



Symmetric mountain pass lemma and sublinear elliptic equations

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Abstract

In this paper, we study the p -Laplace elliptic equations under the Dirichlet boundary condition. We give a general and weak sufficient condition for the existence of a sequence of solutions converging to zero. This result is proved by applying the symmetric mountain pass lemma obtained in our earlier paper. For some elliptic equations with parameters, we decide whether the zero solution is an accumulation point or an isolated point in the set of all solutions.

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1. Introduction

We study the isolation and accumulation of the zero solution for the p -Laplace elliptic equation

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

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where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian with $1 < p < \infty$, $f(x, u)$ is a continuous function which is odd with respect to u , i.e., $f(x, -u) = -f(x, u)$, and Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. In this paper, we apply the symmetric mountain pass lemma to the problem above and prove the existence of a sequence of solutions converging to the zero solution if the nonlinear term $f(x, u)$ is sublinear and odd with respect to u . The definition of sublinearity will be given later on. Moreover, we prove that if f is not sublinear, the zero solution is isolated from other solutions. The oddness assumption on $f(x, u)$ is crucial throughout the paper. Indeed, this assumption is necessary for applying the symmetric mountain pass lemma.

We start with the following obvious assertion. Any elliptic equation which has the zero solution is classified into two types below:

- (A) the zero solution is an accumulation point of the set of all solutions,
- (I) the zero solution is an isolated point of the set of all solutions.

In the above statement, we adopt the $C^1(\overline{\Omega})$ -topology. Hereafter $\|\cdot\|_{C^1(\overline{\Omega})}$ denotes the $C^1(\overline{\Omega})$ norm. Then (A) and (I) are rewritten as

- (A) there exists a sequence of nontrivial solutions for (1.1) whose $C^1(\overline{\Omega})$ -norm converges to zero,
- (I) there exists a constant $C > 0$ such that $\|u\|_{C^1(\overline{\Omega})} \geq C > 0$ for all nontrivial solutions u of (1.1).

In contrast to (A), the existence of a sequence of solutions diverging to infinity is studied in many papers. However, we concentrate on (A) and (I). The most typical example of (A) is a sublinear Emden–Fowler equation,

$$-\Delta u = |u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

with $1 < q < 2$. This problem has a sequence of nontrivial solutions whose $C^2(\overline{\Omega})$ -norm converges to zero (see [1, Example 9] or [14]). These solutions change their signs. Indeed, Brezis and Oswald [6] proved the uniqueness of positive solutions for (1.2) with $1 < q < 2$, and hence the negative solution is also unique. Therefore any solution except for the positive and negative solutions changes its sign. Denoting the right hand side of (1.2) by $f(u) := |u|^{q-2}u$, we see that

$$\lim_{u \rightarrow 0} \frac{f(u)}{u} = \infty. \quad (1.3)$$

Thus the nonlinear term $f(u)$ has a growth order less than one in a neighborhood of $u = 0$, which is the so-called sublinear condition. This condition yields a sequence of solutions converging to zero. More general equations were studied in several papers [2, 4, 11, 23]. Ambrosetti, Brezis and Cerami [2] studied (1.1) with $p = 2$ and $f(u) = \lambda|u|^{q-2}u + |u|^{r-2}u$ with $1 < q < 2 < r \leq 2N/(N-2)$ for $N \geq 3$. They obtained the detailed and important results on the structure of positive solutions and proved the existence of infinitely many solutions. Such a nonlinearity, which is called a concave convex nonlinearity, was studied by Bartsch and Willem [4] and by Hirano [11] also. Ambrosetti and Badiale [1, Example 9] investigated the sublinear elliptic equation (1.2) with $1 < q < 2$ and proved the existence of infinitely many solutions. For the sublinear and superlinear elliptic equation, Wu and An [23] obtained infinitely many solutions.

The most typical example of type (I) is (1.2) with a superlinear exponent $2 < q < \infty$. Indeed, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a nontrivial solution of (1.2), then we take the $L^2(\Omega)$ inner product of (1.2) with u to obtain

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |u|^q dx \leq \|u\|_{\infty}^{q-2} \int_{\Omega} u^2 dx \leq C \|u\|_{\infty}^{q-2} \|\nabla u\|_2^2,$$

with some $C > 0$ independent of u because of the Sobolev embedding, where $\|\cdot\|_r$ denotes the $L^r(\Omega)$ norm. Dividing both sides by $\|\nabla u\|_2^2$, we have $1 \leq C \|u\|_{\infty}^{q-2} \leq C \|u\|_{C^1(\overline{\Omega})}^{q-2}$, which is an a priori lower estimate of all nontrivial solutions. Hence the zero solution of (1.2) with a superlinear exponent q is isolated from other solutions. We note that type (I) includes the case where (1.2) has no nontrivial solutions. Indeed, if Ω is star-shaped and $q \geq 2N/(N-2)$ with $N \geq 3$, then it is known that (1.2) has no nontrivial solutions because of the Pohozaev identity [17].

The sublinearity and oddness of the nonlinear term yield a sequence of solutions converging to zero. To explain this assertion, we state our earlier result [13] on the next equation,

$$-\Delta u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.4)$$

Let $B(x, r)$ denote a ball centered at x with radius r .

Theorem 1.1. (See [13, Theorem 2].) Assume that $f(x, u)$ is a Hölder continuous function on $\overline{\Omega} \times [-\varepsilon, \varepsilon]$ with some $\varepsilon > 0$ and it is odd with respect to u . Moreover, assume that there exist $x_0 \in \Omega$ and $\rho_0 > 0$ such that $B(x_0, \rho_0) \subset \Omega$ and

$$\limsup_{u \rightarrow 0} \left(\inf_{x \in B(x_0, \rho_0)} F(x, u) u^{-2} \right) = \infty, \quad (1.5)$$

$$\liminf_{u \rightarrow 0} \left(\inf_{x \in B(x_0, \rho_0)} F(x, u) u^{-2} \right) > -\infty, \quad (1.6)$$

where

$$F(x, u) := \int_0^u f(x, t) dt.$$

Then there exists a sequence of nontrivial solutions for (1.4) whose $C^2(\overline{\Omega})$ -norm converges to zero. Hence (1.4) is of type (A).

Even if $f(x, u)$ is assumed to be merely continuous and not Hölder continuous, Theorem 1.1 remains valid with the $C^2(\overline{\Omega})$ -norm replaced by the $C^1(\overline{\Omega})$ -norm. Theorem 1.1 can be extended to the p -Laplace equation. Indeed, we have the next result, which will be proved as a corollary of a main theorem.

Theorem 1.2. Assume that $f(x, u)$ is a continuous function on $\overline{\Omega} \times [-\varepsilon, \varepsilon]$ with some $\varepsilon > 0$ and it is odd with respect to u . Moreover, assume that there exist $x_0 \in \Omega$ and $\rho_0 > 0$ such that $B(x_0, \rho_0) \subset \Omega$ and

$$\limsup_{u \rightarrow 0} \left(\inf_{x \in B(x_0, \rho_0)} F(x, u) |u|^{-p} \right) = \infty, \quad (1.7)$$

$$\liminf_{u \rightarrow 0} \left(\inf_{x \in B(x_0, \rho_0)} F(x, u) |u|^{-p} \right) > -\infty. \quad (1.8)$$

Then there exists a sequence of nontrivial solutions for (1.1) whose $C^1(\overline{\Omega})$ -norm converges to zero.

Instead of (1.1), we consider a simple equation,

$$-\Delta_p u = \alpha(x) f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.9)$$

As a direct consequence of Theorem 1.2, we have the next result.

Corollary 1.3. Equation (1.9) has a sequence of nontrivial solutions whose $C^1(\overline{\Omega})$ norm converges to zero if $\alpha(x)$ is continuous on $\overline{\Omega}$ and $f(u)$ is an odd continuous function defined near $u = 0$, $\alpha(x_0) > 0$ at some point $x_0 \in \Omega$ and

$$\lim_{u \rightarrow 0} \frac{F(u)}{|u|^p} = \infty, \quad \text{where } F(u) := \int_0^u f(t) dt. \quad (1.10)$$

It is easy to verify that (1.10) with $\alpha(x_0) > 0$ implies (1.7) and (1.8). Hence Corollary 1.3 follows from Theorem 1.2. We call $f(u)$ p -sublinear (or sublinear for short) if it satisfies (1.10). This is weaker than the condition

$$\lim_{u \rightarrow 0} \frac{f(u)}{|u|^{p-2}u} = \infty. \quad (1.11)$$

Indeed, we can show that (1.11) implies (1.10) by using the L'Hospital rule. The oddness assumption on $f(u)$ is used for applying the symmetric mountain pass lemma. The assumption $\alpha(x_0) > 0$ at some x_0 is essential in Corollary 1.3. Indeed, this is a necessary condition for type (A) when (1.11) holds. We shall show this claim. Condition (1.11) implies that $uf(u) > 0$ for $0 < |u| < \delta$ with a small $\delta > 0$. If $\alpha(x) \leq 0$ for all $x \in \Omega$, we multiply (1.9) by u and integrate it over Ω . Then we obtain

$$\int_{\Omega} |\nabla u|^p dx = \int_{\Omega} \alpha(x) u f(u) dx \leq 0, \quad (1.12)$$

provided that $\|u\|_{\infty} < \delta$. Accordingly, any solution with L^{∞} -norm less than δ identically vanishes in Ω , that is, type (I) occurs. Therefore the assumption $\alpha(x_0) > 0$ at some x_0 is necessary for type (A).

The purpose of the present paper is to weaken the assumptions in Theorems 1.1, 1.2, Corollary 1.3 and to find a general sufficient condition on $f(x, u)$ for type (A). Moreover, for some elliptic equations, we shall give a criterion for the classification of (A) and (I). The next equation is the simplest example to which our theorem is applicable:

$$-\Delta u = a|u|^{q-2}u - d(x)|u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.13)$$

where q, r, a are constants satisfying $q, r > 1$, $a > 0$ and $d(x)$ denotes the distance function from x to $\partial\Omega$, that is,

$$d(x) := \inf\{|x - y| : y \in \partial\Omega\}. \quad (1.14)$$

The first term on the right hand side in (1.13) is sublinear when $1 < q < 2$. If $1 < q < r < 2$, the second term $d(x)|u|^{r-2}u$ decays to zero faster than $a|u|^{q-2}u$ as $u \rightarrow 0$. Therefore (1.13) is of type (A). Denote the right hand side of (1.13) by $f(x, u)$. If $1 < r < q < 2$, then

$$\lim_{u \rightarrow 0} \frac{f(x, u)}{u} = -\infty \quad \text{for all } x \in \Omega. \quad (1.15)$$

Therefore the zero solution of (1.13) seems to be isolated. However, it is an accumulation point of other solutions if q is slightly larger than r . Indeed, we have the next result.

Theorem 1.4. *Let $r > 1$, $1 < q < 2$ and $a > 0$. Then the following assertions hold:*

- (i) *if $1 < q < 2(r+1)/3$, then (1.13) is of type (A),*
- (ii) *if $q > 2(r+1)/3$, then it is of type (I),*
- (iii) *if $q = 2(r+1)/3$ and $a > 0$ is small enough, then it is of type (I),*
- (iv) *if $N \geq 2$, $q = 2(r+1)/3$ and $a > 0$ is large enough, then it is of type (A). Here N is the dimension of the domain Ω .*

In the theorem above, the assertions (i) and (iv) will be proved by using the main theorem, which gives a sufficient condition for type (A). Comparing (i) and (ii) or (iii) and (iv), one finds that our sufficient condition is very close to a necessary and sufficient condition. Theorem 1.4 will be proved in Section 3.

Remark 1.5. Verify that $r < 2(r+1)/3 < 2$ when $1 < r < 2$. Fix $r \in (1, 2)$. If $r < q < 2(r+1)/3$, Theorem 1.4 (i) asserts that (1.13) is of type (A). In this case, (1.15) holds, where $f(x, u)$ denotes the right hand side of (1.13). The function $F(x, u)$ defined after (1.6) is computed as

$$F(x, u) = (a/q)|u|^q - r^{-1}d(x)|u|^r.$$

Assumptions (1.5) and (1.6) are not satisfied because

$$\inf_{x \in B(x_0, \rho_0)} F(x, u)u^{-2} = (a/q)|u|^{-(2-q)} - (D/r)|u|^{-(2-r)} \rightarrow -\infty \quad \text{as } u \rightarrow 0,$$

for any ball $B(x_0, \rho_0)$ in Ω , where $D := \max_{|x-x_0| \leq \rho_0} d(x) > 0$. Therefore Theorem 1.1 is not applicable to this case. However, Theorem 1.4 works well to prove type (A). This is one of strong points in this paper. We shall give a more general theorem in Section 2.

Remark 1.6. We consider what [Theorem 1.4](#) means. Let us begin with an easy equation,

$$-\Delta u = |u|^{q-2}u - |u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.16)$$

Fix $r \in (1, 2)$ and vary q increasingly from 1 to 2. If $q < r$, then $|u|^{q-2}u$ is a dominant term as $u \rightarrow 0$. Hence (1.16) is of type (A). If $r < q$, then $-|u|^{r-2}u$ is dominant, and so it is of type (I). Therefore the number $q_* = r$ divides between types (A) and (I).

[Theorem 1.4](#) says that the division number between (A) and (I) for (1.13) is $q_* = 2(r+1)/3$, that is, if $q < q_*$, then (1.13) is of type (A) and if $q > q_*$, then it is of type (I). As stated in [Remark 1.5](#), [Theorem 1.4](#) (i) ensures that if $1 < r < 2$ and q is slightly larger than r , then type (A) occurs. In this case, (1.15) holds. This condition contradicts (1.3). Recall that (1.3) yields a sequence of solutions converging to zero. We shall explain why this phenomenon happens. Let $1 < r < q < 2$ and put $\theta := q - r > 0$, $\alpha(x, u) := a|u|^\theta - d(x)$. Then (1.13) is rewritten as

$$-\Delta u = \alpha(x, u)|u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

As proved in (1.12), if $\alpha(x, u) \leq 0$ for all x and u , then the equation above has no nontrivial solutions. Hence the assumption that $\alpha(x, u) > 0$ at some x and u is necessary for type (A). If $\alpha(x, u) \geq c > 0$ for all u and $x \in B(x_0, r)$ with some fixed x_0 and $r, c > 0$, then (1.5) and (1.6) are fulfilled. By [Theorem 1.1](#), the equation above is of type (A). Let us consider the case where $\alpha(x, u)$ converges to zero or to a minus value as $u \rightarrow 0$. For each u , let $D(u)$ be the set of points x satisfying $\alpha(x, u) > 0$. If $D(u)$ tends to be rapidly smaller as $u \rightarrow 0$ and the value of $\alpha(x, u)$ in $D(u)$ goes fast to zero, then (A) may not occur. We think that the shrinking speed of the volume of $D(u)$ and the decaying speed of $\alpha(x, u)$ decide which type of (A) or (I) occurs. When $\alpha(x, u) = a|u|^\theta - d(x)$, $D(u)$ is a set of points x whose distance from $\partial\Omega$ is less than $a|u|^\theta$. If q is large (so is θ), then the shrinking speed of $D(u)$ and the decaying speed of $\alpha(x, u)$ in $D(u)$ are fast as $u \rightarrow 0$ and hence (I) occurs. However, if q is less than or equal to r or even if it is slightly larger than r , then the shrinking speed and the decaying speed are slow, and so (A) occurs. [Theorem 1.4](#) proves this fact.

This paper is organized into three sections. In [Section 2](#), we state the main result, which gives a weak and general sufficient condition for type (A). We prove this result by using the symmetric mountain pass lemma, which was obtained in our paper [13]. Moreover, we derive some corollaries from the main theorem and prove [Theorem 1.2](#). In [Section 3](#), we prove [Theorem 1.4](#). Furthermore, we confirm that our theorem, a sufficient condition for type (A), is very close to the necessary and sufficient condition. To this end, we introduce some elliptic equations and decide whether the zero solution is an accumulation point or an isolated point.

2. Main result

In this section, we state the main results and prove them. We shall give a weak and general sufficient condition on $f(x, u)$ which yields type (A). Throughout the paper, we suppose the next condition.

- (f0) $f(x, u)$ is a continuous function defined on $\overline{\Omega} \times [-\varepsilon_0, \varepsilon_0]$ with some $\varepsilon_0 > 0$ and it is odd with respect to u .

We define

$$F(x, u) := \int_0^u f(x, t) dt \quad \text{for } |u| \leq \varepsilon_0. \quad (2.1)$$

We denote a ball centered at x with radius ρ by $B(x, \rho)$. For $\rho > 0$, $x \in \Omega$ satisfying $B(x, \rho) \subset \Omega$ and for $u \in (0, \varepsilon_0)$, we define

$$\overline{F}(x, u, \rho) := \inf\{F(y, u)u^{-p}\rho^p : y \in B(x, \rho)\}, \quad (2.2)$$

$$\underline{F}(x, u, \rho) := \inf\{F(y, tu)u^{-p}\rho^p : y \in B(x, \rho), 0 \leq t \leq 1\}. \quad (2.3)$$

Substituting $t = 0$ into $F(y, tu)u^{-p}\rho^p$, we see that $\underline{F}(x, u, \rho) \leq 0$. We introduce the next assumption.

Assumption 2.1. Let $f(x, u)$ satisfy (f0) and let $\varepsilon_0 > 0$ be as in (f0). We assume that there exists a positive integer k_0 satisfying the condition below. For each $k \geq k_0$, there exist $\mu_k \in (0, \varepsilon_0/2)$, $x_{k,i} \in \Omega$ with $1 \leq i \leq 2k$ and $\rho_k > 0$ such that $B(x_{k,i}, \rho_k) \subset \Omega$, $B(x_{k,i}, \rho_k) \cap B(x_{k,j}, \rho_k) = \emptyset$ for $i \neq j$ and

$$2^{-N} \min_{1 \leq i \leq 2k} \overline{F}(x_{k,i}, \mu_k, \rho_k) + (2 - 2^{-N}) \min_{1 \leq i \leq 2k} \underline{F}(x_{k,i}, \mu_k, \rho_k) > 2^{p+1}/p. \quad (2.4)$$

In (2.4), N is the dimension of the domain Ω . To our knowledge, (2.4) is the weakest assumption in all sublinear conditions on $f(x, u)$. This assumption seems to be difficult to understand, but we shall use it to prove Theorem 1.4. We start with the most general assumption (2.4) and establish a main theorem. After that, we strengthen and simplify the assumption and derive some corollaries, which are readable and handy. We state the main result, which is a new theorem even if $p = 2$, the Laplacian case.

Theorem 2.2. Let $p > 1$. Under Assumption 2.1, (1.1) is of type (A), that is, there exists a sequence of nontrivial solutions for (1.1) whose $C^1(\overline{\Omega})$ -norm converges to zero.

Remark 2.3. We emphasize that in Assumption 2.1, the conditions on the nonlinear term $f(x, u)$ are supposed near $u = 0$ only and there are no conditions for large $|u|$. This is essential and important. Indeed, this assumption allows us to study equations having singularity or supercritical terms as $u \rightarrow \pm\infty$. For example, let us consider the equation

$$-\Delta_p u = \frac{|u|^{q-1}u}{|\sin u|} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with $1 < q < p$. The right hand side has singularities at $n\pi$ with $n \in \mathbb{Z} \setminus \{0\}$, but continuous at $u = 0$. Corollary 1.3 asserts that the equation above is of type (A). Next, we consider the equation

$$-\Delta_p u = \alpha(x)f(u) + |u|^{r-2}u \exp(u^2) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.5)$$

where $N \geq 2$, $p < N$, $1 < p < r$, $\alpha(x)$ is continuous on $\overline{\Omega}$ and $f(u)$ is odd and continuous near $u = 0$. Then the second term on the right hand side is supercritical. However, by [Corollary 2.13](#), which will be stated later on, the equation above is of type (A) if $\alpha(x_0) > 0$ at some $x_0 \in \Omega$ and $f(u)$ satisfies (1.10). Therefore it is important that [Theorem 2.2](#) does not require any conditions on $f(x, u)$ for large $|u|$.

To prove [Theorem 2.2](#), we need the symmetric mountain pass lemma, which gives infinitely many critical values of a functional in a Banach space. There are two types of the symmetric mountain pass lemma. One gives a sequence of critical values diverging to infinity. Another provides a sequence of critical values converging to zero. This paper needs the latter lemma. For the former lemma, refer the readers to [\[3\]](#), [\[18, pp. 53–69\]](#) or [\[19, pp. 108–124\]](#). To state the symmetric mountain pass lemma, we need the notion of Krasnoselskii's genus.

Definition 2.4. Let H be a Banach space and A a subset of H . A is said to be *symmetric* if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the *genus* $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$ (see [\[18, p. 45\]](#)). If there does not exist such a k , we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of H such that $0 \notin A$ and $\gamma(A) \geq k$.

Assumption 2.5. Let H be an infinite dimensional Banach space and I be a real valued C^1 functional defined on H , which satisfies (A1) and (A2) below.

- (A1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais–Smale condition (PS).
- (PS) If u_k is a sequence in H such that $I(u_k)$ is bounded and $I'(u_k) \rightarrow 0$ in H^* as $k \rightarrow \infty$, then it has a convergent subsequence. Here H^* is a dual space of H and $I'(u)$ denotes the Fréchet derivative of $I(u)$.
- (A2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

We state the symmetric mountain pass lemma due to Clark [\[8\]](#) (see also [\[3, 18, 19\]](#)).

Lemma 2.6 (*Symmetric mountain pass lemma*). (See [\[8\]](#).) Suppose that [Assumption 2.5](#) holds. Define c_k by

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} I(u). \quad (2.6)$$

Then each c_k is a critical value of $I(u)$, $c_k \leq c_{k+1} < 0$ for $k \in \mathbb{N}$ and c_k converges to zero. Moreover, if $c_k = c_{k+1} = \cdots = c_{k+n} \equiv c$, then $\gamma(K_c) \geq n + 1$. Here K_c is defined by

$$K_c := \{u \in H : I'(u) = 0, I(u) = c\}.$$

For the proof of the lemma above, we refer the readers to [\[3, 8, 18\]](#) or [\[19\]](#). The convergence of c_k to zero is proved in [\[13\]](#). [Lemma 2.6](#) gives us infinitely many critical values. We state another type of the symmetric mountain pass lemma, which was proved in our paper [\[13\]](#).

Lemma 2.7 (Symmetric mountain pass lemma). (See [13].) Under Assumption 2.5, either (i) or (ii) below holds.

- (i) There exists a sequence u_k such that $I'(u_k) = 0$, $I(u_k) < 0$ and u_k converges to zero.
- (ii) There exist two sequences u_k and v_k such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k \rightarrow \infty} I(v_k) = 0$, and v_k converges to a non-zero limit.

In any case (i) or (ii), there exists a sequence u_k of critical points such that $I'(u_k) = 0$, $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.

Remark 2.8. We shall explain the difference between Lemmas 2.6 and 2.7. Lemma 2.6 ensures the existence of a sequence of critical values converging to zero in \mathbb{R} . On the other hand, Lemma 2.7 guarantees the existence of a sequence of critical points converging to zero in H . Since our aim is to obtain a sequence of solutions for (1.1) converging to zero in $C^1(\overline{\Omega})$, Lemma 2.7 is suitable for our purpose. However, there is a question whether a critical point corresponding to c_k defined by (2.6) converges to zero in H . This is true if the equation

$$I(u) = 0 \quad \text{and} \quad I'(u) = 0 \quad (2.7)$$

has a unique solution $u = 0$. Indeed, under this condition, the Palais–Smale condition (PS) ensures that any critical point corresponding to c_k must converge to zero in H . However, if (2.7) has a nontrivial solution, then a critical point corresponding to c_k does not necessarily converge to zero. Indeed, we constructed an example of a functional I and a Hilbert space H in [13] such that I is defined on H and it satisfies Assumption 2.5, but $\|u\| \geq c > 0$ whenever $I(u) = c_k$ and $I'(u) = 0$, where c is independent of k . Therefore no critical point corresponding to c_k converges to zero. In the present paper, we use a Lagrangian functional I defined by

$$I(u) := \int_{\Omega} \left(\frac{1}{p} |\nabla u|^p - F(x, u) \right) dx. \quad (2.8)$$

For this functional, (2.7) is written as

$$\int_{\Omega} \left(\frac{1}{p} |\nabla u|^p - F(x, u) \right) dx = 0,$$

and

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

However, in general, it is unclear whether the two equations above have no solutions except for the trivial solution. Therefore we employ Lemma 2.7 to get a sequence of solutions converging to zero in H .

To prove Theorem 2.2, we introduce a closed symmetric set as below. Let V_k be a set of points (t_1, \dots, t_{2k}) in \mathbb{R}^{2k} such that $|t_i| \leq 1$ for all $1 \leq i \leq 2k$ and there exist at least k elements t_j with $j = r_1, \dots, r_k$ satisfying $|t_j| = 1$ for $j = r_1, \dots, r_k$, that is,

$$V_k := \{(t_1, \dots, t_{2k}) \in \mathbb{R}^{2k} : |t_i| \leq 1 \text{ for all } i, \sharp\{i : |t_i| = 1\} \geq k\}, \quad (2.9)$$

where $\sharp A$ denotes the cardinal number of a set A . Then V_k is a closed symmetric subset of \mathbb{R}^{2k} which does not contain the origin. Moreover, we have the lemma below.

Lemma 2.9. (See [12, Lemma 4.5].) V_k has the genus of $k + 1$, i.e., $\gamma(V_k) = k + 1$.

To prove Theorem 2.2, we shall extend the definition domain of $f(x, u)$ on the whole set $\overline{\Omega} \times \mathbb{R}$. Let $g(u)$ be an even continuous function in \mathbb{R} such that $g(u) = 1$ for $|u| \leq \varepsilon_0/2$ and $g(u) = 0$ for $|u| \geq \varepsilon_0$, where ε_0 is given in condition (f0). Put $h(x, u) := f(x, u)g(u)$. Then $h(x, u)$ is defined on $\overline{\Omega} \times \mathbb{R}$, odd with respect to u and moreover satisfies Assumption 2.1 because $\mu_k < \varepsilon_0/2$ and $h(x, u) = f(x, u)$ for $|u| \leq \varepsilon_0/2$. We consider the equation

$$-\Delta_p u = h(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We shall prove that the problem above has a sequence u_k of solutions converging to zero in $C^1(\overline{\Omega})$ under Assumption 2.1. If this claim would be proved, then $\|u_k\|_\infty < \varepsilon_0/2$ for k large and hence $h(x, u_k) = f(x, u_k)$. Thus u_k becomes a solution of (1.1), which is the desired conclusion. We rewrite $h(x, u)$ as $f(x, u)$ from now on. Then we may assume that $f(x, u)$ and $F(x, u)$ are defined on $\overline{\Omega} \times \mathbb{R}$ and they are bounded, i.e.,

$$\sup_{\overline{\Omega} \times \mathbb{R}} |f(x, u)| < \infty, \quad \sup_{\overline{\Omega} \times \mathbb{R}} |F(x, u)| < \infty.$$

We need the truncation function h even if $f(x, u)$ was defined for all $u \in \mathbb{R}$ from the beginning. If we do not truncate it, then $F(x, u)$ may diverge rapidly to infinity as $u \rightarrow \pm\infty$ and hence the Lagrangian functional I given by (2.8) cannot be well defined (see (2.5)). We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. As stated above, we may assume that $f(x, u)$ and $F(x, u)$ are bounded in $\overline{\Omega} \times \mathbb{R}$. We employ Lemma 2.7 to prove the theorem. Let $H := W_0^{1,p}(\Omega)$ be a usual Sobolev space and I be defined by (2.8). We shall verify that I satisfies Assumption 2.5. It is clear that $I(0) = 0$ and I is an even functional in $W_0^{1,p}(\Omega)$. Since $F(x, u)$ is bounded in $\overline{\Omega} \times \mathbb{R}$, $I(u)$ is bounded from below. Let us verify the Palais–Smale condition (PS). Let u_k be any sequence in $W_0^{1,p}(\Omega)$ such that $I(u_k)$ is bounded and $I'(u_k) \rightarrow 0$ as $k \rightarrow \infty$. Since $F(x, u)$ is bounded on $\overline{\Omega} \times \mathbb{R}$, the $W_0^{1,p}(\Omega)$ -norm of u_k is also bounded. Therefore u_k has a convergent subsequence in the weak topology of $W_0^{1,p}(\Omega)$. Then it converges in the strong topology as well, which can be proved by a standard argument (see [18] or [19]). Accordingly, the Palais–Smale condition is fulfilled.

Let us show that I satisfies condition (A2). Let μ_k , $x_{k,i}$ and ρ_k with $k \geq k_0$ be given in Assumption 2.1. Since $\Gamma_k \subset \Gamma_{k-1}$ by definition, it is enough to construct an $A_k \in \Gamma_k$ for $k \geq k_0$ such that $\sup_{u \in A_k} I(u) < 0$. Fix $k \geq k_0$. Instead of μ_k , $x_{k,i}$ and ρ_k , we write μ , x_i and ρ for simplicity. Using \overline{F} and \underline{F} given by (2.2) and (2.3), respectively, we define

$$\overline{F}_i := \overline{F}(x_i, \mu, \rho), \quad \underline{F}_i := \underline{F}(x_i, \mu, \rho) \quad \text{for } 1 \leq i \leq 2k.$$

It follows from (2.2) and (2.3) that

$$F(x, \mu) \geq \overline{F}_i \mu^p \rho^{-p} \quad \text{for } x \in B(x_i, \rho), \quad (2.10)$$

$$F(x, t\mu) \geq \underline{F}_i \mu^p \rho^{-p} \quad \text{for } x \in B(x_i, \rho), \quad |t| \leq 1, \quad (2.11)$$

where we have used in (2.11) the fact that $F(x, u)$ is even with respect to u . We define a function $\Phi(t)$ on \mathbb{R} by $\Phi(t) = 1$ for $|t| \leq 1/2$, $\Phi(t) = 2(1 - |t|)$ for $1/2 \leq |t| \leq 1$, $\Phi(t) = 0$ for $|t| \geq 1$. Put $\phi_i(x) := \Phi(|x - x_i|/\rho)$ for $x \in \mathbb{R}^N$. Then $\phi_i \in W^{1,\infty}(\mathbb{R}^N)$. Define $B_i := B(x_i, \rho)$ and $D_i := B(x_i, \rho/2)$. Then $0 \leq \phi_i(x) \leq 1$ in \mathbb{R}^N , $\phi_i(x) = 0$ for $x \in \mathbb{R}^N \setminus B_i$ and

$$\phi_i(x) = 1 \quad \text{for } x \in D_i, \quad |\nabla \phi_i(x)| \leq 2/\rho \quad \text{for } x \in \mathbb{R}^N. \quad (2.12)$$

Let V_k be given by (2.9). We define

$$A_k := \left\{ \mu \sum_{i=1}^{2k} t_i \phi_i(x) : (t_1, \dots, t_{2k}) \in V_k \right\}.$$

Since all the supports of ϕ_i ($1 \leq i \leq 2k$) are disjoint, they are linearly independent. Define $g(t_1, \dots, t_{2k}) := \mu \sum_{i=1}^{2k} t_i \phi_i$. Then g is a mapping from V_k onto A_k and it is an odd homeomorphism. By Lemma 2.9, the genus of V_k is $k + 1$ and so is A_k . Thus $A_k \in \Gamma_k$.

We shall show that $\sup_{A_k} I(u) < 0$. Fix $(t_1, \dots, t_{2k}) \in V_k$ arbitrarily. Then $u := \mu \sum_{i=1}^{2k} t_i \phi_i \in A_k$. Since the support of ϕ_i is \overline{B}_i and $B_i \cap B_j = \emptyset$ for $i \neq j$, we have

$$I(u) = \sum_{i=1}^{2k} \int_{B_i} \left(p^{-1} \mu^p |t_i|^p |\nabla \phi_i|^p - F(x, \mu t_i \phi_i) \right) dx.$$

We denote the volume of the unit ball in \mathbb{R}^N by ω . Then the volume of B_i is $\omega \rho^N$. By the second inequality in (2.12), we have

$$I(u) \leq 2^{p+1} p^{-1} k \omega \mu^p \rho^{N-p} - \sum_{i=1}^{2k} \int_{B_i} F(x, \mu t_i \phi_i) dx. \quad (2.13)$$

To estimate the second term, we define

$$\mathcal{I} := \{i \in \{1, \dots, 2k\} : |t_i| = 1\},$$

$$\mathcal{J} := \{i \in \{1, \dots, 2k\} : |t_i| < 1\}.$$

By the definition of V_k , the cardinal number of \mathcal{I} is greater than or equal to k . We compute the integral of F on B_i for $i \in \mathcal{I}$ and for $i \in \mathcal{J}$, separately. Recall that $F(x, u)$ is even with respect to u and $\phi_i(x) = 1$ on D_i . Clearly, the volume of D_i is $2^{-N} \omega \rho^N$. Then we use (2.10) and (2.11) to obtain for $i \in \mathcal{I}$,

$$\begin{aligned}
\int_{B_i} F(x, \mu t_i \phi_i) dx &= \int_{D_i} F(x, \mu) dx + \int_{B_i \setminus D_i} F(x, \mu t_i \phi_i) dx \\
&\geq 2^{-N} \omega \mu^p \rho^{N-p} \overline{F}_i \\
&\quad + (1 - 2^{-N}) \omega \mu^p \rho^{N-p} \underline{F}_i.
\end{aligned} \tag{2.14}$$

We define

$$\alpha := \min_{1 \leq i \leq 2k} \overline{F}_i, \quad \beta := \min_{1 \leq i \leq 2k} \underline{F}_i.$$

As stated after (2.3), it holds that $\underline{F}_i \leq 0$, and hence $\beta \leq 0$. We rewrite (2.4) as

$$2^{-N} \alpha + (2 - 2^{-N}) \beta > 2^{p+1}/p. \tag{2.15}$$

We reduce (2.14) to

$$\int_{B_i} F(x, \mu t_i \phi_i) dx \geq [2^{-N} \alpha + (1 - 2^{-N}) \beta] \omega \mu^p \rho^{N-p}.$$

The right hand side is positive because of (2.15) with $\beta \leq 0$. Recall that the cardinal number of \mathcal{I} is greater than or equal to k . Summing up both sides of the inequality above over $i \in \mathcal{I}$, we obtain

$$\sum_{i \in \mathcal{I}} \int_{B_i} F(x, \mu t_i \phi_i) dx \geq [2^{-N} \alpha + (1 - 2^{-N}) \beta] k \omega \mu^p \rho^{N-p}. \tag{2.16}$$

Next, by (2.11), we compute for $i \in \mathcal{J}$,

$$\int_{B_i} F(x, \mu t_i \phi_i) dx \geq \omega \mu^p \rho^{N-p} \underline{F}_i \geq \beta \omega \mu^p \rho^{N-p}.$$

Recall that the cardinal number of \mathcal{J} is less than or equal to k . Summing up both sides over $i \in \mathcal{J}$ and using $\beta \leq 0$, we find

$$\sum_{i \in \mathcal{J}} \int_{B_i} F(x, \mu t_i \phi_i) dx \geq k \beta \omega \mu^p \rho^{N-p}. \tag{2.17}$$

The set \mathcal{J} may be empty. In this case, we consider the left hand side to be zero. Then the inequality above is still valid because $\beta \leq 0$. Substituting (2.16) and (2.17) into (2.13) and using (2.15), we obtain

$$I(u) \leq -[2^{-N} \alpha + (2 - 2^{-N}) \beta - 2^{p+1} p^{-1}] k \omega \mu^p \rho^{N-p} < 0,$$

which implies that $\sup_{u \in A_k} I(u) < 0$. Consequently, I satisfies [Assumption 2.5](#).

By Lemma 2.7, I has a sequence u_k of critical points converging to zero in $W_0^{1,p}(\Omega)$. Since $f(x, u)$ is bounded in $\overline{\Omega} \times \mathbb{R}$, the weak solutions $u_k \in W_0^{1,p}(\Omega)$ belong to $C^{1,\theta}(\overline{\Omega})$ with some $\theta \in (0, 1)$ and they are bounded in this space (see [9,21,16] or [20, p. 394, Proposition 2.1]). Here θ is independent of k and $C^{1,\theta}(\overline{\Omega})$ denotes the set of all $C^1(\overline{\Omega})$ functions whose derivatives are Hölder continuous with exponent θ . Since $C^{1,\theta}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, a subsequence of u_k converges to a limit u_∞ in the strong topology of $C^1(\overline{\Omega})$. Since $u_k \rightarrow 0$ in $W_0^{1,p}(\Omega)$, u_∞ must be zero. The uniqueness of the limit u_∞ shows that u_k itself (without extracting a subsequence) converges to zero in $C^1(\overline{\Omega})$. The proof is complete. \square

We give a little simpler condition than Assumption 2.1.

Assumption 2.10. Let $f(x, u)$ satisfy (f0) and let $\varepsilon_0 > 0$ be as in (f0). We assume that there exist sequences $s_n \in (0, \varepsilon_0/2)$, $\rho_n > 0$ and $y_n \in \Omega$ such that $B(y_n, \rho_n) \subset \Omega$ and they satisfy

$$\lim_{n \rightarrow \infty} \overline{F}(y_n, s_n, \rho_n) = \infty, \quad (2.18)$$

$$\liminf_{n \rightarrow \infty} \underline{F}(y_n, s_n, \rho_n) > -\infty. \quad (2.19)$$

Corollary 2.11. Under Assumption 2.10, (1.1) has a sequence of nontrivial solutions whose $C^1(\overline{\Omega})$ -norm converges to zero.

Proof. It is enough to show that Assumption 2.10 implies Assumption 2.1. Impose Assumption 2.10. Then we shall construct μ_k , $x_{k,i}$ and ρ_k satisfying Assumption 2.1. Fix k arbitrarily. Let C_n be the inscribed cube in $B(y_n, \rho_n)$. Then its edge has the length of $2\rho_n/\sqrt{N}$. Let K be the smallest integer that satisfies $K^N \geq 2k$. We divide C_n equally into K^N small cubes by planes parallel to each face of C_n . Denote them by $C_{n,i}$ with $1 \leq i \leq K^N$. More precisely, denote C_n by

$$C_n := [0, a] \times \cdots \times [0, a] \quad \text{with } a := 2\rho_n/\sqrt{N}.$$

Put $I_j := [a(j-1)/K, aj/K]$ with $1 \leq j \leq K$ and define

$$I(j_1, \dots, j_N) := I_{j_1} \times \cdots \times I_{j_N} \quad \text{with } 1 \leq j_1, \dots, j_N \leq K.$$

This is a cube in \mathbb{R}^N and C_n is the union of all these cubes. We rename all $I(j_1, \dots, j_N)$ to $C_{n,i}$ with $1 \leq i \leq K^N$. Then the edge of $C_{n,i}$ has the length of $2\rho_n/K\sqrt{N}$. Denote the inscribed ball in $C_{n,i}$ by $B(x_{n,i}, r_n)$. Then $r_n = \rho_n/K\sqrt{N}$. Since $K^N \geq 2k$, $x_{n,i}$ is defined for all $1 \leq i \leq 2k$. We shall show that Assumption 2.1 is fulfilled with μ_k , $x_{k,i}$ and ρ_k replaced by s_n , $x_{n,i}$ and r_n , respectively, if n is large enough. It is clear that $B(x_{n,i}, r_n) \subset \Omega$ and $B(x_{n,i}, r_n) \cap B(x_{n,j}, r_n) = \emptyset$ when $i \neq j$. Define $M_n := \overline{F}(y_n, s_n, \rho_n)$, which implies that

$$F(x, s_n) s_n^{-p} \rho_n^p \geq M_n \quad \text{for } x \in B(y_n, \rho_n).$$

By (2.19), there exists a $C \geq 0$ such that

$$F(x, ts_n) s_n^{-p} \rho_n^p \geq -C \quad \text{for } x \in B(y_n, \rho_n), \quad 0 \leq t \leq 1.$$

Substituting $\rho_n = K\sqrt{N}r_n$ in the two inequalities above, we have

$$F(x, s_n) s_n^{-p} K^p N^{p/2} r_n^p \geq M_n, \quad F(x, t s_n) s_n^{-p} K^p N^{p/2} r_n^p \geq -C,$$

for $0 \leq t \leq 1$ and $x \in B(y_n, \rho_n)$. Since $B(x_{n,i}, r_n) \subset B(y_n, \rho_n)$, the inequalities above are valid for $x \in B(x_{n,i}, r_n)$ also. Taking the infimum on $x \in B(x_{n,i}, r_n)$, we have

$$\overline{F}(x_{n,i}, s_n, r_n) \geq M_n K^{-p} N^{-p/2}, \quad \underline{F}(x_{n,i}, s_n, r_n) \geq -C K^{-p} N^{-p/2}.$$

Then we get

$$\begin{aligned} & 2^{-N} \min_{1 \leq i \leq 2k} \overline{F}(x_{n,i}, s_n, r_n) + (2 - 2^{-N}) \min_{1 \leq i \leq 2k} \underline{F}(x_{n,i}, s_n, r_n) \\ & \geq 2^{-N} M_n K^{-p} N^{-p/2} - (2 - 2^{-N}) C K^{-p} N^{-p/2}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} M_n = \infty$ by (2.18), the right hand side is larger than $2^{p+1}/p$ for n large enough. The proof is complete. \square

Using Corollary 2.11, we prove Theorem 1.2.

Proof of Theorem 1.2. To prove the theorem, it is enough to show that (1.7) and (1.8) imply (2.18) and (2.19). By (1.7) with the evenness of F , there exists a sequence $s_n > 0$ converging to zero such that

$$\inf_{x \in B(x_0, \rho_0)} F(x, s_n) s_n^{-p} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Put $(y_n, \rho_n) := (x_0, \rho_0)$ for all n . Then the above inequality shows (2.18). By (1.8), there exists a constant $C \geq 0$ such that

$$\inf_{x \in B(x_0, \rho_0)} F(x, u) |u|^{-p} \geq -C \quad \text{for } 0 < |u| \leq 1.$$

Putting $u = t s_n$, we find

$$\inf_{x \in B(x_0, \rho_0)} F(x, t s_n) (t s_n)^{-p} \geq -C \quad \text{for all large } n \text{ and } 0 < t \leq 1,$$

which leads to

$$\inf_{x \in B(x_0, \rho_0)} F(x, t s_n) s_n^{-p} \geq -C t^p \geq -C.$$

Therefore (2.19) holds and the proof is complete. \square

Theorem 1.2 has been proved as a corollary of Theorem 2.2 and obviously Theorem 1.1 follows from Theorem 1.2 with $p = 2$. Therefore Theorem 2.2 is a generalization of all theorems in the present paper and to our knowledge, it is the weakest sufficient condition for type (A). We emphasize that Theorem 2.2 can prove type (A) for many p -Laplace equations, to which Theorems 1.1 and 1.2 are not applicable (see Remark 1.5).

We observe the condition

$$\inf_{x \in B(x_0, \rho_0)} F(x, u) |u|^{-p} \rightarrow \infty \quad \text{as } u \rightarrow 0. \quad (2.20)$$

This condition clearly implies (1.7) and (1.8). Therefore Theorem 1.2 yields the following corollary.

Corollary 2.12. *Let $f(x, u)$ satisfy (f0) and (2.20) with some x_0 and ρ_0 satisfying $B(x_0, \rho_0) \subset \Omega$. Then (1.1) is of type (A).*

Corollary 2.12 ensures that the sublinear condition in a small neighborhood of x_0 yields type (A). If $f(x, u)$ satisfies (2.20), we call $f(x, u)$ *locally sublinear* near x_0 . It is easy to verify that (2.20) follows from the next condition,

$$\inf_{x \in B(x_0, \rho_0)} \frac{f(x, u)}{|u|^{p-2}u} \rightarrow \infty \quad \text{as } u \rightarrow 0. \quad (2.21)$$

Corollary 2.13. *Let $f(x, u)$ satisfy (f0) and (2.21) with some x_0 and ρ_0 . Then (1.1) is of type (A).*

We conclude this section by giving a sufficient condition for (I) in the next theorem, which can be easily proved, however we state it for later use.

Theorem 2.14. *Let λ_1 be the first eigenvalue of the p -Laplacian under the Dirichlet boundary condition. If $f(x, u)$ satisfies*

$$\limsup_{u \rightarrow 0} \left(\sup_{x \in \Omega} \frac{f(x, u)}{|u|^{p-2}u} \right) < \lambda_1, \quad (2.22)$$

then (1.1) is of type (I).

Proof. Since the first eigenvalue λ_1 is the infimum of the Rayleigh quotient (see [10, p. 354, (7.1.4)]), we have

$$\lambda_1 \|u\|_p^p \leq \|\nabla u\|_p^p \quad \text{for } u \in W_0^{1,p}(\Omega). \quad (2.23)$$

By (2.22), we can choose $\lambda > 0$ such that

$$\limsup_{u \rightarrow 0} \left(\sup_{x \in \Omega} \frac{f(x, u)}{|u|^{p-2}u} \right) < \lambda < \lambda_1.$$

Then we take $\varepsilon > 0$ so small that

$$uf(x, u) \leq \lambda |u|^p \quad \text{for } |u| \leq \varepsilon \text{ and } x \in \Omega.$$

Let u be any solution of (1.1). Multiplying (1.1) by u and integrating it over Ω , we have

$$\int_{\Omega} |\nabla u|^p dx = \int_{\Omega} u f(x, u) dx \leq \lambda \int_{\Omega} |u|^p dx,$$

provided that $\|u\|_{\infty} \leq \varepsilon$. Combining this inequality with (2.23), we deduce that $u \equiv 0$ in Ω . Accordingly, if $\|u\|_{\infty} \leq \varepsilon$, then u vanishes in Ω . Clearly, if the C^1 -norm of a solution is small, so is the L^{∞} -norm. Thus (I) occurs. \square

3. Applications

In this section, we study two elliptic equations and apply Theorem 2.2 to them and decide whether each equation is of type (A) or (I). Studying these elliptic equations, we recognize that Assumption 2.1 is a weak sufficient condition for (A) very close to the necessary and sufficient condition.

3.1. Example 1

Let us start with the equation

$$-\Delta_p u = a|u|^{q-2}u - d(x)^s|u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.1)$$

where $1 < p, q, r < \infty$, $a, s > 0$ and $d(x)$ is the distance function defined by (1.14). Denote the right hand side of (3.1) by $f(x, u)$. If $p < q$, then $f(x, u)/|u|^{p-2}u \leq a|u|^{q-p}$ and $f(x, u)$ satisfies (2.22). By Theorem 2.14, (3.1) is of type (I). We consider the case where $1 < q < p$. Then the first term on the right hand side in (3.1) is p -sublinear. If $q > 1$ is very close to 1, then the first term is dominant near $u = 0$, and so this equation is of type (A). If q is large, then the second term is stronger than the first one, and hence it is of type (I). Indeed, we have the next result.

Theorem 3.1. *Let $r > 1$, $1 < q < p$ and $a, s > 0$. Then the following assertions hold:*

- (i) *if $1 < q < p(r+s)/(p+s)$, then (3.1) is of type (A),*
- (ii) *if $q > p(r+s)/(p+s)$, then it is of type (I),*
- (iii) *if $q = p(r+s)/(p+s)$ and $a > 0$ is small enough, then it is of type (I),*
- (iv) *if $N \geq 2$, $q = p(r+s)/(p+s)$ and $a > 0$ is large enough, then it is of type (A).*

We define

$$q_* = q_*(p, r, s) := \frac{p(r+s)}{p+s}.$$

This is the division number between types (A) and (I), that is, if $q < q_*$, then (3.1) is of type (A) and if $q > q_*$, then it is of type (I). If $q = q_*$, then $|u|^{q-2}u$ and $d(x)^s|u|^{r-2}u$ have the same weight, and therefore the coefficient a determines which type of (A) or (I) holds. Using Theorem 3.1, we prove Theorem 1.4 stated in Section 1.

Proof of Theorem 1.4. Putting $p = 2$ and $s = 1$ in Theorem 3.1, we obtain Theorem 1.4. \square

To prove Theorem 3.1, we prepare the next lemma.

Lemma 3.2. Let $r > 1$, $1 < q < p$ and $a, s > 0$. Denote the right hand side of (3.1) by $f(x, u)$. Then $f(x, u)$ satisfies Assumption 2.1 if either (i) or (ii) below holds:

- (i) $1 < q < p(r + s)/(p + s)$,
- (ii) $N \geq 2$, $q = p(r + s)/(p + s)$ and $a > 0$ is large enough.

Proof. We first assume $N \geq 2$ and prove that $f(x, u)$ satisfies Assumption 2.1 if either (i) or (ii) holds. After that, we treat (i) with $N = 1$.

Let $N \geq 2$. There exists a $\delta > 0$ such that for each $k \in \mathbb{N}$, there exist points $\xi_i \in \partial\Omega$ with $1 \leq i \leq 2k$ which satisfy $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$, and δ is independent of k . Indeed, for example, choose a smooth curve on $\partial\Omega$ such that $g : [0, 1] \rightarrow \partial\Omega$ is a C^1 -diffeomorphism from $[0, 1]$ onto $g([0, 1])$. Since g^{-1} is Lipschitz continuous, there exists a $c_0 > 0$ such that $|g(t) - g(s)| \geq c_0|t - s|$ for $t, s \in [0, 1]$. Put $\xi_i := g(i/2k)$ with $1 \leq i \leq 2k$. Then we have for $i \neq j$,

$$|\xi_i - \xi_j| = |g(i/2k) - g(j/2k)| \geq c_0|i - j|/2k \geq c_0/2k.$$

Define $\delta := c_0/8$. Then $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$ and δ is independent of k .

Put $\rho_k := \delta/k$. For each $1 \leq i \leq 2k$, there exists a unique point $x_i \in \Omega$ such that $B(x_i, \rho_k) \subset \Omega$ and $\partial B(x_i, \rho_k) \cap \partial\Omega = \{\xi_i\}$, after replacing δ by a small constant if necessary. Since $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$, $B(x_i, \rho_k) \cap B(x_j, \rho_k) = \emptyset$ for $i \neq j$. Denote the right hand side of (3.1) by $f(x, u)$. The function $F(x, u)$ defined by (2.1) is computed as

$$F(x, u) = (a/q)|u|^q - (1/r)d(x)^s|u|^r.$$

Since $d(x) \leq 2\rho_k$ in $B(x_i, \rho_k)$, we have

$$F(x, u) \geq (a/q)|u|^q - (2^s/r)|u|^r \rho_k^s \quad \text{for } x \in B(x_i, \rho_k). \quad (3.2)$$

Assume condition (i) in the lemma. Then we can choose $\theta > 0$ such that

$$\frac{p}{p-q} < \theta < \frac{qs}{p(q-r)} + 1 \quad \text{when } q > r, \quad (3.3)$$

$$\frac{p}{p-q} < \theta \quad \text{when } q \leq r. \quad (3.4)$$

It follows from (3.3), (3.4) and $q < p(r + s)/(p + s)$ that

$$-(p-q)\theta + p < 0, \quad -(p-q)\theta + p < -(p-r)\theta + p + s. \quad (3.5)$$

We define $\mu_k := \rho_k^\theta$. Let us compute \bar{F} defined by (2.2). Using (3.2), we have

$$\bar{F}(x_i, \mu_k, \rho_k) \geq (a/q)\rho_k^{-(p-q)\theta+p} - (2^s/r)\rho_k^{-(p-r)\theta+p+s} \rightarrow \infty, \quad (3.6)$$

as $k \rightarrow \infty$ because of (3.5). Using (3.2) and $\mu_k = \rho_k^\theta$, we compute

$$F(x, t\mu_k)\mu_k^{-p}\rho_k^p \geq (at^q/q)\rho_k^{-(p-q)\theta+p} - (2^s t^r/r)\rho_k^{-(p-r)\theta+p+s}, \quad (3.7)$$

for $x \in B(x_i, \rho_k)$ and $0 \leq t \leq 1$. We put

$$\alpha_k := a\rho_k^{-(p-q)\theta+p}, \quad \beta_k := 2^s \rho_k^{-(p-r)\theta+p+s}$$

and denote the right hand side of (3.7) by

$$g_k(t) := (\alpha_k/q)t^q - (\beta_k/r)t^r \quad \text{for } t \in [0, 1].$$

We shall show that $g_k(t)$ is bounded from below by a constant independent of k and $t \in [0, 1]$. By (3.6), $\alpha_k/q - \beta_k/r > 0$ for $k \geq k_0$ with a large k_0 . Let $k \geq k_0$. Then $g_k(1) > 0$. We divide the proof into two cases.

Case 1. $q > r$. Then $g_k(t)$ achieves a negative minimum in $[0, 1]$, which is computed as

$$\min_{0 \leq t \leq 1} g_k(t) = -\frac{q-r}{qr} \alpha_k^{-r/(q-r)} \beta_k^{q/(q-r)} = -\frac{q-r}{qr} 2^{qs/(q-r)} a^{-r/(q-r)} \rho_k^v, \quad (3.8)$$

where

$$v = \frac{1}{q-r} (-p(q-r)\theta + pq + qs - pr).$$

Then $v > 0$ because of (3.3). Thus the minimum of g_k converges to zero as $k \rightarrow \infty$.

Case 2. $q \leq r$. Since $t^q \geq t^r$, $g_k(t) \geq [(\alpha_k/q) - (\beta_k/r)]t^q \geq 0$ for $t \in [0, 1]$ and $k \geq k_0$.

By Cases 1 and 2, we have the inequality $g_k(t) \geq -C$ with some $C \geq 0$ independent of k and $t \in [0, 1]$, which shows that $\underline{F}(x_i, \mu_k, \rho_k) \geq -C$ for all $1 \leq i \leq 2k$ and $k \in \mathbb{N}$. This estimate with (3.6) shows (2.4) for all large k .

We assume condition (ii). Let x_i, ρ_k be as in the proof of (i). We put $\theta := p/(p-q)$. By the assumption (ii) of the lemma, we have

$$-(p-q)\theta + p = -(p-r)\theta + p + s = 0.$$

We put $\mu_k := \rho_k^\theta$. Then (3.6) is reduced to

$$\overline{F}(x_i, \mu_k, \rho_k) \geq a/q - 2^s/r. \quad (3.9)$$

Moreover, (3.7) is rewritten as

$$F(x, t\mu_k)\mu_k^{-p}\rho_k^p \geq at^q/q - 2^st^r/r,$$

for $x \in B(x_i, \rho_k)$ and $0 \leq t \leq 1$. Denote the right hand side by $h(t)$. Let a be larger than $2^sq/r$. Then $h(1) > 0$. If $q > r$, then we use the same computation as in (3.8) to get

$$h(t) \geq -Ca^{-r/(q-r)} \quad \text{for } 0 \leq t \leq 1,$$

where $C > 0$ is independent of a and t . If $q \leq r$, then $h(t) \geq [(a/q) - (2^s/r)]t^q \geq 0$ for $a > 2^sq/r$. In any case, we have $h(t) \geq -Ca^{-r/(q-r)}$ for $0 \leq t \leq 1$ if $a > 2^sq/r$. Therefore

$$\underline{F}(x_i, \mu_k, \rho_k) \geq -Ca^{-r/(q-r)} \quad \text{if } a > 2^sq/r. \quad (3.10)$$

We use (3.9) and (3.10) to obtain

$$\begin{aligned} & 2^{-N} \min_{1 \leq i \leq 2k} \overline{F}(x_i, \mu_k, \rho_k) + (2 - 2^{-N}) \min_{1 \leq i \leq 2k} \underline{F}(x_i, \mu_k, \rho_k) \\ & \geq 2^{-N} (a/q - 2^s/r) - (2 - 2^{-N}) C a^{-r/(q-r)} > 2^{p+1}/p, \end{aligned}$$

provided that $a > 0$ is large enough. Thus (2.4) holds.

In the proof above, we chose $\xi_i \in \partial\Omega$ with $1 \leq i \leq 2k$. This is impossible when $N = 1$. Assume condition (i) with $N = 1$, i.e., let $N = 1$, $\Omega = (0, 1)$ and suppose $1 < q < p(r+s)/(p+s)$. Then the distance function is computed as $d(x) = x$ in $[0, 1/2]$ and $d(x) = 1 - x$ in $[1/2, 1]$. Let k_0 be a positive integer satisfying $4k_0 e^{-k_0} < 1/2$. Let $k \geq k_0$. We define

$$\rho_k := e^{-k}, \quad x_i := (2i - 1)e^{-k} \quad \text{with } 1 \leq i \leq 2k. \quad (3.11)$$

Then the ball $B(x_i, \rho_k)$ becomes the interval $(2(i - 1)e^{-k}, 2ie^{-k})$. These intervals are disjoint. We have the estimate $d(x) = x \leq 4ke^{-k} < 1/2$ for $x \in B(x_i, \rho_k)$ with $1 \leq i \leq 2k$. Instead of (3.2), we have

$$F(x, u) \geq (a/q)|u|^q - (1/r)(4ke^{-k})^s |u|^r.$$

Define θ by (3.3) or (3.4) corresponding to $q > r$ or $q \leq r$, respectively. Put $\mu_k := \rho_k^\theta = e^{-k\theta}$. Then we have

$$\begin{aligned} \overline{F}(x_i, \mu_k, \rho_k) & \geq (a/q) \exp([(p - q)\theta - p]k) \\ & \quad - (4^s/r) k^s \exp([(p - r)\theta - p - s]k), \end{aligned}$$

which diverges to infinity as $k \rightarrow \infty$ because of (3.5). Furthermore, we obtain

$$\begin{aligned} F(x, t\mu_k) \mu_k^{-p} \rho_k^p & \geq (a/q) t^q \exp([(p - q)\theta - p]k) \\ & \quad - (4^s/r) t^r k^s \exp([(p - r)\theta - p - s]k). \end{aligned}$$

Using the inequality above, we can prove that $\underline{F}(x_i, \mu_k, \rho_k) \geq -C$ for all i and $k \geq k_0$ with some $C \geq 0$ in the same method as in the case $N \geq 2$. Therefore we have (2.4) and the proof is complete. \square

Proof of Theorem 3.1. By Theorem 2.2 with Lemma 3.2, we obtain the assertions (i) and (iv). We shall show (ii) and (iii). Multiplying (3.1) by u and integrating it over Ω , we have

$$\int_{\Omega} |\nabla u|^p dx = a \int_{\Omega} |u|^q dx - \int_{\Omega} d(x)^s |u|^r dx. \quad (3.12)$$

We shall estimate the first term on the right hand side. Define $\alpha := ps/(p+s)$, $m := (p+s)/s$ and $m' := (p+s)/p$. Using the Hölder inequality, we have

$$\begin{aligned}
\int_{\Omega} |u|^q dx &= \int_{\Omega} |u|^{\alpha} d(x)^{-\alpha} d(x)^{\alpha} |u|^{q-\alpha} dx \\
&\leq \left(\int_{\Omega} |u|^{\alpha m} d(x)^{-\alpha m} dx \right)^{1/m} \left(\int_{\Omega} d(x)^{\alpha m'} |u|^{(q-\alpha)m'} dx \right)^{1/m'} \\
&= \left(\int_{\Omega} |u|^p d(x)^{-p} dx \right)^{1/m} \left(\int_{\Omega} d(x)^s |u|^{\mu} dx \right)^{1/m'}, \tag{3.13}
\end{aligned}$$

where we have put $\mu := (q - \alpha)m'$, i.e.,

$$\mu := (q - \alpha)m' = \frac{qs}{p} + q - s. \tag{3.14}$$

We use the Hardy inequality (see [5, p. 313] or [7,15,22]) to get

$$\int_{\Omega} |u(x)/d(x)|^p dx \leq C_0 \int_{\Omega} |\nabla u|^p dx \quad \text{for } u \in W_0^{1,p}(\Omega),$$

where $C_0 > 0$ depends only on p and Ω . Then (3.13) is rewritten as

$$\int_{\Omega} |u|^q dx \leq C \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/m} \left(\int_{\Omega} d(x)^s |u|^{\mu} dx \right)^{1/m'},$$

with some $C > 0$. We apply the Young inequality

$$xy \leq \frac{1}{m} \varepsilon^m x^m + \frac{1}{m'} \varepsilon^{-m'} y^{m'} \quad \text{for } x, y \geq 0 \text{ and } \varepsilon > 0,$$

where $\varepsilon > 0$ is chosen so small that $\varepsilon^m/m \leq 1/2$. Then we have

$$\begin{aligned}
a \int_{\Omega} |u|^q dx &\leq aC \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/m} \left(\int_{\Omega} d(x)^s |u|^{\mu} dx \right)^{1/m'} \\
&\leq \frac{1}{2} \int_{\Omega} |\nabla u|^p dx + a^{m'} C_1 \int_{\Omega} d(x)^s |u|^{\mu} dx,
\end{aligned}$$

with some C_1 independent of a and u . Substituting the inequality above into (3.12), we obtain

$$(1/2) \int_{\Omega} |\nabla u|^p dx \leq a^{m'} C_1 \int_{\Omega} d(x)^s |u|^{\mu} dx - \int_{\Omega} d(x)^s |u|^r dx. \tag{3.15}$$

We shall show (ii). Let $q > p(r + s)/(p + s)$. Then $\mu > r$. We have

$$(1/2) \int_{\Omega} |\nabla u|^p dx \leq a^{m'} C_1 \|u\|_{\infty}^{\mu-r} \int_{\Omega} d(x)^s |u|^r dx - \int_{\Omega} d(x)^s |u|^r dx \leq 0,$$

if $a^{m'} C_1 \|u\|_{\infty}^{\mu-r} \leq 1$. Consequently, any solution with small L^{∞} -norm vanishes in Ω . Thus the assertion (ii) holds.

We shall show (iii). Let $q = p(r + s)/(p + s)$. Then $\mu = r$. Hence (3.15) is reduce to

$$(1/2) \int_{\Omega} |\nabla u|^p dx \leq (a^{m'} C_1 - 1) \int_{\Omega} d(x)^s |u|^r dx.$$

Recall that C_1 is independent of a . If $a \leq C_1^{-1/m'}$, then the inequality above shows that $u \equiv 0$, that is, there are no nontrivial solutions. Hence (iii) holds. The proof is complete. \square

3.2. Example 2

We consider

$$-\Delta_p u = a|u|^{q-2}u - |x|^s |u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.16)$$

Theorem 3.3. *Let $r > 1$, $1 < q < p$ and $a, s > 0$. Then the following assertions hold:*

- (i) *if $q < r$, then (A) occurs,*
- (ii) *if $0 \notin \overline{\Omega}$ and $q > r$, then (I) occurs,*
- (iii) *if $q = r$ and $|x|^s < a$ at some $x \in \Omega$, then (A) occurs,*
- (iv) *if $q = r$ and $|x|^s \geq a$ for all $x \in \Omega$ (hence $0 \notin \overline{\Omega}$), then (I) occurs,*
- (v) *if $0 \in \overline{\Omega}$ and $q < p(r + s)/(p + s)$, then (A) occurs,*
- (vi) *if $0 \in \partial\Omega$ and $q > p(r + s)/(p + s)$, then (I) occurs,*
- (vii) *if $0 \in \partial\Omega$, $q = p(r + s)/(p + s)$ and $a > 0$ is small, then (I) occurs.*

The theorem above does not include any contradiction. For example, the assumptions in (i) and in (vi) cannot occur simultaneously. Indeed, suppose both $q < r$ and $q > p(r + s)/(p + s)$. These inequalities show $p < r$ and we have

$$q > \frac{p(r + s)}{p + s} > \frac{p(r + s)}{r + s} = p.$$

This contradicts $1 < q < p$, which is assumed first in the lemma. Similarly, (i) and (vii), (iii) and (vi) or (iii) and (vii) are incompatible.

Define $q_* := r$ when $0 \notin \overline{\Omega}$ and $q_* := p(r + s)/(p + s)$ when $0 \in \partial\Omega$. Then Theorem 3.3 says that q_* is the division number between (A) and (I), i.e., if $q < q_*$, then (A) occurs and if $q > q_*$, then (I) occurs. We shall explain why this result holds. When $0 \notin \overline{\Omega}$, we have

$$c \leq |x| \leq C \quad \text{in } \Omega, \quad (3.17)$$

with some $0 < c < C$. Then (3.16) is similar to (1.16) and so $q_* = r$ is the division number. If $0 \in \partial\Omega$, then $|x|$ is like $d(x)$ near $x = 0$ and hence the same result as Theorem 3.1 holds.

Proof of Theorem 3.3. It is easy to prove the assertions (i)–(iv), however we show them to make the paper self-contained. Denote the right hand side of (3.16) by $f(x, u)$. Suppose $q < r$. Let $B(x_0, \rho_0)$ be any ball included in Ω . Then f satisfies (2.21) and hence the assertion (i) follows from Corollary 2.13. Suppose the assumption in (ii). Since $0 \notin \overline{\Omega}$, we have (3.17), which shows that

$$uf(x, u) \leq a|u|^q - c^s|u|^r \leq 0,$$

provided that $|u|$ is small enough. Therefore (2.22) holds and Theorem 2.14 proves the assertion (ii).

Let $q = r$. Then

$$f(x, u) = \alpha(x)|u|^{r-2}u, \quad \alpha(x) := a - |x|^s.$$

The assumption in (iii) implies that $\alpha(x) \geq c_0 > 0$ in a certain ball $B(x_0, r_0)$ with a $c_0 > 0$. Hence (2.21) holds and this is of type (A) by Corollary 2.13. The assumption in (iv) means that $\alpha(x) \leq 0$ for all x . Thus (2.22) holds and this is of type (I).

We shall show (v). To this end, we verify Assumption 2.10. We first deal with the case where the origin 0 is an interior point of Ω . Define $\theta > 0$ by (3.3) or (3.4) corresponding to $q > r$ or $q \leq r$, respectively. Put $y_n := 0$ and let $\rho_n > 0$ be a sequence converging to zero. Then $B(y_n, \rho_n) = B(0, \rho_n) \subset \Omega$ for n large. We have

$$F(x, u) = (a/q)|u|^q - r^{-1}|x|^s|u|^r \geq (a/q)|u|^q - r^{-1}|u|^r\rho_n^s,$$

for $x \in B(0, \rho_n)$. Putting $s_n := \rho_n^\theta$ and using the inequality above, we compute

$$\overline{F}(0, s_n, \rho_n) \geq (a/q)\rho_n^{-(p-q)\theta+p} - (1/r)\rho_n^{-(p-r)\theta+p+s},$$

which diverges to infinity as $n \rightarrow \infty$. Furthermore, we get

$$F(x, tx_n)s_n^{-p}\rho_n^p \geq (at^q/q)\rho_n^{-(p-q)\theta+p} - (t^r/r)\rho_n^{-(p-r)\theta+p+s},$$

for $x \in B(0, \rho_n)$ and $0 \leq t \leq 1$. This is similar to (3.7). In the same method as in the proof of Lemma 3.2 (i), we can prove that $\underline{F}(0, s_n, \rho_n) \geq -C$ with some $C \geq 0$ independent of n . Therefore Assumption 2.10 holds. When $0 \in \partial\Omega$ also, we can choose a small cube $B(y_n, \rho_n)$ near $x = 0$ which is included in Ω . Then the same method as above works well. Therefore Corollary 2.11 proves (v).

We shall show (vi). Let $0 \in \partial\Omega$ and $q > p(r+s)/(p+s)$. Since $0 \in \partial\Omega$, we have $d(x) \leq |x|$ in Ω , where $d(x)$ is the distance function defined by (1.14). Then the Hardy inequality gives

$$\int_{\Omega} (|u(x)|/|x|)^p dx \leq \int_{\Omega} (|u(x)|/d(x))^p dx \leq C \int_{\Omega} |\nabla u|^p dx. \quad (3.18)$$

Let α , m , m' and μ be defined in the proof of [Theorem 3.1](#). In the same computation as in [\(3.13\)](#), we have

$$\begin{aligned} \int_{\Omega} |u|^q dx &\leq \left(\int_{\Omega} |u|^{\alpha m} |x|^{-\alpha m} dx \right)^{1/m} \left(\int_{\Omega} |x|^{\alpha m'} |u|^{(q-\alpha)m'} dx \right)^{1/m'} \\ &= \left(\int_{\Omega} |u|^p |x|^{-p} dx \right)^{1/m} \left(\int_{\Omega} |x|^s |u|^{\mu} dx \right)^{1/m'}. \end{aligned} \quad (3.19)$$

Using [\(3.18\)](#) and [\(3.19\)](#) with the help of the Young inequality, we have

$$a \int_{\Omega} |u|^q dx \leq (1/2) \int_{\Omega} |\nabla u|^p dx + a^{m'} C \int_{\Omega} |x|^s |u|^{\mu} dx. \quad (3.20)$$

Multiplying [\(3.16\)](#) by u and integrating it over Ω , we have

$$\int_{\Omega} |\nabla u|^p dx = a \int_{\Omega} |u|^q dx - \int_{\Omega} |x|^s |u|^r dx. \quad (3.21)$$

Substituting [\(3.20\)](#) into [\(3.21\)](#) and following the lines after [\(3.15\)](#), we obtain the assertion (vi).

We shall show (vii). Assume $q = p(r+s)/(p+s)$. Then $\mu = r$. Substituting [\(3.20\)](#) into [\(3.21\)](#), we find that if $a > 0$ is small enough, then there are no nontrivial solutions. Thus (I) occurs. The proof is complete. \square

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