphi}_\lambda$ is a continuation of the internal global attractor for $\hat{\varphi}_0$ which establishes the uniformity of the continuation. □

As a consequence of Theorem 7 we obtain the following necessary and sufficient condition for the continuation of uniform persistence. This is a local condition and, hence, easier to verify than strong isolation, which is a global concept.

**Corollary 8.** *Let $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I = [0, 1]$, be a (continuous) uniformly dissipative parameterized family of flows. Suppose that $\varphi_0$ is uniformly persistent, and let $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ be a Morse decomposition for $\varphi_0|_M$ where $M$ is the maximal compact invariant set of $\varphi_0$ in $\partial E$. Then the following conditions are equivalent:*

a) *Uniform persistence at $\lambda = 0$ continues uniformly.*
b) *$M_1, \ldots, M_n$ are uniformly externally repelling.*

**Proof.** We shall prove that b) $\Rightarrow$ a), the proof of the converse implication being similar to that in Theorem 7. We use the same notation as before. Suppose that uniform persistence at $\lambda = 0$ does not continue uniformly. Then there is a compact neighborhood $W$ of $\partial E \cup \{\infty\}$ in $\hat{E}$ which is isolating for $\hat{\varphi}_0$ and there are arbitrarily small $\lambda \in I$ and points $x_\lambda \in W \cap \mathring{E}$ such that the trajectory $\gamma_\lambda(x_\lambda)$ for $\hat{\varphi}_\lambda$ is entirely contained in $W$ and hence $\omega_\lambda(x_\lambda) \subset W$. As in the proof of Theorem 7, $W$ is an isolating neighborhood for the repellers $\mathcal{R}_\lambda$ of $\hat{\varphi}_\lambda$, and the sets $\{\infty\}$, $M_1^\lambda, \ldots, M_n^\lambda$ define a Morse decomposition of $\mathcal{R}_\lambda$ for $\lambda$ small. Hence $\omega_\lambda(x_\lambda) \subset \mathcal{R}_\lambda$. Moreover, by the general properties of the Morse decompositions (see [36, Lemma 3.8]), this implies that $\omega_\lambda(x_\lambda) \subset M_i^\lambda$ for some $i$. Suppose $U_i \subset W$ is the neighborhood of $M_i$ given by the fact that $M_i$ is uniformly externally repelling. Since $M_i^\lambda$ is a continuation of $M_i$, $M_i^\lambda \subset U_i$ for $\lambda$ sufficiently small. Hence there are points in $U_i - \partial E$ which stay in $U_i$ for all positive times contrarily to the choice of $U_i$. This contradiction proves the corollary. □

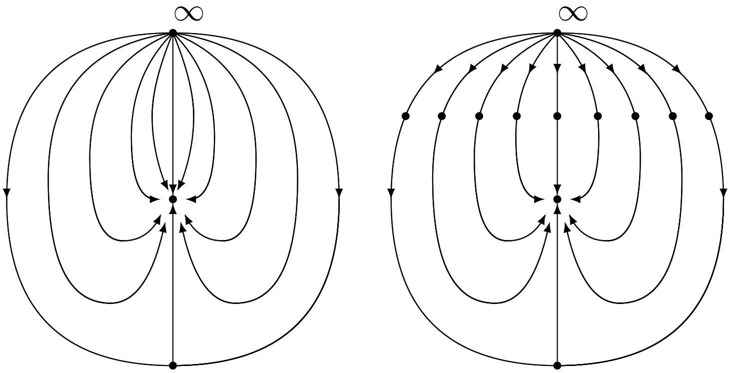


Fig. 2. A uniformly persistent flow $\varphi_0$ and a small perturbation of it.

**Remark 9.** Theorem 7 improves our Theorem 6 in [43] since we only use information about Morse decompositions of the maximal set in the boundary, while in [43] information about all points of $\mathring{E}$ was needed (they were required to visit the compact set $K \subset \mathring{E}$). In Theorem 6 in [43] only dissipativeness was required for the $\varphi_\lambda$. However, Fig. 2 shows that the condition of uniform dissipativeness is essential for a version of this theorem in terms of Morse decompositions.

The figure on the left shows the flow for $\lambda = 0$. The figure on the right shows the flow for $\lambda > 0$. The latter has a line of fixed points which approaches the point at infinity as $\lambda$ approaches 0. This family of flows satisfies:

i) Although all the flows are dissipative, the family is not uniformly dissipative.
ii) $\varphi_0$ is uniformly persistent but uniform persistence at $\lambda = 0$ does not continue uniformly. In fact, for $\lambda > 0$, $\varphi_\lambda$ is not uniformly persistent.
iii) The maximal compact invariant set $M$ of $\varphi_0$ in $\partial E$ is just the lower fixed point, and $M$ is uniformly externally repelling and strongly isolated.

We could ask ourselves to what extent the requirements in Definition 4 of an externally repelling compactum (resp. a uniformly externally repelling family of flows) could be relaxed. We could, in Definition 4, demand only that $\varphi(x,t) \notin U$ (resp. $\varphi_\lambda(x,t) \notin U$) for some $t \in \mathbb{R}$ (*not necessarily positive*), i.e. that $U - \partial E$ does not contain entire trajectories of the flows $\varphi_\lambda$. This would result in a kind of isolation property which is weaker than the strong isolation of Definition 5, which demands, in addition, that the trajectory $\varphi_\lambda(x,\cdot)$ visits the compactum $K$. However, if we require only this weaker condition, then Theorem 7 and Corollary 8 cease to be true. This is illustrated by Fig. 1 again. In spite of this, we still have a form of continuation as the following result shows. We denote by $\gamma_\lambda(x)$ the trajectory of $x$ by the flow $\varphi_\lambda$.

**Theorem 10.** *Let $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I = [0,1]$, be a uniformly dissipative parameterized family of flows, where $\varphi_0$ is uniformly persistent. Let $M$ be the maximal compact invariant set of $\varphi_0|_{\partial E}$. Suppose that there is a neighborhood $U$ of $M$ in $E$ and a $\lambda_0 \in I$ such that for every $x \in U - \partial E$ and every $\lambda \leqslant \lambda_0$ there is a $t \in \mathbb{R}$ (not necessarily positive) such that $\varphi_\lambda(x,t) \notin U$. Then uniform persistence continues uniformly for bounded orbits in the following sense: there exist $\lambda_1 \in I$ and $\beta > 0$ such that $\liminf_{t\to\infty}(d(\varphi_\lambda(x,t), \partial E)) > \beta$ for every $\lambda \leqslant \lambda_1$ and every $x \in \mathring{E}$ such that $\gamma_\lambda(x)$ is bounded.*

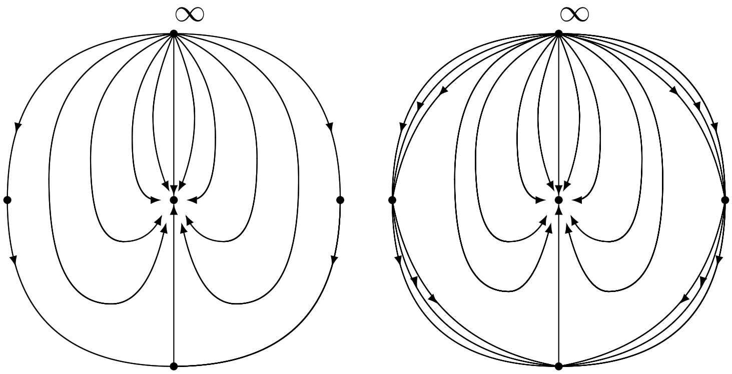


Fig. 3. A uniformly persistent flow $\varphi_0$ and a small perturbation of it.

**Proof.** The proof is very similar to that of Theorem 7 and, for this reason, we only give a sketch. It is sufficient to remark that the hypothesis guarantees that the continuations $M^\lambda$ are contained in the boundary (we still use the same notation as in Theorem 7). Moreover, the Morse decomposition of $\hat{E}$, $\{\infty, M, \mathcal{A}\}$, where $\mathcal{A}$ is the internal global attractor for $\varphi_0$, continues to a Morse decomposition $\{\infty, M^\lambda, \mathcal{A}_\lambda\}$ for $\hat{\varphi}_\lambda$. The bounded orbits of $\varphi_\lambda$ in $\mathring{E}$ are just those linking $M^\lambda$ and $\mathcal{A}_\lambda$, together with the orbits in $\mathcal{A}_\lambda$. The uniform persistence property readily follows from the fact that the continuations $\mathcal{A}_\lambda$ are in a small neighborhood of $\mathcal{A}$. □

It is interesting to remark that the version of Theorem 10 for Morse decompositions of $M$ does not hold, as Fig. 3 illustrates.

The figure on the left shows the flow for $\lambda = 0$. The figure on the right shows the flow for $\lambda > 0$. For $\lambda = 0$ all orbits are attracted by the point in the center, with the exception of three fixed points at the boundary and the four outer orbits, attracted, each, by one of these three points. For $\lambda > 0$ the behavior at $\partial E$ is the same as for $\lambda = 0$ but there are orbits of points in $\mathring{E}$ which are attracted, or repelled, by the three fixed points at the boundary. The set of these orbits shrink as $\lambda$ approaches 0. This family of flows satisfies:

i) The family of flows is uniformly dissipative.
ii) If $M$ is the maximal compact invariant set of $\varphi_0$ in $\partial E$, there is a Morse decomposition $\mathcal{M} = \{M_1, M_2, M_3\}$ for $\varphi_0|_M$, there is a neighborhood $U$ of $M_1 \cup M_2 \cup M_3$ in $E$ and a $\lambda_0 \in I$, such that for every $x \in U - \partial E$ and every $\lambda \leqslant \lambda_0$ there is a $t \in \mathbb{R}$ (not necessarily positive) such that $\varphi_\lambda(x, t) \notin U$.
iii) $\varphi_0$ is uniformly persistent but uniform persistence at $\lambda = 0$ does not continue uniformly for bounded orbits in the sense of Theorem 10.

As a consequence of Theorem 10 we obtain the following corollary. We denote by $W_\lambda^-(U)$ the set of all points $x \in E$ whose $\alpha$-limit for the flow $\varphi_\lambda$ is nonempty and contained in $U$. In the particular case when $U$ is the neighborhood of $M$ in the statement of Theorem 10 then, for $\lambda$ small, $W_\lambda^-(U) = W_\lambda^-(M^\lambda)$ where $M^\lambda$ is the continuation of $M$ for the flow $\varphi_\lambda$.

**Corollary 11.** *With the same hypothesis as in Theorem 10, suppose in addition that the following condition is fulfilled: if $x \in \mathring{E}$ and $x \in W_0^-(U)$ then $x \in W_\lambda^-(U)$ for $\lambda \leqslant \lambda_0$. Then uniform persistence continues uniformly for bounded orbits of $\varphi_0$, i.e. there exist $\beta > 0$ and $\lambda_1 \in I$*

*such that* $\liminf_{t\to\infty} d(\varphi_\lambda(x,t), \partial E) > \beta$ *for every* $\lambda \leqslant \lambda_1$ *and every* $x \in \mathring{E}$ *such that* $\gamma_0(x)$ *is bounded.*

**Proof.** Suppose $x \in \mathring{E}$ and $\gamma_0(x)$ is bounded. If $x \in W_0^-(U)$ then $x \in W_\lambda^-(U)$ and $\gamma_\lambda(x)$ is bounded, hence the statement is a consequence of Theorem 10. If $x \notin W_0^-(U)$ then $x \in \mathcal{A}$ (the internal global attractor of $\varphi_0$) and the result is a consequence of the property of weak continuation of uniform persistence, proved in [43, Theorem 5], according to which, uniform persistence continues uniformly for all compacta contained in $\mathring{E}$. □

## 3. Generalized Poincaré–Andronov–Hopf bifurcations at boundary points of an *n*-manifold

Seibert and Seibert & Florio, following previous work by Marchetti, Negrini, Salvadori and Scalia, studied in a series of papers the bifurcations of dynamical systems resulting from a change in the stability behavior of a fixed equilibrium (see [27,44–46]). In particular they studied those bifurcations which are a consequence of transition from asymptotic stability to complete instability (without requiring that the bifurcating orbits are periodic). Suppose $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in [0,1]$, is a parameterized family of flows and $p \in E$ is a rest point of $\varphi_\lambda$ for every $\lambda \in I$. If $p$ is an attractor for $\lambda = 0$ and a repeller for $\lambda > 0$ then, according to Seibert and Florio [45], a generalized Poincaré–Andronov–Hopf bifurcation takes place at $p$.

We shall see that these bifurcations play a relevant role in the theory of uniformly persistent flows. The following result describes the topological properties of generalized Poincaré–Andronov–Hopf bifurcations occurring at points of the boundary of $\mathbb{R}^n_+$ (or, more generally, of manifolds with boundary). See [42] for more properties of this kind of bifurcations and [31] for a general reference on the Hopf bifurcation.

**Theorem 12.** *Let* $E$ *be a connected* $n$*-manifold with boundary. Consider a continuous family of flows* $\varphi_\lambda : E \times \mathbb{R} \to E$, $\lambda \in I = [0,1]$, *and let* $p \in \partial E$ *be a rest point of* $\varphi_\lambda$ *for every* $\lambda \in I$. *Suppose* $p$ *is an attractor for* $\lambda = 0$ *and a repeller for* $\lambda > 0$. *Then for every compact neighborhood* $V$ *of* $p$ *contained in the basin of attraction of* $p$ *there is a* $\lambda_0 > 0$ *such that*:

1) *For every* $\lambda$ *with* $0 < \lambda \leqslant \lambda_0$ *there is an attractor* $K_\lambda$ *for* $\varphi_\lambda$ *contained in* $\mathring{V} - \{p\}$. *Moreover* $K_\lambda$ *attracts all points in* $V - \{p\}$.
2) $K_\lambda$ *has trivial shape.*
3) *The compactum* $S_\lambda = K_\lambda \cap \partial E$ *has the shape of* $\mathbb{S}^{n-2}$ *and* $K_\lambda / S_\lambda$ *has the shape of* $\mathbb{S}^{n-1}$.
4) $K_\lambda$ *decomposes* $E$ *into two connected components,* $\mathcal{C}_\lambda$ *and* $\mathcal{D}_\lambda$ *with* $p \in \mathcal{C}_\lambda$ *and such that* $\mathcal{C}_\lambda - \{p\}$ *is contained in the basin of attraction of* $K_\lambda$.
5) *The multivalued function* $\Phi : [0,\lambda_0] \to V$ *defined by* $\Phi(0) = \{p\}$, $\Phi(\lambda) = K_\lambda$ *(when* $\lambda \neq 0$*) is upper-semicontinuous.*

**Proof.** In [42] we have proved a similar proposition for interior points of the manifold. However, when $p$ is in the boundary, some substantial differences appear which must be carefully discussed.

Since $E$ is a manifold with boundary and all our claims (except 4)) are relative to a neighborhood of the point $p \in \partial E$, we may assume that we are working in $\mathbb{R}^n_+$. On the other hand, the proof of 4) is an easy consequence of the local arguments and the connectedness of $E$.

It is a basic fact in the Conley index theory that attractors continue. Let $(C_\lambda)$ be the continuation of $\{p\}$ for $\lambda$ small. A property of the continuation is that if $L$ is a compact set contained in the region of attraction of $\{p\}$ for $\varphi_0$ then $L$ is contained in the region of attraction of $C_\lambda$ for $\varphi_\lambda$ with $\lambda$ sufficiently small. This has been proved in [42, Theorem 1]. Thus, if $V$ is a compact neighborhood of $p$ contained in the basin of attraction of $p$ there is a $\lambda_0$ such that $C_\lambda$ attracts $V$ for $0 < \lambda \leqslant \lambda_0$. Since $(C_\lambda)$ is a continuation, we may also assume that $C_\lambda$ is contained in the interior of a closed ball $B[p,\epsilon] \subset \mathring{V}$ for such parameter values. Since $\{p\}$ is now a repeller, its basin of repulsion $\mathcal{R}_\lambda$ (which is an open set in $E$) must be contained in $C_\lambda$. We define $K_\lambda = C_\lambda - \mathcal{R}_\lambda$. The pair $(K_\lambda, \{p\})$ is an attractor–repeller decomposition of $C_\lambda$ (where $K_\lambda$ is an attractor for $\varphi_\lambda|_{C_\lambda}$ and, hence, also for $\varphi_\lambda$). Now, by [24], the inclusion of $K_\lambda$ in its region of attraction is a shape equivalence, so $K_\lambda$ is shape dominated by $B[p,\epsilon] - \{p\}$ (by means of a shape morphism which is an inverse of the inclusion $i : K_\lambda \to B[p,\epsilon] - \{p\}$) since $B[p,\epsilon] - \{p\}$ is contained in such a basin of attraction. Notice that $B[p,\epsilon] - \{p\}$ has trivial shape. Hence $K_\lambda$, being shape dominated by a point, has, in fact, the shape of a point. Moreover, the inclusion

$$i : K_\lambda \cap \partial E \to \big(B[p,\epsilon] - \{p\}\big) \cap \partial E \simeq \mathbb{S}^{n-2}$$

is also a shape equivalence whose inverse is defined by the flow. The argument is exactly the same as in Theorem 1 in [42], where generalized Poincaré–Andronov–Hopf bifurcations in interior points of manifolds are studied. Since $K_\lambda$ and $B^{n-1}$ have trivial shape then by Mardešić's results about the shape of pairs (see [30]) we have that $Sh(K_\lambda, K_\lambda \cap \partial E) = Sh(B^{n-1}, \mathbb{S}^{n-2})$. Hence, also by Mardešić's results about the shape of quotients,

$$Sh(K_\lambda / K_\lambda \cap \partial E) = Sh\big(B^{n-1}/\mathbb{S}^{n-2}\big) = Sh\big(\mathbb{S}^{n-1}\big).$$

Now, by using a general form of Lefschetz duality (see [8]), we have that

$$\check{H}^{n-j}(K_\lambda, K_\lambda \cap \partial E) \to H_j\big(B[p,\epsilon], B[p,\epsilon] - K_\lambda\big)$$

(where $\check{H}^*$ stands for Čech cohomology) is an isomorphism for every $j$. In particular $H_1(B[p,\epsilon], B[p,\epsilon] - K_\lambda) \simeq \mathbb{Z}$ and, using the long homology sequence,

$$\cdots \to H_1\big(B[p,\epsilon]\big) \to H_1\big(B[p,\epsilon], B[p,\epsilon] - K_\lambda\big) \to \tilde{H}_0\big(B[p,\epsilon] - K_\lambda\big) \to \tilde{H}_0\big(B[p,\epsilon]\big) \to \cdots$$

we deduce that $\tilde{H}_0(B[p,\epsilon] - K_\lambda) \simeq \mathbb{Z}$ and, thus, $B[p,\epsilon] - K_\lambda$ is composed of two connected components. Since $E$ is connected, it follows from this that $K_\lambda$ decomposes $E$ into two connected components. The bounded component $\mathcal{C}_\lambda$ is contained in $B[p,\epsilon]$, thus $\mathcal{C}_\lambda - \{p\}$ is contained in the basin of attraction of $K_\lambda$.

The upper-semicontinuity of $\Phi : [0, \lambda_0] \to B_\epsilon(p)$ at 0 follows from the fact that $C_\lambda$ (and hence also $K_\lambda$) is contained in the open ball $B_\delta(p)$ for $\delta$ arbitrarily small provided we take $\lambda$ sufficiently close to 0. More generally, the upper semicontinuity at $\lambda \neq 0$ is a consequence of the upper-semicontinuity properties of the continuation of attractors $C_\lambda$ and the continuation of attractor–repeller decompositions $(K_\lambda, \{p\})$. However, a detailed argument can be given by following exactly the same lines as in the similar proposition for interior points proved in [42, Theorem 1]. □

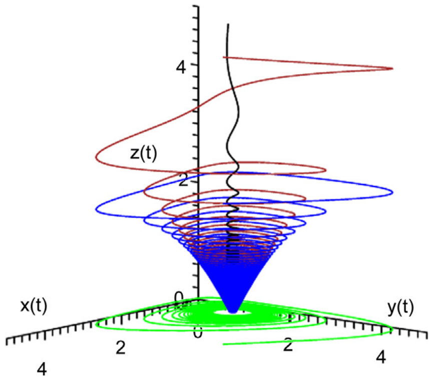


Fig. 4. The flow $\varphi_K$ for $K = 5$.

An important remark, which follows from the proof, is that $S_\lambda = K_\lambda \cap \partial E$ is, in fact, an attractor for the restriction flow $\varphi_\lambda|_{\partial E}$. We call it the *corona* of the Poincaré–Andronov–Hopf bifurcation at the parameter value $\lambda$. It turns out that the permanence properties of the flow after the bifurcation depend heavily on this corona. We shall refer to the attractor $K_\lambda$ produced by the bifurcation as the *Hopf attractor*.

We see here an example of this kind of bifurcations. It is constructed by combining a Holling-type interaction with a pitchfork bifurcation. It consists of a parameterized family of flows $\varphi_\lambda : \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{R}^3_+$ where $\varphi_0$ presents an extreme case of non-permanence: the whole interior is attracted by a point in the boundary. Examples of this kind are common in population dynamics.

Consider the system modeled by the equations:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - y\frac{cx}{a+x},$$

$$\dot{y} = y\left(-d + \frac{bx}{a+x}\right),$$

$$\dot{z} = \frac{25(K-5)z - z^3}{s}$$

in $\mathbb{R}^3_+$, where all parameters are positive. In particular, we fix the values of $a = 3$, $b = 4$, $c = 4$, $d = 1$, $r = 2$, $s = 4 \times 10^3$, and study the behavior of the system as we vary the value of $K$.

According to [21], since $b > d$, whenever $K > \frac{ad}{b-d} = 1$, restricted to the $xy$-plane, the system admits a unique interior fixed point $(x_0, y_0)$ with $x_0 = \frac{ad}{b-d} = 1$. Moreover, $(x_0, y_0)$ is a global attractor for the flow restricted to the open $xy$-quadrant if and only if $K \leqslant a + 2x_0 = 5$. On the other hand, the flow restricted to the $z$-axis has $z = 0$ as a global attractor whenever $K \leqslant 5$. Therefore, if $K \leqslant 5$, the point $(x, y, 0)$ is a global attractor for the flow restricted to $\{(x, y, z) \in \mathbb{R}^3_+ \mid x, y > 0\}$. In particular, for $K = 5$, we have the situation depicted in Fig. 4.

At $K = 5$, the flow restricted to the $xy$-plane experiments a Hopf bifurcation, $(x_0, y_0)$ becomes a source and a limit cycle around it appears. On the other hand, the flow restricted to the $z$-axis suffers a pitchfork bifurcation, $z = 0$ becomes a source and a stable fixed point appears whose attraction basin is all the set $z > 0$. Therefore, the whole 3-dimensional flow experiments

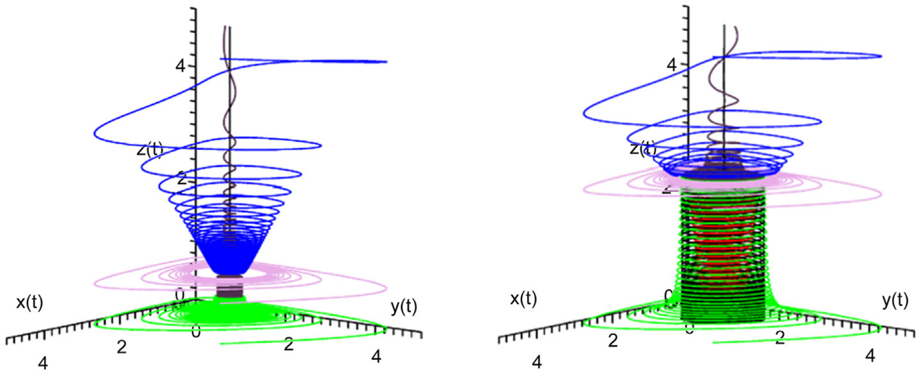


Fig. 5. The flow $\varphi_K$ for $K = 5.02$ and $K = 5.2$.

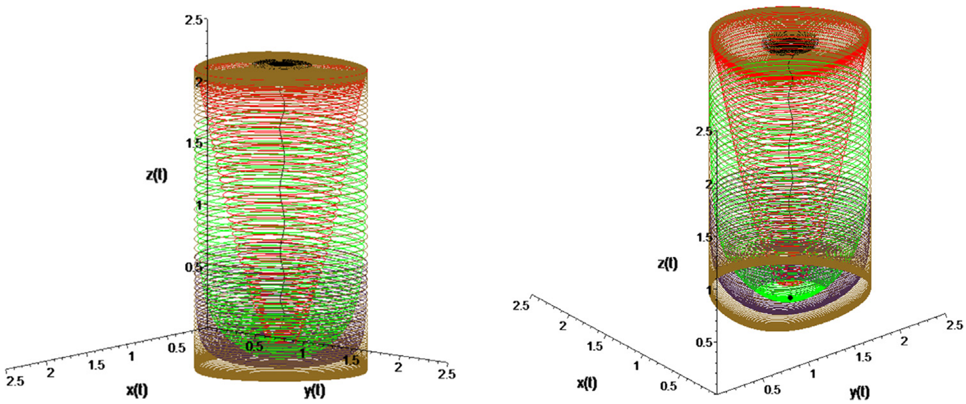


Fig. 6. The Hopf attractor and the corona for $K = 5.2$.

a generalized Poincaré–Andronov–Hopf bifurcation: the point $(x_0, y_0, 0)$ becomes a repeller, while an attractor, for the whole flow in $\mathbb{R}^3_+$ appears.

In Fig. 5 we show the behavior of the flow for some values of $K > 5$.

The attractor appearing for $K > 5$ is the surface of a cylinder without the (open) lower basis (see Fig. 6). It is the Hopf attractor defined after the proof of Theorem 10. Its region of attraction is $\{(x, y, z) \in \mathbb{R}^3_+ \mid x, y > 0\}$ except the fixed unstable point in the $xy$-plane.

On the other hand, the boundary of the lower basis is the corona, which is an attractor for the flow restricted to the boundary of $\mathbb{R}^3_+$. Its region of attraction is the set $\{(x, y, 0) \in \mathbb{R}^3_+ \mid x, y > 0\}$.

We observe in this example that after the Hopf bifurcation takes place, the flow is transformed into a uniformly persistent flow: the top of the cylinder attracts all points in the interior and is an internal global attractor. This illustrates a general fact: uniform persistence can be achieved through Hopf bifurcations.

Notice that the family of flows in the example is not uniformly dissipative. However, if we exclude the coordinate planes $xz$ and $yz$, i.e. if we consider the flows defined only in the manifold $\{(x, y, z) \in \mathbb{R}^3_+ \mid x, y > 0\}$ (which is invariant for all flows) then the family becomes uniformly dissipative. Moreover the fact that $\varphi_\lambda$ is uniformly persistent for $\lambda > 0$ depends only on the behavior of $\varphi_\lambda$ in this invariant manifold. In this form, the example falls as a simple particular case of Theorem 13.

We consider now the general case. We give conditions under which a flow with extreme non-permanence properties becomes uniformly persistent after a Poincaré–Andronov–Hopf bifurcation takes place.

**Theorem 13.** *Let $\varphi_\lambda : E \times \mathbb{R} \to E$ be a uniformly dissipative family of flows defined in the $n$-dimensional manifold with boundary, $E$, and let $M$ be the maximal compact invariant set for $\varphi_0$. Suppose that $M \subset \partial E$ (i.e. the global attractor is in the boundary) and let $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ be a Morse decomposition of $M$, where $M_1 = \{p\}$. Assume that a generalized Poincaré–Andronov–Hopf bifurcation takes place at $p$ and that the following conditions hold*: 1) *$M_2, \ldots, M_n$ are uniformly externally repelling and* 2) *the corona $S_\lambda$ is externally repelling for every $\varphi_\lambda$ with $0 < \lambda \leqslant \lambda_0$. Then $\varphi_\lambda$ is uniformly persistent for $0 < \lambda \leqslant \lambda_0$ and the Hopf attractor $K_\lambda$ has itself an attractor repeller decomposition $(A_\lambda, S_\lambda)$ where $A_\lambda \subset \mathring{E}$ is the internal global attractor and $S_\lambda$ is the corona of the bifurcation. Reciprocally, conditions* 1) *and* 2) *are necessary for the uniform persistence of $\varphi_\lambda$ for $0 < \lambda \leqslant \lambda_0$.*

**Proof.** The maximal compact invariant set, $M$, is an attractor for $\varphi_0$. Consider a continuation $M^\lambda$ of $M$ and also a continuation $\mathcal{M}^\lambda = \{M_1^\lambda, M_2^\lambda, \ldots, M_n^\lambda\}$ of the Morse decomposition $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$. By an argument previously used, the fact that the family $\varphi_\lambda$ is uniformly dissipative implies that $M^\lambda$ is a global attractor for $\varphi_\lambda$. Now, if $x \in \mathring{E}$ then $\omega_\lambda(x) \subset M^\lambda$ and, thus, $\omega_\lambda(x)$ must be contained in some Morse set $M_i^\lambda$. However, since $M_2, \ldots, M_n$ are uniformly externally repelling and $M_2^\lambda, \ldots, M_n^\lambda$ are respectively contained in repelling neighborhoods of those sets, necessarily $\omega_\lambda(x) \subset M_1^\lambda$, i.e. $M_1^\lambda$ attracts $\mathring{E}$. For the same reason $M_2^\lambda, \ldots, M_n^\lambda$ are contained in the boundary $\partial E$. Hence $\{M_1^\lambda \cap \partial E, M_2^\lambda, \ldots, M_n^\lambda\}$ is a Morse decomposition of the compact maximal set in the boundary for $\varphi_\lambda|_{\partial E}$. Now it is easy to see that the external repulsion condition satisfied by the corona $S_\lambda$, together with the fact that $K_\lambda$ attracts $\mathcal{C}_\lambda - \{p\}$, imply that $M_1^\lambda \cap \partial E$ is externally repelling. Hence we can apply the Butler–Waltman theorem to the flow $\varphi_\lambda$ and the Morse decomposition $\{M_1^\lambda \cap \partial E, M_2^\lambda, \ldots, M_n^\lambda\}$ of the compact maximal set in the boundary. According to this $\varphi_\lambda$ is uniformly persistent and its internal global attractor must be in $M_1^\lambda \cap \mathring{E}$. Since $p$ is a repeller for $\varphi_\lambda$ and $K_\lambda$ is its dual attractor for $\varphi_\lambda|_{M_1^\lambda}$, then $A_\lambda$ must necessarily be contained in $K_\lambda$ and, since $S_\lambda$ is invariant and $K_\lambda - S_\lambda$ is attracted by $A_\lambda$, then $S_\lambda$ is the dual repeller of $A_\lambda$ for $\varphi_\lambda|_{K_\lambda}$. On the other hand, conditions 1) and 2) are obviously necessary for the uniform persistence of $\varphi_\lambda$ for $0 < \lambda \leqslant \lambda_0$. □

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## References

[1] N.P. Bhatia, G.P. Szegő, Stability Theory of Dynamical Systems, Grundlehren Math. Wiss., vol. 16, Springer, Berlin, 1970.
[2] K. Borsuk, Theory of Shape, Monogr. Mat., vol. 59, Polish Scientific Publishers, Warsaw, 1975.
[3] G. Butler, H.I. Freedman, P. Waltman, Uniformly persistent systems, Proc. Amer. Math. Soc. 96 (1986) 425–430.
[4] G. Butler, P. Waltman, Persistence in dynamical systems, J. Differential Equations 63 (1986) 255–263.
[5] C. Conley, Isolated Invariant Sets and the Morse Index, CBMS Reg. Conf. Ser. Math., vol. 38, American Mathematical Society, Providence, RI, 1978.

[6] C. Conley, E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations, Comm. Pure Appl. Math. 37 (1984) 207–253.
[7] J.M. Cordier, T. Porter, Shape Theory. Categorical Methods of Approximation, Ellis Horwood Ser.: Math. Appl., Ellis Horwood Ltd./Halsted Press [John Wiley & Sons, Inc.], Chichester/New York, 1989.
[8] A. Dold, Lectures on Algebraic Topology, Springer, 1995.
[9] J. Dydak, J. Segal, Shape Theory. An Introduction, Lecture Notes in Math., vol. 688, Springer, Berlin, 1978.
[10] B.M. Garay, Uniform persistence and chain recurrence, J. Math. Anal. Appl. 139 (1989) 372–381.
[11] B.M. Garay, J. Hofbauer, Robust permanence for ecological differential equations, minimax, and discretizations, SIAM J. Math. Anal. 34 (2003) 1007–1039.
[12] A. Giraldo, M. Morón, F.R. Ruiz del Portal, J.M.R. Sanjurjo, Shape of global attractors in topological spaces, Nonlinear Anal. 60 (5) (2005) 837–847.
[13] A. Giraldo, J.M.R. Sanjurjo, On the global structure of invariant regions of flows with asymptotically stable attractors, Math. Z. 232 (4) (1999) 739–746.
[14] A. Giraldo, J.M.R. Sanjurjo, Singular continuations of attractors, SIAM J. Appl. Dyn. Syst. 8 (2009) 554–575.
[15] B. Günther, J. Segal, Every attractor of a flow on a manifold has the shape of a finite polyhedron, Proc. Amer. Math. Soc. 119 (1993) 321–329.
[16] J.K. Hale, Stability and gradient dynamical systems, Rev. Mat. Complut. 17 (2004) 7–57.
[17] H.M. Hastings, A higher-dimensional Poincaré–Bendixson theorem, Glas. Mat. Ser. III 14 (34) (2) (1979) 263–268.
[18] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.
[19] M.W. Hirsch, H.L. Smith, X.-Q. Zhao, Chain transitivity, attractivity, and strong repellors for semidynamical systems, J. Dynam. Differential Equations 13 (2001) 107–131.
[20] J. Hofbauer, A unified approach to persistence, Acta Appl. Math. 14 (1989) 11–22.
[21] J. Hofbauer, K. Sigmund, The Theory of Evolution and Dynamical Systems. Mathematical Aspects of Selection, London Math. Soc. Stud. Texts, vol. 7, Cambridge University Press, Cambridge, 1988.
[22] S.T. Hu, Theory of Retracts, Wayne State Universite Press, Detroit, 1965.
[23] V. Hutson, The stability under perturbations of repulsive sets, J. Differential Equations 76 (1988) 77–90.
[24] L. Kapitanski, I. Rodnianski, Shape and Morse theory of attractors, Comm. Pure Appl. Math. 53 (2000) 218–242.
[25] D. Li, Morse theory of attractors for infinite dimensional dynamical systems via Lyapunov functions, preprint.
[26] D. Li, A. Qi, Morse equation of attractors for nonsmooth dynamical systems, J. Differential Equations 253 (11) (2012) 3081–3100.
[27] F. Marchetti, P. Negrini, L. Salvadori, M. Scalia, Liapunov direct method in approaching bifurcation problems, Ann. Mat. Pura Appl. (4) 108 (1976) 211–226.
[28] S. Mardešić, J. Segal, Shape Theory. The Inverse System Approach, North-Holland Math. Library, vol. 26, North-Holland Publishing Co., Amsterdam–New York, 1982.
[29] S. Mardešić, Strong Shape and Homology, Springer Monogr. Math., Springer-Verlag, Berlin, 2000.
[30] S. Mardešić, Pairs of compacta and trivial shape, Trans. Amer. Math. Soc. 189 (1971) 329–330.
[31] J.E. Marsden, M. McCracken, The Hopf Bifurcation and Its Applications, Springer, New York, 1976.
[32] M.A. Morón, J.J. Sánchez-Gabites, J.M.R. Sanjurjo, Topology and dynamics of unstable attractors, Fund. Math. 197 (2007) 239–252.
[33] J.W. Robbin, D. Salamon, Dynamical systems, shape theory and the Conley index, in: Charles Conley Memorial Volume, Ergodic Theory Dynam. Systems 8 (1988) 375–393.
[34] J.C. Robinson, Global attractors: topology and finite-dimensional dynamics, J. Dynam. Differential Equations 11 (3) (1999) 557–581.
[35] J.C. Robinson, Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 2001.
[36] D. Salamon, Connected simple systems and the Conley index of isolated invariant sets, Trans. Amer. Math. Soc. 291 (1985) 1–41.
[37] J.J. Sanchez-Gabites, Unstable attractors in manifolds, Trans. Amer. Math. Soc. 362 (2010) 3563–3589.
[38] J.J. Sanchez-Gabites, An approach to the shape Conley index without index pairs, Rev. Mat. Complut. 24 (2011) 95–114.
[39] J.J. Sánchez-Gabites, How strange can an attractor for a dynamical system in a 3-manifold look?, Nonlinear Anal. 74 (17) (2011) 6162–6185.
[40] J.M.R. Sanjurjo, On the structure of uniform attractors, J. Math. Anal. Appl. 192 (1995) 519–528.
[41] J.M.R. Sanjurjo, Morse equations and unstable manifolds of isolated invariant sets, Nonlinearity 16 (2003) 1435–1448.
[42] J.M.R. Sanjurjo, Global topological properties of the Hopf bifurcation, J. Differential Equations 243 (2007) 238–255.

[43] J.M.R. Sanjurjo, On the fine structure of the global attractor of a uniformly persistent flow, J. Differential Equations 252 (2012) 4886–4897.
[44] P. Seibert, Dynamical persistence principles and bifurcation, Univ. Iagel. Acta Math. XXXVI (1998) 65–77.
[45] P. Seibert, J.S. Florio, On the foundations of bifurcation theory, Nonlinear Anal. 22 (1994) 927–944.
[46] P. Seibert, J. Florio, On bifurcations arising from unstable equilibria and invariant sets, J. Differential Equations 150 (1998) 250–263.
[47] S.J. Schreiber, Criteria for $C^r$-robust permanence, J. Differential Equations 162 (2000) 400–426.
[48] H.L. Smith, H.R. Thieme, Dynamical Systems and Population Persistence, Grad. Stud. Math., AMS, Providence, RI, 2011.
[49] E. Spanier, Algebraic Topology, McGraw–Hill, New York, 1966.
[50] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer, Berlin, 1988.
[51] K. Wójcik, Conley index and permanence in dynamical systems, Topol. Methods Nonlinear Anal. 12 (1998) 153–158.