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# The Vlasov–Maxwell–Boltzmann system with a uniform ionic background near Maxwellians

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**Abstract**

The two-species Vlasov–Maxwell–Boltzmann system is an important model for plasma physics describing the time evolution of dilute charged particles consisting of electrons and ions under the influence of the self-consistent internally generated Lorentz forces. In physical situations the ion mass is usually much larger than the electron mass so that the electrons move much faster than the ions. Thus, the ions are often described by a fixed ion background and only the electrons move. For such a simple case, the two-species Vlasov–Maxwell–Boltzmann system is reduced to the one-species Vlasov–Maxwell–Boltzmann system.

Although the one-species Vlasov–Maxwell–Boltzmann system is a simplified model of the two-species Vlasov–Maxwell–Boltzmann system, its global well-posedness theory near a given global Maxwellian in the perturbative framework is more difficult than the two-species case, which is partially due to the slow-decay of the electromagnetic field and up to now, the problem on the construction of global in time solutions near a given global Maxwellian in the perturbative framework for the Cauchy problem of the one-species Vlasov–Maxwell–Boltzmann system with cutoff non-hard sphere intermolecular collisions remains unsolved. It is shown in this paper that the Cauchy problem of the one-species Vlasov–Maxwell–Boltzmann system with cutoff non-hard sphere intermolecular collisions including the cutoff inverse power law potentials is globally well-posed provided that the perturbative initial data satisfies certain regularity, smallness, and integrability conditions. Our analysis is based on a new time-velocity weighted energy method with two key technical parts: one is to introduce the exponentially weighted estimates into the cutoff Boltzmann operator and the other is to design a delicate temporal energy $X(t)$-norm to obtain its uniform bound. As a by-product of our analysis, we can also deduce certain temporal decay estimates on the global solutions constructed above for the same range of intermolecular collisions. Notice that even for the hard sphere

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model, although the global solutions have been successfully constructed in [8], its large time behaviors are unknown, cf. [8].
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## 1. Introduction and main results

The Vlasov–Maxwell–Boltzmann system (called VMB system in the sequel for simplicity) is an important model for plasma physics to describe the time evolution of dilute charged particles (e.g., electrons and ions in the case of two-species) under the influence of the self-consistent internally generated Lorentz forces, cf. [4–6,27,28,31,35]:

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + (E+v\times B)\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_+.F_-),\\
&\partial_t F_- + v\cdot\nabla_x F_- - (E+v\times B)\cdot\nabla_v F_- = Q(F_-,F_+) + Q(F_-.F_-).
\end{aligned}\tag{1.1}$$

Here $F_\pm(t,x,v)\geq 0$ denote the number density functions for the ions $(+)$ and electrons $(-)$ respectively, at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$, and the self-consistent electromagnetic field $[E(t,x),B(t,x)]$ in (1.1) is coupled with $F_\pm(t,x,v)$ through the celebrated Maxwell system:

$$\begin{aligned}
&\partial_t E - \nabla_x\times B = -\int_{\mathbb{R}^3} v\,(F_+ - F_-)\,dv,\\
&\partial_t B + \nabla_x\times E = 0
\end{aligned}\tag{1.2}$$

with constraints

$$\nabla_x\cdot E = \int_{\mathbb{R}^3}(F_+ - F_-)\,dv,\quad \nabla_x\cdot B = 0.\tag{1.3}$$

In physical situations the ion mass is usually much larger than the electron mass so that the electrons move much faster than the ions. Thus, the ions are often described by a fixed ion background $n_b(x)$ and only the electrons move. For such a simple case, the two-species VMB system (1.1), (1.2), (1.3) is then reduced to the following one-species VMB system

$$\begin{aligned}
&\partial_t F + v\cdot\nabla_x F + (E+v\times B)\cdot\nabla_v F = Q(F,F),\\
&\partial_t E - \nabla_x\times B = -\int_{\mathbb{R}^3} vF\,dv,\\
&\partial_t B + \nabla_x\times E = 0,\\
&\nabla_x\cdot E = \int_{\mathbb{R}^3} F\,dv - n_b(x),\quad \nabla_x\cdot B = 0.
\end{aligned}\tag{1.4}$$

Here the unknown $F = F(t,x,v) \geq 0$ stands for the number distribution function of the electrons which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$, $E(t,x)$ and $B(t,x)$ denote the electric and magnetic fields, respectively. $n_b(x)$ is the density of the ionic background and is assumed throughout this paper to be a positive constant which means that it is spatially uniform.

Notice that systems (1.1), (1.2), (1.3), and (1.4) in general contain physical constants such as the magnitude of the charges $e_\pm$, the masses $m_\pm$, and diameters $\sigma_i$ of the two types of particles, the speed of light $c$, etc., cf. [22]. Since our purpose in this paper is to construct global solutions near a given global Maxwellian to the Vlasov–Maxwell–Boltzmann system, without loss of generality, all those physical constants evolved in systems (1.1), (1.2), (1.3), and (1.4) together with some generic positive constants such as $4\pi$ have been normalized to be one for notational simplicity.

The bilinear Boltzmann collision operator $Q(G_1, G_2)$ is defined by (cf. [4–6]):

$$Q(G_1, G_2)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |u - v|^\gamma q_0(\theta) \left\{ G_1(u') G_2(v') - G_1(u) G_2(v) \right\} du d\omega.$$

Here $\omega \in \mathbb{S}^2$, and

$$v' = v - [(v-u)\cdot\omega]\omega, \quad u' = u + [(v-u)\cdot\omega]\omega,$$

which denote velocities after a collision of particles having velocities $v, u$ before the collision and vice versa. Notice that the relation between $v, u$ and $v', u'$ implies the conservation of momentum and kinetic energy during the collision process.

The Boltzmann collision kernel $|u - v|^\gamma q_0(\theta)$ depends only on the relative velocity $|v - u|$ and on the deviation angle $\theta$ given by $\cos\theta = \langle \omega, (v-u)/|v-u| \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual dot product in $\mathbb{R}^3$. Throughout this paper, $q_0(\theta)$ is assumed to satisfy Grad's angular cutoff assumption

$$0 \leq q_0(\theta) \leq C |\cos\theta|$$

for some generic positive constant $C > 0$. The range of the parameter $\gamma$ for physical consideration is $-3 < \gamma \leq 1$ which is called hard potentials when $0 \leq \gamma \leq 1$ with the two special cases $\gamma = 1$ and $\gamma = 0$ being called hard sphere model and Maxwell molecules respectively, while the case $-3 < \gamma < 0$ is called soft potentials.

What we are interested in is the construction of global solutions near a given normalized global Maxwellian $\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ to the Cauchy problems of the two-species VMB system (1.1), (1.2), (1.3) and the one-species VMB system (1.4) with prescribed initial data

$$F_\pm(0,t,x,v) = F_{\pm 0}(x,v), \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x)$$

with constraints

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{+0} - F_{-0})\, dv, \quad \nabla_x \cdot B_0 = 0$$

for the two-species VMB system (1.1), (1.2), (1.3) and with given initial data

$$F(0,x,v)=F_0(v,x),\quad E(0,x)=E_0(x),\quad B(0,x)=B_0(x) \tag{1.5}$$

satisfying the compatibility conditions

$$\nabla_x\cdot E_0=\int_{\mathbb{R}^3}F_0dv-n_b,\quad \nabla_x\cdot B_0=0$$

for the one-species VMB system (1.4), respectively.

Such a problem has been studied extensively recently and some interesting progress has been made. To go directly to the theme of this manuscript, we only review former results concerning the construction of global solutions near Maxwellians to the two-species VMB system (1.1), (1.2), (1.3) and the one-species VMB system (1.4) as follows (for the corresponding results concerning the existence of renormalized solutions of the VMB system with a defect measure in the presence of long-range interactions and the convergence of such a sequence of renormalized solutions of the VMB system under parabolic scaling to the weak solutions of the resulting incompressible viscous electro-magneto-hydrodynamics, the interested readers are referred to [2] and the references cited therein):

- For the Cauchy problem of the two-species VMB system (1.1), (1.2), (1.3), the results for the hard sphere model are quite complete in the perturbative framework: When the initial data $[F_{\pm 0}(x,v),E_0(x),B_0(x)]$ is a periodic function of $x$, the global existence of solutions near a given global Maxwellian was firstly investigated by Guo in [22]. Then, the rate of convergence to such a global Maxwellian with any polynomial speed in large time was shown by Guo and Strain in [25,40] for the two-species VMB system in a periodic box in both the classical and relativistic situations. For the corresponding problem in the whole space, the global in time classical solutions were constructed by Strain in [37] and the optimal temporal decay estimates on the global solution constructed in [37] was studied by Duan and Strain in [14]; For the non-hard sphere intermolecular collisions, the only result available up to now is [12] which shows that the VMB system (1.1), (1.2), (1.3) with physical angular non-cutoff intermolecular collisions including the inverse power law potentials admits a unique global smooth solution which enjoys certain temporal decay estimates provided that the perturbative initial data satisfies suitable smallness, regularity, and integrability conditions;
- For the one-species VMB system (1.4), although it is considered as a simplified model of the two-species VMB system (1.1), (1.2), (1.3), its mathematical theory in the perturbative framework is more difficult than the two-species VMB system (1.1), (1.2), (1.3), which is partially due to the slow of the decay of the solutions, especially the self-consistent electromagnetic field. For its Cauchy problem, the only result availably now in the perturbative framework is obtained by Duan in [8] which deals with the hard sphere model. In [8], although a unique global solution near a given global Maxwellian is constructed for the hard sphere model, since the decay of the solutions of the linearized system is too slow, it is shown in [8] that even for the hard sphere model, it seems impossible to combine the temporal decay of the solution operator of the linearized system with the Duhamel principle to deduce the desired temporal decay of the global solution constructed in [8] as for the two-species VMB system studied in [14].

Thus, a natural question is

- For the Cauchy problem of the one-species VMB system (1.4) near Maxwellians in the whole space with cutoff non-hard sphere intermolecular collisions, how to construct a unique global solution in the perturbative framework and to deduce the desired temporal decay estimates on the global solutions obtained?

The main goal of this manuscript is devoted to such a problem. Since we are concerned with the Cauchy problem (1.4) around the normalized global Maxwellian $\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ and noticing that we can also set $n_b = 1$ without loss of generality, as in [22], if we define the perturbation $f = f(t, x, v)$ by

$$F(t, x, v) = \mu + \mu^{1/2} f(t, x, v),$$

then, the Cauchy problem of the one-species VMB system (1.4) is reformulated as

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f - E \cdot v \mu^{1/2} - \frac{1}{2} E \cdot v f + \mathbf{L} f = \mathbf{\Gamma}(f, f),\\
&\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} v \mu^{1/2} f dv,\\
&\partial_t B + \nabla_x \times E = 0,\\
&\nabla_x \cdot E = \int_{\mathbb{R}^3} \mu^{1/2} f dv, \quad \nabla_x \cdot B = 0
\end{aligned} \tag{1.6}$$

with initial data

$$f(0, x, v) = f_0(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x) \tag{1.7}$$

with constraints

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} \mu^{1/2} f_0 dv, \quad \nabla_x \cdot B_0 = 0. \tag{1.8}$$

Here $\mathbf{L} f = -\mu^{-1/2} \left[ Q\left(\mu, \mu^{1/2} f\right) + Q\left(\mu^{1/2} f, \mu\right) \right]$ and $\mathbf{\Gamma}(f, g) = \mu^{-1/2} Q\left(\mu^{1/2} f, \mu^{1/2} g\right)$ denote the linearized and nonlinear collision operators, respectively.

As in [21], the null space $\mathcal{N}$ of $\mathbf{L}$ is given by

$$\mathcal{N} = \ker \mathbf{L} = \operatorname{span}\left\{ \mu^{1/2}, v_i \mu^{1/2} (1 \le i \le 3), |v^2| \mu^{1/2} \right\}.$$

If we define $\mathbf{P}$ as the orthogonal projection from $L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$, then for any given function $f(t, x, v) \in L^2(\mathbb{R}^3_v)$, one has

$$\mathbf{P} f = \left\{ a(t, x) + v \cdot b(t, x) + \left( |v|^2 - 3 \right) c(t, x) \right\} \mu^{1/2} \tag{1.9}$$

with

$$a=\int_{\mathbb{R}^3}\mu^{1/2}f\,dv,\quad b=\int_{\mathbb{R}^3}v\mu^{1/2}f\,dv,\quad c=\frac{1}{6}\int_{\mathbb{R}^3}\left(|v|^2-3\right)\mu^{1/2}f\,dv. \tag{1.10}$$

Therefore, we have the following macro–micro decomposition with respect to the given global Maxwellian $\mu(v)$ which was introduced in [23]

$$f(t,x,v)=\mathbf{P}f(t,x,v)+\{\mathbf{I}-\mathbf{P}\}f(t,x,v) \tag{1.11}$$

where $\mathbf{I}$ denotes the identity operator. $\mathbf{P}f$ and $\{\mathbf{I}-\mathbf{P}\}f$ are called the macroscopic component and the microscopic component of $f(t,x,v)$, respectively.

Before stating our main results, we first introduce some notations used throughout this manuscript. $C$ denotes some positive constant (generally large) and $\varepsilon$, $\kappa$, $\lambda$ stand for some positive constants (generally small), where $C$, $\varepsilon$, $\kappa$, and $\lambda$ may take different values in different places. $A\lesssim B$ means that there is a generic constant $C>0$ such that $A\le CB$. $A\sim B$ means $A\lesssim B$ and $B\lesssim A$. The multi-indices $\alpha=[\alpha_1,\alpha_2,\alpha_3]$ and $\beta=[\beta_1,\beta_2,\beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial_\beta^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3}$. Similarly, the notation $\partial^\alpha$ will be used when $\beta=0$ and likewise for $\partial_\beta$. The length of $\alpha$ is denoted by $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. $\alpha'\le\alpha$ means that no component of $\alpha'$ is greater than the corresponding component of $\alpha$, and $\alpha'<\alpha$ means that $\alpha'\le\alpha$ and $|\alpha'|<|\alpha|$. And it is convenient to write $\nabla_x^k=\partial^\alpha$ with $|\alpha|=k$. We also use $\langle\cdot,\cdot\rangle$ to denote the $L_v^2$ inner product in $\mathbb{R}_v^3$, with the $L^2$ norm $|\cdot|_2$. For notational simplicity, $(\cdot,\cdot)$ denotes the $L^2$ inner product either in $\mathbb{R}_x^3\times\mathbb{R}_v^3$ or in $\mathbb{R}_x^3$ with the $L^2$ norm $\|\cdot\|$ and $(f(t,\xi,v)\mid g(t,\xi,v))=(f(t,\xi,v),\bar g(t,\xi,v))=\int_{\mathbb{R}_\xi^3\times\mathbb{R}_v^3}f(t,\xi,v)\bar g(t,\xi,v)d\xi dv$. Here $\bar g(t,\xi,v)$ denotes the complex conjugate of $g(t,\xi,v)$.

As in [26], for $s\in\mathbb{R}$, we introduce the operator $\Lambda^s$ by

$$\left(\Lambda^s g\right)(t,x,v)=\int_{\mathbb{R}^3}|\xi|^s\hat g(t,\xi,v)e^{2\pi ix\cdot\xi}d\xi=\int_{\mathbb{R}^3}|\xi|^s\mathcal{F}[g](t,\xi,v)e^{2\pi ix\cdot\xi}d\xi$$

with $\hat g(t,\xi,v)\equiv\mathcal{F}[g](t,\xi,v)$ being the Fourier transform of $g(t,x,v)$ with respect to $x$. The homogeneous Sobolev space $\dot H^s$ is the Banach space consisting of all $g$ satisfying $\|g\|_{\dot H^s}<+\infty$, where

$$\|g(t)\|_{\dot H^s}\equiv\left\|\left(\Lambda^s g\right)(t,x,v)\right\|_{L^2_{x,v}}=\left\||\xi|^s\hat g(t,\xi,v)\right\|_{L^2_{\xi,v}}.$$

Similarly, $\dot H_x^m=\dot H_x^m\left(\mathbb{R}_x^3\right)$ is the homogeneous Sobolev space defined on $\mathbb{R}_x^3$ equipped with norm

$$\|f\|^2_{\dot H^m(\mathbb{R}_x^3)}\equiv\int_{\mathbb{R}_k^3}|k|^{2m}\left|\hat f(k)\right|^2dk.$$

We will also use $H^s\left(\mathbb{R}^3\right)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$. We shall use $\|\cdot\|_p$ to record $L^p$ norms in either $\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ or $\mathbb{R}_x^3$. Letting $w(\xi)\ge1$ be a weight function, we use $\|\cdot\|_{p,w}$ to denote the weighted $L^p$ norms in $\mathbb{R}_x^3\times\mathbb{R}_\xi^3$ or $\mathbb{R}_x^3$. We also use $|\cdot|_p$ to denote $L^p$

norms in $\mathbb{R}^3_\xi$, and $|\cdot|_{p,w}$ for the weighted $L^p$ norms in $\mathbb{R}^3_\xi$. We will use the mixed spatial-velocity spaces, e.g., $L^2_\xi H^s_x = L^2\left(\mathbb{R}^3_\xi; H^s\left(\mathbb{R}^3_x\right)\right)$, etc.

For $l \in \mathbb{R}$, $\langle v\rangle = \sqrt{1+|v|^2}$, $L^2_l$ denotes the weighted space with norm $|f|^2_{L^2_l} \equiv \int_{\mathbb{R}^3_v} |f(v)|^2 \langle v\rangle^{2l} dv$ and set

$$|f|_\nu = |f|_{L^2_{\gamma/2}} = \left|\langle v\rangle^{\frac{\gamma}{2}} f\right|_2, \quad \|f\|_\nu = \left\|\langle v\rangle^{\frac{\gamma}{2}} f\right\|.$$

$B_C \subset \mathbb{R}^3_v$ denotes the ball of radius $C$ centered at the origin, and $L^2(B_C)$ stands for the space $L^2$ over $B_C$ and likewise for other spaces.

As in [12], we introduce the following time-velocity weight function corresponding to the Boltzmann operator

$$w_\ell = w_\ell(t,v) = \langle v\rangle^{-\gamma\ell} e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}, \qquad 0 < q \ll 1. \tag{1.12}$$

For an integer $N \geq 0$, $\ell \in \mathbb{R}$, and a given $f(t,x,v)$, we define the energy functional $\bar{\mathcal{E}}_{N,\ell}(f)$ and the corresponding energy dissipation rate functional $\bar{\mathcal{D}}_{N,\ell}(f)$ by

$$\bar{\mathcal{E}}_{N,\ell}(t) \sim \mathcal{E}_{N,\ell}(t) + \left\|\Lambda^{-s}(f,E,B)\right\|^2 \tag{1.13}$$

and

$$\bar{\mathcal{D}}_{N,\ell}(t) \sim \mathcal{D}_{N,\ell}(t) + \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + \left\|\Lambda^{-s}a\right\|^2 + \left\|\Lambda^{1-s}(a,b,c,E)\right\|^2 + \left\|\Lambda^{2-s}B\right\|^2, \tag{1.14}$$

respectively. Here

$$\mathcal{E}_{N,\ell}(t) \sim \sum_{|\alpha|+|\beta|\leq N} \left\|w_{\ell-|\beta|}\partial^\alpha_\beta f\right\|^2 + \|(E,B)\|^2_{H^N}, \tag{1.15}$$

$$\begin{aligned}\mathcal{D}_{N,\ell}(t) \sim{}& \sum_{1\leq|\alpha|\leq N} \left\|\partial^\alpha(a,b,c)\right\|^2 + \sum_{|\alpha|+|\beta|\leq N} \left\|w_{\ell-|\beta|}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + \|a\|^2 + \|\nabla_x E\|^2_{H^{N-2}} \\ &+ \left\|\nabla^2_x B\right\|^2_{H^{N-3}} + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|\leq N} \left\|\langle v\rangle w_{\ell-|\beta|}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2.\end{aligned} \tag{1.16}$$

In our analysis, we also need to define $\mathcal{E}_N(t)$ (the energy functional without weight), $\mathcal{E}^k_{N_0}(t)$ (the higher order energy functional without weight), and $\mathcal{E}^k_{N_0,\ell}(t)$ (the higher order energy functional with weight) as follows

$$\mathcal{E}_N(t) \sim \sum_{k=0}^N \left\|\nabla^k(f,E,B)\right\|^2, \tag{1.17}$$

$$\mathcal{E}^k_{N_0}(t) \sim \sum_{|\alpha|=k}^{N_0} \left\|\partial^\alpha(f,E,B)\right\|^2, \tag{1.18}$$

$$\mathcal{E}^k_{N_0,\ell}(t) \sim \sum_{\substack{|\alpha|+|\beta|\le N,\\ |\alpha|\ge k}} \left\| w_{\ell-|\beta|}\partial^\alpha_\beta f \right\|^2 + \sum_{|\alpha|=k}^{N_0} \left\| \partial^\alpha (E,B) \right\|^2, \tag{1.19}$$

and the corresponding energy dissipation rate functionals $\mathcal{D}_N(t)$, $\mathcal{D}^k_{N_0}(t)$, and $\mathcal{D}^k_{N_0,\ell}(t)$ are given by

$$\mathcal{D}_N(t) \sim \|a\|^2 + \|\nabla(a,b,c)\|^2_{H^{N-1}} + \|\nabla E\|^2_{H^{N-2}} + \|\nabla^2 B\|^2_{H^{N-3}} + \sum_{|\alpha|\le N} \left\| \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|^2_\nu, \tag{1.20}$$

$$\begin{aligned} \mathcal{D}^k_{N_0}(t) \sim{}& \left\| \nabla^k a \right\|^2 + \sum_{|\alpha|=k+1}^{N_0-1} \left\| \partial^\alpha (\mathbf{P}f, E) \right\|^2 + \sum_{|\alpha|=k+2}^{N_0-1} \left\| \partial^\alpha B \right\|^2 \\ &+ \sum_{|\alpha|=N_0} \left\| \partial^\alpha \mathbf{P} f \right\|^2 + \sum_{k\le|\alpha|\le N_0} \left\| \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|^2_\nu, \end{aligned} \tag{1.21}$$

$$\begin{aligned} \mathcal{D}^k_{N_0,\ell}(t) \sim{}& \left\| \nabla^k a \right\|^2 + \sum_{|\alpha|=k+1}^{N_0-1} \left\| \partial^\alpha (\mathbf{P}f, E) \right\|^2 + \sum_{|\alpha|=k+2}^{N_0-1} \left\| \partial^\alpha B \right\|^2 + \sum_{|\alpha|=N_0} \left\| \partial^\alpha \mathbf{P} f \right\|^2 \\ &+ \sum_{\substack{|\alpha|+|\beta|\le N,\\ |\alpha|\ge k}} \left\| w_{\ell-|\beta|}\partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right\|^2_\nu + (1+t)^{-1-\vartheta} \sum_{\substack{|\alpha|+|\beta|\le N,\\ |\alpha|\ge k}} \left\| \langle v\rangle w_{\ell-|\beta|}\partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right\|^2, \end{aligned} \tag{1.22}$$

respectively.

**Remark 1.1.** Several remarks concerning the introduction of the energy functionals and the corresponding energy dissipation rate functionals are listed below:

- The energy functionals $\mathcal{E}_N(t)$ and $\mathcal{E}^k_{N_0}(t)$ together with the corresponding energy dissipation rate functionals $\mathcal{D}_N(t)$ and $\mathcal{D}^k_{N_0}(t)$ contain only those terms with pure $x$-derivatives.
- The last terms appeared in the right hand side of the energy dissipation rate functionals $\mathcal{D}_{N,\ell}(t)$ and $\mathcal{D}^k_{N_0,\ell}(t)$ represent the extra dissipation effects induced by our introduction of the time-velocity weight $w_\ell(t,v)$ given by (1.12). Such a term plays an essential role in our analysis.
- For the temporal energy functional $\mathcal{E}_{N,\ell}(t)$, if we compare the dissipation rate functional $\mathcal{D}_{N,\ell,\lambda}(t)$ for the two-species VMB system (1.1), (1.2), (1.3) introduced in (1.12) of [12] with $\mathcal{D}_{N,\ell}(t)$ for the one-species VMB system (1.4) defined by (1.16), we can find that $\mathcal{D}_{N,\ell}(t)$ defined by (1.16) does not contain the term $\|(E,\nabla_x B)\|^2$ which implies that dissipative property of the one-species VMB system (1.4) is weaker than two-species case.

With the above preparation in hand, our main result concerning the Cauchy problem (1.6), (1.7) can be stated as follows.

**Theorem 1.1.** *Suppose that* $F_0(x,v)=\mu+\sqrt{\mu}f_0(x,v)\geq 0$, $N_0\geq 6$, $\frac{3}{2}-\frac{3}{4N_0-10}\leq s<\frac{3}{2}$, $\frac{4s+2}{2s-11}<\gamma<0$, $0<\vartheta\leq\min\left\{\frac{4s-2s\gamma+5\gamma-2}{8(1-\gamma)},\frac{4s-2s\gamma+11\gamma+2}{8(1-2\gamma)}\right\}$. *Then there exist two positive constants* $l'$ *and* $\bar{m}$ *which depend only on* $\gamma$, $s$, *and* $N_0$ *such that if we assume further that*

$$Y_0=\sum_{|\alpha|+|\beta|\leq N_0}\left\|w_{l_0+l^*-|\beta|}\partial_\beta^\alpha f_0\right\|+\sum_{|\alpha|+|\beta|\leq N}\left\|w_{l-|\beta|}\partial_\beta^\alpha f_0\right\|+\|(E_0,B_0)\|_{H^N\bigcap\dot{H}^{-s}}+\|f_0\|_{\dot{H}^{-s}}\tag{1.23}$$

*is sufficiently small, the Cauchy problem (1.6), (1.7) admits a unique global solution* $[f(t,x,v),E(t,x),B(t,x)]$ *satisfying* $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\geq 0$. *Here* $N$ *is the smallest integer satisfying* $N>\frac{5(N_0-1)}{3}$ *and* $N\geq\bar{m}+1$ *and the parameters* $l$, $l_0$ *and* $l^*$ *are suitably chosen such that* $l\geq N$, $l_0\geq l+\frac{3}{2}-\frac{1}{\gamma}$, *and* $l^*=l'+(N_0-5)\left(2-\frac{1}{\gamma}\right)+\frac{1}{2}$.

*Moreover, the global solution* $[f(t,x,v),E(t,x),B(t,x)]$ *obtained above enjoys the following temporal decay estimates:*

*(1) Taking* $k=0,1,2,\cdots,N_0-3$, *it follows that*

$$\mathcal{E}_{N_0}^k(t)\lesssim Y_0^2(1+t)^{-\frac{s+k}{2}}.\tag{1.24}$$

*(2) Let* $0\leq i\leq k\leq N_0-5$ *be an integer, it holds that*

$$\mathcal{E}^k_{N_0,l_{0,k}+\frac{i}{2}}(t)\lesssim Y_0^2(1+t)^{-\frac{k+s-i}{2}},\qquad i=0,1,\cdots,k.\tag{1.25}$$

**Remark 1.2.** Several remarks concerning Theorem 1.1 are listed below:

- In Theorem 1.1, the parameter $s$ is assumed to satisfy $s\in\left[\frac{3}{2}-\frac{3}{4N_0-10},\frac{3}{2}\right)$ with $N_0\geq 6$ and the parameter $\bar{m}$ can be taken as

$$\begin{aligned}\bar{m}=\max\Bigg\{&\chi_{N_0\geq 6}\left\{\frac{2s+5}{2s+3}N_0-\frac{4s+15}{2s+3}\right\},\chi_{N_0\geq 6}\left\{\frac{3s+3/2}{3s-1/2}N_0-\frac{3s+13/2}{3s-1/2}\right\},\\&\chi_{N_0\geq 6}\left\{\frac{2s+7}{2s+5}N_0-\frac{6}{2s+5}\right\},\chi_{N_0\geq 7}\left\{\frac{2s+7}{2s+3}N_0-\frac{6s+29}{2s+3}\right\}\Bigg\}\end{aligned}$$

  with $\chi_A$ denoting the indicator function of the set $A$ throughout this manuscript. From which one can deduce that the smallest value of $s$ is $\frac{9}{7}$ when $N_0=6$. It is easy to see that the smallest regularity index we imposed on the initial perturbation is $N=9$.
- In Theorem 1.1, the parameters $s$ and $\gamma$ are assumed to satisfy $\frac{4s+2}{2s-11}<\gamma<0$ with $\frac{9}{7}\leq s<\frac{3}{2}$. It is easy to see that when $s$ is taken sufficiently close to $\frac{3}{2}$, $\gamma$ can be chosen in the range $(-1,0)$. Moreover, although only the case for $\frac{4s+2}{2s-11}<\gamma<0$ with $\frac{9}{7}\leq s<\frac{3}{2}$ is considered in this manuscript, the case of $0\leq\gamma\leq 1$ is simpler and the argument used here can be easily adopted to deal with such a case.
- It is worth pointing out that the largest value of the parameter $k$ appeared in $\mathcal{E}_{N_0}^k(t)$ in (1.24) is $N_0-3$ while the corresponding largest value of the parameter $k$ appeared in $\mathcal{E}^k_{N_0,\ell}(t)$

in (1.25) is $N_0 - 5$. The reason for such a difference is caused by the fact that the term $\sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2$ containing the highest order derivatives of $E$ appeared in (4.40) does not belong to the corresponding dissipation rate functional $\mathcal{D}^k_{N_0,\ell}(t)$.

- In Theorem 1.1, the parameter $s$ is assumed to satisfy $s \in \left[\frac{3}{2} - \frac{3}{4N_0-10}, \frac{3}{2}\right)$ with $N_0 \geq 6$. If the negative Sobolev norm $\|\cdot\|_{\dot{H}^{-s}}$ is replaced by the Besov norm $\|\cdot\|_{\dot{B}^{-s}_{2,\infty}}$ in Theorem 1.1, by combining the argument used in this manuscript with the argument developed in [36], we believe that the corresponding result for $s \in \left[\frac{3}{2} - \frac{3}{4N_0-10}, \frac{3}{2}\right]$ can be obtained.
- Theorem 1.1 deals with the one-species VMB system (1.4) under Grad's cutoff assumption. For the non-cutoff case, since it is now well understood from the recent study of the non-cutoff Boltzmann equation independently by AMUXY [3] and Gressman–Strain [19] that the linearized Boltzmann operator without angular cutoff assumption has the similar anisotropic dissipation phenomenon compared to the Landau, see [20,24]. Based on those estimates established in [3] and [19], some weighted estimates on the corresponding linearized Boltzmann operator and the nonlinear collision term with respect to another similar time-velocity weight $w_{\ell,\lambda}(t,v) = \langle v\rangle^{\gamma\ell} e^{\lambda\langle v\rangle/(1+t)^\vartheta}$ for non-cutoff Boltzmann collision operator are established in [12]. With these weighted estimates in hand, the approach here can also be adapted to treat the one-species VMB system (1.4) for non-cutoff non-hard sphere intermolecular collisions.
- The argument used in this manuscript can be used to treat the one-species Vlasov–Maxwell–Landau system to yield a similar global solvability result and to deduce the similar temporal decay estimates on the global solutions provided that $\gamma \geq -3$.

Now we sketch the main ideas used to deduce our main result. Since the local well-posedness theory for the Cauchy problem of the one-species VMB system (1.4) can be obtained by employing the argument used in [22] for the two-species VMB system (1.1), (1.2), (1.3) in a periodic box for hard sphere model, the main difficulty encountered to extend such a local solution step by step to a global one lies in how to control the possible growth of the solutions induced by the nonlinearity of the system (1.6), especially those terms involving the electromagnetic field $[E(t,x), B(t,x)]$ and nonlinear collisions. More precisely, as pointed out in [12] for the two-species VMB system (1.1), (1.2), (1.3), the main mathematical difficulties to deduce the global solvability result near Maxwellians to either the two-species VMB system (1.1), (1.2), (1.3) or the one-species VMB system (1.4) for soft potentials are the following:

- The degeneration of dissipation at large velocity for the linearized Boltzmann collision operator $\mathbf{L}$ with soft potentials;
- The velocity-growth of the nonlinear term with the velocity-growth rate $|v|$;
- The regularity-loss of the electromagnetic field.

To overcome the above mentioned difficulties, the main arguments used in [12] which are proved to be effective for the two-species VMB system (1.1), (1.2), (1.3) are

- A refined weighted energy method based on the time-velocity weight function $w_\ell(t,v)$ defined in (1.12) is introduced which use essentially the extra dissipative term generated by such a weight function which corresponds to the last terms in the energy dissipation rate functional $\mathcal{D}_{N,\ell,\lambda}(t)$ defined in (1.12) of [12]. A key point to close the analysis is that the

solution $F_{\pm}(t,x,v)$ of the two-species VMB system (1.1), (1.2), (1.3) together with the electromagnetic field $[E(t,x), B(t,x)]$ decay suitably fast with respect to time;

- Motivated by the argument developed in [30] to deduce the temporal decay property of solutions to nonlinear equations of regularity-loss type, a time-weighted energy estimate is designed to close the analysis, which implies that although the $L^2$-norm of terms with the highest order derivative with respect to $x$ of the solutions of the two-species VMB system (1.1), (1.2), (1.3) such as $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ defined in (1.10) of [12] may have a growth with respect to $t$ like

$$\left|\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)\right| \lesssim (1+t)^{\frac{1+\epsilon_0}{2}}$$

for some positive constant $\epsilon_0 >$ as time evolves, the $L^2$-norm of terms with lower order derivatives with respect to $x$ still enjoy some decay rates. In fact, for the two-species VMB system (1.1), (1.2), (1.3), one can show that (cf. [12])
  - the solution operator of the linearized system decays sufficiently fast,
  - as the order of the derivatives of the solution with respect to $x$ increases once, the corresponding temporal decay rate in $L^2$-norm increases $\frac{1}{2}$,

  and the above two properties play an essential role to guarantee that one can combine the temporal decay of the solution operator of the linearized system and the Duhamel principle with the time-velocity splitting argument developed in [39] to close the analysis.

Now let's go back to our problem concerning the one-species VMB system (1.4). For this case, the study on the dissipative property of the one-species linearized VMB system (1.4) in [8] shows that

- the dissipation of the magnetic field in one-species is strictly weaker than the one in two-species and the solution operator of the linearized system decays in $L^2$-norm like $(1+t)^{-\frac{3}{8}}$,
- as the order of the derivatives of the solution with respect to $x$ increases once, the corresponding temporal decay rate in $L^2$-norm increases only $\frac{1}{4}$,

and it is illustrated in [8] that even for the hard sphere model, one can not hope to combine the time-decay rates of the solution operator of the linearized system with the Duhamel principle to yield the time-decay rates of the global solutions of the one-species VMB system (1.4) even though the global existence of solutions to the Cauchy problem of the nonlinear one-species VMB system can be established by constructing some energy functionals, which is motivated by the nonlinear energy method developed recently in [23,33,34,9] for the Boltzmann equation.

Our method to prove Theorem 1.1 is also based on a refined weighted energy method which is based on the time-velocity exponential weight function $w_\ell(t,v) = \langle v\rangle^{-\gamma\ell}\exp\left(\frac{q\langle v\rangle^2}{(1+t)^\vartheta}\right)$ given by (1.12), where the exponential factor $\exp\left(\frac{q\langle v\rangle^2}{(1+t)^\vartheta}\right)$ in the weight function $w_\ell(t,v)$ can generate extra dissipation term which corresponds to the last terms in the energy dissipation rate functional $\mathcal{D}_{N,l}(t)$ defined by (1.16). The most delicate part of our analysis is how to deduce the temporal decay estimates of the solutions of the one-species VMB system (1.4) which, as explained above, is a key step to close the refined weighted energy method. Our analysis is motivated by [26,43] where a new method to establish the optimal time decay rates of the solutions to the Cauchy problem for the compressible Navier–Stokes equations and the Boltzmann equation through the pure energy method is developed. The main points there are the following:

- the negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates,
- a family of scaled energy estimates with minimum derivative counts and interpolations among them are used to deduce the desired temporal decay estimates even without linear decay analysis.

Motivated by [26,43], the main ingredients in our analysis are the following:

- Our first observation that we can add $\|\Lambda^{-s}(f,E,B)\|^2$ and $\|\Lambda^{-s}a\|^2+\|\Lambda^{1-s}(a,b,c,E)\|^2+\|\Lambda^{2-s}B\|^2$ into the energy functional $\bar{\mathcal{E}}_{N,\ell}(t)$ given in (1.13) and the corresponding energy dissipation rate functional $\bar{\mathcal{D}}_{N,\ell}(t)$ given in (1.14) respectively. The resulting terms, especially the new dissipative estimates

$$\int_0^t \left( \left\|\Lambda^{-s}a(\tau)\right\|^2 + \left\|\Lambda^{1-s}(a,b,c,E)(\tau)\right\|^2 + \left\|\Lambda^{2-s}B(\tau)\right\|^2 \right) d\tau, \tag{1.26}$$

  which are not obtained even for the two-species VMB system (1.1), (1.2), (1.3) in [12], play an important role in our analysis;
- Secondly, unlike [12] for the two-species VMB system (1.1), (1.2), (1.3), we use the interpolation technique developed in [26,43] to deduce the decay of the solutions of the one-species VMB system (1.4), especially on temporal decay rates of the electromagnetic field $[E(t,x),B(t,x)]$. Our main observation is that the energy functional $\mathcal{E}^k_{N_0}(t)$ decays faster as $k$ increases. To deduce such a result, based on the a priori assumption

$$X(t)=\sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,l_0+l^*}(\tau)+\mathcal{E}_N(\tau)+\mathcal{E}_{N-1,l}(\tau)+(1+\tau)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N,l}(\tau)\right\}\le M$$

  with $M>0$ being sufficiently small and the parameters $N_0, N, l_0, l^*, l, \epsilon_0$ being suitably chosen, we have by some delicate energy type estimates that the following two types of differential inequalities

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t)+\mathcal{D}^k_{N_0}(t)\le 0,\quad k=0,1,\cdots,N_0-3 \tag{1.27}$$

  and

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^k_{N_0,\ell}(t)+\mathcal{D}^k_{N_0,\ell}(t)\\ &\lesssim \chi_{k\ge1}\sum_{1\le j\le k}\left\|\nabla^j(E,B)\right\|^2_{L^\infty}\left\|\langle v\rangle^{1-\frac{3\gamma}{2}}w_{\ell-N_0+k-1}\nabla_x^{k-j}\nabla_v^{N_0-k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\ &\quad+\sum_{|\alpha|=N_0}\left\|\partial^\alpha E\right\|^2,\quad k=0,1,\cdots,N_0-5\end{aligned} \tag{1.28}$$

  hold for any $t\ge0$. From these two types of differential inequalities, we can deduce the time decay of $\mathcal{E}^k_{N_0}(t)$, from which one can further obtain the time decay of $\mathcal{E}^k_{N_0,l_0}(t)$ with

$\mathcal{E}^1_{N_0,l_0}(t) \in L^1(\mathbb{R}_+)$. Having obtained the time decay of $\mathcal{E}^k_{N_0}(t)$ and $\mathcal{E}^1_{N_0,l_0}(t)$, we can then close the above a priori assumption and then the global solvability result follows immediately from the continuation argument.

It is worth pointing out that the most important part in our analysis is to deduce the temporal decay of the energy functional $\mathcal{E}^k_{N_0}(t)$. But noticing that such an energy functional $\mathcal{E}^k_{N_0}(t)$ contains only those terms involving the pure $x$-derivatives and due to the fact that for the one-species VMB system (1.6) with cutoff potentials, the corresponding energy dissipation rate functional $\mathcal{D}^k_{N_0}(t)$ contains also only those terms involving the pure $x$-derivatives, the problem to deduce the differential inequality (1.27) lies in how to control the terms involving the electromagnetic field $(E + v \times B) \cdot \nabla_v f$ and the nonlinear collision term $\mathbf{\Gamma}(f, f)$. Fortunately these terms are nonlinear and by some delicate Sobolev interpolations among the spatial regularity, we can indeed obtain (1.27). Here again the fact that the energy functional $\bar{\mathcal{E}}_{N,\ell}(t)$ given in (1.13) contains the term $\|\Lambda^{-s}(f, E, B)\|^2$ and the corresponding energy dissipation rate functional $\bar{\mathcal{D}}_{N,\ell}(t)$ given in (1.14) contains the term $\|\Lambda^{-s}a\|^2 + \|\Lambda^{1-s}(a, b, c, E)\|^2 + \|\Lambda^{2-s}B\|^2$ plays an essential role in our analysis. For details, see the analysis in Section 3.

Before concluding this section, it is worth to pointing out that, in the perturbative context, there have been extensive investigations on the global solvability of the Boltzmann and related equations near Maxwellians, see the first result by Ukai [42] and also [3–5,7–21,26,22–24,29,25,32–34,38,39,41,37,40,43,44] and reference therein for the Landau equation, the Vlasov–Poisson–Landau system, the Vlasov–Maxwell–Landau system, the Vlasov–Poisson–Boltzmann system, and the Vlasov–Maxwell–Boltzmann, etc.

The rest of this paper is organized as follows. In Section 2, we list some basic lemmas which will be used later. Section 3 is devoted to deducing certain energy type estimates and the main purpose here is to deduce the desired estimates which are sufficient to yield the differential inequality for the energy functional $\mathcal{E}^k_{N_0}(t)$ and the proof of Theorem 1.1 will be given in Section 4.

## 2. Preliminary

In this section, we cite some fundamental results concerning the weighted energy type estimates on the linearized collision operator $\mathbf{L}$ and the nonlinear term $\mathbf{\Gamma}$ for cutoff cases, whose proofs can be found in [16,39,44].

Recall that $\mathbf{L} = \nu - K$ with the collision frequency $\nu(v)$ and the nonlocal integral operator $K$ defined by

$$\nu(v) = \int_{\mathbb{R}^3\times S^2} |v-u|^\gamma q_0(\theta)\mu(u)d\omega du \sim (1+|v|)^\gamma$$

and

$$\begin{aligned} Kf(v) = &\int_{\mathbb{R}^3\times S^2} |v-u|^\gamma q_0(\theta)\mu(u)^{1/2}\mu(u')^{1/2} f(v')d\omega du \\ &+ \int_{\mathbb{R}^3\times S^2} |v-u|^\gamma q_0(\theta)\mu(u)^{1/2}\mu(v')^{1/2} f(u')d\omega du \end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^3\times S^2} |v-u|^{\gamma} q_0(\theta)\mu(u)^{1/2}\mu(v)^{1/2} f(u)\,d\omega du \\
&= \int_{\mathbb{R}^3} K(v,u) f(u)\,du,
\end{aligned}
$$

we list in the following lemma some velocity weighted estimates on the collision frequency $\nu(v)$ and integral operator K with respect to the velocity weighted function

$$
w_l(t,v) = \langle v\rangle^{-\gamma l} e^{\tilde{q}\langle v\rangle^2/(1+t)^{\vartheta}}, \quad l\in\mathbb{R}, \quad 0\le \tilde{q} \ll 1.
$$

The first lemma concerns the coercivity estimate on the linearized collision operators $\mathbf{L}$.

**Lemma 2.1.** *(Cf. [39].) Let $-3<\gamma<0$, $l\in\mathbb{R}$, and $0\le\tilde{q}\ll 1$. If $|\beta|>0$, then for any $\eta>0$, there is $C_\eta>0$ such that*

$$
\begin{aligned}
\int_{\mathbb{R}^3} w_l^2(v)\partial_\beta(\nu(v)f)\partial_\beta f\,dv \ge{}& \int_{\mathbb{R}^3}\nu(v)w_l^2(v)\left|\partial_\beta f\right|^2 dv - \eta\sum_{|\beta_1|\le|\beta|}\int_{\mathbb{R}^3}\nu(v)w_l^2(v)\left|\partial_{\beta_1} f\right|^2 dv \\
&- C_\eta\int_{\mathbb{R}^3\times\mathbb{R}^3}\chi_{|v|\le 2C_\eta}\langle v\rangle^{-2\gamma l}|f|^2 dv.
\end{aligned}
\tag{2.1}
$$

*Furthermore, if $|\beta|\ge 0$, then for any $\eta>0$, there is $C_\eta>0$ such that*

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} w_l^2(v)\partial_\beta(Kf)g\,dv\right| \le{}& \left\{\eta\sum_{|\beta_1|\le|\beta|}\left(\int_{\mathbb{R}^3}\nu(v)w_l^2(v)\left|\partial_{\beta_1} f\right|^2 dv\right)^{1/2}\right. \\
&\left. + C_\eta\left(\int_{\mathbb{R}^3}\chi_{|v|\le 2C_\eta}\langle v\rangle^{-2\gamma l}|f|^2 dv\right)^{1/2}\right\}\left(\int_{\mathbb{R}^3}\nu(v)w_l^2(v)|g|^2 dv\right)^{1/2}.
\end{aligned}
\tag{2.2}
$$

The following two lemmas are concerned with the corresponding estimates on nonlinear term $\mathbf{\Gamma}$.

**Lemma 2.2.** *(Cf. [39,44].) Let $-3<\gamma<0$, $N\ge 4$, $g_i=g_i(x,\xi)$ $(i=1,2,3)$, $\beta_1+\beta_2=\beta$ and $\alpha_1+\alpha_2=\alpha$, we have*

*(i) When $|\beta_1|\le\frac{N}{2}$, we have*

$$
\begin{aligned}
&\left\langle w_{l-|\beta|}^2\mathbf{\Gamma}\left(\partial_{\beta_1}^{\alpha_1} g_1, \partial_{\beta_2}^{\alpha_2} g_2\right), \partial_\beta^\alpha g_3\right\rangle \\
&\lesssim \sum_{m\le 2}\left|w_{l-m-|\beta_1|}\nabla_v^m\partial_{\beta_1}^{\alpha_1} g_1\right|_{L^2}\left|w_{l-|\beta_2|}\partial_{\beta_2}^{\alpha_2} g_2\right|_{L^2_v}\left|w_{l-|\beta|}\partial_\beta^\alpha g_3\right|_{L^2_v}
\end{aligned}
\tag{2.3}
$$

*or*

$$\begin{aligned}&\left\langle w_{l-|\beta|}^2\mathbf{\Gamma}\left(\partial_{\beta_1}^{\alpha_1}g_1,\partial_{\beta_2}^{\alpha_2}g_2\right),\partial_\beta^\alpha g_3\right\rangle\\&\lesssim\sum_{m\le 2}\left|w_{l-m-|\beta_1|}\nabla_v^m\partial_{\beta_1}^{\alpha_1}g_1\right|_{L^2_\nu}\left|w_{l-|\beta_2|}\partial_{\beta_2}^{\alpha_2}g_2\right|_{L^2}\left|w_{l-|\beta|}\partial_\beta^\alpha g_3\right|_{L^2_\nu};\end{aligned}\tag{2.4}$$

*(ii) When $|\beta_2|\le\frac{N}{2}$, we have*

$$\begin{aligned}&\left\langle w_{l-|\beta|}^2\mathbf{\Gamma}\left(\partial_{\beta_1}^{\alpha_1}g_1,\partial_{\beta_2}^{\alpha_2}g_2\right),\partial_\beta^\alpha g_3\right\rangle\\&\lesssim\sum_{m\le 2}\left|w_{l-|\beta_1|}\partial_{\beta_1}^{\alpha_1}g_1\right|_{L^2_\nu}\left|w_{l-m-|\beta_2|}\nabla_v^m\partial_{\beta_2}^{\alpha_2}g_2\right|_{L^2}\left|w_{l-|\beta|}\partial_\beta^\alpha g_3\right|_{L^2_\nu}\end{aligned}\tag{2.5}$$

*or*

$$\begin{aligned}&\left\langle w_{l-|\beta|}^2\mathbf{\Gamma}\left(\partial_{\beta_1}^{\alpha_1}g_1,\partial_{\beta_2}^{\alpha_2}g_2\right),\partial_\beta^\alpha g_3\right\rangle\\&\lesssim\sum_{m\le 2}\left|w_{l-|\beta_1|}\partial_{\beta_1}^{\alpha_1}g_1\right|_{L^2}\left|w_{l-m-|\beta_2|}\nabla_v^m\partial_{\beta_2}^{\alpha_2}g_2\right|_{L^2_\nu}\left|w_{l-|\beta|}\partial_\beta^\alpha g_3\right|_{L^2_\nu}.\end{aligned}\tag{2.6}$$

*(iii) Furthermore, set $w=\langle v\rangle^{-\gamma}$, $l\ge 0$, it holds that*

$$\begin{aligned}\left|w^l\mathbf{\Gamma}(g_1,g_2)\right|_{L^2}^2&\lesssim\sum_{|\beta|\le 2}\left|w^{l-|\beta|}\partial_\beta g_1\right|_{L^2_\nu}^2\left|w^l g_2\right|_{L^2_\nu}^2,\\\left|w^l\mathbf{\Gamma}(g_1,g_2)\right|_{L^2}^2&\lesssim\sum_{|\beta|\le 2}\left|w^l g_1\right|_{L^2_\nu}^2\left|w^{l-|\beta|}\partial_\beta g_2\right|_{L^2_\nu}^2.\end{aligned}$$

In what follows, we will collect some analytic tools which will be used in this paper. The first one is on the Sobolev interpolation among the spatial regularity.

**Lemma 2.3.** *(Cf. [26,1].) Let $2\le p<\infty$ and $k,\ell,m\in\mathbb{R}$, then we have*

$$\left\|\nabla^k f\right\|_{L^p}\lesssim\left\|\nabla^\ell f\right\|_{L^2}^{\theta}\left\|\nabla^m f\right\|_{L^2}^{1-\theta}.\tag{2.7}$$

*Here $0\le\theta\le 1$ and $\ell$ satisfy*

$$\frac{1}{p}-\frac{k}{3}=\left(\frac{1}{2}-\frac{\ell}{3}\right)\theta+\left(\frac{1}{2}-\frac{m}{3}\right)(1-\theta).\tag{2.8}$$

*Moreover, we have that*

$$\left\|\nabla^k f\right\|_{L^\infty}\lesssim\left\|\nabla^\ell f\right\|_{L^2}^{\theta}\left\|\nabla^m f\right\|_{L^2}^{1-\theta},\tag{2.9}$$

*where* $0 \le \theta \le 1$ *and* $\ell$ *satisfy*

$$-\frac{k}{3} = \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta), \quad \ell \le k+1, \quad m \ge k+2. \tag{2.10}$$

To deduce an estimate on $\|\Lambda^{-s} f\|$, we need the following $L^p - L^q$ inequality for the operator $\Lambda^{-s}$.

**Lemma 2.4.** *(Cf. [26,1].) Let* $0 < s < 3$, $1 < p < q < \infty$, $\frac{1}{q} + \frac{s}{3} = \frac{1}{p}$, *then we have*

$$\left\|\Lambda^{-s} f\right\|_{L^q} \lesssim \|f\|_{L^p}. \tag{2.11}$$

In many places, we will use Minkowski's integral inequality to interchange the orders of integration over $x$ and $v$.

**Lemma 2.5.** *(Cf. [26].) For* $1 \le p \le q \le \infty$, *we have*

$$\|f\|_{L^q_z L^p_y} \le \|f\|_{L^p_y L^q_z}. \tag{2.12}$$

Before concluding this section, we list below some identities concerning the macroscopic part of $f(t,x,v)$. To this end, for any solution $f(t,x,v)$ of the VMB system (1.6), by applying the macro–micro decomposition (1.11) introduced first in [23] for the Boltzmann equation and by defining moment functions $\Theta_{mj}(f)$ and $\Xi_j(f)$, $1 \le m, j \le 3$, by

$$\Theta_{mj}(f) = \int_{\mathbb{R}^3} (v_m v_j - 1)\mu^{1/2} f\,dv, \quad \Xi_j(f) = \frac{1}{10}\int_{\mathbb{R}^3} (|v|^2 - 5) v_j \mu^{1/2} f\,dv$$

as in [9], one can derive form (1.6) a fluid-type system of equations

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x (a + 2c) - E + \nabla_x \cdot \Theta(\{\mathbf{I} - \mathbf{P}\} f) = Ea + b \times B, \\ \partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{5}{3}\nabla \cdot \Xi(\{\mathbf{I} - \mathbf{P}\} f) = \frac{1}{3} E \cdot b, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \partial_t \Theta_{ij}(\{\mathbf{I} - \mathbf{P}\} f) + \partial_j b_i + \partial_i b_j - \frac{3}{2}\delta_{ij}\nabla_x \cdot b - \frac{10}{3}\delta_{ij}\nabla_x \cdot \Xi(\{\mathbf{I} - \mathbf{P}\} f) \\ \qquad = \Theta_{ij}(r+g) - \frac{2}{3}\delta_{ij} E \cdot b, \qquad 1 \le i, j \le 3, \\ \partial_t \Xi_j(\{\mathbf{I} - \mathbf{P}\} f) + \partial_j c = B_j(r+g), \end{cases} \tag{2.14}$$

where

$$r = -v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} f - \mathbf{L} f, \quad g = \frac{1}{2} v \cdot Ef - (E + v \times B) \cdot \nabla_v f + \mathbf{\Gamma}(f, f). \tag{2.15}$$

## 3. Energy estimates

This section is devoted to deducing certain uniform-in-time a priori estimates on the solution $[f(t,x,v), E(t,x), B(t,x)]$ to the Cauchy problem (1.6) and (1.7) and the main purpose is to get the desired differential inequality (1.27) for the energy functional $\mathcal{E}^k_{N_0}(t)$ for certain $k \le N_0$. For this purpose, suppose that the Cauchy problem (1.6) and (1.7) admits a smooth solution $f(t,x,v)$ defined on $[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$ for some $T\in(0,+\infty)$, we now turn to deduce certain energy type a priori estimates on the solution $[f(t,x,v), E(t,x), B(t,x)]$ to the Cauchy problem (1.6) and (1.7), which are sufficient to yield (1.27). The first one is concerned with the $k$-th order energy type estimate on $f$ without weight.

**Lemma 3.1.** *Suppose that $N_0\ge 6$, $\frac{3}{2}-\frac{3}{4N_0-10}\le s<\frac{3}{2}$, then there exist positive constants $\hat{l}$ and $\bar{m}$ which depend only on $N_0$, $s$, and $\gamma$ such that the following estimates hold:*

*(1) For $k=0,1,\cdots,N_0-3$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^k f\right\|^2+\left\|\nabla^k(E,B)\right\|^2\right)+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
\lesssim&\max\{\mathcal{E}_{\bar{m},\bar{m}}(t),\bar{\mathcal{E}}_{k,l'}(t)\}\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^{k+2}B\right\|^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{k+1}f\right\|_\nu^2\right)\\
&+\varepsilon\left\|\nabla^{k+1}f\right\|_\nu^2.
\end{aligned}\tag{3.1}$$

*(2) If $k=N_0-2$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0-2}f\right\|^2+\left\|\nabla^{N_0-2}(E,B)\right\|^2\right)+\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
\lesssim&\max\{\mathcal{E}_{\bar{m},\bar{m}}(t),\bar{\mathcal{E}}_{N_0-2,l'}(t)\}\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2+\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_\nu}+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)\\
&+\varepsilon\left(\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right).
\end{aligned}\tag{3.2}$$

*(3) If $k=N_0-1$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0-1}f\right\|^2+\left\|\nabla^{N_0-1}(E,B)\right\|^2\right)+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
\lesssim&\max\{\mathcal{E}_{\bar{m},\bar{m}}(t),\bar{\mathcal{E}}_{N_0-1,l'}(t)\}\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2+\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_\nu}+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)\\
&+\varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right).
\end{aligned}\tag{3.3}$$

*(4) If $k = N_0$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0} f\right\|^2 + \left\|\nabla^{N_0}(E,B)\right\|^2\right) + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{\bar m,\bar m}(t), \bar{\mathcal{E}}_{N_0,l'}(t)\right\} \\
&\quad\times\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2 + \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x L^2_\nu} + \left\|\nabla^{N_0-1} f\right\|^2_{H^1_x L^2_\nu}\right) \\
&\quad+\varepsilon\left(\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-1} f\right\|^2_{H^1_x L^2_\nu}\right).
\end{aligned} \tag{3.4}$$

*Here recall that throughout this manuscript $\varepsilon > 0$ denotes some positive constant which can be chosen sufficiently small, $n > \frac{2N_0-5}{3}$, $l' \geq \max\left\{\frac{3}{2} - \frac{\gamma}{2}, -\frac{\hat l}{\gamma}\right\}$, and the parameter $\bar m$ can be taken as*

$$\begin{aligned}
\bar m = \max\Bigg\{&\chi_{N_0\geq 6}\left\{\frac{2s+5}{2s+3}N_0 - \frac{4s+15}{2s+3}\right\}, \chi_{N_0\geq 6}\left\{\frac{3s+3/2}{3s-1/2}N_0 - \frac{3s+13/2}{3s-1/2}\right\}, \\
&\chi_{N_0\geq 6}\left\{\frac{2s+7}{2s+5}N_0 - \frac{6}{2s+5}\right\}, \chi_{N_0\geq 7}\left\{\frac{2s+7}{2s+3}N_0 - \frac{6s+29}{2s+3}\right\}\Bigg\},
\end{aligned}$$

*and $\hat l = \max\limits_{1\leq j\leq 21}\left\{\hat l_j\right\}$, where $\hat l_j$ will be specified in the proof of Lemma 3.1.*

**Proof.** Before proving Lemma 3.1 in detail, we first use several sentences to illustrate the main ideas. Since the main purpose of Lemma 3.1 and the coming lemma is to derive the differential inequality (1.27) for the energy functional $\mathcal{E}^k_{N_0}(t)$. To this end, as pointed out in the introduction, since such an energy functional $\mathcal{E}^k_{N_0}(t)$ contains only those terms involving the pure $x$-derivatives and consequently for the one-species VMB system (1.6) with cutoff potentials, the corresponding energy dissipation rate functional $\mathcal{D}^k_{N_0}(t)$ contains also only those terms involving the pure $x$-derivatives and the problem to deduce the differential inequality (1.27) lies in how to control the terms involving the electromagnetic field $(E + v\times B)\cdot\nabla_v f$ and the nonlinear collision term $\mathbf{\Gamma}(f,f)$ suitably. Our main observation is that these terms involved are nonlinear and hence we can control them suitably by some delicate Sobolev interpolations among the spatial regularity. More precisely, we can show that those terms appeared in the parenthesis in the right hand side of the estimates (3.1), (3.2), (3.3), and (3.4) contain only terms involving the pure $x$-derivatives with suitable order of derivatives, from which we can indeed obtain (1.27) by suitably linear combinations.

Now we turn to prove Lemma 3.1. To this end, we first deal with the case $k = 0$ and to do so, we have by multiplying (1.6) by $f$ and integrating the resulting identity with respect to $x$ and $v$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$ that

$$\frac{d}{dt}\left(\|f\|^2 + \|(E,B)\|^2\right) + \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 \lesssim |(v\cdot Ef, f)| + |(\mathbf{\Gamma}(f,f), \{\mathbf{I}-\mathbf{P}\}f)|. \tag{3.5}$$

For the first term in the right hand of (3.5), we can obtain that

$$
\begin{aligned}
|(v\cdot Ef,f)| &\lesssim |(v\cdot Ef,\mathbf{P}f)|+|(v\cdot E\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f)|+|(v\cdot E\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f)|\\
&\lesssim \|E\|_{L^2_x}\left\|\mu^{\delta}f\right\|_{L^3_xL^2_v}\left\|\mu^{\delta}f\right\|_{L^6_xL^2_v}+\|E\|_{L^6_x}\left\|\langle v\rangle^{-\frac{\gamma}{2}+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^3_xL^2_v}\\
&\quad\times\left\|\langle v\rangle^{\frac{\gamma}{2}}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_xL^2_v}\\
&\lesssim \|E\|\left\|\mu^{\delta}f\right\|_{L^2_vL^3_x}\left\|\mu^{\delta}f\right\|_{L^2_vL^6_x}+\|E\|_{L^2_vL^6_x}\left\|\langle v\rangle^{-\frac{\gamma}{2}+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^3_x}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\nu}\\
&\lesssim \left\|\Lambda^{-\frac12}E\right\|^{\frac23}\|\nabla_xE\|^{\frac13}\left\|\Lambda^{-\frac12}f\right\|^{\frac13}\|\nabla_xf\|_{\nu}^{\frac23}\|\nabla_xf\|_{\nu}\\
&\quad+\|\nabla_xE\|\left\|\langle v\rangle^{-\frac{\gamma}{2}+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\nu}\\
&\lesssim \bar{\mathcal{E}}_{1,\frac12-\frac1\gamma}(t)\left(\|\nabla_xE\|^2+\|\nabla_xf\|_{\nu}^2\right)+\varepsilon\left(\|\nabla_xf\|_{\nu}^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_{\nu}^2\right).
\end{aligned}
$$

Here we need to ask that $s\geq\frac12$.

The other term can be estimated as follows:

$$
\begin{aligned}
|(\mathbf{\Gamma}(f,f),\{\mathbf{I}-\mathbf{P}\}f)| &\lesssim \int_{\mathbb{R}^3_x}|f|_{H^2_v}|f|_{L^2_v}|\{\mathbf{I}-\mathbf{P}\}f|_{L^2_v}dx\\
&\lesssim \int_{\mathbb{R}^3_x}|f|_{H^2_v}|f|_{L^2_v}|\{\mathbf{I}-\mathbf{P}\}f|_{L^2_v}dx\\
&\lesssim \|f\|_{H^2_vL^2_x}\|f\|_{L^2_vL^\infty_x}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\nu}\\
&\lesssim \|f\|^2_{H^2_vL^2_x}\|\nabla_xf\|^2_{L^2_vH^1_x}+\varepsilon\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\nu}\\
&\lesssim \mathcal{E}_{2,2}(t)\|\nabla_xf\|^2_{H^1_xL^2_v}+\varepsilon\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\nu}
\end{aligned}
$$

Collecting the above estimates gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\|f\|^2+\|(E,B)\|^2\right)+\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\nu}\\
&\lesssim\bar{\mathcal{E}}_{2,\max\left\{2,\frac12-\frac1\gamma\right\}}(t)\left(\|\nabla_xE\|^2+\|\nabla_xf\|^2_{\nu}+\left\|\nabla_x^2f\right\|^2_{\nu}\right)+\varepsilon\|\nabla_xf\|^2_{\nu},
\end{aligned}
\tag{3.6}
$$

which gives (3.1) with $k=0$.

For $k=1,2,\cdots,N_0-3$, we have by applying $\nabla^k$ to (1.6), multiplying the resulting identity by $\nabla^kf$, and then integrating the final result with respect to $x$ and $v$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$ that

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^kf\right\|^2+\left\|\nabla^k(E,B)\right\|^2\right)+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\nu}\\
&\lesssim\underbrace{\left|\left(\nabla^k(v\cdot Ef),\nabla^kf\right)\right|}_{I_1}+\underbrace{\left|\left(\nabla^k(E\cdot\nabla_vf),\nabla^kf\right)\right|}_{I_2}\\
&\quad+\underbrace{\left|\left(\nabla^k((v\times B)\cdot\nabla_vf),\nabla^kf\right)\right|}_{I_3}+\underbrace{\left|\left(\nabla^k\mathbf{\Gamma}(f,f),\nabla^kf\right)\right|}_{I_4}.
\end{aligned}
\tag{3.7}
$$

By applying the macro–micro decomposition (1.11), one can deduce that

$$I_1 \lesssim \underbrace{\left|\left(\nabla^k(v\cdot Ef), \nabla^k \mathbf{P}f\right)\right|}_{I_{1,1}} + \underbrace{\left|\left(\nabla^k(v\cdot E\mathbf{P}f), \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{1,2}} + \underbrace{\left|\left(\nabla^k(v\cdot E\{\mathbf{I}-\mathbf{P}\}f), \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{1,3}}. \tag{3.8}$$

$I_{1,1}$ can be dominated by

$$\begin{aligned} I_{1,1} &\lesssim \sum_{j\le k-1} \left\|\left(\nabla^j E\nabla^{k-1-j}\left(\mu^\delta f\right)\right)\right\| \left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\| \\ &\lesssim \sum_{j\le k-1} \left\|\nabla^j E\right\|_{L^6_x} \left\|\nabla^{k-1-j}\left(\mu^\delta f\right)\right\|_{L^3_x L^2_v} \left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\| \\ &\lesssim \sum_{j\le k-1} \left\|\Lambda^{-\frac{1}{2}} E\right\|^{\frac{2k-2j}{2k+3}} \left\|\nabla^{k+1}E\right\|^{\frac{2j+3}{2k+3}} \left\|\Lambda^{-\frac{1}{2}}\left(\mu^\delta f\right)\right\|^{\frac{2j+3}{2k+3}} \left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\|^{\frac{2k-2j}{2k+3}} \\ &\quad \times \left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\| \\ &\lesssim \bar{\mathcal{E}}_{0,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2 + \left\|\nabla^{k+1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{k+1}f\right\|_\nu^2. \end{aligned}$$

Here we also need to ask that $s \ge \frac{1}{2}$.

Similarly, for $I_{1,2}$, we can get that

$$I_{1,2} \lesssim \bar{\mathcal{E}}_{0,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2 + \left\|\nabla^{k+1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{k+1}f\right\|_\nu^2.$$

For $I_{1,3}$, it holds from the Cauchy inequality and the Hölder inequality that

$$\begin{aligned} I_{1,3} &\lesssim \sum_{j\le k} \left\|\left(\nabla^j E\cdot v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f, \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right)\right\| \\ &\lesssim \sum_{j\le k} \left\|\nabla^j E\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\| \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\| \\ &\lesssim \underbrace{\sum_{j\le k} \left\|\nabla^j E\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|^2}_{I_{1,3,1}} + \varepsilon\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

To deal with $I_{1,3,1}$, when $j=0$, we have from Lemma 2.3 that

$$
\begin{aligned}
I_{1,3,1} &\lesssim \left(\|E\|_{L^\infty_x}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|\right)^2\\
&\lesssim \left(\left\|\Lambda^{-s}E\right\|^{1-\theta}\left\|\nabla^{k+1}E\right\|^{\theta}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{1-\theta}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\hat{l}_1}\right\|^{\theta}\right)^2\\
&\lesssim \bar{\mathcal{E}}_{k,-\frac{\hat{l}_1}{\gamma}}(t)\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right).
\end{aligned}
$$

Here $\theta=\frac{3+2s}{2(k+1+s)}$ and $\hat{l}_1=\frac{\gamma}{2}-\frac{\gamma-1}{\theta}=\frac{\gamma}{2}-\frac{2(\gamma-1)(k+1+s)}{3+2s}$.

While for the case $j\neq 0$, we can deduce from Lemma 2.3 that

$$
\begin{aligned}
I_{1,3,1} &\lesssim \sum_{1\le j\le k}\left(\left\|\nabla^j E\right\|_{L^6_x}\left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|_{L^2_vL^3_x}\right)^2\\
&\lesssim \sum_{1\le j\le k}\left(\left\|\Lambda^{-s}E\right\|^{1-\theta_j}\left\|\nabla^{k+1}E\right\|^{\theta_j}\left\|\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|^{1-\alpha_j}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|^{\alpha_j}\right)^2\\
&\lesssim \sum_{1\le j\le k}\left(\left\|\Lambda^{-s}E\right\|^{1-\theta_j}\left\|\nabla^{k+1}E\right\|^{\theta_j}\left\|\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|^{1-\alpha_j}\right.\\
&\quad\left.\times\left(\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\hat{l}_{2j}}\right\|^{1-\beta_j}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{\beta}\right)^{\alpha_j}\right)^2\\
&\lesssim \sum_{1\le j\le k}\left(\left\|\Lambda^{-s}E\right\|^{1-\theta_j}\left\|\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^{1-\alpha_j}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\hat{l}_{2j}}\right\|^{\alpha_j(1-\beta_j)}\right.\\
&\quad\left.\times\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{\alpha_j\beta_j}\left\|\nabla^{k+1}E\right\|^{\theta_j}\right)^2\\
&\lesssim \bar{\mathcal{E}}_{k,-\frac{\hat{l}_2}{\gamma}}(t)\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right).
\end{aligned}
$$

Here we can deduce from Lemma 2.3 that $\theta_j=\frac{j+s+1}{k+1+s}$, $\alpha_j=\frac{2k-2j+1}{2k}$, $\frac{\gamma}{2}\cdot\beta_j+\hat{l}_{2j}(1-\beta_j)=1-\frac{\gamma}{2}$. If we set $\alpha_j\beta_j=1-\theta_j$, then we can get that $\hat{l}_{2j}=\frac{\gamma}{2}-\frac{\gamma-1}{1-\beta_j}=\frac{\gamma}{2}-\frac{(\gamma-1)(2k-2j+1)(k+1+s)}{(2k-2j+1)(k+1+s)-2k(k-j)}$ with $1\le j\le k$. And we set $\hat{l}_2=\max\limits_{1\le j\le k}\left\{\hat{l}_{2j}\right\}$.

Consequently

$$
\begin{aligned}
I_1 \lesssim{}& \bar{\mathcal{E}}_{k,-\frac{\max\{\hat{l}_1,\hat{l}_2\}}{\gamma}}(t)\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^{k+1}f\right\|_\nu^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right)\\
&+\varepsilon\left(\left\|\nabla^{k+1}f\right\|_\nu^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right).
\end{aligned}
$$

For $I_2$ in the right hand side of (3.7), we can get that

$$
\begin{aligned}
I_2 &\lesssim \left|\left(E\cdot\nabla_v\nabla^k f,\nabla^k f\right)\right| + \sum_{1\le j\le k}\left|\left(\nabla^j E\cdot\nabla_v\nabla^{k-j} f,\nabla^k f\right)\right| \\
&= \sum_{1\le j\le k}\left|\left(\nabla^j E\cdot\nabla_v\nabla^{k-j} f,\nabla^k f\right)\right| \\
&= \underbrace{\sum_{1\le j\le k}\left|\left(\nabla^j E\cdot\nabla_v\nabla^{k-j}\mathbf{P} f,\nabla^k f\right)\right|}_{I_{2,1}} \\
&\quad + \underbrace{\sum_{1\le j\le k}\left|\left(\nabla^j E\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f,\nabla^k\mathbf{P} f\right)\right|}_{I_{2,2}} \\
&\quad + \underbrace{\sum_{1\le j\le k}\left|\left(\nabla^j E\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f,\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right)\right|}_{I_{2,3}},
\end{aligned}
\tag{3.9}
$$

and $I_{2,1}$ and $I_{2,2}$ in the right hand side of (3.9) can be bounded by

$$
I_{2,1}+I_{2,2}\lesssim \bar{\mathcal{E}}_{0,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^{k+1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{k+1}f\right\|_\nu^2.
$$

For $I_{2,3}$, when $j=k$, we have

$$
\begin{aligned}
I_{2,3} &\lesssim \left\|\nabla^k E\right\|_{L_x^6}\left\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\} f\right\|_{L_v^2L_x^3}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|_{L_v^2L_x^2} \\
&\lesssim \left\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\} f\right\|_{L_v^2L_x^3}\left\|\nabla^{k+1}E\right\|\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\
&\lesssim \bar{\mathcal{E}}_{2,\frac{3}{2}}(t)\left\|\nabla^{k+1}E\right\|^2+\varepsilon\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2,
\end{aligned}
\tag{3.10}
$$

while for the case $1\le j\le k-1$, we can deduce that

$$
\begin{aligned}
I_{2,3} &\lesssim \sum_{1\le j\le k-1}\left\|\nabla^j E\right\|_{L_x^\infty}\left\|\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\| \\
&\lesssim \sum_{1\le j\le k-1}\left\|\Lambda^{-s}E\right\|^{\frac{2k-2j-1}{2(k+1+s)}}\left\|\nabla^{k+1}E\right\|^{\frac{2j+2s+3}{2(k+1+s)}}\left\|\nabla_v^{m_{1j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{1}{m_{1j}+1}} \\
&\quad\times\left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{m_{1j}}{m_{1j}+1}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\| \\
&\lesssim \sum_{1\le j\le k-1}\left\|\Lambda^{-s}E\right\|^{\frac{2k-2j-1}{2(k+1+s)}}\left\|\nabla^{k+1}E\right\|^{\frac{2j+2s+3}{2(k+1+s)}}\left\|\nabla_v^{m_{1j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{1}{m_{1j}+1}} \\
&\quad\times\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+s)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{m_{1j}(k-j+s)}{(m_{1j}+1)(k+s)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{1\le j\le k-1} \left\|\Lambda^{-s}E\right\|^{\frac{2k-2j-1}{2(k+1+s)}} \left\|\nabla^{k+1}E\right\|^{\frac{2j+2s+3}{2(k+1+s)}} \left\|\nabla_v^{m_{1j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{1j}+1}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+s)}} \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}(k-j+s)\beta_j}{(m_{1j}+1)(k+s)}} \\
&\quad\times \left\|\langle v\rangle^{\hat{l}_{3j}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}(k-j+s)(1-\beta_j)}{(m_{1j}+1)(k+s)}} \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta_j} \left\|\langle v\rangle^{\hat{l}_{3j}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta_j} \\
&\lesssim \max\left\{\mathcal{E}_{k+m_1,1+m_1}(t), \bar{\mathcal{E}}_{k,-\frac{\hat{l}_3}{\gamma}}(t)\right\}\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right) \\
&\quad+\varepsilon\left\|\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \qquad (3.11)
\end{aligned}
$$

Here we have used the fact that there exist positive constants $m_{1j}$ and $\beta_j\in(0,1)$ such that

$$
\frac{2j+2s+3}{2(k+1+s)}+\left(\frac{m_{1j}(k-j+s)}{(m_{1j}+1)(k+s)}+1\right)\beta_j=2,\quad 1\le j\le k-1.
$$

Since the choice of the constant $m_{1j}$ will determine the minimal regularity index we imposed on the initial perturbation, we need to determine its value as small as possible. Due to fact that we will encounter such a kind of problem several times in the following analysis, we will show how to find such an $m_{1j}$ in details.

Firstly it is easy to see that a necessary and sufficient condition to guarantee the existence of such a $\beta$ is

$$
\frac{2j+2s+3}{2(k+1+s)}+\frac{m_{1j}(k-j+s)}{(m_{1j}+1)(k+s)}+1>2,\quad 1\le j\le k-1,
$$

from which one can deduce that $m_{1j}>\frac{2k^2+2sk-2jk-2js-k-s}{2ks+3k+2s^2+3s-2j}$ holds for $1\le j\le k-1$. Noticing that $1<s<\frac{3}{2}$, we can deduce from the above inequalities that it is enough to take $m_{1j}$ as

$$
\begin{aligned}
m_1=\max_{1\le j\le k-1} m_{1j}&=\frac{2k}{2s+3}>\frac{2k^2-3s}{2ks+3k+2s^2+3s-2}\\
&\ge \max_{1\le j\le k-1}\frac{2k^2+2sk-2jk-2js-k-s}{2ks+3k+2s^2+3s-2j}.
\end{aligned}
$$

Consequently, $k-j+1+m_{1j}\le k+m_1=\frac{2s+5}{2s+3}k\le\frac{2s+5}{2s+3}N_0-\frac{6s+15}{2s+3}$ with $N_0\ge 6$.

Moreover, since $\hat{l}_{3j}$ satisfies $\frac{\gamma}{2}\beta_j+\hat{l}_{3j}(1-\beta_j)=0$, we can get that

$$
\begin{aligned}
\hat{l}_{3j}&=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}\\
&=\frac{\gamma}{2}-\frac{\gamma[m_{1j}(2k-j+2s)+k+s](k+1+s)}{(4k-2j+2s+1)(m_{1j}+1)(k+s)-2[m_{1j}(2k-j+2s)+k+s](k+1+s)},
\end{aligned}
$$

and we set $\hat{l}_3=\max\limits_{1\le j\le k-1}\left\{\hat{l}_{3j}\right\}$.

Applying the similar trick as (3.9), the term $I_3$ containing $B$ in the right hand side of (3.7) can be bounded by

$$\begin{aligned}
I_3 &\lesssim \left|\left((v\times B)\cdot\nabla_v\nabla^k f,\nabla^k f\right)\right| + \sum_{1\le j\le k}\left|\left((v\times\nabla^j B)\cdot\nabla_v\nabla^{k-j} f,\nabla^k f\right)\right| \\
&= \sum_{1\le j\le k}\left|\left((v\times\nabla^j B)\cdot\nabla_v\nabla^{k-j} f,\nabla^k f\right)\right| \\
&= \underbrace{\sum_{1\le j\le k}\left|\left((v\times\nabla^j B)\cdot\nabla_v\nabla^{k-j}\mathbf{P} f,\nabla^k f\right)\right|}_{I_{3,1}} \\
&\quad + \underbrace{\sum_{1\le j\le k}\left|\left((v\times\nabla^j B)\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f,\nabla^k\mathbf{P} f\right)\right|}_{I_{3,2}} \\
&\quad + \underbrace{\sum_{1\le j\le k}\left|\left((v\times\nabla^j B)\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f,\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right)\right|}_{I_{3,3}},
\end{aligned}\tag{3.12}$$

where we have used the fact that $\left|\left((v\times B)\cdot\nabla_v\nabla^k f,\nabla^k f\right)\right| = 0$.

$I_{3,1}$ and $I_{3,2}$ in the right hand side of (3.12) can be estimated as follows:

$$\begin{aligned}
I_{3,1}+I_{3,2} &\lesssim \sum_{1\le j\le k}\left\|\nabla^j B\right\|_{L^2_x}\left\|\nabla^{k-j}\left(\mu^\delta f\right)\right\|_{L^2_vL^3_x}\left\|\nabla^k\left(\mu^\delta f\right)\right\|_{L^2_vL^6_x} \\
&\lesssim \sum_{1\le j\le k}\left\|\nabla^j B\right\|\left\|\nabla^{k-j}\left(\mu^\delta f\right)\right\|_{L^2_vL^3_x}\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\| \\
&\lesssim \sum_{1\le j\le k}\left\|\Lambda^{-s} B\right\|^{\frac{k+2-j}{k+s+2}}\left\|\nabla^{k+2} B\right\|^{\frac{s+j}{k+s+2}}\left\|\Lambda^{-s}\left(\mu^\delta f\right)\right\|^{\frac{2j+1}{2(k+s+1)}} \\
&\quad\times\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\|^{\frac{2k+2s-2j+1}{2(k+s+1)}}\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\| \\
&\lesssim \bar{\mathcal{E}}_{k+1,0}(t)\left(\left\|\nabla^{k+2} B\right\|^2+\left\|\nabla^{k+1} f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{k+1} f\right\|_\nu^2.
\end{aligned}$$

Here we need to ask that $\frac{s+j}{k+s+2}+\frac{2k+2s-2j+1}{2(k+s+1)}\ge 1$ for $1\le j\le k$, which implies that $s\ge\frac{3}{2}-\frac{3}{4N_0-10}$.

As to $I_{3,3}$, we can deduce from Lemma 2.3 that

$$\begin{aligned}
I_{3,3} &\lesssim \sum_{1\le j\le k}\left\|\nabla^j B\right\|_{L^\infty_x}\left\|\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|\left\|\langle v\rangle\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\| \\
&\lesssim \sum_{1\le j\le k}\left\|\Lambda^{-s} B\right\|^{\frac{2k-2j+1}{2(k+2+s)}}\left\|\nabla^{k+2} B\right\|^{\frac{2j+3+2s}{2(k+2+s)}}\left\|\nabla_v^{m_{2j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\} f\right\|^{\frac{1}{m_{2j}+1}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}}{m_{2j}+1}} \left\|\langle v\rangle\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
\lesssim &\sum_{1\le j\le k} \left\|\Lambda^{-s}B\right\|^{\frac{2k-2j+1}{2(k+2+s)}} \left\|\nabla^{k+2}B\right\|^{\frac{2j+3+2s}{2(k+2+s)}} \left\|\nabla_v^{m_{2j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{2j}+1}} \\
&\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}j}{(m_{2j}+1)(k+s)}} \left\|\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}(k-j+s)}{(m_{2j}+1)(k+s)}} \left\|\langle v\rangle\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
\lesssim &\sum_{1\le j\le k} \left\|\Lambda^{-s}B\right\|^{\frac{2k-2j+1}{2(k+2+s)}} \left\|\nabla^{k+2}B\right\|^{\frac{2j+3+2s}{2(k+2+s)}} \\
&\times \left\|\nabla_v^{m_{2j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{2j}+1}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}j}{(m_{2j}+1)(k+s)}} \\
&\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}(k-j+s)\beta_j}{(m_{2j}+1)(k+s)}} \left\|\langle v\rangle^{\hat{l}_{4j}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{2j}(k-j+s)(1-\beta_j)}{(m_{2j}+1)(k+s)}} \\
&\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta_j} \left\|\langle v\rangle^{\hat{l}_{5j}}\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta_j} \\
\lesssim &\max\left\{\mathcal{E}_{k+m_2,1+m_2}(t), \bar{\mathcal{E}}_{k,-\frac{\hat{l}_5}{\gamma}}(t)\right\}\left(\left\|\nabla^{k+2}B\right\|^2+\left\|\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^2\right) \\
&+\varepsilon\left\|\nabla^{k}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^2. \qquad (3.13)
\end{aligned}
$$

Here we have used the fact that if we choose $m_{2j}$ suitably, there exist positive constants $\beta_j\in(0,1)$ such that $\frac{2j+2s+3}{2(k+2+s)}+\frac{m_{2j}(k-j+s)\beta_j}{(m_{2j}+1)(k+s)}+\beta_j=2$ holds for $1\le j\le k$. Similar to the way to determine $m_{1j}$, we can take $m_{2j}$ as $m_2=\max_{1\le j\le k}m_{2j}=\frac{2k}{2s+3}+\frac{2s-1}{2s+3}>\frac{2k^2+2ks-k-s}{2ks+3k+2s^2+3s-4}$. Consequently, $k-j+1+m_{2j}\le k+m_2=\frac{2s+5}{2s+3}k+\frac{2s-1}{2s+3}\le\frac{2s+5}{2s+3}N_0-\frac{4s+16}{2s+3}$ with $N_0\ge 4$. On the other hand, since $\hat{l}_{4j}$ and $\hat{l}_{5j}$ satisfy respectively $\frac{\gamma}{2}\beta_j+\hat{l}_{4j}(1-\beta_j)=0$ and $\frac{\gamma}{2}\beta_j+\hat{l}_{5j}(1-\beta_j)=1$ with $0<\beta_j<1$, one can deduce that, $\hat{l}_{4j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$ and $\hat{l}_{5j}=\frac{\gamma}{2}-\frac{\gamma-2}{2(1-\beta_j)}$. It is easy to see that $\hat{l}_{5j}>\hat{l}_{4j}$, $\beta_j=\frac{(4k+5+2s-2j)(2k+1-2j)(m_{2j}+1)(k+s)}{(2k+4+2s)(3k+3s+4+2km_{2j}+2sm_{2j}-jm_{2j})}$ with $1\le j\le k$. If we set $\hat{l}_5=\max\limits_{1\le j\le k}\left\{\hat{l}_{5j}\right\}$, we have $\hat{l}_5\ge\hat{l}_4$.

For the last term $I_4$ in the right hand side of (3.7), it follows that

$$
\begin{aligned}
I_4\lesssim &\underbrace{\sum_{1\le j\le k}\left|\left(\mathbf{\Gamma}(\nabla^j\mathbf{P}f,\nabla^{k-j}f),\nabla^k\{\mathbf{I}-\mathbf{P}\}\right)\right|}_{4,1} \\
&+\underbrace{\sum_{1\le j\le k}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{k-j}\mathbf{P}f),\nabla^k\{\mathbf{I}-\mathbf{P}\}\right)\right|}_{I_{4,2}} \\
&+\underbrace{\sum_{1\le j\le k}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f),\nabla^k\{\mathbf{I}-\mathbf{P}\}\right)\right|}_{I_{4,3}}. \qquad (3.14)
\end{aligned}
$$

$I_{4,1}$ can be estimated from Lemma 2.3 as follows:

$$\begin{aligned} I_{4,1} &\lesssim \sum_{1\le j\le k} \left\|\nabla^j \mathbf{P} f\right\|_{L_x^6 H_v^2} \left\|\nabla^{k-j} f\right\|_{L_x^3 L_v^2} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \sum_{1\le j\le k} \left\|\nabla^j \mathbf{P} f\right\|_{H_v^2 L_x^6} \left\|\nabla^{k-j} f\right\|_{L_v^2 L_x^3} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \sum_{1\le j\le k} \left\|\mu^\delta \nabla^{m'} f\right\|^{1-\theta_j} \left\|\mu^\delta \nabla^{k+1} f\right\|^{\theta_j} \|f\|_\nu^{\theta_j} \\ &\quad \times \left\|\nabla^{k+1} f\right\|_\nu^{1-\theta_j} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \mathcal{E}_{N_0-1}(t) \left\|\nabla^{k+1} f\right\|_\nu^2 + \varepsilon \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2. \end{aligned} \tag{3.15}$$

Here $\theta_j = \frac{2j+1}{2(k+1)}$, $m' = \frac{k+1}{2j+1} \le k+1 \le N_0 - 1$.

Similarly, from Lemma 2.3, $I_{4,2}$ can be bounded by

$$I_{4,2} \lesssim \mathcal{E}_{N_0-1}(t) \left\|\nabla^{k+1} f\right\|_\nu^2 + \varepsilon \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2.$$

For the third term $I_{4,3}$ in the right hand side of (3.14), when $j = 0$ or $j = k$, one has

$$\begin{aligned} I_{4,3} &\lesssim \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \|\{\mathbf{I}-\mathbf{P}\} f\|_{H_v^2 L_x^\infty} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \|f\|_{H_v^2 L_x^\infty} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t) \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 + \varepsilon \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2, \end{aligned}$$

while for the case $j = 1$ or $k-1$, we have

$$\begin{aligned} I_{4,3} &\lesssim \sum_{j=k-1} \left\|\nabla^{k-1} \{\mathbf{I}-\mathbf{P}\} f\right\|_{L_x^6 L_v^2} \|\nabla \{\mathbf{I}-\mathbf{P}\} f\|_{H_v^2 L_x^3} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \|\nabla \{\mathbf{I}-\mathbf{P}\} f\|_{H_v^2 H_x^1} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t) \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 + \varepsilon \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2. \end{aligned}$$

For the last case $2 \le j \le k-2$, we have

$$\begin{aligned} I_{4,3} &\lesssim \sum_{2\le j\le k-2} \left\|\nabla^j \{\mathbf{I}-\mathbf{P}\} f\right\|_{L_x^2 H_v^2} \left\|\nabla^{k-j} \{\mathbf{I}-\mathbf{P}\} f\right\|_{L_x^\infty L_v^2} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\ &\lesssim \underbrace{\sum_{2\le j\le k-2} \left\|\nabla^j \{\mathbf{I}-\mathbf{P}\} f\right\|_{L_x^2 L_v^2} \left\|\nabla^{k-j} \{\mathbf{I}-\mathbf{P}\} f\right\|_{L_v^2 L_x^\infty} \left\|\nabla^k \{\mathbf{I}-\mathbf{P}\} f\right\|_\nu}_{I_{4,3,1}} \end{aligned}$$

$$+\underbrace{\sum_{2\le j\le k-2}\left\|\nabla_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_xL^2_v}\left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^\infty_x}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{4,3,2}}$$

$$+\underbrace{\sum_{2\le j\le k-2}\left\|\nabla_v^2\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_xL^2_x}\left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^\infty_x}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{4,3,3}}. \quad (3.16)$$

$I_{4,3,3}$ is the most difficult term among these three terms in the right hand side of the above inequality which can be bounded as follows

$$\begin{aligned}
I_{4,3,3}\lesssim&\sum_{2\le j\le k-2}\left\|\nabla_v^{m_{3j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{3j}+2}}\left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}}{m_{3j}+2}}\\
&\times\left\|\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^\infty_x}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\\
\lesssim&\sum_{2\le j\le k-2}\left\|\nabla_v^{m_{3j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{3j}+2}}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}(k-j)}{(m_{3j}+2)(k+s)}}\\
&\times\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}(j+s)}{(m_{3j}+2)(k+s)}}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2j-3}{2(k+s)}}\\
&\times\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^{\frac{2k+2s-2j+3}{2(k+s)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\\
\lesssim&\sum_{2\le j\le k-2}\left\|\nabla_v^{m_{3j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{3j}+2}}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}(k-j)}{(m_{3j}+2)(k+s)}}\\
&\times\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}(j+s)\beta_j}{(m_{3j}+2)(k+s)}}\left\|\langle v\rangle^{\hat{l}_{6j}}\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{3j}(j+s)(1-\beta_j)}{(m_{3j}+2)(k+s)}}\\
&\times\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2j-3}{2(k+s)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^{\frac{2k+2s-2j+3}{2(k+s)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\\
\lesssim&\max\left\{\mathcal{E}_{k+m_3,2+m_3}(t),\bar{\mathcal{E}}_{k,-\frac{\hat{l}_6}{\gamma}}(t)\right\}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \quad (3.17)
\end{aligned}$$

Here we need to ask that there exist positive constants $m_{3j}$ and $\beta_j\in(0,1)$ such that $\frac{m_{3j}(j+s)\beta_j}{(m_{3j}+2)(k+s)}+\frac{2k+2s-2j+3}{2(k+s)}=1$ holds for $2\le j\le k-2$. Similar to the way to determine $m_{1j}$ and $m_{2j}$, a sufficient condition to guarantee such a requirement is to take $m_{3j}$ as $m_3=\max_{2\le j\le k-2}m_{3j}=\frac{4k}{2s+3}-\frac{8}{2s+3}>\frac{4k^2+4sk-14k-2s}{2ks+3k+2s^2+3s}$. Consequently, $j+2+m_{3j}\le k+m_3\le\frac{2s+7}{2s+3}k-\frac{8}{2s+3}\le\frac{2s+7}{2s+3}N_0-\frac{6s+29}{2s+3}$ with $N_0\ge7$. Moreover, since $\hat{l}_{6j}$ satisfies $\frac{\gamma}{2}\beta_j+\hat{l}_{6j}(1-\beta_j)=0$, one can deduce that $\hat{l}_{6j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$, where $\beta_j=\frac{(2j-3)(m_{3j}+2)(k+s)}{m_{3j}(2k+2s)(j+s)}$ with $2\le j\le k-2$. And we set $\hat{l}_6=\max\limits_{2\le j\le k-2}\left\{\hat{l}_{6j}\right\}$.

$I_{4,3,1}$ and $I_{4,3,2}$ can be bounded by the upper bound of $I_{4,3,3}$.

Collecting the above estimates gives (3.1).

When $k = N_0 - 2 \geq 4$, similar to that of (3.7), we can get that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0-2} f\right\|^2 + \left\|\nabla^{N_0-2}(E,B)\right\|^2\right) + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \underbrace{\left|\left(\nabla^{N_0-2}(E\cdot v f), \nabla^{N_0-2} f\right)\right|}_{I_5} + \underbrace{\left|\left(\nabla^{N_0-2}(E\cdot \nabla_v f), \nabla^{N_0-2} f\right)\right|}_{I_6} \\
&\quad + \underbrace{\left|\left(\nabla^{N_0-2}((v\times B)\cdot \nabla_v f), \nabla^{N_0-2} f\right)\right|}_{I_7} + \underbrace{\left|\left(\nabla^{N_0-2}\mathbf{\Gamma}(f,f), \nabla^{N_0-2} f\right)\right|}_{I_8}.
\end{aligned} \tag{3.18}$$

Applying the similar trick as (3.8), one has

$$I_5 \lesssim \bar{\mathcal{E}}_{N_0-1,l'}(t)\left(\left\|\nabla^{N_0-1}E\right\|^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2. \tag{3.19}$$

For $I_7$ in the right hand side of (3.18), we can get that

$$\begin{aligned}
I_7 &\lesssim \sum_{1\leq j\leq N_0-2}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-2-j}\nabla_v f, \nabla^{N_0-2} f\right)\right| \\
&\lesssim \underbrace{\sum_{1\leq j\leq N_0-2}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-2-j}\nabla_v f, \nabla^{N_0-2}\mathbf{P} f\right)\right|}_{I_{7,1}} \\
&\quad + \underbrace{\sum_{1\leq j\leq N_0-2}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-2-j}\nabla_v\mathbf{P} f, \nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{7,2}} \\
&\quad + \underbrace{\sum_{1\leq j\leq N_0-2}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-2-j}\nabla_v\{\mathbf{I}-\mathbf{P}\}f, \nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{7,3}}.
\end{aligned} \tag{3.20}$$

Applying Lemma 2.3, one has

$$I_{7,1} + I_{7,2} \lesssim \bar{\mathcal{E}}_{N_0-1,0}(t)\left(\left\|\nabla^{N_0-1}B\right\|^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2.$$

As to $I_{7,3}$, when $j = N_0 - 2$, it follows from Lemma 2.3 that

$$\begin{aligned}
I_{7,3} &\lesssim \left\|\nabla^{N_0-2}B\right\|_{L_x^6}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^3}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_x^2L_v^2} \\
&\lesssim \left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2H_x^1}\left\|\nabla^{N_0-1}B\right\|\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \mathcal{E}_{2,\frac{3}{2}-\frac{1}{\gamma}}(t)\left\|\nabla^{N_0-1}B\right\|^2 + \varepsilon\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,
\end{aligned}$$

while for the case $1 \le j \le N_0 - 3$, one has

$$
\begin{aligned}
I_{7,3} &\lesssim \sum_{1\le j\le N_0-3} \left\|\nabla^j B\right\|_{L_x^\infty} \left\|\nabla^{N_0-2-j}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\| \left\|\langle v\rangle\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{4j}+1}\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{4j}+1}} \\
&\quad\times \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}}{m_{4j}+1}} \left\|\langle v\rangle\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{4j}+1}\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{4j}+1}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}(j-1)}{(m_{4j}+1)(N_0-3+s)}} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}(N_0-2+s-j)}{(m_{4j}+1)(N_0-3+s)}} \left\|\langle v\rangle\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{4j}+1}\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{4j}+1}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}(j-1)}{(m_{4j}+1)(N_0-3+s)}} \left\|\langle v\rangle^{\hat{l}_{7j}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}(N_0-1+s-j)(1-\beta_j)}{(m_{4j}+1)(N_0-2+s)}} \\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{4j}(N_0-1+s-j)\beta_j}{(m_{4j}+1)(N_0-2+s)}} \left\|\langle v\rangle^{\hat{l}_{8j}}\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta_j} \\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta_j} \\
&\lesssim \max\left\{\mathcal{E}_{N_0-2+m_4,1+m_4}(t), \bar{\mathcal{E}}_{N_0-2,-\frac{\hat{l}_8}{\gamma}}(t)\right\}\left(\left\|\nabla^{N_0-1}B\right\|^2 + \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right) \\
&\quad+\varepsilon\left(\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right). \qquad (3.21)
\end{aligned}
$$

Here we have used the fact that there exist positive constants $m_{4j}$ and $\beta_j \in (0,1)$ such that $\frac{2j+2s+3}{2(N_0-1+s)} + \frac{m_{4j}(N_0-2+s-j)\beta_j}{(m_{4j}+1)(N_0-3+s)} + \beta = 2$ holds for $1 \le j \le N_0 - 3$. Similar to the analysis used in determining $m_{3j}$ and by noticing that $N_0 \ge 6$ and $\frac{9}{7} \le s < \frac{3}{2}$, it is easy to see that we can take

$$
\begin{aligned}
m_4 &= \max_{1\le j\le N_0-3} m_{4j} = \frac{2N_0}{2s+3} - \frac{9}{2s+3} > \frac{2N_0^2+(2s-9)N_0+21-7s}{2sN_0+5N_0+2s^2-s-4j-11} \\
&\ge \max_{1\le j\le N_0-3} \frac{2N_0^2+(2s-11-2j)N_0+6j-2sj+15-5s}{2sN_0+5N_0+2s^2-s-4j-11}.
\end{aligned}
$$

Consequently, $N_0 - 1 - j + m_{4j} \le N_0 - 2 + m_4 \le \frac{2s+5}{2s+3}N_0 - \frac{4s+15}{2s+3}$ with $N_0 \ge 6$. Moreover, since $\hat{l}_{7j}$ and $\hat{l}_{8j}$ satisfy respectively $\frac{\gamma}{2}\beta_j + \hat{l}_{7j}(1-\beta_j) = 0$ and $\frac{\gamma}{2}\beta_j + \hat{l}_{8j}(1-\beta_j) = 1$ with $0 < \beta_j < 1$, one can deduce that, $\hat{l}_{7j} = \frac{\gamma}{2} - \frac{\gamma}{2(1-\beta_j)}$ and $\hat{l}_{8j} = \frac{\gamma}{2} - \frac{\gamma-2}{2(1-\beta_j)}$ from which we can

see that $\hat{l}_{8j} > \hat{l}_{7j}$, where $\beta_j = \frac{(4N_0+4s-7-2j-s)(N_0-3+s)(m_{4j}+1)}{(2N_0-2+2s)(N_0-3+s+m_{4j}(2N_0+2s-j-5))}$ with $1 \le j \le N_0-3$. If we take $\hat{l}_8 = \max\limits_{1\le j\le N_0-3}\left\{\hat{l}_{8j}\right\}$, we can deduce that $\hat{l}_8 \ge \hat{l}_7$.

Using the same trick as $I_7$ gives

$$I_6 \lesssim \max\left\{\mathcal{E}_{N_0-2+m_4,1+m_4}(t), \bar{\mathcal{E}}_{N_0-1,\max\left\{\frac{3}{2}-\frac{1}{\gamma},-\frac{\hat{l}_8}{\gamma}\right\}}(t)\right\}\left(\left\|\nabla^{N_0-1}E\right\|^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2 + \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^1_xL^2_\nu}\right) + \varepsilon\left(\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^1_xL^2_\nu} + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right).$$

For the last term $I_8$ in the right hand side of (3.18), applying the similar way as (3.14), one can get that

$$\begin{aligned} I_8 \lesssim & \underbrace{\sum_{j\le N_0-2}\left|\left(\mathbf{\Gamma}(\nabla^j\mathbf{P}f, \nabla^{N_0-2-j}f), \nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{8,1}} \\ & + \underbrace{\sum_{j\le N_0-2}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f, \nabla^{N_0-2-j}\mathbf{P}f), \nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{8,2}} \\ & + \underbrace{\sum_{j\le N_0-2}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f, \nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f), \nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{8,3}}. \end{aligned} \tag{3.22}$$

We only treat the term $I_{8,3}$ for brevity. For this purpose, when $j = 0$ or $j = N_0 - 2$ one has

$$\begin{aligned} I_{8,3} &\lesssim \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vL^\infty_x}\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|f\|_{H^2_vL^\infty_x}\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \varepsilon\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2, \end{aligned}$$

while for the case $j = 1$ or $N_0 - 3$, we have

$$\begin{aligned} I_{8,3} &\lesssim \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^6_xL^2_\nu} \|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vL^3_x}\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vH^1_x}\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \varepsilon\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2, \end{aligned}$$

while for the case $j = 2$ or $N_0 - 4$, we have

$$\begin{aligned} I_{8,3} &\lesssim \left\|\nabla^{N_0-4}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x L^2_v} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \mathcal{E}_{4,2}(t) \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^2_v L^2_x} + \varepsilon \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu. \end{aligned}$$

For the last case $3 \le j \le N_0 - 5$, we have

$$\begin{aligned} I_{8,3} &\lesssim \sum_{3\le j\le N_0-5} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x H^2_v} \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \underbrace{\sum_{3\le j\le N_0-5} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_v} \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{8,3,1}} \\ &\quad + \underbrace{\sum_{3\le j\le N_0-5} \left\|\nabla_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_v} \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{8,3,2}} \\ &\quad + \underbrace{\sum_{3\le j\le N_0-5} \left\|\nabla^2_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_x} \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{8,3,3}}. \end{aligned} \tag{3.23}$$

$I_{8,3,3}$ is the most difficult term among these three terms in the right hand side of the above inequality, which can be bounded as follows

$$\begin{aligned} I_{8,3,3} &\lesssim \sum_{3\le j\le N_0-5} \left\|\nabla_v^{m_{5j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{5j}+2}} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}}{m_{5j}+2}} \\ &\quad\times \left\|\nabla^{N_0-2-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \sum_{3\le j\le N_0-5} \left\|\nabla_v^{m_{5j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{5j}+2}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}(N_0-3-j)}{(m_{5j}+2)(N_0-3+s)}} \\ &\quad\times \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}(j+s)}{(m_{5j}+2)(N_0-3+s)}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2j-5}{2(N_0-3+s)}} \\ &\quad\times \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2N_0+2s-2j-1}{2(N_0-3+s)}}_\nu \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{3\le j\le N_0-5} \left\|\nabla_v^{m_{5j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{5j}+2}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}(N_0-3-j)}{(m_{5j}+2)(N_0-3+s)}}\\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}(j+s)\beta_j}{(m_{5j}+2)(N_0-3+s)}} \left\|\langle v\rangle^{\hat{l}_{9j}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{5j}(j+s)(1-\beta_j)}{(m_{5j}+2)(N_0-3+s)}}\\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2j-5}{2(N_0-3+s)}} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^{\frac{2N_0+2s-2j-1}{2(N_0-3+s)}} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\\
&\lesssim \max\left\{\mathcal{E}_{N_0-3+m_5,m_5+2}(t), \bar{\mathcal{E}}_{N_0-2,-\frac{\hat{l}_9}{\gamma}}(t)\right\}\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&\quad+\varepsilon\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}
\tag{3.24}
$$

Here we have used the fact that there exist positive constants $m_{5j}$ and $\beta_j\in(0,1)$ such that $\frac{m_{5j}(j+s)\beta_j}{(m_{5j}+2)(N_0-3+s)}+\frac{2N_0+2s-2j-1}{2(N_0-3+s)}=1$ holds for $3\le j\le N_0-5$. Noticing that $N_0\ge 8$ and $\frac{9}{7}\le s<\frac{3}{2}$, it is easy to see that we can take

$$
\begin{aligned}
m_5=\max_{3\le j\le N_0-5} m_{5j}&=\frac{4N_0}{2s+3}-\frac{28}{2s+3}>\frac{4N_0^2+4sN_0-42N_0+90-30s}{(N_0-3+s)(2s+5)}\\
&\ge \max_{3\le j\le N_0-5}\frac{2(2jN_0-5N_0-6j+15+2sj-5s)}{(N_0-3+s)(2s+5)}.
\end{aligned}
$$

Consequently, $j+2+m_{5j}\le N_0-3+m_5\le\frac{2s+7}{2s+3}N_0-\frac{6s+37}{2s+3}$ with $N_0\ge 8$. Moreover, since $\hat{l}_{9j}$ satisfies $\frac{\gamma}{2}\beta_j+\hat{l}_{9j}(1-\beta_j)=0$, one can deduce that $\hat{l}_{9j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$, where $\beta_j=\frac{(2j-5)(m_{5j}+2)}{2m_{5j}(j+s)}$ with $3\le j\le N_0-5$. And we take $\hat{l}_9=\max\limits_{3\le j\le N_0-5}\left\{\hat{l}_{9j}\right\}$.

Apply the same way as the estimate on $I_{8,3,3}$, the term $I_{8,3,1}$ and $I_{8,3,2}$ can be dominated by the upper bound of $I_{8,3,3}$.

Collecting the above estimates gives (3.2).

When $k=N_0-1\ge 5$, similar to that of (3.7), we have that

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0-1}f\right\|^2+\left\|\nabla^{N_0-1}(E,B)\right\|^2\right)+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&\lesssim \underbrace{\left|\left(\nabla^{N_0-1}(E\cdot vf),\nabla^{N_0-1}f\right)\right|}_{I_9}+\underbrace{\left|\left(\nabla^{N_0-1}(E\cdot\nabla_v f),\nabla^{N_0-1}f\right)\right|}_{I_{10}}\\
&\quad+\underbrace{\left|\left(\nabla^{N_0-1}((v\times B)\cdot\nabla_v f),\nabla^{N_0-1}f\right)\right|}_{I_{11}}+\underbrace{\left|\left(\nabla^{N_0-1}\Gamma(f,f),\nabla^{N_0-1}f\right)\right|}_{I_{12}}.
\end{aligned}
\tag{3.25}
$$

Applying the similar trick as (3.8), one has

$$
I_9\lesssim\bar{\mathcal{E}}_{N_0-1,l'}(t)\left(\left\|\nabla^{N_0-1}E\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2.
\tag{3.26}
$$

For $I_{11}$ in the right hand side of (3.25), we can get that

$$\begin{aligned} I_{11} &\lesssim \sum_{1\le j\le N_0-1} \left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-1-j}\nabla_v f, \nabla^{N_0-1} f\right)\right| \\ &\lesssim \underbrace{\sum_{1\le j\le N_0-1} \left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-1-j}\nabla_v f, \nabla^{N_0-1}\mathbf{P} f\right)\right|}_{I_{11,1}} \\ &\quad + \underbrace{\sum_{1\le j\le N_0-1} \left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-1-j}\nabla_v \mathbf{P} f, \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\} f\right)\right|}_{I_{11,2}} \\ &\quad + \underbrace{\sum_{1\le j\le N_0-1} \left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-1-j}\nabla_v \{\mathbf{I}-\mathbf{P}\} f, \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\} f\right)\right|}_{I_{11,3}}. \end{aligned} \tag{3.27}$$

Applying Lemma 2.3, one has

$$I_{11,1}+I_{11,2}\lesssim \bar{\mathcal{E}}_{N_0-1,0}(t)\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2.$$

As to $I_{11,3}$, when $j=N_0-2$ and $j=N_0-1$, it follows from Lemma 2.3 that

$$\begin{aligned} I_{11,3} &\lesssim \left\|\nabla^{N_0-2}B\right\|_{L_x^6}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_v\nabla_x\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^3}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_x^2L_v^2} \\ &\quad + \left\|\nabla^{N_0-1}B\right\|_{L_x^2}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_v\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^\infty}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_x^2L_v^2} \\ &\lesssim \left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2H_x^1}\left\|\nabla^{N_0-1}B\right\|\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \mathcal{E}_{2,\frac{3}{2}-\frac{1}{\gamma}}(t)\left\|\nabla^{N_0-1}B\right\|^2+\varepsilon\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

Moreover, for the case $1\le j\le N_0-3$, one has

$$\begin{aligned} I_{11,3} &\lesssim \sum_{1\le j\le N_0-3}\left\|\nabla^j B\right\|_{L_x^\infty}\left\|\nabla^{N_0-1-j}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|\left\|\langle v\rangle\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\| \\ &\lesssim \sum_{1\le j\le N_0-3}\left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}}\left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}}\left\|\nabla_v^{m_{6j}+1}\nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{6j}+1}} \\ &\quad\times\left\|\nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}}{m_{6j}+1}}\left\|\langle v\rangle\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{6j}+1}\nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{6j}+1}}\\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}(j-1)}{(m_{6j}+1)(N_0-2+s)}} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}(N_0-1+s-j)}{(m_{6j}+1)(N_0-2+s)}} \left\|\langle v\rangle\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|\\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{6j}+1}\nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{6j}+1}}\\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}(j-1)}{(m_{6j}+1)(N_0-1+s)}} \left\|\langle v\rangle^{\hat{l}_{10j}}\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}(N_0-1+s-j)(1-\beta_j)}{(m_{6j}+1)(N_0-2+s)}}\\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{6j}(N_0-1+s-j)\beta_j}{(m_{6j}+1)(N_0-2+s)}} \left\|\langle v\rangle^{\hat{l}_{11j}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta_j}\\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta_j}\\
&\lesssim \max\left\{\mathcal{E}_{N_0-1+m_6,m_6+1}(t),\bar{\mathcal{E}}_{N_0-1,-\frac{\hat{l}_{11}}{\gamma}}(t)\right\}\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right)\\
&\quad+\varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right).
\end{aligned}
\tag{3.28}
$$

Here we have used the fact that there exist positive constants $m_{6j}$ and $\beta_j\in(0,1)$ such that $\frac{2j+2s+3}{2(N_0-1+s)}+\frac{m_{6j}(N_0-1+s-j)\beta_j}{(m_{6j}+1)(N_0-2+s)}+\beta_j=2$ holds for $1\le j\le N_0-3$. Noticing that $\frac{9}{7}\le s<\frac{3}{2}$ and $N_0\ge 6$, it is easy to see that we can take

$$
\begin{aligned}
m_6=\max_{1\le j\le N_0-3} m_{6j}&=\frac{2N_0}{3s-1/2}-\frac{7}{3s-1/2}>\frac{2N_0^2+2sN_0+14-11N_0-7s}{3sN_0+3s^2-6s}\\
&\ge \max_{1\le j\le N_0-3}\frac{2N_0^2+2sN_0+4j+10-9N_0-2jN_0-2sj-5s}{3sN_0+3s^2-2j-s-6}.
\end{aligned}
$$

Consequently, $N_0-j+m_{6j}\le N_0-1+m_6\le\frac{3s+3/2}{3s-1/2}N_0-\frac{3s+13/2}{3s-1/2}$ with $N_0\ge 6$. Moreover, since $\hat{l}_{10j}$ and $\hat{l}_{11j}$ satisfy respectively $\frac{\gamma}{2}\beta_j+\hat{l}_{10j}(1-\beta_j)=0$ and $\frac{\gamma}{2}\beta_j+\hat{l}_{11j}(1-\beta_j)=1$ with $0<\beta_j<1$, one can deduce that, $\hat{l}_{10j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$ and $\hat{l}_{11j}=\frac{\gamma}{2}-\frac{\gamma-2}{2(1-\beta_j)}$ from which we can see that $\hat{l}_{11j}>\hat{l}_{10j}$, where $\beta_j=\frac{(4N_0+4s-7-2j-s)(N_0-2+s)(m_{4j}+1)}{(2N_0-2+2s)(N_0-1+s-j+m_{4j}(2N_0+3s-j-3))}$ with $1\le j\le N_0-3$. And we take $\hat{l}_{11}=\max\limits_{1\le j\le N_0-3}\left\{\hat{l}_{11j}\right\}$.

Applying the same trick as $I_{11}$ yields that

$$
\begin{aligned}
I_{10}\lesssim\max&\left\{\mathcal{E}_{N_0-j+m_6,m_6+1}(t),\mathcal{E}_{2,\frac{3}{2}-\frac{1}{\gamma}}(t),\bar{\mathcal{E}}_{N_0-1,-\frac{\hat{l}_{11}}{\gamma}}(t)\right\}\left(\left\|\nabla^{N_0-1}E\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right.\\
&\left.+\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right)+\varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right).
\end{aligned}
$$

For the last term $I_{12}$ in the right hand side of (3.25), one has by applying the similar argument to deal with (3.14) that

$$\begin{aligned} I_{12} \lesssim & \underbrace{\sum_{j\le N_0-1}\left|\left(\mathbf{\Gamma}(\nabla^j\mathbf{P}f,\nabla^{N_0-1-j}f),\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{12,1}} \\ & + \underbrace{\sum_{j\le N_0-1}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0-1-j}\mathbf{P}f),\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{12,2}} \\ & + \underbrace{\sum_{j\le N_0-1}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f),\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{12,3}}. \end{aligned} \tag{3.29}$$

We only estimate the term $I_{12,3}$ for brevity. To this end, when $j=0$ or $j=N_0-1$ one has

$$\begin{aligned} I_{12,3} &\lesssim \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_\nu L^\infty_x}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|f\|_{H^2_\nu L^\infty_x}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

For the case $j=1$ or $N_0-2$, we have

$$\begin{aligned} I_{12,3} &\lesssim \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^6_xL^2_\nu}\|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_\nu L^3_x}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_\nu H^1_x}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

For the case $j=2$ or $N_0-3$, we have

$$\begin{aligned} I_{12,3} &\lesssim \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_xL^2_\nu}\|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_\nu L^2_x}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_\nu L^2_x}\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_\nu}\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

For the case $j = 3$ or $N_0 - 4$, we have

$$\begin{aligned} I_{12,3} &\lesssim \left\| \nabla^{N_0-4}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^\infty_x L^2_v} \|\nabla^3\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu \\ &\lesssim \|\nabla^3\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\| \nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f \right\|_{H^1_x L^2_v} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu \\ &\lesssim \mathcal{E}_{5,2}(t) \left\| \nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f \right\|^2_{H^1_x L^2_v} + \varepsilon \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|^2_\nu. \end{aligned}$$

For the last case $4 \le j \le N_0 - 5$, we have

$$\begin{aligned} I_{12,3} &\lesssim \sum_{4\le j\le N_0-5} \left\| \nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_x H^2_v} \left\| \nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^\infty_x L^2_v} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu \\ &\lesssim \underbrace{\sum_{4\le j\le N_0-5} \left\| \nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_x L^2_v} \left\| \nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_v L^\infty_x} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu}_{I_{12,3,1}} \\ &\quad + \underbrace{\sum_{4\le j\le N_0-5} \left\| \nabla_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_x L^2_v} \left\| \nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_v L^\infty_x} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu}_{I_{12,3,2}} \\ &\quad + \underbrace{\sum_{4\le j\le N_0-5} \left\| \nabla_v^2\nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_x L^2_x} \left\| \nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_v L^\infty_x} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu}_{I_{12,3,3}}. \end{aligned} \tag{3.30}$$

$I_{12,3,3}$ is the most difficult term among those three terms in the right hand side of the above inequality, which can be bounded as follows

$$\begin{aligned} I_{12,3,3} &\lesssim \sum_{4\le j\le N_0-5} \left\| \nabla_v^{m_{7j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|^{\frac{2}{m_{7j}+2}} \left\| \nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|^{\frac{m_{7j}}{m_{7j}+2}} \\ &\quad \times \left\| \nabla^{N_0-1-j}\{\mathbf{I}-\mathbf{P}\}f \right\|_{L^2_v L^\infty_x} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu \\ &\lesssim \sum_{4\le j\le N_0-5} \left\| \nabla_v^{m_{7j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f \right\|^{\frac{2}{m_{7j}+2}} \left\| \Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f \right\|^{\frac{m_{7j}(N_0-3-j)}{(m_{7j}+2)(N_0-3+s)}} \\ &\quad \times \left\| \nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f \right\|^{\frac{m_{7j}(j+s)}{(m_{7j}+2)(N_0-3+s)}} \left\| \Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu^{\frac{2j-5}{2(N_0-3+s)}} \\ &\quad \times \left\| \nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu^{\frac{2N_0+2s-2j-1}{2(N_0-3+s)}} \left\| \nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f \right\|_\nu \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{4\le j\le N_0-5} \left\|\nabla_v^{m_{7j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{7j}+2}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{7j}(N_0-3-j)}{(m_{7j}+2)(N_0-3+s)}} \\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{7j}(j+s)\beta_j}{(m_{7j}+2)(N_0-3+s)}} \left\|\langle v\rangle^{\hat{l}_{12j}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{7j}(j+s)(1-\beta_j)}{(m_{7j}+2)(N_0-3+s)}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2j-5}{2(N_0-3+s)}} \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^{\frac{2N_0+2s-2j-1}{2(N_0-3+s)}} \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \max\left\{\mathcal{E}_{N_0-3+m_7,2+m_7}(t), \bar{\mathcal{E}}_{N_0-1,-\frac{\hat{l}_{12}}{\gamma}}(t)\right\}\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\quad+\varepsilon\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}
\tag{3.31}
$$

Here we have used the fact that there exist positive constants $m_{7j}$ and $\beta_j\in(0,1)$ such that $\frac{m_{7j}(j+s)\beta_j}{(m_{7j}+2)(N_0-3+s)}+\frac{2N_0+2s-2j-1}{2(N_0-3+s)}=1$ holds for $4\le j\le N_0-5$. A necessary and sufficient condition to guarantee the existence of such a $\beta$ is $\frac{m_{7j}(j+s)}{(m_{7j}+2)(N_0-3+s)}+\frac{2N_0+2s-2j-1}{2(N_0-3+s)}>1$, $4\le j\le N_0-5$, from which one can deduce that $m_{7j}>\frac{4j-10}{2s+5}$ holds for $4\le j\le N_0-5$. It is easy to see that we can take

$$
m_7=\max_{4\le j\le N_0-5} m_{7j}=\frac{4}{2s+5}N_0-\frac{29}{2s+5}>\frac{4N_0-30}{2s+5}=\max_{4\le j\le N_0-5}\frac{4j-10}{2s+5}.
$$

Consequently, $j+2+m_{7j}\le N_0-3+m_7=\frac{2s+9}{2s+5}N_0-\frac{6s+45}{2s+5}$ with $N_0\ge 9$. Moreover, since $\hat{l}_{12j}$ satisfies $\frac{\gamma}{2}\beta_j+\hat{l}_{12j}(1-\beta_j)=0$, one can deduce that $\hat{l}_{12j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$, where $\beta_j=\frac{(2j-5)(m_{7j}+2)}{2m_{7j}(j+s)}$ with $4\le j\le N_0-5$. And set $\hat{l}_{12}=\max\limits_{4\le j\le N_0-5}\left\{\hat{l}_{12j}\right\}$.

Similarly, $I_{12,3,1}$ and $I_{12,3,2}$ can be controlled by the upper bound of $I_{12,3,3}$.

Collecting the above estimates gives (3.3).

Finally, we turn to the case of $k=N_0\ge 6$. For such a case, we have similar to that of (3.7) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0}f\right\|^2+\left\|\nabla^{N_0}E\right\|^2\right)+\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \underbrace{\left|\left(\nabla^{N_0}(E\cdot vf),\nabla^{N_0}f\right)\right|}_{I_{13}}+\underbrace{\left|\left(\nabla^{N_0}(E\cdot\nabla_v f),\nabla^{N_0}f\right)\right|}_{I_{14}} \\
&\quad+\underbrace{\left|\left(\nabla^{N_0}((v\times B)\cdot\nabla_v f),\nabla^{N_0}f\right)\right|}_{I_{15}}+\underbrace{\left|\left(\nabla^{N_0}\mathbf{\Gamma}(f,f),\nabla^{N_0}f\right)\right|}_{I_{16}}.
\end{aligned}
\tag{3.32}
$$

For the first term $I_{13}$ in the right hand of the above inequality, one has

$$
\begin{aligned}
I_{13} \lesssim & \underbrace{\left|\left(E\cdot v\nabla^{N_0} f, \nabla^{N_0} f\right)\right|}_{I_{13,1}} + \underbrace{\left|\left(\nabla E\cdot v\nabla^{N_0-1} f, \nabla^{N_0} f\right)\right|}_{I_{13,2}} \\
& + \underbrace{\left|\left(\nabla^{N_0-1} E\cdot v\nabla f, \nabla^{N_0} f\right)\right|}_{I_{13,3}} + \underbrace{\left|\left(\nabla^{N_0} E\cdot v f, \nabla^{N_0} f\right)\right|}_{I_{13,4}} \\
& + \underbrace{\sum_{2\le j\le N_0-1}\left|\left(\nabla^{j} E\cdot v\nabla^{N_0-j} f, \nabla^{N_0} f\right)\right|}_{I_{13,5}}. \qquad (3.33)
\end{aligned}
$$

For the first term $I_{13,1}$ in the right hand side of (3.33), using the Hölder inequality with exponents $(p,q,r)=(+\infty,2,2)$ gives

$$
\begin{aligned}
I_{13,1} &\lesssim \|E\|_{L^\infty}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla^{N_0} f\right\|\left\|\nabla^{N_0} f\right\|_\nu \\
&\lesssim \left\|\Lambda^{-s}E\right\|^{1-\theta}\left\|\nabla^{N_0-1}E\right\|^{\theta}\left\|\langle v\rangle^{\hat{l}_{13}}\nabla^{N_0} f\right\|^{\theta}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0} f\right\|^{1-\theta}\left\|\nabla^{N_0} f\right\|_\nu \\
&\lesssim \bar{\mathcal{E}}_{N_0,-\frac{\hat{l}_{13}}{\gamma}}(t)\left(\left\|\nabla^{N_0-1}E\right\|^2+\left\|\nabla^{N_0} f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0} f\right\|_\nu^2.
\end{aligned}
$$

Here we can deduce from Lemma 2.3 that $\theta=\frac{3+2s}{2(N_0-1+s)}$, $\hat{l}_{13}\theta+\frac{\gamma}{2}(1-\theta)=1-\frac{\gamma}{2}$ which gives $\hat{l}_{13}=\frac{\gamma}{2}-\frac{\gamma-1}{\theta}=\frac{\gamma}{2}-\frac{2(\gamma-1)(N_0-1+s)}{3+2s}$.

Similarly, $I_{13,2}$ can be bounded by

$$
\begin{aligned}
I_{13,2} &\lesssim \|\nabla E\|_{L^3}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla^{N_0-1} f\right\|_{L^6}\left\|\nabla^{N_0} f\right\|_\nu \\
&\lesssim \left\|\Lambda^{-s}E\right\|^{1-\theta}\left\|\nabla^{N_0-1}E\right\|^{\theta}\left\|\langle v\rangle^{\hat{l}_{14}}\nabla^{N_0} f\right\|^{\theta}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0} f\right\|^{1-\theta}\left\|\nabla^{N_0} f\right\|_\nu \\
&\lesssim \bar{\mathcal{E}}_{N_0,-\frac{\hat{l}_{14}}{\gamma}}(t)\left(\left\|\nabla^{N_0-1}E\right\|^2+\left\|\nabla^{N_0} f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0} f\right\|_\nu^2.
\end{aligned}
$$

Here we can deduce from Lemma 2.3 that $\theta=\frac{3+2s}{2(N_0-1+s)}$, $\hat{l}_{14}\theta+\frac{\gamma}{2}(1-\theta)=1-\frac{\gamma}{2}$ which gives $\hat{l}_{14}=\frac{\gamma}{2}-\frac{\gamma-1}{\theta}=\frac{\gamma}{2}-\frac{2(\gamma-1)(N_0-1+s)}{3+2s}$.

For $I_{13,3}$, we have by using the Hölder inequality with exponents $(p,q,r)=(2,+\infty,2)$ that

$$
I_{13,3}\lesssim\left\|\nabla^{N_0-1}E\right\|\left\|\langle v\rangle^{1-\gamma/2}\nabla f\right\|_{L^\infty}\left\|\nabla^{N_0} f\right\|_\nu\lesssim\mathcal{E}_{3,\frac{1}{2}-\frac{1}{\gamma}}(t)\left\|\nabla^{N_0-1}E\right\|^2+\varepsilon\left\|\nabla^{N_0} f\right\|_\nu^2.
$$

Applying Lemma 2.3, the last term $I_{13,5}$ in the right hand side of (3.33) can be dominated by

$$
\begin{aligned}
I_{13,5} &\lesssim \sum_{2\le j\le N_0-2}\left\|\nabla^{j}E\right\|_{L^6}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla^{N_0-j} f\right\|_{L^3}\left\|\nabla^{N_0} f\right\|_\nu \\
&\lesssim \sum_{2\le j\le N_0-2}\left\|\Lambda^{-s}E\right\|^{1-\theta_j}\left\|\nabla^{N_0-1}E\right\|^{\theta_j}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla^{N_0-1} f\right\|^{\alpha_j}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{1-\alpha_j} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \sum_{2\le j\le N_0-2} \left\| \Lambda^{-s} E \right\|^{1-\theta_j} \left\| \nabla^{N_0-1} E \right\|^{\theta_j} \left( \left\| \langle v\rangle^{\hat{l}_{15j}} \nabla^{N_0-1} f \right\|^{1-\beta_j} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\beta_j} \right)^{\alpha_j} \\
&\times \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{1-\alpha_j} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \sum_{2\le j\le N_0-2} \left\| \Lambda^{-s} E \right\|^{1-\theta_j} \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{1-\alpha_j} \left\| \langle v\rangle^{\hat{l}_{15j}} \nabla^{N_0-1} f \right\|^{\alpha_j-\alpha_j\beta_j} \\
&\times \left\| \nabla^{N_0-1} E \right\|^{\theta_j} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\alpha_j\beta_j} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \bar{\mathcal{E}}_{N_0-1,-\frac{\hat{l}_{15}}{\gamma}}(t) \left( \left\| \nabla^{N_0-1} E \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_\nu^2 \right) + \varepsilon \left\| \nabla^{N_0} f \right\|_\nu^2 .
\end{aligned} \tag{3.34}
$$

Here we deduce from Lemma 2.3 that $\theta_j = \frac{j+s+1}{N_0-1+s}$, $\alpha_j = \frac{2N_0-2j+1}{2(N_0-1)}$, and $\frac{\gamma}{2}\cdot\beta_j + \hat{l}_{15j}(1-\beta_j) = 1-\frac{\gamma}{2}$. If we set $\alpha_j\beta_j = 1-\theta_j$ which gives that $\hat{l}_{15j} = \frac{\gamma}{2} - \frac{\gamma-1}{1-\beta_j} = \frac{\gamma}{2} - \frac{(\gamma-1)(2N_0-2j+1)(N_0-1+s)}{(2N_0-2j+1)(N_0-1+s)-2(N_0-2-j)(N_0-1)}$. And we take $\hat{l}_{15} = \max\limits_{2\le j\le N_0-2}\left\{\hat{l}_{15j}\right\}$.

For the most important term $I_{13,4}$, if $n > \frac{2}{3}N_0 - \frac{5}{3}$, we have from Lemma 2.3 that

$$
\begin{aligned}
I_{13,4} \lesssim & \left\| \nabla^{N_0-1} E \right\|^{\frac{n}{n+1}} \left\| \nabla^{N_0+n} E \right\|^{\frac{1}{n+1}} \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{\frac{2N_0-5}{2N_0-2}} \left\| \langle v\rangle^{1-\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\frac{3}{2N_0-2}} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \left\| \nabla^{N_0-1} E \right\|^{\frac{n}{n+1}} \left\| \nabla^{N_0+n} E \right\|^{\frac{1}{n+1}} \left( \left\| \langle v\rangle^{\hat{l}_{16}} \nabla^{N_0-1} f \right\|^{1-\alpha} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\alpha} \right)^{\frac{3}{2N_0-2}} \\
&\times \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{\frac{2N_0-5}{2N_0-2}} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \left\| \nabla^{N_0-1} E \right\|^{\frac{n}{n+1}} \left\| \nabla^{N_0+n} E \right\|^{\frac{1}{n+1}} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\frac{3\alpha}{2N_0-2}} \\
&\times \left\| \langle v\rangle^{\hat{l}_{16}} \nabla^{N_0-1} f \right\|^{\frac{3(1-\alpha)}{2N_0-2}} \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{\frac{2N_0-5}{2N_0-2}} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \left\| \nabla^{N_0-1} E \right\|^{\frac{n}{n+1}} \left\| \nabla^{N_0+n} E \right\|^{\frac{1}{n+1}} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\frac{3\alpha}{2N_0-2}} \\
&\times \left\| \langle v\rangle^{\hat{l}_{16}} \nabla^{N_0-1} f \right\|^{\frac{3(1-\alpha)}{2N_0-2}} \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|^{\frac{2N_0-5}{2N_0-2}} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \left\| \nabla^{N_0-1} E \right\|^{\frac{n}{n+1}} \left\| \nabla^{N_0+n} E \right\|^{\frac{1}{n+1}} \left\| \langle v\rangle^{\frac{\gamma}{2}} \nabla^{N_0-1} f \right\|^{\frac{1}{n+1}} \\
&\times \left\| \langle v\rangle^{\hat{l}_{16}} \nabla^{N_0-1} f \right\|^{\frac{3(1-\alpha)}{2N_0-2}} \left\| \langle v\rangle^{-\frac{\gamma}{2}-s+1} f \right\|^{\frac{2N_0-5}{2N_0-2}} \left\| \nabla^{N_0} f \right\|_\nu \\
\lesssim & \max\left\{ \mathcal{E}_{N_0+n}(t), \bar{\mathcal{E}}_{N_0-1,-\frac{\hat{l}_{16}}{\gamma}}(t) \right\} \left( \left\| \nabla^{N_0-1} E \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_\nu^2 \right) + \varepsilon \left\| \nabla^{N_0} f \right\|_\nu^2 .
\end{aligned} \tag{3.35}
$$

Here we require that $\frac{3\alpha}{2N_0-2}=\frac{1}{n+1}$ and $\frac{\gamma}{2}\cdot\alpha+\hat{l}_{16}(1-\alpha)=1-\frac{\gamma}{2}$. From which deduce that $\alpha=\frac{2N_0-2}{3(n+1)}$ and $\hat{l}_{16}=\frac{1-\frac{\gamma}{2}-\frac{\gamma}{2}\cdot\alpha}{1-\alpha}=\frac{\gamma}{2}+\frac{1-\gamma}{1-\alpha}=\frac{\gamma}{2}+\frac{3(1-\gamma)(n+1)}{3n-2N_0+1}$.

For the second term $I_{15}$ in the right hand side of (3.32), due to

$$\begin{aligned}
I_{15}\lesssim&\sum_{1\le j\le N_0-1}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-j}\nabla_v f,\nabla^{N_0}f\right)\right|+\left|\left((v\times\nabla^{N_0}B)\cdot\nabla_v f,\nabla^{N_0}f\right)\right|\\
\lesssim&\underbrace{\sum_{1\le j\le N_0-1}\left|\left((v\times\nabla^j B)\cdot\nabla_v f,\nabla^{N_0}\mathbf{P}f\right)\right|}_{I_{15,1}}\\
&+\underbrace{\sum_{1\le j\le N_0-1}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-j}\nabla_v\mathbf{P}f,\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{15,2}}\\
&+\underbrace{\sum_{1\le j\le N_0-1}\left|\left((v\times\nabla^j B)\cdot\nabla^{N_0-j}\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{15,3}}.\\
&+\underbrace{\left|\left((v\times\nabla^{N_0}B)\cdot\nabla_v\mathbf{P}f,\nabla^{N_0}f\right)\right|}_{I_{15,4}}+\underbrace{\left|\left((v\times\nabla^{N_0}B)\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0}\mathbf{P}f\right)\right|}_{I_{15,5}}\\
&+\underbrace{\left|\left((v\times\nabla^{N_0}B)\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{15,6}}.
\end{aligned}\tag{3.36}$$

Applying Lemma 2.3, one has

$$I_{15,1}+I_{15,2}\lesssim\bar{\mathcal{E}}_{N_0,0}(t)\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0}f\right\|_\nu^2.$$

As to $I_{15,3}$, when $j=N_0-1$ and $j=N_0-2$, it follows from Lemma 2.3 that

$$\begin{aligned}
I_{15,3}\lesssim&\left\|\nabla^{N_0-1}B\right\|_{L^2_x}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_x\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^\infty_x}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_xL^2_v}\\
&+\left\|\nabla^{N_0-2}B\right\|_{L^6_x}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla^2_x\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vL^3_x}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_xL^2_v}\\
\lesssim&\left\|\langle v\rangle^{1-\frac{\gamma}{2}}\nabla_x\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_vH^2_x}\left\|\nabla^{N_0-1}B\right\|\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\\
\lesssim&\mathcal{E}_{4,\frac{3}{2}-\frac{1}{\gamma}}(t)\left\|\nabla^{N_0-1}B\right\|^2+\varepsilon\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}$$

As for the case $1\le j\le N_0-3$, one has

$$
\begin{aligned}
I_{15,3} &\lesssim \sum_{1\le j\le N_0-3} \left\|\nabla^j B\right\|_{L_x^\infty} \left\|\nabla^{N_0-j}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\| \left\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{8j}+1}\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{8j}+1}} \\
&\quad\times \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}}{m_{8j}+1}} \left\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{8j}+1}\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{8j}+1}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}(j-1)}{(m_{8j}+1)(N_0-1+s)}} \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}(N_0+s-j)}{(m_{8j}+1)(N_0-1+s)}} \left\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\| \\
&\lesssim \sum_{1\le j\le N_0-3} \left\|\Lambda^{-s}B\right\|^{\frac{2N_0-2j-5}{2(N_0-1+s)}} \left\|\nabla^{N_0-1}B\right\|^{\frac{2j+2s+3}{2(N_0-1+s)}} \left\|\nabla_v^{m_{8j}+1}\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{8j}+1}} \\
&\quad\times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}(j-1)}{(m_{8j}+1)(N_0-1+s)}} \left\|\langle v\rangle^{\hat{l}_{17j}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}(N_0+s-j)(1-\beta_j)}{(m_{8j}+1)(N_0-1+s)}} \\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{8j}(N_0+s-j)\beta_j}{(m_{8j}+1)(N_0-1+s)}} \left\|\langle v\rangle^{\hat{l}_{18j}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta_j} \\
&\quad\times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta_j} \\
&\lesssim \max\left\{\mathcal{E}_{N_0+m_8,m_8+1}(t), \bar{\mathcal{E}}_{N_0,-\frac{\hat{l}_{18}}{\gamma}}(t)\right\}\left(\left\|\nabla^{N_0-1}B\right\|^2 + \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right) \\
&\quad+\varepsilon\left(\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right). \qquad (3.37)
\end{aligned}
$$

Here we have used the fact that there exist positive constants $m_{8j}$ and $\beta_j \in (0,1)$ such that $\frac{2j+2s+3}{2(N_0-1+s)} + \frac{m_{8j}(N_0+s-j)\beta_j}{(m_{8j}+1)(N_0-1+s)} + \beta_j = 2$ holds for $1 \le j \le N_0-3$. Similar to the way to determine of $m_{1j}$, it is suffice to take

$$
m_8 = \max_{1\le j\le N_0-3} m_{8j} = \frac{2N_0-6}{2s+5} > \frac{2N_0-7}{2s+5} = \max_{1\le j\le N_0-3}\frac{2N_0-2j-5}{2s+5}.
$$

Consequently, $N_0 - j + 1 + m_{8j} \le N_0 + m_8 \le \frac{2s+7}{2s+5}N_0 - \frac{6}{2s+5}$ with $N_0 \ge 6$. Moreover, since $\hat{l}_{17j}$ and $\hat{l}_{18j}$ satisfy respectively $\frac{\gamma}{2}\beta_j + \hat{l}_{17j}(1-\beta_j) = 0$ and $\frac{\gamma}{2}\beta_j + \hat{l}_{18j}(1-\beta_j) = 1$ with $0 < \beta_j < 1$, one can deduce that $\hat{l}_{17j} = \frac{\gamma}{2} - \frac{\gamma}{2(1-\beta_j)}$ and $\hat{l}_{18j} = \frac{\gamma}{2} - \frac{\gamma-2}{2(1-\beta_j)}$ from which we can see that $\hat{l}_{18j} > \hat{l}_{17j}$, where $\beta_j = \frac{(4N_0+2s-7-2j)(m_{8j}+1)}{2m_{8j}(N_0+s-j)}$ with $1 \le j \le N_0-3$. And we take $\hat{l}_{18} = \max\limits_{1\le j\le N_0-3}\left\{\hat{l}_{18j}\right\}$.

Applying the similar trick as $I_{13}$, we have

$$I_{15,4}+I_{15,5}\lesssim \mathcal{E}_{N_0+n}(t)\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0}f\right\|_\nu^2. \tag{3.38}$$

For $I_{15,6}$, by employing similar argument, we can get that

$$\begin{aligned}
I_{15,6}&\lesssim \left\|\nabla^{N_0+n}B\right\|^{\frac{1}{1+n}}\left\|\nabla^{N_0-1}B\right\|^{\frac{n}{1+n}}\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|_{L^\infty_x L^2_v}\left\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|\\
&\lesssim \|\nabla^{N_0+n}B\|^{\frac{1}{1+n}}\left\|\nabla^{N_0-1}B\right\|^{\frac{n}{1+n}}\|\nabla_x\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|_{H^1_xL^2_v}\left\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|\\
&\lesssim \left\|\nabla^{N_0+n}B\right\|^{\frac{1}{1+n}}\left\|\nabla^{N_0-1}B\right\|^{\frac{n}{1+n}}\left\|\nabla_x\nabla_v^{m_9+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{1}{1+m_9}}\\
&\quad\times\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{H^1_xL^2_v}^{\frac{m_9}{1+m_9}}\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\|\\
&\lesssim \left\|\nabla^{N_0+n}B\right\|^{\frac{1}{1+n}}\left\|\nabla^{N_0-1}B\right\|^{\frac{n}{1+n}}\left\|\nabla_x\nabla_v^{m_9+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{1}{1+m_9}}\\
&\quad\times\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{m_9(N_0-4)}{(1+m_9)(N_0-3+s)}}\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{m_9(1+s)}{(1+m_9)(N_0-3+s)}}\|\langle v\rangle\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\|\\
&\lesssim \left\|\nabla^{N_0+n}B\right\|^{\frac{1}{1+n}}\left\|\nabla^{N_0-1}B\right\|^{\frac{n}{1+n}}\left\|\nabla_x\nabla_v^{m_9+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{1}{1+m_9}}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{m_9(N_0-4)}{(1+m_9)(N_0-3+s)}}\\
&\quad\times\left\|\langle v\rangle^{\hat{l}_{19}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{m_9\beta(1+s)}{(1+m_9)(N_0-3+s)}}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_v}^{\frac{m_9(1-\beta)(1+s)}{(1+m_9)(N_0-3+s)}}\\
&\quad\times\left\|\langle v\rangle^{\hat{l}_{20}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^{1-\beta}\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\beta}\\
&\lesssim \max\left\{\mathcal{E}_{N_0+n}(t),\mathcal{E}_{3+m_9,1+m_9}(t),\bar{\mathcal{E}}_{N_0,-\frac{\hat{l}_{20}}{\gamma}}(t)\right\}\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right)\\
&\quad+\varepsilon\left(\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right).
\end{aligned}\tag{3.39}$$

Here we have used the fact that there exist positive constants $m_9$ and $\beta\in(0,1)$ such that $\frac{n}{1+n}+\frac{m_9(1+s)\beta}{(1+m_9)(N_0-3+s)}+\beta=2$ holds for $1\le j\le N_0-3$. Similar to the discussions above, it is suffice to take $m_9=\frac{N_0-2+s}{sn+n+4-N_0}$. Then, if we choose $n$ be the smallest value which is strictly greater than $\frac{2}{3}N_0-\frac{5}{3}$ and use the facts that $N_0\ge 6$, $s\in\left[\frac{9}{7},\frac{3}{2}\right)$, we can deduce that $3+m_9\le 3+\frac{N_0-2+s}{(s+1)n+4-N_0}\le 3+\frac{36}{24s-13}\le 6$. Moreover, since $\hat{l}_{19}$ and $\hat{l}_{20}$ satisfy respectively $\frac{\gamma}{2}\beta+\hat{l}_{19}(1-\beta)=0$ and $\frac{\gamma}{2}\beta+\hat{l}_{20}(1-\beta)=1$ with $0<\beta<1$, one can deduce that, $\hat{l}_{19}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta)}$ and $\hat{l}_{20}=\frac{\gamma}{2}-\frac{\gamma-2}{2(1-\beta)}$ from which we can see that $\hat{l}_{20}>\hat{l}_{19}$, where $\beta=\frac{(2+n)(N_0-3+s)(m_9+1)}{(1+n)m_9(N_0-3+s+m_9(N_0+2s-2))}$.

$I_{14}$ can be dominated by using the same trick as $I_{15}$

$$\begin{aligned} I_{14} \lesssim & \max\left\{\mathcal{E}_{N_0+m_8,m_8+1}(t), \mathcal{E}_{4,\frac{3}{2}-\frac{1}{\gamma}}(t), \bar{\mathcal{E}}_{N_0,-\frac{\hat{\imath}_{18}}{\gamma}}(t), \mathcal{E}_{N_0+n}(t), \mathcal{E}_{3+m_9,1+m_9}(t), \bar{\mathcal{E}}_{N_0,-\frac{\hat{\imath}_{20}}{\gamma}}(t)\right\} \\ & \times\left(\left\|\nabla^{N_0-1}E\right\|^2+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0}f\right\|_\nu^2\right) \\ & +\varepsilon\left(\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0}f\right\|_\nu^2\right). \end{aligned} \tag{3.40}$$

For $I_{16}$, we have

$$\begin{aligned} I_{16} \lesssim & \underbrace{\sum_{j\le N_0}\left|\left(\mathbf{\Gamma}(\nabla^j\mathbf{P}f,\nabla^{N_0-j}f),\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{16,1}} \\ & +\underbrace{\sum_{j\le N_0}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0-j}\mathbf{P}f),\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{16,2}} \\ & +\underbrace{\sum_{j\le N_0}\left|\left(\mathbf{\Gamma}(\nabla^j\{\mathbf{I}-\mathbf{P}\}f,\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f),\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{16,3}}, \end{aligned} \tag{3.41}$$

and we only estimate the term $I_{16,3}$ for brevity. To this end, when $j=0$ or $j=N_0$, one has

$$\begin{aligned} I_{16,3} &\lesssim \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu\|\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vL^\infty_x}\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|f\|_{H^2_vL^\infty_x}\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

For the case $j=1$ or $N_0-1$, we have

$$\begin{aligned} I_{16,3} &\lesssim \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^6_xL^2_v}\|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vL^3_x}\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\ &\lesssim \|\nabla\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_vH^1_x}\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\ &\lesssim \mathcal{E}_{4,2}(t)\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\varepsilon\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \end{aligned}$$

For the case $j=2$ or $N_0-2$, we have

$$
\begin{aligned}
I_{16,3} &\lesssim \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|\nabla^2\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x L^2_v} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \mathcal{E}_{4,2}(t) \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^1_x L^2_v} + \varepsilon \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu.
\end{aligned}
$$

For the case $j = 3$ or $N_0 - 3$, we have

$$
\begin{aligned}
I_{16,3} &\lesssim \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \|\nabla^3\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|\nabla^3\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x L^2_v} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \mathcal{E}_{5,2}(t) \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^1_x L^2_v} + \varepsilon \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu.
\end{aligned}
$$

For the case $j = 4$ or $N_0 - 4$, we have

$$
\begin{aligned}
I_{16,3} &\lesssim \left\|\nabla^{N_0-4}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \|\nabla^4\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|\nabla^4\{\mathbf{I}-\mathbf{P}\}f\|_{H^2_v L^2_x} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_x L^2_v} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \mathcal{E}_{6,2}(t) \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{H^1_x L^2_v} + \varepsilon \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu.
\end{aligned}
$$

For the last case $5 \le j \le N_0 - 5$, we have

$$
\begin{aligned}
I_{16,3} &\lesssim \sum_{5\le j\le N_0-5} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x H^2_v} \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^\infty_x L^2_v} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \underbrace{\sum_{5\le j\le N_0-5} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_v} \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{16,3,1}} \\
&\quad + \underbrace{\sum_{5\le j\le N_0-5} \left\|\nabla_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_v} \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{16,3,2}} \\
&\quad + \underbrace{\sum_{5\le j\le N_0-5} \left\|\nabla^2_v\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_x L^2_x} \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_v L^\infty_x} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu}_{I_{16,3,3}}.
\end{aligned}
\tag{3.42}
$$

$I_{16,3,3}$ is the most difficult term among those three terms in the right hand side of the above inequality, which can treated as follows

$$
\begin{aligned}
I_{16,3,3} \lesssim & \sum_{5\le j\le N_0-5} \left\|\nabla_v^{m_{10j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{10j}+2}} \left\|\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}}{m_{10j}+2}} \\
& \times \left\|\nabla^{N_0-j}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^\infty} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_v \\
\lesssim & \sum_{5\le j\le N_0-5} \left\|\nabla_v^{m_{10j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{10j}+2}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}(N_0-3-j)}{(m_{10j}+2)(N_0-3+s)}} \\
& \times \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}(j+s)}{(m_{10j}+2)(N_0-3+s)}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^{\frac{2j-9}{2(N_0-3+s)}} \\
& \times \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^{\frac{2N_0+2s-2j+3}{2(N_0-3+s)}} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_v \\
\lesssim & \sum_{5\le j\le N_0-5} \left\|\nabla_v^{m_{10j}+2}\nabla^j\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{m_{10j}+2}} \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}(N_0-3-j)}{(m_{10j}+2)(N_0-3+s)}} \\
& \times \left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}(j+s)\beta_j}{(m_{10j}+2)(N_0-3+s)}} \left\|\langle v\rangle^{\hat{l}_{21j}}\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{10j}(j+s)(1-\beta_j)}{(m_{10j}+2)(N_0-3+s)}} \\
& \times \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^{\frac{2j-9}{2(N_0-3+s)}} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^{\frac{2N_0+2s-2j+3}{2(N_0-3+s)}} \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_v \\
\lesssim & \max\left\{\mathcal{E}_{N_0-3+m_{10},2+m_{10}}(t), \bar{\mathcal{E}}_{N_0,-\frac{\hat{l}_{21}}{\gamma}}(t)\right\} \left\|\nabla^{N_0-3}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^2 \\
& + \varepsilon\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_v^2. \qquad (3.43)
\end{aligned}
$$

Here we have used the fact that there exist positive constants $m_{10j}$ and $\beta_j \in (0,1)$ such that $\frac{m_{10j}(j+s)\beta_j}{(m_{10j}+2)(N_0-3+s)} + \frac{2N_0+2s-2j+3}{2(N_0-3+s)} = 1$ holds for $5 \le j \le N_0-5$. A necessary and sufficient condition to guarantee the existence of such a $\beta$ is

$$
\frac{m_{10j}(j+s)}{(m_{10j}+2)(N_0-3+s)} + \frac{2N_0+2s-2j+3}{2(N_0-3+s)} > 1, \quad 5 \le j \le N_0-5,
$$

from which one can deduce that $m_{10j} > \frac{4j-18}{2s+9}$ holds for $5 \le j \le N_0-5$. It is easy to see that we can take

$$
m_{10} = \max_{5\le j\le N_0-5} m_{10j} = \frac{4}{2s+9}N_0 - \frac{37}{2s+9} > \frac{4N_0-38}{2s+9} = \max_{5\le j\le N_0-5}\frac{4j-18}{2s+9}.
$$

Consequently, $j+2+m_{10j} \le N_0-3+m_{10} \le \frac{2s+13}{2s+9}N_0 - \frac{6s+64}{2s+9}$ with $N_0 \ge 10$. Moreover, since $\hat{l}_{21j}$ satisfies $\frac{\gamma}{2}\beta_j + \hat{l}_{21j}(1-\beta_j) = 0$, one can deduce that $\hat{l}_{21j} = \frac{\gamma}{2} - \frac{\gamma}{2(1-\beta_j)}$, where $\beta_j = \frac{(2j-9)(m_{10j}+2)}{2m_{10j}(j+s)}$ with $5 \le j \le N_0-5$. And we take $\hat{l}_{21} = \max_{5\le j\le N_0-5}\left\{\hat{l}_{21j}\right\}$.

The term $I_{16,3,1}$ and $I_{16,3,2}$ can be dominated by the upper bound of $I_{16,3,3}$.

Thus collecting the above estimates gives (3.4) and the proof of Lemma 3.1 is complete. □

The follow lemma is concerned with the macroscopic dissipation estimates on $f(t,x,v)$.

**Lemma 3.2.** *For the macroscopic dissipation estimates on $f(t,x,v)$, we have the following results:*

*(i) For $k=0,1,2\cdots,N_0-3$, there exists an interactive energy functionals $G_f^k(t)$ satisfying*

$$G_f^k(t)\lesssim \left\|\nabla^k(f,E,B)\right\|^2+\left\|\nabla^{k+1}(f,E,B)\right\|^2+\left\|\nabla^{k+2}E\right\|^2$$

*such that*

$$\begin{aligned}&\frac{d}{dt}G_f^k(t)+\left\|\nabla^k a\right\|^2+\left\|\nabla^{k+1}(\mathbf{P}f,E)\right\|^2+\left\|\nabla^{k+2}B\right\|^2\\&\lesssim \bar{\mathcal{E}}_{N_0-1,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2+\left\|\nabla^{k+2}B\right\|^2+\left\|\nabla^{k+1}f\right\|_\nu^2\right)+\left\|\nabla^{k+2}(a,c,E)\right\|^2\\&\quad+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{k+2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2;\end{aligned}\tag{3.44}$$

*(ii) For $k=N_0-2,N_0-1$, there exist interactive energy functionals $G_f^{N_0-2}(t)$ and $G_f^{N_0-1}(t)$ satisfying*

$$G_f^{N_0-2}(t)+G_f^{N_0-1}(t)\lesssim \left\|\nabla^{N_0-2}(f,E,B)\right\|^2+\left\|\nabla^{N_0-1}(f,E,B)\right\|^2+\left\|\nabla^{N_0}(f,E)\right\|^2$$

*such that*

$$\begin{aligned}&\frac{d}{dt}\left\{G_f^{N_0-2}(t)+G_f^{N_0-1}(t)\right\}+\left\|\nabla^{N_0-1}\mathbf{P}f\right\|_{H^1}^2+\left\|\nabla^{N_0-2}a\right\|^2+\left\|\nabla^{N_0-1}(E,B)\right\|^2\\&\lesssim \bar{\mathcal{E}}_{N_0,0}(t)\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)+\left\|\nabla^{N_0-2}b\right\|^2\\&\quad+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2;\end{aligned}\tag{3.45}$$

*(iii) As in [10], there exists an interactive energy functional $\mathcal{E}_N^{int}(t)$ and $\mathcal{D}_N^{mac}(t)$ satisfying*

$$\mathcal{E}_N^{int}(t)\lesssim\sum_{|\alpha|\le N}\left\|\partial^\alpha(f,E,B)\right\|^2$$

*and*

$$\mathcal{D}_N^{mac}(t)\sim\|a\|^2+\|\nabla(a,b,c)\|_{H^{N-1}}^2+\|\nabla E\|_{H^{N-2}}^2+\|\nabla^2 B\|_{H^{N-3}}^2,$$

*respectively such that*

$$\frac{d}{dt}\mathcal{E}_N^{int}(t)+\mathcal{D}_{N,mac}(t)\lesssim\sum_{|\alpha|\le N}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\mathcal{E}_N(t)\mathcal{D}_N(t)\tag{3.46}$$

*holds for any $t\in[0,T]$.*

**Proof.** For the case of $k = 0, 1, \cdots, N_0 - 2$, as in [24], by repeating the arguments used in Lemma 3.1, and by using the local conservation laws (2.13) and the macroscopic equations (2.14) which are derived from the so-called macro–micro decomposition (1.11), we can deduce that there exists an interactive functional $G^k_{\bar{f}}(t)$ such that

$$\begin{aligned}
&\frac{d}{dt}G^k_{\bar{f}}(t) + \left\|\nabla^{k+1}\mathbf{P}f\right\|^2 + \left\|\nabla^k a\right\|^2 \\
&\lesssim \underbrace{\left\|\left\langle \nabla^k\left(\mathbf{\Gamma}(f,f) + \frac{1}{2}v\cdot Ef + (E + v\times B)\cdot\nabla_v f\right),\ \mu^\delta\right\rangle\right\|^2}_{I_{21}} \\
&\quad + \varepsilon\|\nabla^{k+1}E\|^2 + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + \left\|\nabla^{k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu. \qquad (3.47)
\end{aligned}$$

Similar to that of Lemma 3.1, $I_{21}$ can be bounded by

$$I_{21} \lesssim \bar{\mathcal{E}}_{N_0-1,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2 + \left\|\nabla^{k+2}B\right\|^2 + \left\|\nabla^{k+1}f\right\|^2_\nu\right) + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu,$$

which yields that

$$\begin{aligned}
&\frac{d}{dt}G^k_{\bar{f}}(t) + \left\|\nabla^{k+1}\mathbf{P}f\right\|^2 + \left\|\nabla^k a\right\|^2 \\
&\lesssim \bar{\mathcal{E}}_{N_0-1,0}(t)\left(\left\|\nabla^{k+1}E\right\|^2 + \left\|\nabla^{k+2}B\right\|^2 + \left\|\nabla^{k+1}f\right\|^2_\nu\right) + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu \\
&\quad + \left\|\nabla^{k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu, \qquad (3.48)
\end{aligned}$$

where

$$G^k_{\bar{f}}(t) \lesssim \left\|\nabla^k f\right\|^2 + \left\|\nabla^{k+1}f\right\|^2.$$

When $k = N_0 - 1$, we have by employing the same argument that

$$\begin{aligned}
&\frac{d}{dt}G^{N_0-1}_{\bar{f}}(t) + \left\|\nabla^{N_0}\mathbf{P}f\right\|^2 + \left\|\nabla^{N_0-1}a\right\|^2 \\
&\lesssim \bar{\mathcal{E}}_{N_0,0}(t)\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2 + \left\|\nabla^{N_0-1}f\right\|^2_\nu\right) \\
&\quad + \varepsilon\|\nabla^{N_0-1}E\|^2 + \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu, \qquad (3.49)
\end{aligned}$$

where

$$G^{N_0-1}_{\bar{f}}(t) \lesssim \left\|\nabla^{N_0-1}f\right\|^2 + \left\|\nabla^{N_0}f\right\|^2.$$

By the similar procedure as in the proof of Lemma 3.1, we have for $k = 0, 1, \cdots, N_0 - 3$ that

$$\begin{aligned}
&\frac{d}{dt}G^k_{E,B}(t) + \left\|\nabla^{k+1}E\right\|^2 + \left\|\nabla^{k+2}B\right\|^2 \\
&\lesssim \left\|\nabla^{k+1}\{Ea + b\times B\}\right\|^2 + \left\|\nabla^{k+1}b\right\|^2 + \left\|\nabla^{k+2}(a,c,E)\right\|^2 + \left\|\nabla^{k+2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu \\
&\lesssim \bar{\mathcal{E}}_{N_0,0}(t)\left(\left\|\nabla^{k+1}(E,a,b)\right\|^2 + \left\|\nabla^{k+2}B\right\|^2\right) + \left\|\nabla^{k+1}b\right\|^2 + \left\|\nabla^{k+2}(a,c,E)\right\|^2 \\
&\quad + \left\|\nabla^{k+2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu
\end{aligned} \tag{3.50}$$

where

$$G^k_{E,B}(t) \lesssim \left\|\nabla^{k+1}(E,b,B)\right\|^2 + \left\|\nabla^{k+2}B\right\|^2.$$

While for the case of $k = N_0 - 2$,

$$\begin{aligned}
&\frac{d}{dt}G^{N_0-2}_{E,B}(t) + \left\|\nabla^{N_0-1}(E,B)\right\|^2 \\
&\lesssim \left\|\nabla^{N_0-1}\{Ea + b\times B\}\right\|^2 + \|\nabla^{N_0-2}b\|^2 + \left\|\nabla^{N_0}(a,b,c)\right\|^2 + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu \\
&\lesssim \bar{\mathcal{E}}_{N_0,0}(t)\left\|\nabla^{N_0-1}(E,B,a,b)\right\|^2 + \|\nabla^{N_0-2}b\|^2 + \left\|\nabla^{N_0}(a,b,c)\right\|^2 \\
&\quad + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu,
\end{aligned} \tag{3.51}$$

where

$$G^{N_0-2}_{E,B}(t) \lesssim \left\|\nabla^{N_0-1}(E,b,B)\right\|^2 + \left\|\nabla^{N_0-2}B\right\|^2.$$

With the above estimates in hand, we now turn to deduce (3.44), (3.45), and (3.46). In fact for the case of $k = 0, 1, \cdots, N_0 - 3$, a proper linear combination of (3.48) and (3.50) gives (3.44) with $G^k_f(t) \sim G^k_{\bar{f}}(t) + \kappa_1 G^k_{E,B}(t)$, $0 < \kappa_1 \ll 1$, while (3.45) follows by a suitable linear combination of (3.49), (3.51) and (3.48) with $k = N_0 - 2$, where $G^{N_0-2}_f(t) + G^{N_0-1}_f(t) \sim G^{N_0-2}_{\bar{f}}(t) + G^{N_0-1}_{\bar{f}}(t) + \kappa_2 G^{N_0-2}_{E,B}(t)$, $0 < \kappa_2 \ll 1$.

Finally, (3.46) follows by the same trick as in [12], we omit the details for brevity. Thus we have completed the proof of Lemma 3.2. □

## 4. The proof of our main result

This section is devoted to proving our main result based on the continuation argument. For this purpose, suppose that the Cauchy problem (1.6) and (1.7) admits a unique local solution $f(t,x,v)$ defined on the time interval $0 \le t \le T$ for some $0 < T < \infty$ and the solution $f(t,x,v)$ satisfies the a priori assumption

$$X(t)=\sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,l_0+l^*}(\tau)+\mathcal{E}_N(\tau)+\mathcal{E}_{N-1,l}(\tau)+(1+\tau)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N,l}(\tau)\right\}\le M, \tag{4.1}$$

where the parameters $N_0, N, l, l_0$, and $l^*$ are given in Theorem 1.1 and $M$ is a sufficiently small positive constant. Then to use the continuation argument to extend such a solution step by step to a global one, one only need to deduce certain uniform-in-time energy type estimates on $f(t,x,v)$ such that the a priori assumption (4.1) can be closed.

For this purpose, we first deduce the temporal decay of the energy functional $\mathcal{E}^k_{N_0}(t)$ in the following lemma.

**Lemma 4.1.** *Under the a priori assumption (4.1), then we have for $k=0,1,2,\cdots,N_0-3$ that*

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t)+\mathcal{D}^k_{N_0}(t)\le 0. \tag{4.2}$$

*Furthermore, we can get that*

$$\mathcal{E}^k_{N_0}(t)\lesssim\sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,\frac{k+s}{2}}(\tau),\mathcal{E}_{N_0+\frac{k+s}{2}}(\tau)\right\}(1+t)^{-\frac{k+s}{2}}. \tag{4.3}$$

**Proof.** From Lemma 3.1 and Lemma 3.2, we can obtain for $k=0,1,2,\cdots,N_0-3$ that

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t)+\mathcal{D}^k_{N_0}(t)\le 0.$$

To get the time decay, we first deduce from Lemma 2.3 that

$$\left\|\nabla^k(\mathbf{P}f,E,B)\right\|\le\left\|\nabla^{k+2}(\mathbf{P}f,E,B)\right\|^{\frac{k+s}{k+s+2}}\left\|\Lambda^{-s}(\mathbf{P}f,E,B)\right\|^{\frac{2}{k+s+2}}$$

and

$$\left\|\nabla^{k+1}B\right\|\le\left\|\nabla^{k+2}B\right\|^{\frac{k+s}{k+s+2}}\left\|\Lambda^{\frac{k+2-s}{2}}B\right\|^{\frac{2}{k+s+2}}.$$

Secondly, for other terms evolving the microscopic component of $f(t,x,v)$, we have from the Hölder inequality that

$$\sum_{k\le|\alpha|\le N_0}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|\le\sum_{k\le|\alpha|\le N_0}\left\|\langle v\rangle^{\frac{\gamma}{2}}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{k+s}{k+s+2}}\left\|\langle v\rangle^{-\frac{\gamma(k+s)}{2}}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{2}{k+s+2}}.$$

Thirdly, we can get from Lemma 2.3 that

$$\left\|\nabla^{N_0}(E,B)\right\|\lesssim\left\|\nabla^{N_0-1}(E,B)\right\|^{\frac{k+s}{k+s+2}}\left\|\nabla^{N_0+\frac{k+s}{2}}(E,B)\right\|^{\frac{2}{k+s+2}}.$$

Consequently we can deduce that

$$\mathcal{E}^k_{N_0}(t) \le \left(\mathcal{D}^k_{N_0}(t)\right)^{\frac{k+s}{k+s+2}} \left(\sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,\frac{k+s}{2}}(\tau), \mathcal{E}_{N_0+\frac{k+s}{2}}(\tau)\right\}\right)^{\frac{2}{k+s+2}}.$$

Hence, we deduce that

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t) + \left(\sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,\frac{k+s}{2}}(\tau), \mathcal{E}_{N_0+\frac{k+s}{2}}(\tau)\right\}\right)^{-\frac{2}{k+s}} \left(\mathcal{E}^k_{N_0}(t)\right)^{1+\frac{2}{k+s}} \le 0.$$

Solving this inequality directly, we get

$$\mathcal{E}^k_{N_0}(t) \lesssim \sup_{0\le\tau\le t}\left\{\bar{\mathcal{E}}_{N_0,\frac{k+s}{2}}(\tau), \mathcal{E}_{N_0+\frac{k+s}{2}}(\tau)\right\}(1+t)^{-\frac{k+s}{2}}.$$

This completes the proof of Lemma 4.1. □

Bases on the time decay of $\mathcal{E}^k_{N_0}(t)$ obtained in the above lemma, we are ready to deduce the Lyapunov inequality for $\mathcal{E}_N(t)$.

**Lemma 4.2.** *Suppose that $N_0 \ge 5$, $N \le 2N_0$, $\vartheta \le \frac{4s-2s\gamma+5\gamma-2}{8(1-\gamma)}$ and $l_0 \ge \frac{3}{2} - \frac{1}{\gamma}$, then there exist energy functionals $\mathcal{E}_N(t)$, $\mathcal{E}^1_{N_0,l_0}(t)$ and the corresponding energy dissipation rate functionals $\mathcal{D}_N(t)$, $\mathcal{D}_{N,l}(t)$, which satisfy (1.17), (1.22), (1.20), and (1.16) such that*

$$\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t) \lesssim \frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N,l}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t). \tag{4.4}$$

**Proof.** First of all, it is straightforward to establish the energy identities

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le N}\left(\left\|\partial^\alpha f\right\|^2 + \left\|\partial^\alpha(E,B)\right\|^2\right) + \sum_{1\le|\alpha|\le N}\left(\mathbf{L}\partial^\alpha f, \partial^\alpha f\right)\\
&= \underbrace{\sum_{1\le|\alpha|\le N}\left(\partial^\alpha\left(\frac{1}{2}E\cdot v f\right), \partial^\alpha f\right)}_{J_1} - \underbrace{\sum_{1\le|\alpha|\le N}\left(\partial^\alpha\left((E+v\times B)\cdot\nabla_v f\right), \partial^\alpha f\right)}_{J_2}\\
&\quad + \underbrace{\sum_{1\le|\alpha|\le N}\left(\partial^\alpha\mathbf{\Gamma}(f,f), \partial^\alpha f\right)}_{J_3}.
\end{aligned}$$

For the $\partial^\alpha$ derivative term related to $(E,B)$ with $1 \le |\alpha| \le N$, i.e., the estimates on $J_1$ and $J_2$, one has

$$
\begin{aligned}
J_1 \lesssim\ & \|E\|_{L^\infty} \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \\
& + \sum_{\substack{1\le|\alpha_1|\le N_0-1,\\ \alpha_1\neq\alpha}} \left\|\partial^{\alpha_1}E\right\|_{L^6} \left\| \langle v\rangle^{1/2}\partial^{\alpha-\alpha_1} f \right\|_{L^3} \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \\
& + \sum_{\substack{|\alpha_1|=N_0,\\ \alpha_1\neq\alpha}} \left\|\partial^{\alpha_1}E\right\| \left\| \langle v\rangle^{1/2}\partial^{\alpha-\alpha_1} f \right\|_{L^\infty} \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \\
& + \sum_{\substack{|\alpha_1|\ge N_0+1,\\ \alpha_1\neq\alpha}} \left\|\partial^{\alpha_1}E\right\|_{L^6} \left\| \langle v\rangle^{1-\frac{\gamma}{2}}\partial^{\alpha-\alpha_1} f \right\|_{L^3} \left\|\partial^\alpha f\right\|_\nu \\
& + \left\|\partial^\alpha E\right\| \left\| \langle v\rangle^{1-\frac{\gamma}{2}} f \right\|_{L^\infty} \left\|\partial^\alpha f\right\|_\nu \\
\lesssim\ & \left( \|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}} + \left\|\nabla^2 E\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}} \right) \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t) + \varepsilon\mathcal{D}_N(t).
\end{aligned} \tag{4.5}
$$

Here we choose $l_0 \ge \frac{1}{2} - \frac{1}{\gamma}$ and we also need to assume that $N - N_0 - 1 + 1 \le N_0$ and $N_0 - 3 \ge 2$, which yield that $N \le 2N_0$ and $N_0 \ge 5$ respectively. It is worth to pointing out that the most subtle part is the estimation on the first term in the right hand side of the above first inequality which is carried out as follows

$$
\begin{aligned}
& \|E\|_{L^\infty} \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \left\| \langle v\rangle^{1/2}\partial^\alpha f \right\| \\
\lesssim\ & \|E\|_{L^\infty} \left\| \langle v\rangle^{\gamma/2}\partial^\alpha f \right\|^{\frac{2}{2-\gamma}} \left\| \langle v\rangle\partial^\alpha f \right\|^{\frac{2(1-\gamma)}{2-\gamma}} \\
\lesssim\ & \|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}} \|\langle v\rangle\partial^\alpha f\|^2 + \varepsilon\|\partial^\alpha f\|_\nu^2,
\end{aligned} \tag{4.6}
$$

and the other terms are easier in some sense.

For $J_2$, due to $((v\times B)\cdot\partial^\alpha\nabla_v f, \partial^\alpha f) = 0$, we can deduce by employing the same argument to deal with $J_1$ that

$$
J_2 \lesssim \left( \|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}} + \left\|\nabla_x^2(E,B)\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}} \right) \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t) + \varepsilon\mathcal{D}_N(t),
$$

where we choose $l_0 \ge \frac{3}{2} - \frac{1}{\gamma}$.

By applying Lemma 2.3, it is easy to get that

$$
J_3 \lesssim \mathcal{E}^{1/2}_{N_0,l'}(t)\mathcal{D}_N(t) + \mathcal{E}^{1/2}_N(t)(\mathcal{E}^1_{N_0,l_0}(t))^{1/2}\mathcal{D}^{1/2}_N(t).
$$

Collecting the above estimates gives

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le N}\Big(\big\|\partial^\alpha f\big\|^2+\big\|\partial^\alpha(E,B)\big\|^2\Big)+\sum_{|\alpha|\le N}\big\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_\nu^2\\
\lesssim&\left(\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}}+\Big\|\nabla_x^2(E,B)\Big\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}}\right)\mathcal{D}_{N,l}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t)\\
&+\Big(\mathcal{E}^{1/2}_{N_0,l'}(t)+\varepsilon\Big)\mathcal{D}_N(t).
\end{aligned}\tag{4.7}$$

Here by using the temporal decay of $\|E\|_{H^{N_0}}$ and $\|\nabla B\|_{H^{N_0-1}}$ obtained in (4.3), we need to ask that $\frac{3+2s}{8}\cdot\frac{2-\gamma}{1-\gamma}>1$ which yields that $\vartheta\le\frac{4s-2s\gamma+5\gamma-2}{8(1-\gamma)}$.

Combing (3.46) with (4.7) gives (4.11).

For the estimates on the microscopic component $\{\mathbf{I}-\mathbf{P}\}f$, we first apply the micro projection $\{\mathbf{I}-\mathbf{P}\}$ to the first equation of (1.6) to get that

$$\partial_t\{\mathbf{I}-\mathbf{P}\}f+v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+\mathbf{L}f=\{\mathbf{I}-\mathbf{P}\}g+\mathbf{P}v\cdot\nabla_x f-v\cdot\nabla_x\mathbf{P}f.\tag{4.8}$$

After acting $\partial^\alpha_\beta$ with $|\alpha|+|\beta|\le N_0$ and $|\beta|\ge1$ to the above equation and further multiplying it by $\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$, then direct energy method gives the identity

$$\begin{aligned}
&\frac12\frac{d}{dt}\Big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\|^2+\Big(\partial^\alpha_\beta\mathbf{L}f,\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big)\\
=&\Big(\partial^\alpha_\beta\{-v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f\},\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big)\\
&+\Big(\partial^\alpha_\beta\{\{\mathbf{I}-\mathbf{P}\}g+\mathbf{P}v\cdot\nabla_x f-v\cdot\nabla_x\mathbf{P}f\},\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big).
\end{aligned}\tag{4.9}$$

By the similar way as before, we can obtain that

$$\begin{aligned}
&\frac{d}{dt}\sum_{m=1}^{N_0-k}C_m\sum_{|\beta|=m}\Big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\|^2+\sum_{|\beta|=m}\Big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\|_\nu^2\\
\lesssim&\bar{\mathcal{E}}_{N_0,l'}(t)\left(\Big\|\nabla^{\min\{k+1,N_0-1\}}E\Big\|^2+\Big\|\nabla^{\min\{k+2,N_0-1\}}B\Big\|^2+\Big\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\Big\|^2_{L^2_xH^{N_0-k}_\nu}\right)\\
&+\Big\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\Big\|_\nu^2+\Big\|\nabla^{k+1}\mathbf{P}f\Big\|^2.
\end{aligned}\tag{4.10}$$

Thus we have completed the proof of Lemma 4.2. □

Now we turn to the weighted energy estimates on $f(t,x,v)$, i.e. the estimate on $\mathcal{E}_{N,l}(t)$.

**Lemma 4.3.** *Suppose that* $\frac32-\frac{3}{4N_0-10}\le s<\frac32$, $N_0\ge6$, $N\le2N_0$, $\frac{4s+2}{2s-11}<\gamma<0$, $l\ge N$, $l_0\ge l+\frac32-\frac1\gamma$, *and* $0<\vartheta\le\min\left\{\frac{4s-2s\gamma+5\gamma-2}{8(1-\gamma)},\frac{4s-2s\gamma+11\gamma+2}{8(1-2\gamma)}\right\}$, *then there exists an energy functional* $\mathcal{E}_{N,l}(t)$ *satisfying* (1.15) *such that*

$$\frac{d}{dt}\mathcal{E}_{N,l}(t)+\mathcal{D}_{N,l}(t)\lesssim \mathcal{E}_{N,l}(t)\mathcal{E}^1_{N_0,l_0}(t)+\sum_{|\alpha|=N}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|. \tag{4.11}$$

*Here $\mathcal{D}_{N,l}(t)$ and $\mathcal{E}^1_{N_0,l_0}(t)$ are constructed in Lemma 4.2.*

**Proof.** The proof of this lemma is divided into two steps. The first step is to deduce the desired energy type estimates on the derivatives of $f(t,x,v)$ with respect to the $x$-variable. For this purpose, the standard energy estimate on $\partial^\alpha f$ with $|\alpha|=N$ weighted by the time-velocity dependent function $w_l=w_l(t,v)$ gives

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=N}\left\|w_l\partial^\alpha f\right\|^2+\sum_{|\alpha|=N}\left\|w_l\partial^\alpha f\right\|_\nu^2+\frac{\vartheta q}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_l\partial^\alpha f\right\|^2\\
&\lesssim \sum_{|\alpha|=N}\left\|\partial^\alpha f\right\|_\nu^2+\sum_{|\alpha|=N}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|+\underbrace{\sum_{|\alpha|=N}\left|\left(\partial^\alpha(E\cdot vf),w_l^2\partial^\alpha f\right)\right|}_{J_4}\\
&\quad+\underbrace{\sum_{|\alpha|=N}\left|\left(\partial^\alpha[(E+v\times B)\cdot\nabla_v f],w_l^2\partial^\alpha f\right)\right|}_{J_5}+\underbrace{\sum_{|\alpha|=N}\left|\left(\partial^\alpha\mathbf{\Gamma}(f,f),w_l^2\partial^\alpha f\right)\right|}_{J_6}.
\end{aligned}\tag{4.12}$$

As to the estimates on $J_4$, $J_5$, and $J_6$, we can deduce by following exactly the argument used above to control $J_1$, $J_2$, and $J_3$ that

$$\begin{aligned}
J_4&\lesssim \|E\|_{L^\infty}\left\|\langle v\rangle^{1/2}w_l\partial^\alpha f\right\|_{L^2}\left\|\langle v\rangle^{1/2}w_l\partial^\alpha f\right\|\\
&\quad+\sum_{1\le|\alpha_1|\le N_0-1}\left\|\partial^{\alpha_1}E\right\|_{L^6}\left\|\langle v\rangle^{1/2}w_l\partial^{\alpha-\alpha_1}f\right\|_{L^3}\left\|\langle v\rangle^{1/2}w_l\partial^\alpha f\right\|\\
&\quad+\sum_{|\alpha_1|=N_0}\left\|\partial^{\alpha_1}E\right\|\left\|\langle v\rangle^{1/2}w_l\partial^{\alpha-\alpha_1}f\right\|_{L^\infty}\left\|\langle v\rangle^{1/2}w_l\partial^\alpha f\right\|\\
&\quad+\sum_{\substack{|\alpha_1|\ge N_0+1,\\ \alpha_1\ne\alpha}}\left\|\partial^{\alpha_1}E\right\|_{L^6}\left\|\langle v\rangle^{1-\frac{\gamma}{2}}w_l\partial^{\alpha-\alpha_1}f\right\|_{L^3}\left\|w_l\partial^\alpha f\right\|_\nu\\
&\quad+\left\|\partial^\alpha E\right\|\left\|\langle v\rangle^{1-\frac{\gamma}{2}}w_l f\right\|_{L^\infty}\left\|w_l\partial^\alpha f\right\|_\nu\\
&\lesssim(1+t)^{1+\vartheta}\left(\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}}+\left\|\nabla^2E\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}}\right)\mathcal{D}_{N,l}(t)\\
&\quad+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t)+\varepsilon\mathcal{D}_{N,l}(t),
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
J_5&\lesssim(1+t)^{1+\vartheta}\left(\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}}+\left\|\nabla_x^2E\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}}+\|\nabla_xB\|_{L^\infty}^{\frac{2-\gamma}{1-2\gamma}}+\left\|\nabla_x^3B\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-2\gamma}}\right)\mathcal{D}_{N,l}(t)\\
&\quad+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t)+\varepsilon\mathcal{D}_{N,l}(t),
\end{aligned}\tag{4.14}$$

and

$$J_6 \lesssim \mathcal{E}^{1/2}_{N_0,l_0}(t)\mathcal{D}_{N,l}(t) + \mathcal{E}^{1/2}_{N,l}(t)\left(\mathcal{E}^{1}_{N_0,l_0}(t)\right)^{1/2}\mathcal{D}^{1/2}_{N,l}(t). \tag{4.15}$$

Here $l_0 \geq l + \frac{3}{2} - \frac{1}{\gamma}$ and it is worth emphasizing that to deduce the estimate on $J_5$, we need to ask that $N_0 - 3 \geq 3$ which implies $N_0 \geq 6$ and we have also used the following result which used the fact that $-1 < \gamma < 0$

$$\begin{aligned}
&\sum_{|\alpha_1|=1}\Big(\big(v\times\partial^{\alpha_1}B\big)\times\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f, w_l^2\partial^\alpha f\Big)\\
&\lesssim \|\nabla_x B\|_{L^\infty}\|\langle v\rangle^{\frac{1-\gamma}{2}} w_l\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|\|\langle v\rangle^{\frac{1-\gamma}{2}} w_l\partial^\alpha f\|\\
&\lesssim \|\nabla_x B\|_{L^\infty}\|\langle v\rangle w_{l-1}\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|^{\frac{1-2\gamma}{2-\gamma}}\|\langle v\rangle^{\frac{\gamma}{2}} w_{l-1}\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|^{\frac{1+\gamma}{2-\gamma}}\\
&\quad\times\|\langle v\rangle w_l\partial^\alpha f\|^{\frac{1-2\gamma}{2-\gamma}}\|\langle v\rangle^{\frac{\gamma}{2}} w_l\partial^\alpha f\|^{\frac{1+\gamma}{2-\gamma}}\\
&\lesssim \|\nabla_x B\|_{L^\infty}^{\frac{2-\gamma}{1-2\gamma}}\|\langle v\rangle w_{l-1}\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|\|\langle v\rangle w_l\partial^\alpha f\| + \varepsilon\|w_{l-1}\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|_\nu\\
&\quad\times\|w_l\partial^\alpha f\|_\nu\\
&\lesssim (1+t)^{1+\vartheta}\|\nabla_x B\|_{L^\infty}^{\frac{2-\gamma}{1-2\gamma}}\mathcal{D}_{N,l}(t) + \varepsilon\mathcal{D}_{N,l}(t).
\end{aligned}\tag{4.16}$$

Collecting the above estimates gives the following desired weighted energy type estimates on the derivatives of $f(t,x,v)$ with respect to the $x$-variables only

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=N}\big\|w_l\partial^\alpha f\big\|^2 + \sum_{|\alpha|=N}\big\|w_l\partial^\alpha f\big\|_\nu^2 + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\big\|\langle v\rangle w_l\partial^\alpha f\big\|^2\\
&\lesssim \sum_{|\alpha|=N}\big\|\partial^\alpha f\big\|_\nu^2 + \sum_{|\alpha|=N}\big\|\partial^\alpha E\big\|\big\|\mu^\delta\partial^\alpha f\big\| + \Big(\mathcal{E}^{1/2}_{N_0,l_0}(t)+\varepsilon\Big)\mathcal{D}_{N,l}(t)\\
&\quad + (1+t)^{1+\vartheta}\left(\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}} + \big\|\nabla_x^2 E\big\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}} + \|\nabla_x B\|_{L^\infty}^{\frac{2-\gamma}{1-2\gamma}} + \big\|\nabla_x^3 B\big\|_{H^{N_0-3}}^{\frac{2-\gamma}{1-2\gamma}}\right)\mathcal{D}_{N,l}(t)\\
&\quad + \mathcal{E}_{N,l}(t)\mathcal{E}^1_{N_0,l_0}(t).
\end{aligned}\tag{4.17}$$

Now we turn to get the desired energy type estimates on the mixed derivatives with respect to both $x$ and $v$ variables. Due to

$$\Big\|w_{l-|\beta|}\partial^\alpha_\beta\mathbf{P}f\Big\| \lesssim \big\|\partial^\alpha(a,b,c)\big\|$$

and notice that $\|\partial^\alpha(a,b,c)\|$ has been estimated in (4.17), we only need to control $\Big\|w_{l-|\beta|}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\|$ suitably and for this purpose, we need to deduce first the equations which the microscopic components of $f(t,x,v)$ satisfy. In fact, by applying the microscopic projection $\{\mathbf{I}-\mathbf{P}\}$ to the first equation of (1.6), we can get that

$$\partial_t\{\mathbf{I}-\mathbf{P}\}f + v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f + \mathbf{L}f = \{\mathbf{I}-\mathbf{P}\}g + \mathbf{P}(v\cdot\nabla_x f) - v\cdot\nabla_x\mathbf{P}f. \tag{4.18}$$

For the weighted energy estimate on $\{\mathbf{I}-\mathbf{P}\}\partial_\beta^\alpha f$ with $|\alpha|+|\beta|\le N$ and $0\le|\alpha|\le N-1$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{\substack{|\alpha|\le N-1,\\ |\alpha|+|\beta|\le N}} C_{\alpha,\beta}\left\|w_{l-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right)\\
&+\kappa\sum_{\substack{|\alpha|\le N-1,\\ |\alpha|+|\beta|\le N}}\left(\left\|w_{l-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\frac{q}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_{l-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right)\\
\lesssim&\sum_{|\alpha|=N}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\sum_{|\alpha|\le N-1}\left\|\nabla_x\partial^\alpha\mathbf{P}f\right\|^2+\left(\mathcal{E}_{N_0,l_0}^{1/2}(t)+\varepsilon\right)\mathcal{D}_{N,l}(t)\\
&+(1+t)^{1+\vartheta}\left(\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}}+\left\|\nabla_x^2E\right\|_{H^{N_0-2}}^{\frac{2-\gamma}{1-\gamma}}+\|\nabla_xB\|_{L^\infty}^{\frac{2-\gamma}{1-2\gamma}}+\left\|\nabla_x^3B\right\|_{H^{N_0-3}}^{\frac{2-\gamma}{1-2\gamma}}\right)\mathcal{D}_{N,l}(t)\\
&+\mathcal{E}_{N,l}(t)\mathcal{E}_{N_0,l_0}^1(t).
\end{aligned}\tag{4.19}$$

Here we used the fact that $\left((v\times B)\cdot\partial_\beta^\alpha\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\ w_{l-|\beta|}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)=0$ and the similar trick as (4.16).

Therefore, a proper linear combination of (4.4), (4.17) and (4.19) implies

$$\frac{d}{dt}\mathcal{E}_{N,l}(t)+\mathcal{D}_{N,l}(t)\lesssim\mathcal{E}_{N,l}(t)\mathcal{E}_{N_0,l_0}^1(t)+\sum_{|\alpha|=N}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|.$$

Here by using temporal decay of $\|E\|_{H^{N_0}}$ and $\|\nabla B\|_{H^{N_0-1}}$ obtained in (4.3), we need to ask that $\frac{3+2s}{8}\cdot\frac{2-\gamma}{1-\gamma}>1$ and $\frac{5+2s}{8}\cdot\frac{2-\gamma}{1-2\gamma}>1$, which imply that $\vartheta\le\min\left\{\frac{4s-2s\gamma+5\gamma-2}{8(1-\gamma)},\frac{4s-2s\gamma+11\gamma+2}{8(1-2\gamma)}\right\}$. Meanwhile, we also need to assume that $\vartheta>0$ and from which we deduce that $\gamma>\frac{4s-2}{2s-11}$ by using the fact that $\frac{3}{2}-\frac{3}{4N_0-10}\le s<\frac{3}{2}$ with $N_0\ge6$. Thus we have completed the proof of Lemma 4.3. □

Applying the similar trick as Lemma 4.3, we have the following lemma:

**Lemma 4.4.** *Under the assumptions of Lemma 4.3, then*

$$\frac{d}{dt}\mathcal{E}_{N-1,l}(t)+\mathcal{D}_{N-1,l}(t)\lesssim\mathcal{E}_{N-1,l}(t)\mathcal{E}_{N_0,l_0}^1(t)+\sum_{|\alpha|=N-1}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|\tag{4.20}$$

*holds for all* $0\le t\le T$.

To close the first part of the a priori assumption (4.1), we need to deduce the desired estimates in certain negative Sobolev spaces on $[f(t,x,v),E(t,x),B(t,x)]$ and the first one is on the estimate of $\|(f,E,B)(t)\|_{\dot H^{-s}}$.

**Lemma 4.5.** *Under the assumptions stated above, if we assume further that $N_0 \geq 5$, we have for $s \in \left[\frac{3}{2} - \frac{3}{4N_0-10}, \frac{3}{2}\right)$ that*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\Lambda^{-s} f\right\|^2 + \left\|\Lambda^{-s}(E,B)\right\|^2\right) + \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \left\{\bar{\mathcal{E}}_{0,0}(t)\right\}^{1/2}\left(\left\|\mu^\delta f\right\|^2 + \left\|\mu^\delta \Lambda^{1-s} f\right\|^2\right) \\
&\quad + \bar{\mathcal{E}}_{2,2}(t)\left(\left\|\Lambda^{\frac{3}{2}-s}\mathbf{P}f\right\|^2 + \left\|\langle v\rangle^{1-\frac{\gamma}{2}}\{\mathbf{I}-\mathbf{P}\}f\right\|^2 + \left\|\langle v\rangle^{-\frac{\gamma}{2}}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right. \\
&\quad \left. + \left\|\Lambda^{\frac{3}{2}-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right). \qquad (4.21)
\end{aligned}$$

**Proof.** We have by taking Fourier transform of (1.6) with respect to $x$, multiplying the resulting identity by $|\xi|^{-2s}\bar{\hat{f}}$ with $\bar{\hat{f}}$ being the complex conjugate of $\hat{f}$, and integrating the final result with respect to $\xi$ and $v$ over $\mathbb{R}^3_\xi \times \mathbb{R}^3_v$ that we can get

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\Lambda^{-s} f\right\|^2 + \left\|\Lambda^{-s}(E,B)\right\|^2\right) + \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \underbrace{\left|\left(\mathcal{F}[E\cdot\nabla_v f] \mid |\xi|^{-2s}\hat{f}\right)\right|}_{J_7} + \underbrace{\left|\left(v\cdot\mathcal{F}[Ef] \mid |\xi|^{-2s}\hat{f}\right)\right|}_{J_8} \\
&\quad + \underbrace{\left|\left(\mathcal{F}[v\times B\cdot\nabla_v f] \mid |\xi|^{-2s}\hat{f}\right)\right|}_{J_9} + \underbrace{\left|\left(\mathcal{F}[\Gamma(f,f)] \mid |\xi|^{-2s}\hat{f}\right)\right|}_{J_{10}}. \qquad (4.22)
\end{aligned}$$

For $J_7$, we have from Lemma 2.3, Lemma 2.4, and Lemma 2.5 that

$$\begin{aligned}
J_7 &\lesssim \left|\left(\mathcal{F}[E\cdot\nabla_v f] \mid |\xi|^{-2s}\mathbf{P}\hat{f}\right)\right| + \left|\left(\mathcal{F}[E\cdot\nabla_v\mathbf{P}f] \mid |\xi|^{-2s}\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\right| \\
&\quad + \left|\left(\mathcal{F}[E\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f] \mid |\xi|^{-2s}\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\right| \\
&\lesssim \left\|\Lambda^{-1-s}\left(E\cdot\mu^\delta f\right)\right\|\left\|\mu^\delta\Lambda^{1-s}f\right\| + \left\|\Lambda^{-s}\left(E\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right)\right\|\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|E\cdot\mu^\delta f\right\|_{L_x^{\frac{6}{5+2s}}}\left\|\mu^\delta\Lambda^{1-s}f\right\| + \left\|E\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|_{L_x^{\frac{6}{3+2s}}}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|E\|_{L_x^{\frac{6}{2+2s}}}\left\|\mu^\delta f\right\|_{L_x^2}\left\|\mu^\delta\Lambda^{1-s}f\right\| + \|E\|^2_{L_x^{\frac{3}{s}}}\left\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2 + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \left\|\Lambda^{\frac{1}{2}-s}E\right\|\left\|\mu^\delta f\right\|\left\|\mu^\delta\Lambda^{1-s}f\right\| + \left\|\Lambda^{\frac{3}{2}-s}E\right\|^2\left\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2 + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \left\{\bar{\mathcal{E}}_{0,0}(t)\right\}^{1/2}\left(\left\|\mu^\delta f\right\|^2 + \left\|\mu^\delta\Lambda^{1-s}f\right\|^2\right) + \bar{\mathcal{E}}_{1,1}(t)\left\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2 \\
&\quad + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}$$

For $J_8$ and $J_9$, we have by repeating the argument used in deducing the estimate on $J_6$ that

$$J_8 + J_9 \lesssim \bar{\mathcal{E}}_{1,1}(t)\left(\left\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2 + \left\|\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|^2\right) + \{\bar{\mathcal{E}}_{0,0}(t)\}^{1/2}\left(\left\|\mu^\delta f\right\|^2 + \left\|\mu^\delta\Lambda^{1-s}f\right\|^2\right) + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.$$

As to $J_{10}$, we have from Lemma 2.3, Lemma 2.4, and Lemma 2.5 that

$$\begin{aligned}
J_{10} &\lesssim \left\|\Lambda^{-s}\left(\langle v\rangle^{-\frac{\gamma}{2}}\mathbf{\Gamma}(f,f)\right)\right\|\left\|\Lambda^{-s}\left(\langle v\rangle^{\frac{\gamma}{2}}\{\mathbf{I}-\mathbf{P}\}f\right)\right\| \\
&\lesssim \left\|\Lambda^{-s}\left(\langle v\rangle^{-\frac{\gamma}{2}}\mathbf{\Gamma}(f,f)\right)\right\|\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|\langle v\rangle^{-\frac{\gamma}{2}}\mathbf{\Gamma}(f,f)\right\|_{L^2_vL^p_x}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|\langle v\rangle^{-\frac{\gamma}{2}}\mathbf{\Gamma}(f,f)\right\|_{L^p_xL^2_v}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|\left|\langle v\rangle^{-\frac{\gamma}{2}}f\right|_{L^2_v}\left|\langle v\rangle^{-\frac{\gamma}{2}}f\right|_{H^2_v}\right\|_{L^p_x}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|\langle v\rangle^{-\frac{\gamma}{2}}f\right\|_{L^{\frac{3}{s}}_xL^2_v}\left\|\langle v\rangle^{-\frac{\gamma}{2}}f\right\|_{L^2_xH^2_v}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|f\|_{L^2_vL^{\frac{3}{s}}_x}\|f\|_{L^2_xH^2_v}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \left\|\Lambda^{\frac{3}{2}-s}f\right\|\|f\|_{L^2_xH^2_v}\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|f\|_{L^2_xH^2_v}\left(\left\|\Lambda^{\frac{3}{2}-s}\mathbf{P}f\right\| + \left\|\Lambda^{\frac{3}{2}-s}\{\mathbf{I}-\mathbf{P}\}f\right\|\right)\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu \\
&\lesssim \|f\|_{L^2_xH^2_v}\left(\left\|\Lambda^{\frac{3}{2}-s}\mathbf{P}f\right\|^2 + \left\|\Lambda^{\frac{3}{2}-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right) + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\lesssim \bar{\mathcal{E}}_{2,2}(t)\left(\left\|\Lambda^{\frac{3}{2}-s}\mathbf{P}f\right\|^2 + \left\|\Lambda^{\frac{3}{2}-s}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right) + \varepsilon\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}$$

Collecting the above estimates gives (4.21). Thus we have completed the proof of Lemma 4.5. □

(4.21) tells us that to deduce an estimate on $\|(f,E,B)(t)\|_{\dot{H}^{-s}}$, one has to estimate $\left\|\Lambda^{1-s}f\right\|$, $\left\|\Lambda^{\frac{3}{2}-s}f\right\|$, $\left\|\langle v\rangle^{1-\gamma/2}\{\mathbf{I}-\mathbf{P}\}f\right\|$, and $\left\|\langle v\rangle^{-\gamma/2}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|$ first. Due to the fact that the assumptions $N_0 \geq 5$ and $s \in \left[\frac{3}{2} - \frac{3}{4N_0-10}, \frac{3}{2}\right)$ imply that $1-s<0$, $-s<1-s$, and $\frac{3}{2}-s \geq 0$. These observations together with the fact that the nonlinear energy method developed by [23] and [33] for the Boltzmann type equations can indeed yield a nice weighted estimate on the microscopic component $\{\mathbf{I}-\mathbf{P}\}f$ together with its integer-order derivatives with respect to $x$ and $v$. That is, one can control $\left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|$, $\left\|\Lambda^{\frac{3}{2}-s}\{\mathbf{I}-\mathbf{P}\}f\right\|$, $\left\|\langle v\rangle^{1-\gamma/2}\{\mathbf{I}-\mathbf{P}\}f\right\|$, and $\left\|\langle v\rangle^{-\gamma/2}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|$ suitably. Thus to estimate the terms $\left\|\Lambda^{1-s}f\right\|$,

$\left\|\Lambda^{\frac{3}{2}-s}f\right\|$, $\left\|\langle v\rangle^{1-\gamma/2}\{\mathbf{I}-\mathbf{P}\}f\right\|$, and $\left\|\langle v\rangle^{-\gamma/2}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|$ is then reduced to estimate $\left\|\Lambda^{\frac{3}{2}-s}\mathbf{P}f\right\|$ and $\left\|\Lambda^{1-s}\mathbf{P}f\right\|$ suitably. To this end, since $\frac{3}{2}-s>1-s$, the standard interpolation technique together with the fact that the nonlinear energy method developed by [23] and [33] for the Boltzmann type equations can also yield an estimate on $\nabla_x^k\mathbf{P}f$ for positive integer $k$ imply that we only need to deduce certain estimates on $\|\Lambda^{1-s}\mathbf{P}f\|^2$. This is exactly what we want to do in the following lemma.

**Lemma 4.6.** *Let $s\in\left[\frac{3}{2}-\frac{3}{4N_0-10},\frac{3}{2}\right)$ and $N_0\geq 5$, then there exists an interactive functional $G_f^{-s}(t)$ satisfying*

$$G_f^{-s}(t)\lesssim\left\|\Lambda^{-s}(f,E,B)\right\|^2+\left\|\Lambda^{1-s}(f,E,B)\right\|^2+\left\|\Lambda^{2-s}E\right\|$$

*such that*

$$\begin{aligned}&\frac{d}{dt}G_f^{-s}(t)+\left\|\Lambda^{1-s}(a,b,c)\right\|^2+\left\|\Lambda^{-s}a\right\|^2+\left\|\Lambda^{1-s}E\right\|^2+\left\|\Lambda^{2-s}B\right\|^2\\&\lesssim\left\|\Lambda^{2-s}E\right\|^2+\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{2-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\&\quad+\bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t)\end{aligned}\tag{4.23}$$

*holds for any $0\leq t\leq T$.*

**Proof.** Through Fourier transform, we can obtain that from (2.13), (2.14) that

$$\begin{cases}\partial_t\hat{a}+i\xi\cdot\hat{b}=0,\\ \partial_t\hat{b}+i\xi\left(\hat{a}+2\hat{c}\right)-\hat{E}+i\xi\cdot\Theta\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)=\widehat{Ea}+\widehat{b\times B},\\ \partial_t\hat{c}+\frac{1}{3}i\xi\cdot\hat{b}+\frac{5}{3}i\xi\cdot\Xi\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)=\frac{1}{3}\widehat{E\cdot b},\end{cases}\tag{4.24}$$

and

$$\begin{cases}\partial_t\Theta_{mj}\left(\{\mathbf{I}-\mathbf{P}\}f\right)+i\xi_j\hat{b}_m+i\xi_m\hat{b}_j-\frac{2}{3}\delta_{mj}i\xi\cdot\hat{b}-\frac{10}{3}\delta_{mj}i\xi\cdot\Xi\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\\ \qquad=\Theta_{mj}\left(\hat{r}+\hat{g}\right)-\frac{2}{3}\delta_{mj}\widehat{E\cdot b},\qquad 1\leq m,j\leq 3,\\ \partial_t\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)+i\xi_j\hat{c}=B_j\left(\hat{r}+\hat{g}\right),\end{cases}\tag{4.25}$$

where

$$\hat{r}=-i\xi\cdot v\{\mathbf{I}-\mathbf{P}\}\hat{f}-\mathbf{L}\hat{f},\quad \hat{g}=\frac{1}{2}v\cdot\widehat{Ef}-\mathcal{F}[(E+v\times B)\cdot\nabla_v f]+\mathcal{F}[\mathbf{\Gamma}(f,f)].\tag{4.26}$$

Furthermore, we can deduce from $(1.6)_2$, $(1.6)_3$, and $(1.6)_4$ that

$$\begin{cases}\partial_t\hat{E}-i\xi\times\hat{B}=-\hat{b},\\ \partial_t\hat{B}+i\xi\times\hat{E}=0,\\ i\xi\cdot\hat{E}=\hat{a},\quad i\xi\cdot\hat{B}=0.\end{cases}\tag{4.27}$$

Taking the $L^2$ inner product of $(4.25)_2$ with $i\xi_j|\xi|^{-2s}\bar{\hat{c}}$, we can deduce the bound of $c$ as in the following

$$\begin{aligned}\left\|\Lambda^{1-s}c\right\|^2 &= \left\||\xi|^{1-s}\hat{c}\right\|^2 = \sum_{j=1}^{3}\left(i\xi_j\hat{c}\mid i\xi_j\hat{c}|\xi|^{-2s}\right)\\ &\lesssim -\sum_{j=1}^{3}\left(\partial_t\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j|\xi|^{-2s}\hat{c}\right)+\sum_{j=1}^{3}\left(\Xi_j\left(\hat{r}+\hat{g}\right)\mid i\xi_j\hat{c}|\xi|^{-2s}\right)\\ &= -\frac{d}{dt}\sum_{j=1}^{3}\left(\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j\hat{c}|\xi|^{-2s}\right)+\sum_{j=1}^{3}\left(\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j\partial_t\hat{c}|\xi|^{-2s}\right)\\ &\quad+\sum_{j=1}^{3}\left(\Xi_j\left(\hat{r}+\hat{g}\right)\mid i\xi_j\hat{c}|\xi|^{-2s}\right)\end{aligned}\tag{4.28}$$

For the second term in the right hand of the above inequality, we have by using $(4.24)_3$ that

$$\begin{aligned}&\left(\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j\partial_t\hat{c}|\xi|^{-2s}\right)\\ &=\left(\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j|\xi|^{-2s}\left\{\frac{1}{3}\widehat{E\cdot b}-\frac{1}{3}i\xi\cdot\hat{b}-\frac{5}{3}i\xi\cdot\Xi\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\right\}\right)\\ &\lesssim\varepsilon\left\|\Lambda^{1-s}b\right\|^2+\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t),\end{aligned}$$

while for the last term in the right hand of (4.28), one has

$$\begin{aligned}&\left(\Xi_j\left(\hat{r}+\hat{g}\right)\mid i\xi_j\hat{c}|\xi|^{-2s}\right)\\ &\lesssim\varepsilon\left\|\Lambda^{1-s}c\right\|^2+\left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t).\end{aligned}$$

Thus we can obtain that

$$\begin{aligned}\frac{d}{dt}G_c^{-s}(t)+\left\|\Lambda^{1-s}c\right\|^2\lesssim\varepsilon\left\|\Lambda^{1-s}b\right\|^2+\left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\ +\bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t).\end{aligned}\tag{4.29}$$

Here

$$G_c^{-s}(t)\equiv\sum_{j=1}^{3}\Re\left(\Xi_j\left(\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)\mid i\xi_j\hat{c}|\xi|^{-2s}\right).$$

Noticing the identity

$$\int_{\mathbb{R}^3} \left| |\xi|^{-s} \left( \xi_j \hat{b}_m + \xi_m \hat{b}_j - \frac{2}{3} \delta_{mj} \xi \cdot \hat{b} \right) \right|^2 d\xi = 2 \sum_{mj} \left\| |\xi|^{-s} \xi_m \hat{b}_j \right\|^2 + \frac{2}{3} \left\| |\xi|^{-s} \xi \cdot \hat{b} \right\|^2$$
$$= 2 \left\| \Lambda^{1-s} b \right\|^2 + \frac{2}{3} \left\| \Lambda^{-s} (\nabla_x \cdot b) \right\|^2,$$

and by using $(4.25)_1$ and $(4.24)_2$, we can get by repeating the argument used in deducing (4.29) that

$$\frac{d}{dt} G_b^{-s}(t) + \left\| \Lambda^{1-s} b \right\|^2 \lesssim \varepsilon \left\| \Lambda^{1-s} (a, c, E) \right\|^2 + \left\| \Lambda^{-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2$$
$$+ \left\| \Lambda^{1-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t) \bar{\mathcal{D}}_{2,0}(t). \tag{4.30}$$

Next, we estimate $a$. For this purpose, similar to that of (4.29), we have by using $(4.24)_2$ and $(4.24)_1$ that

$$\frac{d}{dt} G_a^{-s}(t) + \left\| \Lambda^{-s} a \right\|^2 + \left\| \Lambda^{1-s} a \right\|^2$$
$$\lesssim \left\| \Lambda^{1-s} (b, c) \right\|^2 + \left\| \Lambda^{1-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t) \bar{\mathcal{D}}_{2,0}(t). \tag{4.31}$$

Setting

$$G_{\tilde{f}}^{-s}(t) = G_b^{-s}(t) + \kappa_1 G_c^{-s}(t) + \kappa_2 G_a^{-s}(t), \quad 0 < \kappa_2 \ll \kappa_1 \ll 1,$$

we can therefore deduce by combing (4.29) and (4.30) with (4.31) that

$$\frac{d}{dt} G_{\tilde{f}}^{-s}(t) + \left\| \Lambda^{-s} a \right\|^2 + \left\| \Lambda^{1-s} (a, b, c) \right\|^2$$
$$\lesssim \varepsilon \left\| \Lambda^{1-s} E \right\|^2 + \left\| \Lambda^{-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2 + \left\| \Lambda^{1-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t) \bar{\mathcal{D}}_{2,0}(t). \tag{4.32}$$

To deduce an estimate on the electromagnetic field $[E(t,x), B(t,x)]$, we first get from $(4.24)_2$ and $(4.27)_1$ that

$$\left\| \Lambda^{1-s} E \right\|^2 = \left( \hat{E} \mid \hat{E} |\xi|^{2-2s} \right)$$
$$\lesssim \left( \partial_t \hat{b} + i\xi \left( \hat{a} + 2\hat{c} \right) + i\xi \cdot \Theta \left( \{\mathbf{I} - \mathbf{P}\} \hat{f} \right) - \widehat{Ea} + \widehat{b \times B} \mid \hat{E} |\xi|^{2-2s} \right)$$
$$\lesssim \frac{d}{dt} \left( \hat{b} \mid \hat{E} |\xi|^{2-2s} \right) - \left( \hat{b} \mid \partial_t \hat{E} |\xi|^{2-2s} \right) + \varepsilon \left\| \Lambda^{1-s} E \right\|^2 + \left\| \Lambda^{2-s} (a, c) \right\|^2$$
$$+ \left\| \Lambda^{2-s} \{\mathbf{I} - \mathbf{P}\} f \right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t) \bar{\mathcal{D}}_{2,0}(t)$$

$$\begin{aligned}
&\lesssim \frac{d}{dt}\left(\hat{b} \mid \hat{E}|\xi|^{2-2s}\right) - \left(\hat{b} \mid \left(i\xi \times \hat{B} - \hat{b}\right)|\xi|^{2-2s}\right) + \varepsilon \left\|\Lambda^{1-s} E\right\|^2 + \left\|\Lambda^{2-s}(a,c)\right\|^2 \\
&\quad + \left\|\Lambda^{2-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t) \\
&\lesssim \frac{d}{dt}\left(\hat{b} \mid \hat{E}|\xi|^{2-2s}\right) + \left\|\Lambda^{1-s} b\right\|^2 + \left\|\Lambda^{2-s}(a,c)\right\|^2 + \varepsilon \left\|\Lambda^{1-s} E\right\|^2 + \varepsilon \left\|\Lambda^{2-s} B\right\|^2 \\
&\quad + \left\|\Lambda^{2-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t). \qquad (4.33)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&-\frac{d}{dt}\Re\left(\hat{b} \mid \hat{E}|\xi|^{2-2s}\right) + \left\|\Lambda^{1-s} E\right\|^2 \\
&\lesssim \left\|\Lambda^{1-s} b\right\|^2 + \left\|\Lambda^{2-s}(a,c)\right\|^2 + \varepsilon \left\|\Lambda^{2-s} B\right\|^2 + \left\|\Lambda^{2-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\quad + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t). \qquad (4.34)
\end{aligned}$$

On the other hand, for $B$, it follows that

$$\begin{aligned}
\left\|\Lambda^{2-s} B\right\|^2 &= \left(\hat{B} \mid \hat{B}|\xi|^{4-2s}\right) = \left(i\xi \times \hat{B} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) = \left(\partial_t \hat{E} + \hat{b} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) \\
&= \frac{d}{dt}\left(\hat{E} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) - \left(\hat{E} \mid i\xi \times \partial_t \hat{B}|\xi|^{2-2s}\right) + \left(\hat{b} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) \\
&= \frac{d}{dt}\left(\hat{E} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) + \left(\hat{E} \mid \xi \times \left(\xi \times \hat{E}\right)|\xi|^{2-2s}\right) + \left(\hat{b} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) \\
&\lesssim \frac{d}{dt}\left(\hat{E} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) + \left\|\Lambda^{2-s} E\right\|^2 + \left\|\Lambda^{1-s} b\right\|^2 + \varepsilon \left\|\Lambda^{2-s} B\right\|^2.
\end{aligned}$$

That is

$$\begin{aligned}
&-\frac{d}{dt}\Re\left(\hat{E} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right) + \left\|\Lambda^{2-s} B\right\|^2 \\
&\lesssim \left\|\Lambda^{2-s} E\right\|^2 + \left\|\Lambda^{1-s} b\right\|^2 + \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2. \qquad (4.35)
\end{aligned}$$

Combining (4.32) and (4.34) with (4.35) gives

$$\begin{aligned}
&\frac{d}{dt} G_f^{-s}(t) + \left\|\Lambda^{1-s}(a,b,c)\right\|^2 + \left\|\Lambda^{-s} a\right\|^2 + \left\|\Lambda^{1-s} E\right\|^2 + \left\|\Lambda^{2-s} B\right\|^2 \\
&\lesssim \left\|\Lambda^{2-s}(a,c,E)\right\|^2 + \left\|\Lambda^{-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\Lambda^{1-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\Lambda^{2-s}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\quad + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t). \qquad (4.36)
\end{aligned}$$

Here $G_f^{-s}(t) = G_f^{-s}(t) - \tilde{\kappa}_1 \Re\left(\hat{b} \mid \hat{E}|\xi|^{2-2s}\right) - \tilde{\kappa}_2 \Re\left(\hat{E} \mid i\xi \times \hat{B}|\xi|^{2-2s}\right)$ with $\kappa_i$ $(i = 1, 2)$ being suitably chosen such that $0 < \tilde{\kappa}_2 \ll \tilde{\kappa}_1 \ll 1$. This completes the proof of Lemma 4.6. □

**Lemma 4.7.** *For any $\ell$ with $\ell = l_0 + l^*$ with $l^* = l' + (N_0 - 5)\left(2 - \frac{1}{\gamma}\right) + \frac{1}{2}$ with $l'$ being given in Lemma 3.1, there is an energy functional $\bar{\mathcal{E}}_{N_0,\ell}(t)$ such that*

$$\frac{d}{dt}\bar{\mathcal{E}}_{N_0,\ell}(t) + \bar{\mathcal{D}}_{N_0,\ell}(t) \lesssim \sum_{|\alpha|=N_0} \varepsilon \|\partial^\alpha E\|^2. \tag{4.37}$$

*Furthermore, we have*

$$\sup_{0\le\tau\le t} \bar{\mathcal{E}}_{N_0,l_0+l^*}(\tau) \lesssim Y_0^2 + \varepsilon \int_0^t \|\nabla^{N_0} E\|^2 ds. \tag{4.38}$$

**Proof.** Noticing that we have obtained the temporal decay estimates on $\mathcal{E}_{N_0}(t)$ and due to

$$\sum_{|\alpha|=N_0} \left(\partial^\alpha E \cdot v\mu^{1/2}, w_\ell \partial^\alpha f\right) \lesssim \sum_{|\alpha|=N_0} \left(\varepsilon \left\|\partial^\alpha E\right\|^2 + C_\varepsilon \left\|\mu^\delta \partial^\alpha f\right\|^2\right),$$

we can deduce by employing the argument used to yield the energy inequality (4.11) for $\mathcal{E}_{N,l}(t)$ that

$$\frac{d}{dt}\mathcal{E}_{N_0,\ell}(t) + \mathcal{D}_{N_0,\ell}(t) \lesssim \varepsilon \sum_{|\alpha|=N_0} \left\|\partial^\alpha E\right\|^2. \tag{4.39}$$

A proper linear combination of (4.21), (4.23), and (4.39) gives (4.37), therefore (4.38) follows by the time integration of (4.37). This completes the proof of Lemma 4.7. □

To close $\mathcal{E}_N(t)$ and $\mathcal{E}_{N,l}(t)$ in the a priori assumption (4.1), we need the time decay of $\mathcal{E}^1_{N_0,l_0}(t)$.

**Lemma 4.8.** *Let $l_0 \le \ell \le l_0 + (N_0 - 5)\left(2 - \frac{1}{\gamma}\right) + \frac{1}{2}$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}^k_{N_0,\ell}(t) + \mathcal{D}^k_{N_0,\ell}(t) \\
&\lesssim \sum_{|\alpha|=N_0} \left\|\partial^\alpha E\right\|^2 + \chi_{k\ge 1} \sum_{1\le j\le k} \left\|\nabla^j (E,B)\right\|^2_{L^\infty} \left\|\langle v\rangle^{1-\frac{3\gamma}{2}} w_{\ell-N_0+k-1} \nabla_x^{k-j} \nabla_v^{N_0-k+1}\{\mathbf{I}-\mathbf{P}\} f\right\|^2
\end{aligned} \tag{4.40}$$

*for any $0 \le t \le T$, where $k = 0, 1, \cdots, N_0 - 5$.*

**Proof.** Let $l'$ be given in Lemma 3.1, we first deduce certain weighted energy type estimates on the derivatives of $f(t,x,v)$ with respect to the $x$ variable only. For this purpose, take $k+1 \le$

$|\alpha| \le N_0 - 1$, by applying $\partial^\alpha$ to (1.6) and taking inner product of the resulting identity with $w_\ell^2 \partial^\alpha f$ over $\mathbb{R}^3_x \times \mathbb{R}^3_v$, we have by using the argument used in deducing Lemma 3.1 that

$$\begin{aligned}
&\frac{d}{dt}\left\|w_\ell \partial^\alpha f\right\|^2 + \left\|w_\ell \partial^\alpha f\right\|_\nu^2 + \frac{1}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_\ell \partial^\alpha f\right\|^2 \\
&\lesssim \left\|\nabla^{|\alpha|} E\right\|^2 + \bar{\mathcal{E}}_{N_0,\ell+l'}(t)\left(\left\|w_\ell \nabla^{|\alpha|} f\right\|_\nu^2 + \sum_{\substack{|\alpha'|=|\alpha|-1,\\ |\beta'|\le 2}}\left\|w_{\ell-|\beta'|}\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right. \\
&\quad \left. + \left\|\nabla^{|\alpha|} E\right\|^2 + \left\|\nabla^{|\alpha|+1} B\right\|^2\right),
\end{aligned} \tag{4.41}$$

while for the case of $|\alpha| = N_0$, one has

$$\begin{aligned}
&\frac{d}{dt}\left\|w_\ell \partial^\alpha f\right\|^2 + \left\|w_\ell \partial^\alpha f\right\|_\nu^2 + \frac{1}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_\ell \partial^\alpha f\right\|^2 \\
&\lesssim \left\|\partial^\alpha E\right\|^2 + \max\left\{\mathcal{E}_N(t), \bar{\mathcal{E}}_{N_0,\ell+l'}(t)\right\}\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2\right. \\
&\quad \left.\left\|w_{\ell-|\beta'|}\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-1} f\right\|_\nu^2 + \left\|\nabla^{N_0} f\right\|_\nu^2\right).
\end{aligned} \tag{4.42}$$

As for the case of $k = 1, \cdots, N_0 - 5$, we have

$$\begin{aligned}
&\frac{d}{dt}\left\|w_\ell \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2 + \left\|w_\ell \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \frac{1}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_\ell \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
&\lesssim \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{k+1} f\right\|_\nu^2 + \left\|\nabla^{k+1} E\right\|^2 + \left\|\nabla^{k+2} B\right\|^2 \\
&\quad + \bar{\mathcal{E}}_{N_0-1,\ell+l'}(t)\left(\left\|w_\ell \nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \sum_{\substack{|\alpha'|=k,\\ |\beta'|\le 2}}\left\|w_{\ell-|\beta'|}\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right. \\
&\quad \left. + \left\|\nabla^{k+1} E\right\|^2 + \left\|\nabla^{k+2} B\right\|^2\right).
\end{aligned} \tag{4.43}$$

As to the weighted energy type estimates on the mixed derivatives of $f(t,x,v)$ with respect to $x$ and $v$ variables, as pointed out in Lemma 4.3, we only need to deduce an estimate on $\left\|w_{l-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|$. To do so, applying $\partial_\beta^\alpha$ with $|\alpha|+|\beta| \le N_0$ and $k \le |\alpha| \le N_0 - 1$ to (4.18) and then taking the $L^2$ inner product of the resulting identity with $w_{\ell-|\beta|}^2 \partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\|w_{\ell-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\left\|w_{\ell-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\frac{1}{(1+t)^{1+\vartheta}}\left\|\langle v\rangle w_{\ell-|\beta|}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\lesssim\mathcal{E}_{N_0-1,\ell+l'}(t)\Bigg(\sum_{|\alpha|+|\beta_1|\le N_0}\left\|w_{\ell-|\beta_1|}\nabla_x^{|\alpha|}\partial_{\beta_1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla_x^{|\alpha|+1}f\right\|_\nu^2\\
&\quad+\sum_{m=\min\{N_0-1,|\alpha|+1\}}\left\|\nabla_x^m E\right\|^2+\sum_{m=\min\{N_0-1,|\alpha|+2\}}\left\|\nabla_x^m B\right\|^2\Bigg)\\
&\quad+\sum_{|\beta'|<|\beta|}\left\|w_{\ell-|\beta|}\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|w_{\ell-|\beta|+1}\partial_{\beta-e_i}^{\alpha+e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\nabla_x^{|\alpha|+1}f\right\|_\nu^2\\
&\quad+\sum_{m=\min\{N_0-1,|\alpha|+1\}}\left\|\nabla_x^m E\right\|^2+\sum_{m=\min\{N_0-1,|\alpha|+2\}}\left\|\nabla_x^m B\right\|^2+\sum_{k\le|\alpha|\le N_0\le N_0}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&\quad+\chi_{k\ge1}\sum_{1\le j\le k}\left\|\nabla^j(E,B)\right\|_{L^\infty}^2\left\|\langle v\rangle^{1-\frac{3\gamma}{2}}w_{\ell-N_0+k-1}\nabla_x^{k-j}\nabla_v^{N_0-k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2. \qquad (4.44)
\end{aligned}
$$

Therefore, a proper linear combination of (4.2) (4.12), (4.43) and (4.44) implies

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{N_0,\ell}^k(t)+\mathcal{D}_{N_0,\ell}^k(t)\\
&\lesssim\sum_{|\alpha|=N_0}\left\|\partial^\alpha E\right\|^2+\chi_{k\ge1}\sum_{1\le j\le k}\left\|\nabla^j(E,B)\right\|_{L^\infty}^2\left\|\langle v\rangle^{1-\frac{3\gamma}{2}}w_{\ell-N_0+k-1}\nabla_x^{k-j}\nabla_v^{N_0-k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2. \qquad (4.45)
\end{aligned}
$$

This completes the proof of Lemma 4.8. □

Based on the above lemma, we can obtain the time decay of $\mathcal{E}_{N_0,\ell}^k(t)$.

**Lemma 4.9.** *Suppose $l_{0,N_0-5}=l_0$ and $l_{0,k}+2-\frac{1}{\gamma}=l_{0,k-1}$ with $0\le k\le N_0-5$, it holds that*

$$
\mathcal{E}_{N_0,l_{0,k}+\frac{i}{2}}^k(t)\lesssim X(t)(1+t)^{-\frac{k+s-i}{2}},\qquad i=0,1,\cdots,k. \qquad (4.46)
$$

**Proof.** Firstly, from Lemma 4.8, multiplying (4.40) with $\ell=l_{0,0}$ by $(1+t)^{\frac{s}{2}+\epsilon}$ gives

$$
\begin{aligned}
&\frac{d}{dt}\left[(1+t)^{\frac{s}{2}+\epsilon}\mathcal{E}_{N_0,l_{0,0}}^0(t)\right]+(1+t)^{\frac{s}{2}+\epsilon}\mathcal{D}_{N_0,l_{0,0}}^0(t)\\
&\lesssim\sum_{|\alpha|=N_0}(1+t)^{\frac{s}{2}+\epsilon}\|\partial^\alpha E\|^2+(1+t)^{\frac{s}{2}-1+\epsilon}\mathcal{E}_{N_0,l_{0,0}}^k(t). \qquad (4.47)
\end{aligned}
$$

Here $\epsilon$ is taken as a sufficiently small positive constant.

Secondly, if we take $\ell=l_{0,0}+1/2$ in (4.40), it holds that

$$
\frac{d}{dt}\mathcal{E}_{N_0,l_{0,0}+1/2}^0(t)+\mathcal{D}_{N_0,l_{0,0}+1/2}^0(t)\lesssim\sum_{|\alpha|=N_0}\|\partial^\alpha E\|^2. \qquad (4.48)
$$

A proper linear combination of (4.47) and (4.10) yields

$$\begin{aligned}&\frac{d}{dt}\Big[C_0(1+t)^{\frac{s}{2}+\epsilon}\mathcal{E}^0_{N_0,l_{0,0}}(t)+C_1\mathcal{E}^0_{N_0,l_{0,0}+1/2}(t)\Big]\\&\quad+(1+t)^{\frac{s}{2}+\epsilon}\mathcal{D}^0_{N_0,l_{0,0}}(t)+\mathcal{D}^0_{N_0,l_{0,0}+1/2}(t)\\&\lesssim\sum_{|\alpha|=N_0}(1+t)^{\frac{s}{2}+\epsilon}\|\partial^\alpha E\|^2+(1+t)^{\frac{s}{2}-1+\epsilon}\|(\mathbf{P}f,E,B,\nabla B)\|^2.\end{aligned}\tag{4.49}$$

Integrating (4.49) with respect to $t$ over $[0,t]$ yields

$$\begin{aligned}&(1+t)^{\frac{s}{2}+\epsilon}\mathcal{E}^0_{N_0,l_{0,0}}(t)+\mathcal{E}^0_{N_0,l_{0,0}+1/2}(t)\\&\quad+\int_0^t\Big((1+\tau)^{\frac{s}{2}+\epsilon}\mathcal{D}^0_{N_0,l_{0,0}}(\tau)+\mathcal{D}^0_{N_0,l_{0,0}+1/2}(\tau)\Big)d\tau\\&\lesssim X(t)(1+t)^{\epsilon}.\end{aligned}\tag{4.50}$$

Combining (4.49) with (4.50), we can deduce that

$$\mathcal{E}^0_{N_0,l_{0,0}}(t)\lesssim X(t)(1+t)^{-\frac{s}{2}}.$$

Based on the time decay of $\mathcal{E}^0_{N_0,l_{0,0}}(t)$, we can deduce that

$$\mathcal{E}^0_{N_0,l_{0,1}}(t)\lesssim X(t)(1+t)^{-\frac{s+1}{2}}.$$

By using the principle of mathematical induction, we can get that

$$\mathcal{E}^0_{N_0,l_{0,k}}(t)\lesssim X(t)(1+t)^{-\frac{s+k}{2}},$$

where $l_{0,k}+2-\frac{1}{\gamma}=l_{0,k-1}$. Thus we complete the proof of Lemma 4.9. □

Now we are ready to obtain the closed estimate on $\mathcal{E}_N(t)$ and $\mathcal{E}_{N,l}(t)$ of the time-weighted energy norm $X(t)$ in the following:

**Lemma 4.10.** *It holds that*

$$\sup_{0\le s\le t}\Big\{\mathcal{E}_N(s)+(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N,l}(t)\Big\}+\int_0^t\mathcal{D}_N(s)ds\lesssim Y_0^2.\tag{4.51}$$

**Proof.** Firstly, by multiplying (4.4) by $(1+t)^{-\epsilon_0}$, it holds that

$$\begin{aligned}&\frac{d}{dt}\big[(1+t)^{-\epsilon_0}\mathcal{E}_N(t)\big]+\epsilon_0(1+t)^{-1-\epsilon_0}\mathcal{E}_N(t)+(1+t)^{-\epsilon_0}\mathcal{D}_N(t)\\&\lesssim\frac{\delta}{(1+t)^{1+\epsilon_0+\vartheta}}\mathcal{D}_{N,l}(t)+(1+t)^{-\epsilon_0}\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0}(t).\end{aligned}\tag{4.52}$$

Secondly, multiplying (4.11) by $(1+t)^{-(1+\epsilon_0)/2}$, one has

$$\begin{aligned}
&\frac{d}{dt}\left[(1+t)^{-\frac{\epsilon_0+1}{2}}\mathcal{E}_{N,l}(t)\right]+\frac{\epsilon_0+1}{2}(1+t)^{-\frac{\epsilon_0+3}{2}}\mathcal{E}_{N,l}(t)+(1+t)^{-\frac{\epsilon_0+1}{2}}\mathcal{D}_{N,l}(t)\\
&\lesssim (1+t)^{-\frac{\epsilon_0+1}{2}}\mathcal{E}_{N,l}(t)\mathcal{E}^1_{N_0,l_0}(t)+(1+t)^{-\frac{\epsilon_0+1}{2}}\left\|\nabla^N E\right\|\left\|\mu^\delta\nabla^N f\right\|\\
&\lesssim (1+t)^{-\frac{\epsilon_0+1}{2}}\mathcal{E}_{N,l}(t)\mathcal{E}^1_{N_0,l_0}(t)+(1+t)^{-\epsilon_0-1}\left\|\nabla^N E\right\|^2+\varepsilon\left\|\mu^\delta\nabla^N f\right\|^2. \qquad (4.53)
\end{aligned}$$

A proper linear combination of (4.4), (4.52) and (4.53) and taking the time integration yield

$$\begin{aligned}
&\mathcal{E}_N(t)+(1+t)^{-\frac{\epsilon_0+1}{2}}\mathcal{E}_{N,l}(t)+\int_0^t\left[(1+s)^{-1-\epsilon_0}\mathcal{E}_N(s)+\mathcal{D}_N(s)\right]ds\\
&+\int_0^t\left[(1+s)^{-\frac{\epsilon_0+3}{2}}\mathcal{E}_{N,l}(s)+(1+s)^{-\frac{\epsilon_0+1}{2}}\mathcal{D}_{N,l}(s)\right]ds\lesssim Y_0^2.
\end{aligned}$$

Here we have taken $l_0=l_{0,N_0-5}\le l_{0,1}$ and have used the fact that

$$\mathcal{E}^1_{N_0,l_0}(t)\le\mathcal{E}^1_{N_0,l_{0,1}}(t)\le(1+t)^{-\frac{1+s}{2}}X(t)$$

obtained in Lemma 4.9. This completes the proof of Lemma 4.10. □

With the above preparations in hand, we now turn to prove Theorem 1.1. To this end, recalling the definition of the $X(t)$-norm and by putting the estimates (4.38), (4.20), and (4.51) together, we can finally arrive at

$$X(t)\lesssim Y_0^2, \qquad (4.54)$$

thus the global solvability result follows immediately from the well-established local existence result for the Cauchy problem of the one-species VMB system (1.6), (1.7) and by employing the continuation argument in the usual way.

Based on (4.54), for $k=0,1,2,\cdots,N_0-3$, we have from Lemma 4.1 that

$$\mathcal{E}^k_{N_0}(t)\lesssim Y_0^2(1+t)^{-\frac{k+s}{2}},$$

this gives (1.24).

On the other hand, let $0\le i\le k\le N_0-5$ be an integer, we have from Lemma 4.9 that

$$\mathcal{E}^k_{N_0,l_{0,k}+\frac{i}{2}}(t)\lesssim Y_0^2(1+t)^{-\frac{k+s-i}{2}},\qquad i=0,1,\cdots,k.$$

Consequently, (1.25) follows from the above inequality. Thus the proof of Theorem 1.1 is complete.

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## References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, 1985.
[2] D. Arsenio, L. Saint-Raymond, Solutions of the Vlasov–Maxwell–Boltzmann system with long-range interactions, C. R. Math. Acad. Sci. Paris 351 (9–10) (2013) 357–360.
[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, The Boltzmann equation without angular cutoff in the whole space: I. Global existence for soft potential, J. Funct. Anal. 263 (3) (2012) 915–1010.
[4] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Appl. Math. Sci., vol. 106, Springer-Verlag, New York, 1994.
[5] C. Cercignani, The Boltzmann Equation and Its Applications, Springer-Verlag, New York, 1988.
[6] S. Chapman, T.G. Cowling, The Mathematical Theory of Nonuniform Gases. An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases, in co-operation with D. Burnett; revised reprint of the third edition; with a foreword by Carlo Cercignani Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990.
[7] L. Desvillettes, C. Villani, On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation, Invent. Math. 159 (2005) 245–316.
[8] R.-J. Duan, Dissipative property of the Vlasov–Maxwell–Boltzmann system with a uniform ionic background, SIAM J. Math. Anal. 43 (6) (2011) 2732–2757.
[9] R.-J. Duan, On the Cauchy problem for the Boltzmann equation in the whole space: global existence and uniform stability in $L^2_\xi(H^N_x)$, J. Differential Equations 244 (12) (2008) 3204–3234.
[10] R.-J. Duan, Global smooth dynamics of a fully ionized plasma with long-range collisions, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 31 (4) (2014) 751–778.
[11] R.-J. Duan, S.-Q. Liu, The Vlasov–Poisson–Boltzmann system without angular cutoff, Comm. Math. Phys. 324 (1) (2013) 1–45.
[12] R.-J. Duan, S.-Q. Liu, T. Yang, H.-J. Zhao, Stability of the nonrelativistic Vlasov–Maxwell–Boltzmann system for angular non-cutoff potentials, Kinet. Relat. Models 6 (1) (2013) 159–204.
[13] R.-J. Duan, R.M. Strain, Optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal. 199 (1) (2011) 291–328.
[14] R.-J. Duan, R.M. Strain, Optimal large-time behavior of the Vlasov–Maxwell–Boltzmann system in the whole space, Comm. Pure Appl. Math. 24 (11) (2011) 1497–1546.
[15] R.-J. Duan, T. Yang, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system in the whole space: the hard potential case, J. Differential Equations 252 (12) (2012) 6356–6386.
[16] R.-J. Duan, T. Yang, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system for soft potentials, Math. Methods Models Appl. Sci. 23 (6) (2013) 979–1028.
[17] R.-J. Duan, T. Yang, H.-J. Zhao, Global solutions to the Vlasov–Poisson–Landau system, see also arXiv:1112.3261, 2011.
[18] Robert T. Glassey, The Cauchy Problem in Kinetic Theory, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
[19] P.T. Gressman, R.M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off, J. Amer. Math. Soc. 24 (3) (2011) 771–847.
[20] Y. Guo, The Landau equation in a periodic box, Comm. Math. Phys. 231 (2002) 391–434.
[21] Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (9) (2002) 1104–1135.
[22] Y. Guo, The Vlasov–Maxwell–Boltzmann system near Maxwellians, Invent. Math. 153 (3) (2003) 593–630.
[23] Y. Guo, The Boltzmann equation in the whole space, Indiana Univ. Math. J. 53 (2004) 1081–1094.
[24] Y. Guo, The Vlasov–Poisson–Landau system in a periodic box, J. Amer. Math. Soc. 25 (2012) 759–812.
[25] Y. Guo, R.M. Strain, Momentum regularity and stability of the relativistic Vlasov–Maxwell–Boltzmann system, Comm. Math. Phys. 310 (2012) 649–673.

[26] Y. Guo, Y.-J. Wang, Decay of dissipative equation and negative Sobolev spaces, Comm. Partial Differential Equations 37 (2012) 2165–2208.
[27] P. Helander, D.J. Sigmar, Collisional Transport in Magnetized Plasmas, Cambridge University Press, 2002.
[28] F.L. Hinton, Collisional transport in plasma, in: M.N. Rosenbluth, R.Z. Sagdeev (Eds.), Handbook of Plasma Physics, vol. I: Basic Plasma Physics, North-Holland Publishing Company, 1983, p. 147.
[29] Juhi Jang, Vlasov–Maxwell–Boltzmann diffusive limit, Arch. Ration. Mech. Anal. 194 (2) (2009) 531–584.
[30] T. Hosono, S. Kawashima, Decay property of regularity-loss type and application to some nonlinear hyperbolic–elliptic system, Math. Models Methods Appl. Sci. 16 (11) (2006) 1839–1859.
[31] N.A. Krall, A.W. Trivelpiece, Principles of Plasma Physics, McGraw–Hill, 1973.
[32] P.-L. Lions, On Boltzmann and Landau equations, Philos. Trans. R. Soc. Lond. A 346 (1994) 191–204.
[33] T.-P. Liu, T. Yang, S.-H. Yu, Energy method for the Boltzmann equation, Phys. D 188 (2004) 178–192.
[34] T.-P. Liu, S.-H. Yu, Boltzmann equation: micro–macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246 (2004) 133–179.
[35] P.A. Markowich, C.A. Ringhofer, C. Schmeiser, Semiconductor Equations, Springer-Verlag, Vienna, 1990, x+248 pp.
[36] V. Sohinger, R.M. Strain, The Boltzmann equation, Besov spaces, and optimal time decay rates in the whole space, Adv. Math. 261 (2014) 274–332.
[37] R.M. Strain, The Vlasov–Maxwell–Boltzmann system in the whole space, Comm. Math. Phys. 268 (2) (2006) 543–567.
[38] R.M. Strain, Y. Guo, Stability of the relativistic Maxwellian in a collisional plasma, Comm. Math. Phys. 251 (2) (2004) 263–320.
[39] R.M. Strain, Y. Guo, Exponential decay for soft potentials near Maxwellian, Arch. Ration. Mech. Anal. 187 (2) (2008) 287–339.
[40] R.M. Strain, Y. Guo, Almost exponential decay near Maxwellian, Comm. Partial Differential Equations 31 (1–3) (2006) 417–429.
[41] R.M. Strain, K.-Y. Zhu, The Vlasov–Poisson–Landau System in $\mathbb{R}^3_x$, Arch. Ration. Mech. Anal. 210 (2) (2013) 615–671.
[42] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, Proc. Japan Acad. 50 (1974) 179–184.
[43] Y.-J. Wang, Global solution and time decay of the Vlasov–Poisson–Landau system in $\mathbb{R}^3_x$, SIAM J. Math. Anal. 44 (5) (2012) 3281–3323.
[44] Q.-H. Xiao, L.-J. Xiong, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system with angular cutoff for soft potentials, J. Differential Equations 255 (6) (2013) 1196–1232.