



# Large time behavior of solution to a fully parabolic chemotaxis–haptotaxis model in higher dimensions

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## Abstract

This paper deals with the chemotaxis–haptotaxis model of cancer invasion

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + u(1 - \mu u - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0 \end{cases}$$

in a bounded smooth domain  $\Omega \subset \mathbb{R}^n$  with zero-flux boundary conditions, where  $\chi$ ,  $\xi$  and  $\mu$  are positive parameters. It is shown that if  $\mu/\chi$  is suitably large then for all sufficiently smooth initial data, the associated initial–boundary–value problem possesses a unique global-in-time classical solution that is bounded in  $\Omega \times (0, \infty)$ , and if the initial data  $w_0$  is small,  $w$  becomes asymptotically negligible. Moreover, we prove that when domain  $\Omega$  is convex,  $(\frac{1}{\mu}, \frac{1}{\mu}, 0)$  is globally asymptotically stable provided that  $u_0 \not\equiv 0$  and thereby extends the result of Hillen et al. (2013) [18] to the higher space dimensions.

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## 1. Introduction

Cell migration plays an important role in a wide variety of physiological and pathophysiological processes, including embryo development, skin wound healing, cancer invasion and metastasis [23]. As a significant mechanism of directional migration of cells, chemotaxis is the movement of cells in response to concentration gradients of a chemical signal emitted by the cells themselves in many biological process. A well-known chemotaxis model was proposed by Keller and Segel [22] in the 1970s, which describes the aggregation processes of the cellular slime mold *Dictyostelium discoideum*. During the past few decades, a number of variations of the Keller–Segel model are proposed and studied, see [17,19,30] for a wide survey on various chemotaxis models.

Recently, there is an increasing biological and mathematical interest in mathematical models of cancer invasion, and chemotactic mechanisms have been detected to be crucial in the process of cancer invasion [2,7,8,13,27,32,36,41]. As we know, cancer invasion is associated with the degradation of the extracellular matrix (ECM). In fact, ECM is degraded by matrix degrading enzymes (MDE) such as the urokinase-type plasminogen activator (uPA) secreted by tumor cells. This degradation allows the cells migration following the gradients of uPA (chemotaxis) [4,5]. In addition, tumor cells interact with the fibers of the extracellular matrix, the corresponding tumor cells movement in response to gradients of non-diffusible macromolecules such as vitronectin (haptotaxis) [2]. The classical Keller–Segel model has been extended by Chaplain and Lolas [7,8] to describe processes of cancer invasion, where, in addition to random movement, cancer cells bias their movement both towards a gradient of the diffusible MDE by chemotaxis, and towards a gradient of the non-diffusible ECM through detecting the ECM material vitronectin VN adhered therein by haptotaxis. According to the model proposed in [7,8], we are concerned with the chemotaxis–haptotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + ru(1 - \mu u - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega \end{cases} \quad (1.1)$$

in a bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n \geq 3$ ) with smooth boundary  $\partial\Omega$ , where  $\partial/\partial\nu$  denotes the outward normal derivative on  $\partial\Omega$ , the variables  $u$ ,  $v$ ,  $w$  describe the cancer cell density, the MDE concentration and the ECM density. The parameters  $\chi$  and  $\xi$  measure the chemotactic and haptotactic sensitivities respectively, parameter  $r$  and  $\mu$  are the proliferation rate and the reciprocal of carrying capacity, respectively. Throughout this paper, the initial data  $(u_0, v_0, w_0)$  are assumed that for some  $\vartheta \in (0, 1)$

$$\begin{cases} u_0 \in C^1(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega, \ u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in C^{2+\vartheta}(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.2)$$

As a subsystem, the model (1.1) contains a Keller–Segel–type chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ru(1 - u), & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (1.3)$$

This chemotaxis-only system has been extensively studied through the past decades, where the main issue of the investigation is whether the solutions of the models are bounded or blow-up (see e.g., [10,11,21,30,42,43,45]). In particular, when  $n \geq 2$ , solutions to system (1.3) with  $r = 0$  may blow up in finite time [16,28]. Anyhow, the logistic source in (1.3) has a certain blow-up preventing effect on chemotaxis models. For instance, arbitrarily small  $r > 0$  guarantees the boundedness of solutions when  $n \leq 2$  [29], and suitably large  $r$  (compared to the chemotactic sensitivity  $\chi$ ) can rule out blow-up in the case  $n \geq 3$  [43]. However, it should be remarked that chemotaxis models do not exclude the possibility of the finite-time explosions even in the presence of certain logistic-type growth inhibitions, provided the latter are suitably weak [45].

A second subsystem is obtained upon focusing on the cross-diffusive interaction with  $w$ , resulting a haptotaxis-only system

$$\begin{cases} u_t = \Delta u - \xi \nabla \cdot (u \nabla w) + ru(1 - \mu u - w), & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega. \end{cases} \quad (1.4)$$

Here the third equation in (1.4) is only an ordinary differential equation (ODE), no regularizing effect on  $w$  can be expected. Correspondingly, the mathematical literature on it is comparatively thin. As for the global existence and asymptotic behavior of solution to the variant of (1.4), we refer to [25,26,31,41].

From a theoretical point of view, due to the fact that the chemotaxis and haptotaxis terms require different  $L^p$ -estimate techniques, the problem related to the chemotaxis–haptotaxis models of cancer invasion presents an important mathematical challenging. To the best of our knowledge, there exist some boundedness and stabilization results on the simplified parabolic–elliptic–ODE chemotaxis–haptotaxis model [35,37,38]. Indeed, according to the fact that the diffusion rate of the MDE is much larger than that of cancer cells in realistic situations, one may follow an approach of quasi-steady-state approximation for the second equation in (1.1) and hence concentrate on the simplified chemotaxis–haptotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + ru(1 - u - w), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), w(x, 0) = w_0(x), & x \in \Omega. \end{cases} \quad (1.5)$$

The global boundedness of solutions to this system has been proved for any  $r > 0$  in two dimensions and for large  $r$  in three dimensions [35] and for  $r > \frac{(n-2)_+}{n}\chi$  in higher dimensions [37]. As to the parabolic–parabolic–ODE chemotaxis–haptotaxis system (1.1), only little appears to be known. In fact, the global existence of classical solutions to (1.1) has been proved for any  $r > 0$  in two dimensions [34], for large  $r$  in three dimensions [36], however, the result on the global boundedness of solutions to (1.1) are made only in space dimensions  $n \leq 3$  and is still left in higher dimensions [6,18,33].

The purpose of the present work is to investigate the convergence of all solution components in (1.1) under some conditions, possibly involving the initial data or the interaction between chemotactic cross-diffusion and the limitation of cell growth. Having in mind this focusing, we henceforth only consider (1.1) with  $r = 1$  for simplicity in presentation.

The first object of the present paper is to address the global boundedness of solutions to (1.1). Our main result in this respect is the following.

**Theorem 1.1.** *Let  $3 \leq n \leq 8$ ,  $\chi > 0$ ,  $\xi > 0$ . Then there exists some positive constant  $\theta_0$  such that for  $\frac{\mu}{\chi} > \theta_0$  and  $(u_0, v_0, w_0)$  fulfilling (1.2) with some  $\vartheta \in (0, 1)$ , problem (1.1) with  $r = 1$  admits a unique classical solution which is globally bounded in  $\Omega \times (0, \infty)$ .*

Note that for the simplified chemotaxis–haptotaxis system (1.5), under an explicit smallness condition on  $w_0$ , the third solution component  $w$  will become asymptotically negligible [37,38]. It is also worth to remark that recently Painter and Winkler [18] have shown that under an explicit smallness condition on  $w_0$ , the complicated spatio-temporal patterns of (1.1) in one-dimensional domain are organized by that of the corresponding chemotaxis system with logistic source. However, it is still left as an open problem in the higher-dimensional case. Our result in this direction can be stated as follows

**Theorem 1.2.** *Assume the hypothesis of Theorem 1.1 holds. Moreover, if*

$$1 - \|w_0\|_{L^\infty(\Omega)} > \frac{\chi^2}{4\mu^2} + \xi^2(\|\nabla w_0\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{4e} + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{2\mu}), \quad (1.6)$$

*then the global-in-time solution  $(u, v, w)$  of (1.1) with  $r = 1$  has the property*

$$\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\alpha t} \quad \text{for all } t > 0 \quad (1.7)$$

*with some positive constants  $\alpha$  and  $C$ .*

Going beyond this, we note that in very recent paper [3], Tao and Winkler claim that if  $n \leq 3$ ,  $(u, v, w)$  is a bounded global classical solution of (1.1) with  $\mu = 1$ , then  $(u, v, w)$  converges to  $(1, 1, 0)$  exponentially provided that  $r > \frac{\chi^2}{8}$ . On the other hand, recently in [46] it was found that all the solutions of the corresponding chemotaxis subsystem of (1.1) with  $r = 1$  converge to  $(\frac{1}{\mu}, \frac{1}{\mu})$  exponentially for suitable large value of  $\frac{\mu}{\chi}$  and convex domain  $\Omega$ . Hence it is natural to expect that when domain  $\Omega$  is convex, the similar convergence result is valid for the chemotaxis–haptotaxis (1.1) with  $r = 1$  provided that  $\frac{\mu}{\chi}$  is suitably large.

**Theorem 1.3.** Assume the hypothesis of [Theorem 1.2](#) holds and  $\Omega$  is a convex domain. Then there exists constant  $\theta_1 > 0$  with the following property: if  $\frac{\mu}{\chi} > \theta_1$ , one can find  $\gamma > 0$  and  $C > 0$  such that the global classical solution  $(u, v, w)$  of [\(1.1\)](#) satisfies

$$\|u(\cdot, t) - \frac{1}{\mu}\|_{L^\infty(\Omega)} \leq C e^{-\gamma t}, \quad (1.8)$$

$$\|v(\cdot, t) - \frac{1}{\mu}\|_{L^\infty(\Omega)} \leq C e^{-\gamma t} \quad (1.9)$$

as well as

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\alpha t} \quad (1.10)$$

as  $t \rightarrow \infty$ .

The structure of this paper is as follows. In [Section 2](#), we recall some preliminary results and state the local existence of the classical solutions of [\(1.1\)](#). In [Section 3](#), we develop some  $L^p$ -estimate techniques to raise the a priori estimate of solutions from  $L^1(\Omega) \rightarrow L^{n-1}(\Omega) \rightarrow L^{n+1}(\Omega)$ , and then use the standard Alikakos–Moser iteration (see e.g. [\[1\]](#) and Lemma A.1 of [\[40\]](#)) to show [Theorem 1.1](#). [Section 4](#) is devoted to show the exponential decay of  $w$  if  $w_0$  is small in the sense of [\(1.6\)](#) and  $\frac{\mu}{\chi}$  is suitably large. Furthermore, when domain  $\Omega$  is convex, we prove that  $(\frac{1}{\mu}, \frac{1}{\mu}, 0)$  is globally asymptotically stable provided that  $u_0 \not\equiv 0$ .

## 2. Preliminaries

To derive some estimates, we shall use the following Gagliardo–Nirenberg interpolation inequality [\[12\]](#): Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary, let  $l, k$  be any integers satisfying  $0 \leq l < k$ , and let  $1 \leq q, r \leq \infty$ , and  $p \in \mathbb{R}^+$ ,  $\frac{1}{k} \leq \theta \leq 1$  such that

$$\frac{1}{p} - \frac{l}{n} = \theta \left( \frac{1}{q} - \frac{k}{n} \right) + (1 - \theta) \frac{1}{r}. \quad (2.1)$$

Then for any  $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$ , there exist two positive constants  $c_1$  and  $c_2$  depending only on  $\Omega, q, k, r$  and  $n$  such that the following inequality holds:

$$\|D^l u\|_{L^p(\Omega)} \leq c_1 \|D^k u\|_{L^q(\Omega)}^\theta \|u\|_{L^r(\Omega)}^{1-\theta} + c_2 \|u\|_{L^r(\Omega)} \quad (2.2)$$

with the following exception: If  $1 < q < \infty$  and  $k - l - \frac{n}{q}$  is a non-negative integer, then [\(2.1\)](#) holds for  $\theta$  satisfying  $\frac{1}{k} \leq \theta < 1, r > 1$  (see also [\[14, Theorem 3.4, page 20\]](#) and [\[35\]](#)).

We also need some fundamental estimates for solutions to the following problem for inhomogeneous linear heat equations:

$$\begin{cases} v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (2.3)$$

which can be derived from a standard regularity argument involving the variation-of-constants formula for  $v$  and  $L^q - L^p$  estimates for the heat semigroup (see [20, Lemma 4.1] for instance).

**Lemma 2.1.** (See [21, Lemma 2.1] [47, Lemma 2.2].) Let  $T > 0$ ,  $1 \leq p \leq \infty$ ,  $v_0 \in L^p(\Omega)$  and  $u \in L^1(0, T; L^p(\Omega))$ . Then (2.3) has a unique solution  $v \in C([0, T]; L^p(\Omega))$  given by

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds, \quad t \in [0, T],$$

where  $e^{t\Delta}$  is the semigroup generated by the Neumann Laplacian. In addition, let  $1 \leq r \leq \infty$ ,  $1 \leq q \leq p < \frac{nq}{n-q}$ ,  $v_0 \in W^{1,p}(\Omega) \cap W^{2,r}(\Omega)$  and  $u \in L^\infty(0, T; L^q(\Omega)) \cap L^r(0, T; L^r(\Omega))$ . Then for every  $t \in (0, T)$ ,

$$\|v(t)\|_{L^p(\Omega)} \leq \|v_0\|_{L^p(\Omega)} + c_3 \|u\|_{L^\infty((0,T); L^q(\Omega))}, \quad (2.4)$$

$$\|\nabla v(t)\|_{L^p(\Omega)} \leq \|\nabla v_0\|_{L^p(\Omega)} + c_3 \|u\|_{L^\infty((0,T); L^q(\Omega))}, \quad (2.5)$$

$$\int_0^t \int_\Omega e^{rs} |\Delta v(x, s)|^r dx ds \leq c_3 \int_0^t \int_\Omega e^{rs} |u(x, s)|^r dx ds + c_3 \|v_0\|_{W^{2,r}(\Omega)}, \quad (2.6)$$

where  $c_3$  is a positive constant depending on  $p$ ,  $q$ ,  $r$  and  $n$ .

In order to investigate the large-time behavior of solutions to (1.1), we frequently use some known smooth estimates for the heat semigroup  $(e^{\tau\Delta})_{\tau \geq 0}$  under Neumann boundary condition [44,46]: there exist positive constants  $k_1, k_2, k_3$  such that

$$\|\nabla e^{\tau\Delta} \varphi\|_{L^p(\Omega)} \leq k_1 \|\nabla \varphi\|_{L^p(\Omega)} \quad \text{for all } \tau > 0 \text{ and any } \varphi \in W^{1,p}(\Omega)$$

and

$$\|\nabla e^{\tau\Delta} \varphi\|_{L^p(\Omega)} \leq k_2 (1 + \tau^{-\frac{1}{2}}) \|\varphi\|_{L^\infty(\Omega)} \quad \text{for all } \tau > 0 \text{ and each } \varphi \in L^\infty(\Omega)$$

as well as

$$\|e^{\tau\Delta} \nabla \cdot \varphi\|_{L^\infty(\Omega)} \leq k_3 (1 + \tau^{-\frac{1}{2} - \frac{n}{2p}}) \|\varphi\|_{L^p(\Omega)} \quad (2.7)$$

for all  $\tau > 0$  and all  $\varphi \in C^1(\bar{\Omega}; \mathbb{R}^n)$  fulfilling  $\varphi \cdot \nu = 0$  on  $\partial\Omega$ .

We state a result concerning the local existence of classical solutions, along with a convenient extensibility criterion, which can be proved by well-established methods involving standard parabolic regularity theory and an appropriate fixed point framework, we refer the reader to [38, Lemma A.1] and [39, Lemma 2.1] or also [9, Theorem 2.1] for the details.

**Lemma 2.2.** Let  $\chi > 0$ ,  $\xi > 0$  and  $\mu > 0$ , and assume that  $u_0, v_0$  and  $w_0$  satisfy (1.2) with some  $\vartheta \in (0, 1)$ . Then problem (1.1) admits a unique classical solution

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \\ v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L_{loc}^\infty([0, T_{\max}); W^{1,q}(\Omega)) (q > n) \\ w \in C^{2,1}(\bar{\Omega} \times [0, T_{\max})) \end{cases}$$

with  $u \geq 0$ ,  $v \geq 0$  and  $0 \leq w \leq \|w_0\|_{L^\infty(\Omega)}$ , where  $T_{\max}$  denotes the maximal existence time. In addition, if  $T_{\max} < +\infty$ , then

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

### 3. Boundedness of solutions

According to Lemma 2.2, we need only to establish a uniform bound of  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  in order to obtain the boundedness of the solutions to (1.1). Here one essential analytic difficulty stems from the fact that the chemotaxis and haptotaxis terms in the first equation of (1.1) require different  $L^p$ -estimate techniques, since ECM density satisfies an ordinary differential equation (ODE) whereas MDE concentration satisfies a parabolic equation (PDE). A fundamental step toward an adequate treatment of the haptotaxis term is to establish a one-side pointwise estimate which connects  $\Delta w$  to  $v$ . This paper develops some  $L^p$ -estimate techniques to raise the a priori estimate of a solution from  $L^1(\Omega) \rightarrow L^{n-1}(\Omega) \rightarrow L^{n+1}(\Omega)$ .

Before proving the main results in this paper, we shall introduce some notations. For simplicity, the variable of integration in an integral will be omitted without ambiguity, e.g. the integral  $\int_\Omega f(x)dx$  is written as  $\int_\Omega f(x)$ . In what follows,  $c_i$  ( $i = 4, 5, \dots$ ) denotes various constants which are independent of  $t$ .

According to the local existence theory,  $(u(\cdot, s), v(\cdot, s), w(\cdot, s)) \in (C^2(\bar{\Omega}))^3$  for any  $s \in (0, T_{\max})$ . Hence without loss of generality, we can assume that there exists a constant  $M > 0$  such that

$$\|u_0\|_{C^2(\bar{\Omega})} + \|v_0\|_{C^2(\bar{\Omega})} + \|w_0\|_{C^2(\bar{\Omega})} \leq M.$$

The following properties of solutions of (1.1) are well known.

**Lemma 3.1.** *Let  $(u, v, w)$  be the solution of (1.1). Then*

- (i)  $\|u(\cdot, t)\|_{L^1(\Omega)} \leq m := \max\{\frac{|\Omega|}{\mu}, \|u_0\|_{L^1(\Omega)}\}$  for all  $t \in (0, T_{\max})$ ;
- (ii)  $\|v(\cdot, t)\|_{L^1(\Omega)} \leq \max\{m, \|v_0\|_{L^1(\Omega)}\}$  for all  $t \in (0, T_{\max})$ ;
- (iii)  $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}^2 \leq \frac{5m}{4\mu} + \|\nabla v_0\|_{L^2(\Omega)}^2$  for all  $t \in (0, T_{\max})$ ;
- (iv)  $\|v(\cdot, t)\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq c_1(\frac{5m}{4\mu} + \|\nabla v_0\|_{L^2(\Omega)}^2)^{\frac{1}{2}} + c_2 \max\{m, \|v_0\|_{L^1(\Omega)}\}$  for all  $t \in (0, T_{\max})$ .

**Proof.** (i) Integrating the first equation in (1.1) with respect to  $x \in \Omega$ , we have

$$\frac{d}{dt} \int_\Omega u(x, t) \leq \int_\Omega u(x, t) - \mu \int_\Omega u^2(x, t), \quad (3.1)$$

since  $w \geq 0$  by Lemma 2.2. Moreover, by the Cauchy–Schwarz inequality, we get

$$\frac{d}{dt} \int_{\Omega} u(x, t) + \int_{\Omega} u(x, t) \leq \frac{|\Omega|}{\mu}, \quad (3.2)$$

which implies that  $\|u(\cdot, t)\|_{L^1(\Omega)} \leq \max\{\frac{|\Omega|}{\mu}, \|u_0\|_{L^1(\Omega)}\}$ .

(ii) Integrating the second equation in (1.1) with respect to  $x \in \Omega$  yields

$$\frac{d}{dt} \int_{\Omega} v(x, t) + \int_{\Omega} v(x, t) \leq \int_{\Omega} u(x, t) \leq \sup_{0 \leq t \leq T_{\max}} \int_{\Omega} u(x, t). \quad (3.3)$$

So (ii) follows from the non-negativity of  $v$  and (i).

(iii) Multiplying the second equation in (1.1) by  $-\Delta v$  and integrating over  $\Omega$ , we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v(x, t)|^2 + \int_{\Omega} |\Delta v(x, t)|^2 + \int_{\Omega} |\nabla v(x, t)|^2 &= - \int_{\Omega} u \Delta v \\ &\leq \int_{\Omega} |\Delta v(x, t)|^2 + \frac{1}{4} \int_{\Omega} u^2(x, t) \end{aligned}$$

and thus

$$\frac{d}{dt} \int_{\Omega} |\nabla v(x, t)|^2 + 2 \int_{\Omega} |\nabla v(x, t)|^2 \leq \frac{1}{2} \int_{\Omega} u^2(x, t).$$

Combining this with (3.1), we obtain

$$\frac{d}{dt} \int_{\Omega} (u(x, t) + 2\mu |\nabla v(x, t)|^2) + 2 \int_{\Omega} (u(x, t) + 2\mu |\nabla v(x, t)|^2) \leq 3 \int_{\Omega} u(x, t).$$

This, together with the Gronwall lemma and estimate (i), yields

$$\int_{\Omega} u(x, t) + 2\mu \int_{\Omega} |\nabla v(x, t)|^2 \leq \frac{5m}{2} + 2\mu \|\nabla v_0\|_{L^2(\Omega)}^2$$

and thereby (iii) holds.

(iv) It results from (2.2), (ii) and (iii) immediately.  $\square$

According to an explicit representation formula of  $w$

$$w(x, t) = w_0(x) e^{-\int_0^t v(x, s) ds} \quad (3.4)$$

for  $x \in \Omega$ ,  $t \in (0, T)$ , one has

**Lemma 3.2.** (See [6, Lemma 2.2].) Let  $(u_0, v_0, w_0)$  satisfy (1.2) and  $(u, v, w)$  be a classical solution of (1.1) in  $\Omega \times (0, T)$ . Then



$$\nabla w(x, t) = \nabla w_0(x) e^{-\int_0^t v(x, s) ds} - w_0(x) e^{-\int_0^t v(x, s) ds} \int_0^t \nabla v(x, s) ds \quad (3.5)$$

and

$$\begin{aligned} \Delta w(x, t) &\geq \Delta w_0(x) e^{-\int_0^t v(x, s) ds} - 2e^{-\int_0^t v(x, s) ds} \nabla w_0(x) \cdot \int_0^t \nabla v(x, s) ds \\ &\quad - \frac{1}{e} w_0(x) - w_0(x) v(x, t) e^{-\int_0^t v(x, s) ds} \end{aligned} \quad (3.6)$$

for all  $x \in \Omega$  and  $t \in (0, T)$ .

**Lemma 3.3.** Let  $(u, v, w)$  be a classical solution of (1.1) with  $n = 3$ . Then there exist some positive constant  $\theta_2$  and  $C > 0$  such that

$$\|u(\cdot, t)\|_{L^4(\Omega)} \leq C \quad (3.7)$$

for all  $t \in (0, T_{\max})$  and  $\frac{\mu}{\chi} > \theta_2$ .

**Proof.** Multiplying the first equation in (1.1) by  $u^{p-1}$  ( $p > 1$ ) and applying the Young inequality, we obtain

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+1} \\ &\leq (p-1) \chi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + (p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \int_{\Omega} u^p \\ &= -\frac{p-1}{p} \chi \int_{\Omega} u^p \Delta v + (p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \int_{\Omega} u^p \\ &\leq \varepsilon_1 \int_{\Omega} u^{p+1} + \varepsilon_1^{-p} \chi^{p+1} \int_{\Omega} |\Delta v|^{p+1} + (p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \int_{\Omega} u^p. \end{aligned} \quad (3.8)$$

On the other hand, by (3.6), we have

$$\begin{aligned} &(p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \\ &= -\frac{p-1}{p} \xi \int_{\Omega} u^p \Delta w \\ &\leq -\frac{p-1}{p} \xi \int_{\Omega} u^p \left( \Delta w_0(x) e^{-\int_0^t v(x, s) ds} - 2e^{-\int_0^t v(x, s) ds} \nabla w_0(x) \cdot \int_0^t \nabla v(x, s) ds - \right. \end{aligned}$$

$$\begin{aligned}
& \frac{1}{e} w_0(x) - w_0(x) v(x, t) e^{-\int_0^t v(x, s) ds} \Big) \\
& \leq 2M\xi \int_{\Omega} u^p + M\xi \int_{\Omega} u^p v - \frac{2(p-1)}{p} \xi \int_{\Omega} u^p \nabla e^{-\int_0^t v(x, s) ds} \cdot \nabla w_0(x) \\
& = 2M\xi \int_{\Omega} u^p + M\xi \int_{\Omega} u^p v \\
& \quad + \frac{2(p-1)}{p} \xi \int_{\Omega} (u^p e^{-\int_0^t v(x, s) ds} \Delta w_0(x) + \nabla u^p \cdot \nabla w_0(x) e^{-\int_0^t v(x, s) ds}) \\
& \leq 4M\xi \int_{\Omega} u^p + M\xi \int_{\Omega} u^p v + 2M(p-1)\xi \int_{\Omega} u^{p-1} |\nabla u|. \tag{3.9}
\end{aligned}$$

Inserting (3.9) into (3.8) yields

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+1} \\
& \leq \varepsilon_1 \int_{\Omega} u^{p+1} + \varepsilon_1^{-p} \chi^{p+1} \int_{\Omega} |\Delta v|^{p+1} + (1+4M\xi) \int_{\Omega} u^p + M\xi \int_{\Omega} u^p v \\
& \quad + 2M(p-1)\xi \int_{\Omega} u^{p-1} |\nabla u| \\
& \leq (\varepsilon_1 + 2\varepsilon_2) \int_{\Omega} u^{p+1} + \varepsilon_1^{-p} \chi^{p+1} \int_{\Omega} |\Delta v|^{p+1} + C(\varepsilon_2, p, \xi, M) + \varepsilon_2^{-p} (M\xi)^{p+1} \int_{\Omega} v^{p+1} \\
& \quad + \frac{p-1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + 2(p-1)(M\xi)^2 \int_{\Omega} u^p \\
& \leq (\varepsilon_1 + 3\varepsilon_2) \int_{\Omega} u^{p+1} + \varepsilon_1^{-p} \chi^{p+1} \int_{\Omega} |\Delta v|^{p+1} + C(\varepsilon_2, p, \xi, M) \\
& \quad + \varepsilon_2^{-p} (M\xi)^{p+1} \int_{\Omega} v^{p+1} + \frac{p-1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2
\end{aligned}$$

with constant  $C(\varepsilon_2, p, \xi, M)$  depending only upon  $\varepsilon_2$ ,  $p$ ,  $\xi$  and  $M$ , which together with the Young inequality implies that

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} u^p + (p+1) \int_{\Omega} u^p + \frac{p(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + p(\mu - 4\varepsilon_2 - \varepsilon_1) \int_{\Omega} u^{p+1} \\
& \leq \varepsilon_1^{-p} p \chi^{p+1} \int_{\Omega} |\Delta v|^{p+1} + C(\varepsilon_2, p, \xi, M) + \varepsilon_2^{-p} p (M\xi)^{p+1} \sup_{0 \leq t < T_{\max}} \|v(\cdot, t)\|_{L^{p+1}(\Omega)}^{p+1} \tag{3.10}
\end{aligned}$$

for all  $t \in (0, T_{\max})$ .

Now taking  $p = 4$  in (3.10) and using Lemma 3.1(iv), we get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u^4 + 5 \int_{\Omega} u^4 + 6 \int_{\Omega} u^2 |\nabla u|^2 + 4(\mu - \varepsilon_1 - 4\varepsilon_2) \int_{\Omega} u^5 \\ & \leq 4\chi^5 \varepsilon_1^{-4} \int_{\Omega} |\Delta v|^5 + C(\varepsilon_2, \xi, M). \end{aligned}$$

Applying the variation-of-constants formula to the above inequality gives

$$\begin{aligned} & \int_{\Omega} u^4(\cdot, t) + 4(\mu - \varepsilon_1 - 4\varepsilon_2) \int_0^t \int_{\Omega} e^{-5(t-s)} u^5(\cdot, s) ds \\ & \leq \int_{\Omega} u_0^4 + C(\varepsilon_2, \xi, M) + 4\chi^5 \varepsilon_1^{-4} \int_0^t \int_{\Omega} e^{-5(t-s)} |\Delta v(\cdot, s)|^5 ds. \end{aligned} \quad (3.11)$$

On the other hand, by (2.6) in Lemma 2.1, it follows that

$$\int_0^t \int_{\Omega} e^{-5(t-s)} |\Delta v(\cdot, s)|^5 ds \leq c_3 \int_0^t \int_{\Omega} e^{-5(t-s)} u^5(\cdot, s) ds + c_3 \|v_0\|_{W^{2,5}(\Omega)}^5. \quad (3.12)$$

So inserting (3.12) into (3.11) yields

$$\begin{aligned} & \int_{\Omega} u^4(\cdot, t) + 4(\mu - \varepsilon_1 - 4\varepsilon_2) \int_0^t \int_{\Omega} e^{-5(t-s)} u^5(\cdot, s) ds \\ & \leq \int_{\Omega} u_0^4 + C(\varepsilon_2, \xi, M) + 4\chi^5 \varepsilon_1^{-4} c_3 \int_0^t \int_{\Omega} e^{-5(t-s)} u^5(\cdot, s) ds + 4\chi^5 \varepsilon_1^{-4} c_3 \|v_0\|_{W^{2,5}(\Omega)}^5, \end{aligned}$$

and thereby (3.7) is valid with  $\theta_2 = 5 \cdot 4^{-\frac{4}{5}} c_3^{\frac{1}{5}}$  by simply minimizing  $(0, \infty) \ni x \mapsto x + \chi^5 c_3 x^{-4}$ .  $\square$

**Lemma 3.4.** *Let  $(u, v, w)$  be a classical solution of (1.1) with  $4 \leq n \leq 8$ . Then there exist some positive constant  $\theta_3$  and  $C > 0$  such that*

$$\|u(\cdot, t)\|_{L^{n+1}(\Omega)} \leq C$$

for all  $t \in (0, T_{\max})$  and  $\frac{\mu}{\chi} > \theta_3$ .

**Proof.** We mainly devote to the proof of this lemma in the case of  $n = 4$ , since the proof in the other cases can be proceeded analogously.

When  $n = 4$ , taking  $p = 3$  in (3.10) and using Lemma 3.1(iv) yield

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u^3 + 4 \int_{\Omega} u^3 + 3(\mu - \varepsilon_1 - 4\varepsilon_2) \int_{\Omega} u^4 \\ & \leq 3\chi^4 \varepsilon_1^{-3} \int_{\Omega} |\Delta v|^4 + C(\varepsilon_2, \xi, M). \end{aligned}$$

Applying the variation-of-constants formula to the above inequality yields

$$\begin{aligned} & \int_{\Omega} u^3(\cdot, t) + 3(\mu - \varepsilon_1 - 4\varepsilon_2) \int_0^t \int_{\Omega} e^{-4(t-s)} u^4(\cdot, s) ds \\ & \leq \int_{\Omega} u_0^3 + C(\varepsilon_2, \xi, M) + 3\chi^4 \varepsilon_1^{-3} \int_0^t \int_{\Omega} e^{-4(t-s)} |\Delta v(\cdot, s)|^4 ds. \end{aligned} \quad (3.13)$$

On the other hand, in view of (2.6) of Lemma 2.1, we have

$$\int_0^t \int_{\Omega} e^{-4(t-s)} |\Delta v(\cdot, s)|^4 ds \leq c_3 \int_0^t \int_{\Omega} e^{-4(t-s)} u^4(\cdot, s) ds + c_3 \|v_0\|_{W^{2,4}(\Omega)}^4. \quad (3.14)$$

Thus, from (3.13) and (3.14) we see that

$$\begin{aligned} & \int_{\Omega} u^3(\cdot, t) + 3(\mu - \varepsilon_1 - 4\varepsilon_2) \int_0^t \int_{\Omega} e^{-4(t-s)} u^4(\cdot, s) ds \\ & \leq \int_{\Omega} u_0^3 + C(\varepsilon_2, \xi, M) + 3\chi^4 \varepsilon_1^{-3} c_3 \int_0^t \int_{\Omega} e^{-4(t-s)} u^4(\cdot, s) ds + 3\chi^4 \varepsilon_1^{-3} c_3 \|v_0\|_{W^{2,4}(\Omega)}^4, \end{aligned}$$

which implies that there exists a constant  $\theta_4 > 0$  such that if  $\frac{\mu}{\chi} > \theta_4$ , then

$$\|u(\cdot, t)\|_{L^3(\Omega)} \leq C$$

for all  $t \in (0, T_{\max})$ .

Now in view of (2.5) of Lemma 2.1 and the Sobolev imbedding theorem, we have

$$\|v(\cdot, t)\|_{L^s(\Omega)} \leq C$$

for  $s \in (1, \infty)$  and  $t \in (0, T_{\max})$ .

Therefore taking  $p = 5$  in (3.10) and using Lemma 3.1(iv) again, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u^5 + 6 \int_{\Omega} u^5 + 5(\mu - \varepsilon_1 - 4\varepsilon_2) \int_{\Omega} u^6 \\ & \leq 5\chi^6 \varepsilon_1^{-5} \int_{\Omega} |\Delta v|^6 + C(\varepsilon_2, \xi, M). \end{aligned}$$

Based on Lemma 2.1, a reasoning in the argument above shows that there exists a constant  $\theta_3 > 0$  such that if  $\frac{\mu}{\chi} > \theta_3$ ,

$$\|u(\cdot, t)\|_{L^5(\Omega)} \leq C$$

for all  $t \in (0, T_{\max})$ , which implies that this lemma is valid for  $n = 4$ .

As for  $n \in [5, 8]$ , by the argument above, one can raise the a priori estimate of  $u(\cdot, t)$  and  $v(\cdot, t)$  from  $\|u(\cdot, t)\|_{L^{p_1}(\Omega)} \rightarrow \|v(\cdot, t)\|_{L^{q_1}(\Omega)} \rightarrow \|u(\cdot, t)\|_{L^{p_2}(\Omega)} \rightarrow \|v(\cdot, t)\|_{L^{q_2}(\Omega)} \rightarrow \cdots \rightarrow \|u(\cdot, t)\|_{L^{p_k}(\Omega)} \rightarrow \|v(\cdot, t)\|_{L^{q_k}(\Omega)} \rightarrow \|u(\cdot, t)\|_{L^{p_{k+1}}(\Omega)}$ , where  $p_1 = 1$ ,  $q_1 = \frac{2n}{n-2}$ ,  $p_2 = q_1 - 1$ ,  $q_2 = \frac{np_2}{n-2p_2}$ ,  $p_k = q_{k-1} - 1$ ,  $q_k = \frac{np_k}{n-2p_k}$ ,  $k = 2$  if  $n = 5$ ,  $k = 3$  if  $n = 6, 7$  and  $k = 5$  if  $n = 8$ . Hence the proof of this lemma is completed by the Hölder inequality.  $\square$

In light of the estimate (2.5) of Lemma 2.1 for the Neumann heat semigroup, a direct consequence of Lemmas 3.3 and 3.4 is as follows.

**Lemma 3.5.** *Under the same assumptions as in Theorem 1.1, there exists  $C > 0$  independent of  $T_{\max}$  such that the solution  $(u, v, w)$  of (1.1) satisfies*

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \quad (3.15)$$

Note that  $\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)}$  is bounded by (3.15), however  $\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}$  might become unbounded in light of (3.4). Therefore, Lemma A.1 in [40] can not be directly applied to the first equation in (1.1) to get the boundedness of  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ . We perform a straightforward adaptation of the well-known Moser–Alikakos  $L^p$  iteration to derive  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  for  $t \in (0, T_{\max})$ . Since the procedure is rather standard, we may confine ourselves to an outline.

**Lemma 3.6.** *Under the assumptions of Theorem 1.1, there exists  $C > 0$  independent of  $T_{\max}$  such that the classical solution  $(u, v, w)$  of (1.1) satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \quad (3.16)$$

**Proof.** Noticing  $w \geq 0$  and multiplying the first equation in (1.1) by  $u^{p-1}$  ( $p > 1$ ), we get

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+1} \\ & \leq (p-1)\chi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + (p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \int_{\Omega} u^p \end{aligned}$$

$$\leq \frac{(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{(p-1)\chi^2}{2} \int_{\Omega} u^p |\nabla v|^2 + (p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \int_{\Omega} u^p.$$

This in conjunction with (3.9) and (3.15) yields

$$\frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p + \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 \leq c_4 p^2 \int_{\Omega} u^p. \quad (3.17)$$

Now according to the Gagliardo–Nirenberg inequality (see (2.2)), one has

$$\begin{aligned} c_4 p^2 \int_{\Omega} u^p &= c_4 p^2 \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\ &\leq c_5 p^2 \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2n}{n+2}} \cdot \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{4}{n+2}} + c_5 p^2 \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2, \end{aligned}$$

which, together with the Young inequality, yields

$$c_4 p^2 \int_{\Omega} u^p \leq \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 + c_6 p^2 \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2 + c_6 p^{n+2} \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2. \quad (3.18)$$

Inserting (3.18) into (3.17), we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p &\leq c_7 p^{n+2} \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2 \\ &\leq c_7 p^{n+2} (\max\{1, \|u^{\frac{p}{2}}\|_{L^1(\Omega)}\})^2. \end{aligned} \quad (3.19)$$

Let  $p_k := 2^k$  and  $M_k(T) := \max\{1, \sup_{t \in (0, T)} \int_{\Omega} u^{p_k}(\cdot, t)\}$  for  $T \in (0, T_{\max})$  and  $k = 1, 2, \dots$ .

Then (3.19) implies that for any  $t \in (0, T)$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^{p_k} + \int_{\Omega} u^{p_k} &\leq c_7 p_k^{n+2} (\max\{1, \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}\})^2 \\ &= c_7 p_k^{n+2} M_{k-1}^2(T), \end{aligned}$$

which, upon a comparison argument, entails the existence of  $b > 1$  independent of  $k$  such that

$$M_k(T) \leq \max\{b^k M_{k-1}^2(T), |\Omega| \|u_0\|_{L^\infty(\Omega)}^{p_k}\}.$$

Now if  $b^k M_{k-1}^2(T) \leq \|u_0\|_{L^\infty(\Omega)}^{p_k}$  for infinitely many  $k \geq 1$ , then (3.16) with  $C = \|u_0\|_{L^\infty(\Omega)}$ . Conversely, if  $b^k M_{k-1}^2(T) > \|u_0\|_{L^\infty(\Omega)}^{p_k}$  for all sufficiently large  $k$ , then

$$M_k(T) \leq b^k M_{k-1}^2(T) \quad (3.20)$$

for all sufficiently large  $k$ , and thereby (3.20) is still valid for all  $k \geq 1$  upon enlarging  $b$  if necessary. Hence  $\ln M_k(T) \leq k \ln b + 2 \ln M_{k-1}(T)$  for all  $k \geq 1$ . By a straightforward induction, we get

$$\ln M_k(T) \leq (k+2) \ln b + 2^k (\ln M_0 + 2 \ln b)$$

and thus

$$M_k(T) \leq b^{k+2+2^{k+1}} M_0^{2^k}. \quad (3.21)$$

From (3.21) and Lemma 3.1(i), it follows that (3.16) is valid with  $C = b^2 \max\{1 + |\Omega|, \|u_0\|_{L^1(\Omega)}\}$  by taking  $T \rightarrow T_{\max}$ .  $\square$

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** In view of Lemma 3.4,  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  is bounded uniformly with respect to  $t \in (0, T_{\max})$ . Hence the statement of global existence and boundedness of classical solutions to (1.1) is a straightforward consequence of Lemma 2.2.  $\square$

#### 4. Asymptotic behavior

In this section, we concentrate on the suitably large  $\frac{\mu}{\chi}$  under which we have just asserted global boundedness of solution to (1.1). This section goes further to make sure that under a suitable smallness condition on  $w_0$ , all the solutions  $(u, v, w)$  of (1.1) with  $r = 1$  converge to  $(\frac{1}{\mu}, \frac{1}{\mu}, 0)$  exponentially whenever  $\frac{\mu}{\chi}$  is sufficiently large. To this end, we first show that the solution component  $w(\cdot, t)$  will decay exponentially as  $t \rightarrow \infty$ . In view of the representations (3.4) and (3.5), it seems favorable to show an appropriate lower bound for  $\int_0^t v(x, s) ds$ . Fortunately, in the same way as in the proof of Hillen–Painter–Winkler [18, Lemma 3.1], one can obtain the pointwise estimate

$$\begin{aligned} (e^{t\Delta} z)(x) &\geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d^2}{4t}} \int_{\Omega} z \\ &\geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d^2}{4t}} \int_{\Omega} z \quad \text{for all nonnegative } z \in C(\overline{\Omega}) \end{aligned}$$

for the Neumann heat semigroup  $e^{t\Delta}$  with  $d = \text{diam}\Omega$ , and thereby establish the analogue of [18, Corollary 3.3] in the multi-dimensional case.

**Lemma 4.1.** *If  $(u, v, w)$  is a global classical solution of (1.1), then there exist constant  $\Gamma > 0$  and  $C_1^*$  such that*

$$\int_0^t v(x, s) ds \geq \Gamma \cdot \int_0^t \int_{\Omega} u(y, s) dy ds - C_1^* \quad \text{for all } x \in \Omega, t > 0,$$

where  $\Gamma$  and  $C_1^*$  only depends  $\text{diam}\Omega$  and  $\|u_0\|_{L^1(\Omega)}$ , respectively.

Let us recall the following simple statement on the eventual validity of the appropriate bounds for  $\int_{\Omega} u(x, t) dx$ ,  $\int_{\Omega} v(x, t) dx$ .

**Lemma 4.2.** *Let  $(u, v, w)$  be a global classical solution of (1.1). Then*

- (i)  $\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^1(\Omega)} \leq \frac{|\Omega|}{\mu}$  ;  
(ii)  $\limsup_{t \rightarrow \infty} \|v(\cdot, t)\|_{L^1(\Omega)} \leq \frac{|\Omega|}{\mu}$ .

**Proof.** (i) By (3.2), it is easy to see that  $\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^1(\Omega)} \leq \frac{|\Omega|}{\mu}$ .

(ii) From (i), it follows that for any  $\varepsilon > 0$ , there exists  $t_1 := t_1(\varepsilon)$  such that for all  $t \geq t_1$ ,

$$\|u(\cdot, t)\|_{L^1(\Omega)} < \frac{|\Omega|}{\mu} + \varepsilon.$$

Furthermore, by (3.3), we have

$$\begin{aligned} \int_{\Omega} v(\cdot, t) dx &= e^{-(t-t_1)} \int_{\Omega} v(\cdot, t_1) dx + \int_{t_1}^t \int_{\Omega} e^{-(t-s)} u(\cdot, s) dx ds \\ &\leq \frac{|\Omega|}{\mu} + \varepsilon + e^{-(t-t_1)} \int_{\Omega} v(\cdot, t_1) dx, \end{aligned}$$

which, along with the arbitrariness of  $\varepsilon$ , implies that  $\limsup_{t \rightarrow \infty} \|v(\cdot, t)\|_{L^1(\Omega)} \leq \frac{|\Omega|}{\mu}$ .  $\square$

Our argument in the proof of Lemma 4.4 will involve an upper bound for  $\int_{\Omega} |\nabla w(\cdot, t)|^2$ , which is provided by the following lemma.

**Lemma 4.3.** *If  $(u, v, w)$  is a global classical solution of (1.1), then*

$$\int_{\Omega} |\nabla w(\cdot, t)|^2 \leq 2 \int_{\Omega} |\nabla w_0|^2 + \frac{|\Omega|}{2e} \|w_0\|_{L^\infty(\Omega)}^2 + \|w_0\|_{L^\infty(\Omega)}^2 \cdot \int_{\Omega} v(\cdot, t) \quad (4.1)$$

for all  $t > 0$ .

**Proof.** The proof of this lemma proceeds along the idea of the arguments of Lemma 4.1 in [37]. By (3.5), we see that

$$|\nabla w(x, t)|^2 \leq 2|\nabla w_0(x)|^2 e^{-2 \int_0^t v(x, s) ds} + 2w_0^2(x) e^{-2 \int_0^t v(x, s) ds} \left| \int_0^t \nabla v(x, s) ds \right|^2$$



and thus

$$\int_{\Omega} |\nabla w(\cdot, t)|^2 \leq 2 \int_{\Omega} |\nabla w_0|^2 + 2 \|w_0\|_{L^\infty(\Omega)}^2 \int_{\Omega} e^{-2 \int_0^t v(\cdot, s) ds} \left| \int_0^t \nabla v(\cdot, s) ds \right|^2. \quad (4.2)$$

Employing the integration of parts, we furthermore obtain that

$$\begin{aligned} \int_{\Omega} e^{-2 \int_0^t v(\cdot, s) ds} \left| \int_0^t \nabla v(\cdot, s) ds \right|^2 &= -\frac{1}{2} \int_{\Omega} \nabla e^{-2 \int_0^t v(\cdot, s) ds} \cdot \left( \int_0^t \nabla v(\cdot, s) ds \right) \\ &= \frac{1}{2} \int_{\Omega} e^{-2 \int_0^t v(\cdot, s) ds} \cdot \left( \int_0^t \Delta v(\cdot, s) ds \right) \\ &= \frac{1}{2} \int_{\Omega} e^{-2 \int_0^t v(\cdot, s) ds} \cdot \left( \int_0^t (v_t(\cdot, s) + v(\cdot, s) - u(\cdot, s)) ds \right) \\ &\leq \frac{1}{2} \int_{\Omega} e^{-2 \int_0^t v(\cdot, s) ds} \cdot \left( \int_0^t v(\cdot, s) ds + v(\cdot, t) \right) \\ &\leq \frac{|\Omega|}{4e} + \frac{1}{2} \int_{\Omega} v(\cdot, t), \end{aligned}$$

which, together with (4.2), yields (4.1).  $\square$

**Lemma 4.4.** *Let  $(u, v, w)$  be a global classical solution of (1.1). Then if*

$$1 - \|w_0\|_{L^\infty(\Omega)} > \frac{\chi^2}{4\mu^2} + \xi^2 (\|\nabla w_0\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{4e} + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{2\mu}),$$

there exist  $\beta > 0$  and  $C_2^* > 0$  such that

$$\int_0^t \int_{\Omega} u(\cdot, s) ds \geq \beta t - C_2^* \text{ for all } t > 0. \quad (4.3)$$

**Proof.** According to the strong maximum principle and hypothesis  $u_0 \not\equiv 0$  in (1.2),  $u$  is positive in  $\bar{\Omega} \times (0, \infty)$ , and thereby we may multiply the first equation in (1.1) by  $\frac{1}{u}$  and use the Young inequality to obtain

$$\frac{d}{dt} \int_{\Omega} \ln u = \int_{\Omega} \frac{|\nabla u|^2}{u^2} - \chi \int_{\Omega} \frac{1}{u} \nabla u \cdot \nabla v - \xi \int_{\Omega} \frac{1}{u} \nabla u \cdot \nabla w + |\Omega| - \mu \int_{\Omega} u - \int_{\Omega} w$$

$$\geq -\frac{\chi^2}{2} \int_{\Omega} |\nabla v|^2 - \frac{\xi^2}{2} \int_{\Omega} |\nabla w|^2 + |\Omega| - \mu \int_{\Omega} u - |\Omega| \|w_0\|_{L^\infty(\Omega)}. \quad (4.4)$$

Let  $c^* = \frac{1}{2}(1 - \|w_0\|_{L^\infty(\Omega)} - \frac{\chi^2}{4\mu^2} - \xi^2(\|\nabla w_0\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{4e} + \frac{\|w_0\|_{L^\infty(\Omega)}^2}{2\mu}))$  and  $\varepsilon = \frac{4|\Omega|\mu c^*}{\chi^2 + 2\mu\xi^2\|w_0\|_{L^\infty(\Omega)}^2}$ . By Lemma 4.2, there exists  $t_2 > 0$  such that for all  $t \geq t_2$ ,

$$\int_{\Omega} u(\cdot, t) \leq \frac{|\Omega|}{\mu} + \varepsilon, \quad \int_{\Omega} v(\cdot, t) \leq \frac{|\Omega|}{\mu} + \varepsilon, \quad (4.5)$$

and thus inequality (3.1) gives

$$\begin{aligned} \mu \int_{t_2}^t \int_{\Omega} u^2(\cdot, s) ds &\leq \int_{t_2}^t \int_{\Omega} u(\cdot, s) ds + \int_{\Omega} u(\cdot, t_2) \\ &\leq \frac{|\Omega|}{\mu} + \varepsilon + \left(\frac{|\Omega|}{\mu} + \varepsilon\right)(t - t_2). \end{aligned}$$

Furthermore, multiplying the second equation in (1.1) by  $v$  and using the Young inequality, we obtain

$$\frac{d}{dt} \int_{\Omega} v^2(\cdot, t) + 2 \int_{\Omega} |\nabla v(\cdot, t)|^2 + \int_{\Omega} v^2(\cdot, t) \leq \int_{\Omega} u^2(\cdot, t)$$

and thereby

$$\begin{aligned} \int_{t_2}^t \int_{\Omega} |\nabla v(\cdot, s)|^2 ds &\leq \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{1}{2} \int_{t_2}^t \int_{\Omega} u^2(\cdot, s) ds \\ &\leq \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu} + \left(\frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu}\right)(t - t_2). \end{aligned} \quad (4.6)$$

Therefore, along with (4.1), (4.5) and (4.6), integrating (4.4) over  $(t_2, t)$  leads to

$$\begin{aligned} &\int_{\Omega} \ln u(\cdot, t) - \int_{\Omega} \ln u(\cdot, t_2) + \mu \int_{t_2}^t \int_{\Omega} u(\cdot, s) ds \\ &\geq |\Omega|(1 - \|w_0\|_{L^\infty(\Omega)})(t - t_2) - \frac{\chi^2}{2} \int_{t_2}^t \int_{\Omega} |\nabla v|^2 - \frac{\xi^2}{2} \int_{t_2}^t \int_{\Omega} |\nabla w|^2 \\ &\geq (2|\Omega|c^* - \frac{\varepsilon\chi^2}{4\mu} - \frac{\varepsilon\xi^2}{2}\|w_0\|_{L^\infty(\Omega)}^2)(t - t_2) - \frac{\chi^2}{2} \left\{ \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu} \right\} \end{aligned}$$

for all  $t > t_2$ .

By the special choice of  $\varepsilon$  and the non-negativity of  $u$ , we have

$$\begin{aligned}
 \mu \int_0^t \int_{\Omega} u(\cdot, s) ds &\geq |\Omega| c^*(t - t_2) + \int_{\Omega} \ln u(\cdot, t_2) - \int_{\Omega} \ln u(\cdot, t) \\
 &\quad - \frac{\chi^2}{2} \left\{ \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu} \right\} \\
 &\geq |\Omega| c^*(t - t_2) + \int_{\Omega} \ln u(\cdot, t_2) - \int_{\Omega} u(\cdot, t) \\
 &\quad - \frac{\chi^2}{2} \left\{ \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu} \right\} \\
 &\geq |\Omega| c^*(t - t_2) + \int_{\Omega} \ln u(\cdot, t_2) - \frac{|\Omega|}{\mu} - \varepsilon \\
 &\quad - \frac{\chi^2}{2} \left\{ \frac{1}{2} \int_{\Omega} v^2(\cdot, t_2) + \frac{|\Omega|}{2\mu^2} + \frac{\varepsilon}{2\mu} \right\} \\
 &\geq |\Omega| c^* t - C_3^*.
 \end{aligned}$$

On the other hand, when  $t \leq t_2$ , by the non-negativity of  $u$ , we have

$$\mu \int_0^t \int_{\Omega} u(\cdot, s) ds \geq 0 \geq |\Omega| c^* t - |\Omega| c^* t_2.$$

Therefore, (4.3) is valid with

$$\beta := \frac{|\Omega| c^*}{\mu}$$

and

$$C_2^* := \max\{\beta t_2, \frac{C_3^*}{\mu}\}.$$

Combining the information above with (3.4) and (3.5), we can easily obtain an exponential decay of  $w$ .  $\square$

**Proof of Theorem 1.2.** From Lemma 4.1 and Lemma 4.4, we know that

$$\int_0^t v(x, s) ds \geq \Gamma \cdot \int_0^t \int_{\Omega} u(y, s) dy ds - C_1^* \geq \Gamma \beta t - \Gamma C_2^* - C_1^* \quad \text{for all } x \in \Omega, t > 0, \quad (4.7)$$

which, along with (3.4), implies that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4^* e^{-2\alpha t} \quad (4.8)$$

with  $\alpha = \frac{\Gamma\beta}{2}$  and  $C_4^* = \|w_0\|_{L^\infty(\Omega)} e^{\Gamma C_2^* + C_1^*}$ .

On the other hand, from Theorem 1.1 and Lemma 3.5, it follows that there exists  $B > 0$  such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq B \quad \text{and} \quad \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \leq B \quad \text{for all } t \geq 0. \quad (4.9)$$

Now in light of (3.5), (4.7) and (4.9), we have

$$\begin{aligned} \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} &\leq \|\nabla w_0\|_{L^\infty(\Omega)} e^{\Gamma C_2^* + C_1^*} e^{-2\alpha t} + \|w_0\|_{L^\infty(\Omega)} B e^{\Gamma C_2^* + C_1^*} e^{-2\alpha t} \\ &\leq C e^{-\alpha t} \quad \text{for all } t > 0, \end{aligned} \quad (4.10)$$

where  $C = e^{\Gamma C_2^* + C_1^*} (\|\nabla w_0\|_{L^\infty(\Omega)} + \frac{B}{e\alpha} \|w_0\|_{L^\infty(\Omega)})$  and we have used the elementary inequality  $ze^{-z} \leq \frac{1}{e}$  for all  $z \geq 0$ . Hence (1.7) results from (4.8) and (4.10).

In order to show the global asymptotic stability of  $(\frac{1}{\mu}, \frac{1}{\mu}, 0)$ , we introduce

$$U(x, t) = u(x, t) - \frac{1}{\mu} \quad \text{and} \quad V(x, t) = v(x, t) - \frac{1}{\mu} \quad \text{for } x \in \Omega, t \geq 0$$

throughout the sequel. Accordingly,  $(U, V)$  satisfies

$$\begin{cases} U_t = \Delta U - \chi \nabla \cdot (u \nabla V) - \xi \nabla \cdot (u \nabla w) - U - \mu U^2 - uw, & x \in \Omega, t > 0, \\ V_t = \Delta V - V + U, & x \in \Omega, t > 0, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ U(x, 0) = u_0(x) - \frac{1}{\mu}, \quad V(x, 0) = v_0(x) - \frac{1}{\mu}, & x \in \Omega. \end{cases} \quad (4.11)$$

For the proof of Theorem 1.3 below, we fix any number  $\lambda \in (0, 1)$  and define the linear operator  $A = -\Delta + \lambda$  in  $L^p(\Omega)$  ( $p > 1$ ) with the domain  $D(A) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0\}$ . Then it is well-known [15] that  $A$  is sectorial and possesses closed fractional powers  $A^\kappa$  for any  $\kappa > 0$ , and the corresponding domains  $D(A^\kappa)$  have the embedding property

$$D(A^\kappa) \hookrightarrow W^{2,\infty}(\Omega) \quad \text{if} \quad 2\kappa - \frac{n}{p} > 2. \quad (4.12)$$

Moreover, if  $(e^{-tA})_{t \geq 0}$  denotes the corresponding analytical semigroup, then for  $p > 1$ , there exists  $K(p, \kappa)$  such that

$$\|A^\kappa e^{-tA} \varphi\|_{L^p(\Omega)} \leq K(p, \kappa) t^{-\kappa} \|\varphi\|_{L^p(\Omega)} \quad \text{for } t > 0 \text{ and } \varphi \in L^p(\Omega). \quad (4.13)$$

Since the higher regularity estimates for  $v$  are involved in the proof of Theorem 1.3 below, we establish the following pointwise estimate for  $\Delta v$  by using the parabolic regularization argument.  $\square$

**Lemma 4.5.** Suppose that the assumptions of [Theorem 1.2](#) hold and  $(u, v, w)$  is a global solution of [\(1.1\)](#), then there exists  $C > 0$  such that for all  $t > 0$ ,

$$\limsup_{t \rightarrow \infty} \|v(\cdot, t)\|_{W^{2,\infty}(\Omega)} \leq C. \quad (4.14)$$

**Proof.** For any  $\beta \in (0, \frac{1}{2})$ , according to a variation-of-constants formula associated with the first equation in [\(1.1\)](#), we have

$$\begin{aligned} u(\cdot, t) &= e^{(t-t_0)(\Delta-1)}u(\cdot, t_0) - \chi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) ds \\ &\quad - \xi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s)) ds \\ &\quad + \int_{t_0}^t e^{(t-s)(\Delta-1)} u(\cdot, s) (2 - \mu u(\cdot, s) - w(\cdot, s)) ds \end{aligned}$$

and thus

$$\begin{aligned} \|A^\beta u(\cdot, t)\|_{L^p(\Omega)} &\leq e^{-(t-t_0)(1-\lambda)} \|A^\beta e^{-(t-t_0)A} u(\cdot, t_0)\|_{L^p(\Omega)} \\ &\quad + \chi \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \|A^\beta e^{-\frac{t-s}{2}A} e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^p(\Omega)} ds \\ &\quad + \xi \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \|A^\beta e^{-\frac{t-s}{2}A} e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s))\|_{L^p(\Omega)} ds \\ &\quad + \int_{t_0}^t e^{-(t-s)(1-\lambda)} \|A^\beta e^{-(t-s)A} u(\cdot, s) (2 - \mu u(\cdot, s) - w(\cdot, s))\|_{L^p(\Omega)} ds \end{aligned} \quad (4.15)$$

for all  $t > t_0$ .

Here by [\(4.13\)](#), we have

$$\begin{aligned} &e^{-(t-t_0)(1-\lambda)} \|A^\beta e^{-(t-t_0)A} u(\cdot, t_0)\|_{L^p(\Omega)} \\ &\leq K(p, \beta) (t - t_0)^{-\beta} e^{-(t-t_0)(1-\lambda)} \|u(\cdot, t_0)\|_{L^p(\Omega)}. \end{aligned} \quad (4.16)$$

By [Lemma 3.5](#), [Lemma 3.6](#) and [\(4.13\)](#), we get

$$\chi \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \|A^\beta e^{-\frac{t-s}{2}A} e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^p(\Omega)} ds$$

$$\begin{aligned}
&\leq \chi K(p, \beta) \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \left(\frac{t-s}{2}\right)^{-\beta} \|e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s))\|_{L^p(\Omega)} ds \\
&\leq \chi K(p, \beta) c_2(p) \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \left(\frac{t-s}{2}\right)^{-\beta} \left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right) \|u(\cdot, s) \nabla v(\cdot, s)\|_{L^p(\Omega)} ds \\
&\leq \chi K(p, \beta) c_2(p) \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \left(\frac{t-s}{2}\right)^{-\beta} \left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right) \|u(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla v(\cdot, s)\|_{L^p(\Omega)} ds \\
&\leq \chi K(p, \beta) c_2(p) C \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \left(\frac{t-s}{2}\right)^{-\beta} \left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right) ds \\
&= \chi K(p, \beta) c_2(p) C \int_0^\infty e^{-(1-\frac{\lambda}{2})\sigma} \left(\frac{\sigma}{2}\right)^{-\beta} \left(1 + \left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right) d\sigma.
\end{aligned} \tag{4.17}$$

By Lemma 3.6 and Theorem 1.2, we can also get the following result

$$\begin{aligned}
&\xi \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \|A^\beta e^{-\frac{t-s}{2}A} e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s))\|_{L^p(\Omega)} ds \\
&\leq \xi K(p, \beta) c_2(p) \int_{t_0}^t e^{-(t-s)(1-\frac{\lambda}{2})} \left(\frac{t-s}{2}\right)^{-\beta} \left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right) \|u(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla w(\cdot, s)\|_{L^p(\Omega)} ds \\
&\leq \xi K(p, \beta) c_2(p) C \int_0^\infty e^{-(1-\frac{\lambda}{2})\sigma} \left(\frac{\sigma}{2}\right)^{-\beta} \left(1 + \left(\frac{\sigma}{2}\right)^{-\frac{1}{2}}\right) d\sigma.
\end{aligned} \tag{4.18}$$

In addition, by Lemma 3.6 and (3.5), we have

$$\begin{aligned}
&\int_{t_0}^t e^{-(t-s)(1-\lambda)} \|A^\beta e^{-(t-s)A} u(\cdot, s) (2 - \mu u(\cdot, s) - w(\cdot, s))\|_{L^p(\Omega)} ds \\
&\leq K(p, \beta) \int_{t_0}^t e^{-(t-s)(1-\lambda)} (t-s)^{-\beta} \|u(\cdot, s) (2 - \mu u(\cdot, s) - w(\cdot, s))\|_{L^p(\Omega)} ds \\
&\leq K(p, \beta) C \int_0^\infty e^{-\sigma(1-\lambda)} \sigma^{-\beta} d\sigma.
\end{aligned} \tag{4.19}$$

Collecting (4.15)–(4.19), we find that

$$\limsup_{t \rightarrow \infty} \|A^\beta u(\cdot, t)\|_{L^p(\Omega)} \leq C.$$

Now according to a variation-of-constants formula associated with the second equation in (1.1), we can write

$$\begin{aligned} v(\cdot, t) &= e^{(t-t_0)(\Delta-1)} v(\cdot, t_0) + \int_{t_0}^t e^{(t-s)(\Delta-1)} u(\cdot, s) ds \\ &= e^{-(1-\lambda)(t-t_0)} e^{-(t-t_0)A} v(\cdot, t_0) + \int_{t_0}^t e^{-(1-\lambda)(t-s)} e^{-(t-s)A} u(\cdot, s) ds \end{aligned}$$

and hence use (4.12) to estimate

$$\begin{aligned} \|v(\cdot, t)\|_{W^{2,\infty}(\Omega)} &\leq C \|A^\kappa v(\cdot, s)\|_{L^p(\Omega)} \\ &\leq C e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} v(\cdot, t_0)\|_{L^p(\Omega)} \\ &\quad + C \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^\kappa e^{-(t-s)A} u(\cdot, s)\|_{L^p(\Omega)} ds \\ &\leq C e^{-(1-\lambda)(t-t_0)} \|A^\kappa e^{-(t-t_0)A} v(\cdot, t_0)\|_{L^p(\Omega)} \\ &\quad + C \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^{\kappa-\beta} e^{-(t-s)A} A^\beta u(\cdot, s)\|_{L^p(\Omega)} ds \end{aligned}$$

with  $\kappa \in (1, \frac{3}{2})$ . Therefore by using (4.13), we can verify (4.14) and refer the reader to the proof of [46, Lemma 5.1] for the details.  $\square$

**Proof of Theorem 1.3.** By Theorem 1.2, we need only to show (1.8) and (1.9). Since the proof proceeds along the alternative reasoning of Theorem 1.1 in [46], we give details only in places which are characteristic of the present setting. In particular, the treatment of haptotaxis term in the first equation of (1.1). For clarity, the proof is divided into following several steps.

Step 1: When  $\rho := \frac{n\chi}{4\mu} < 1$ ,

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{1}{(1-\rho)\mu}. \quad (4.20)$$

According to (4.11) and using  $\nabla V \cdot \nabla \Delta V = \frac{1}{2} \Delta |\nabla V|^2 - |D^2 V|^2$ , one can find that  $z(x, t) := U(x, t) + \frac{\chi}{2} |\nabla V(x, t)|^2$  satisfies

$$z_t = \Delta z - \chi |D^2 V|^2 - \chi u \Delta V - U - \mu U^2 - uw - \chi |\nabla V|^2 - \xi \nabla U \cdot \nabla w - \xi u \Delta w.$$

Now since  $-\chi u \Delta V \leq \chi |D^2 V|^2 + \frac{n\chi}{4} u^2$ , we further get

$$\begin{aligned} z_t - \Delta z + z &\leq \frac{n\chi}{4} u^2 - \mu U^2 - uw - \frac{\chi}{2} |\nabla V|^2 - \xi \nabla U \cdot \nabla w - \xi u \Delta w \\ &\leq \frac{n\chi}{4} u^2 - \mu U^2 - uw - \xi u \Delta w - \xi \nabla(z - \frac{\chi}{2} |\nabla V|^2) \cdot \nabla w \\ &= -\xi \nabla z \cdot \nabla w + \frac{n\chi}{4} u^2 - \mu U^2 - uw - \xi u \Delta w + \frac{\xi\chi}{2} \nabla(|\nabla V|^2) \cdot \nabla w \\ &\leq -\xi \nabla z \cdot \nabla w + \frac{n\chi}{\mu(4\mu - n\chi)} - uw - \xi u \Delta w + \frac{\xi\chi}{2} \nabla(|\nabla V|^2) \cdot \nabla w. \end{aligned} \quad (4.21)$$

On the other hand, from (3.4), it follows that

$$\begin{aligned} \Delta w(x, t) &= \Delta w_0(x) e^{-\int_0^t v(x,s) ds} - 2e^{-\int_0^t v(x,s) ds} \nabla w_0(x) \cdot \int_0^t \nabla v(x, s) ds \\ &\quad + w_0(x) e^{-\int_0^t v(x,s) ds} \left| \int_0^t \nabla v(x, s) ds \right|^2 - w_0(x) e^{-\int_0^t v(x,s) ds} \int_0^t \Delta v(x, s) ds. \end{aligned}$$

Hence by Lemma 4.5 and done as in Theorem 1.2, we can conclude that there exists some  $C > 0$  such that

$$\|w(\cdot, t)\|_{W^{2,\infty}(\Omega)} \leq C e^{-\alpha t} \quad (4.22)$$

for all  $t > 0$ . Furthermore, by Lemma 3.6, Lemma 4.5 and (4.22), we can see that

$$\limsup_{t \rightarrow \infty} \|uw + \xi u \Delta w - \frac{\xi\chi}{2} \nabla(|\nabla V|^2) \cdot \nabla w\|_{L^\infty(\Omega)} \leq C e^{-\alpha t}. \quad (4.23)$$

From (4.21) and (4.23), it follows that for any  $\varepsilon > 0$ , there exists  $t_3 := t_3(u, v, w) > 0$  such that for all  $t \geq t_3$ ,

$$z_t - \Delta z + \xi \nabla z \cdot \nabla w + z \leq \frac{n\chi}{\mu(4\mu - n\chi)} + \varepsilon.$$

Noticing that since  $\Omega$  is convex and  $\frac{\partial V}{\partial \nu} = 0$  on  $\partial\Omega$ ,  $\frac{\partial |\nabla V|^2}{\partial \nu} \leq 0$  on  $\partial\Omega$  by a well-known result [24] and hence  $\frac{\partial z}{\partial \nu} \leq 0$  on  $\partial\Omega$ . Therefore by a comparison principle, we have

$$z(x, t) \leq y(t) \quad \text{for all } x \in \Omega, t \geq t_3, \quad (4.24)$$

where  $y(t)$  is the solution of the initial-value problem

$$\begin{cases} y'(t) + y(t) = \frac{n\chi}{\mu(4\mu - n\chi)} + \varepsilon, \\ y(t_3) = \|z(\cdot, t_3)\|_{L^\infty(\Omega)}. \end{cases} \quad (4.25)$$



Upon the explicit solution to problem (4.25), we can see that

$$\lim_{t \rightarrow \infty} y(t) = \frac{n\chi}{\mu(4\mu - n\chi)} + \varepsilon. \quad (4.26)$$

Combining (4.26) with (4.24) yields

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} &\leq \frac{1}{\mu} + \lim_{t \rightarrow \infty} y(t) \\ &= \frac{1}{(1-\rho)\mu} + \varepsilon \end{aligned}$$

with  $\rho = \frac{n\chi}{4\mu}$  and hence

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{1}{(1-\rho)\mu}$$

by the arbitrariness of  $\varepsilon$ .

Step 2: For any  $p > 1$ , there exists  $C(p) > 0$  such that  $\limsup_{t \rightarrow \infty} \|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leq \frac{C(p)}{(1-\rho)\mu}$ .

Since the equation for  $V$  in (4.11) and the second equation in (2.3) of [46] are identical, they have the same estimate for  $\nabla v$ .

Step 3: For all  $p > 1$  and  $\beta \in (0, \frac{1}{2})$ , there exists  $C(p, \beta) > 0$  such that

$$\limsup_{t \rightarrow \infty} \|A^\beta U(\cdot, t)\|_{L^p(\Omega)} \leq \frac{C(p, \beta)}{(1-\rho)^2\mu}.$$

The proof is very similar to that of Lemma 4.2 of [46] and Lemma 4.5, hence we omit it here.

Step 4: There exists constant  $c_8 > 0$  such that

$$\limsup_{t \rightarrow \infty} \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{c_8}{(1-\rho)^2\mu}. \quad (4.27)$$

As for its proof, we refer the reader to that of Lemma 5.1 in [46].

Step 5: There exists constant  $c_9 > 0$  such that

$$\liminf_{t \rightarrow \infty} (\inf_{x \in \Omega} u(x, t)) \geq \frac{1}{\mu} - \frac{c_9\rho}{(1-\rho)^2\mu}. \quad (4.28)$$

By Theorem 1.2 and the estimate (4.27), so that for any  $\varepsilon > 0$  we can find  $t_4 > 0$  such that

$$\|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{2c_8}{(1-\rho)^2\mu} \quad \text{and} \quad \|w(\cdot, t)\|_{L^\infty(\Omega)} < \varepsilon$$

for all  $t \geq t_4$ . Therefore, from the first equation in (1.1), it follows that

$$\begin{aligned} u_t &= \Delta u - \chi \nabla u \cdot \nabla v - \chi u \Delta v + u - \mu u^2 - u w \\ &\geq \Delta u - \chi \nabla u \cdot \nabla v - \chi u \frac{2c_8}{(1-\rho)^2\mu} - \mu u^2 - \varepsilon u \end{aligned}$$

$$= \Delta u - \chi \nabla u \cdot \nabla v + \left(1 - \frac{2c_8\chi}{(1-\rho)^2\mu} - \varepsilon\right)u - \mu u^2 \quad \text{for } x \in \Omega \text{ and } t > t_4.$$

Thereupon, the comparison principle asserts that

$$u(x, t) \geq y(t) \quad \text{for all } x \in \Omega \text{ and } t > t_4,$$

where  $y(t)$  is the solution of the initial-value problem

$$\begin{cases} y'(t) = \left(1 - \frac{2c_8\chi}{(1-\rho)^2\mu} - \varepsilon\right)y - \mu y^2, \\ y(t_3) = \inf_{x \in \Omega} u(x, t_4) > 0, \end{cases}$$

and hence (4.28) is valid with  $c_9 = \frac{8c_8}{n}$  by the arbitrariness of  $\varepsilon$ .

Step 6: Combining (4.20) with (4.28), one can find

$$\limsup_{t \rightarrow \infty} \|U(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{c_{10}\rho}{(1-\rho)^2\mu}$$

with some  $c_{10} > 0$ . We refer the reader to the proof of Corollary 6.2 in [46] for the details.

Step 7: For any fixed  $\gamma \in (0, \min\{\alpha, 1\})$  with  $\alpha > 0$  given by Theorem 1.2, there exists  $\rho_1 = \rho_1(\gamma)$  such that if  $\rho := \frac{n\chi}{4\mu} < \rho_1$ , one can find  $C > 0$  such that  $U(x, t)$  satisfies

$$\|U(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\gamma t}$$

for all  $t > 0$ .

According to a variation-of-constants formula associated with the first equation in (4.11), we get

$$\begin{aligned} U(\cdot, t) &= e^{(t-t_0)(\Delta-1)} U(\cdot, t_0) - \chi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla V(\cdot, s)) ds \\ &\quad - \xi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s)) ds - \mu \int_{t_0}^t e^{(t-s)(\Delta-1)} U^2(\cdot, s) ds \\ &\quad - \int_{t_0}^t e^{(t-s)(\Delta-1)} u(\cdot, s) w(\cdot, s) ds. \end{aligned}$$

Hence according to the proof of Lemma 7.1 in [46], it is sufficient to show that for all  $t \geq t_0$ ,

$$\|\xi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s)) ds\|_{L^\infty(\Omega)} \leq C e^{-\gamma(t-t_0)}$$

and

$$\left\| \int_{t_0}^t e^{(t-s)(\Delta-1)} u(\cdot, s) w(\cdot, s) ds \right\|_{L^\infty(\Omega)} \leq C e^{-\gamma(t-t_0)}$$

with some constant  $C > 0$ . Indeed, from (1.7), (2.7) and (4.20), it follows that for any fixed  $p > n$ ,

$$\begin{aligned} & \left\| \xi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s)) ds \right\|_{L^\infty(\Omega)} \\ & \leq k_3 \xi \int_{t_0}^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}) e^{-(t-s)} \|u(\cdot, s) \nabla w(\cdot, s)\|_{L^p(\Omega)} ds \\ & \leq \frac{c_{11} \xi}{(1-\rho)\mu} \int_{t_0}^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}) e^{-(t-s)} e^{-\gamma(s-t_0)} ds \\ & = \frac{c_{11} \xi}{(1-\rho)\mu} \left( \int_0^{t-t_0} (1 + \sigma^{-\frac{1}{2}-\frac{n}{2p}}) e^{-(1-\gamma)\sigma} d\sigma \right) e^{-\gamma(t-t_0)} \\ & \leq \frac{c_{11} \xi}{(1-\rho)\mu} \left( \int_0^\infty (1 + \sigma^{-\frac{1}{2}-\frac{n}{2p}}) e^{-(1-\gamma)\sigma} d\sigma \right) e^{-\gamma(t-t_0)}. \end{aligned}$$

Next, by the maximum principle together with (1.7) and (4.20), we can see that

$$\begin{aligned} & \left\| \int_{t_0}^t e^{(t-s)(\Delta-1)} u(\cdot, s) w(\cdot, s) ds \right\|_{L^\infty(\Omega)} \\ & \leq \int_{t_0}^t e^{-(t-s)} \|u(\cdot, s) w(\cdot, s)\|_{L^\infty(\Omega)} ds \\ & \leq \frac{c_{12}}{(1-\rho)\mu} \int_{t_0}^t e^{-(t-s)} e^{-\gamma(s-t_0)} ds \\ & \leq \frac{c_{12}}{(1-\rho)\mu} \left( \int_0^\infty e^{-(1-\gamma)\sigma} d\sigma \right) e^{-\gamma(t-t_0)}. \end{aligned}$$

Step 8: Since the equation for  $V$  in (4.11) and in (2.3) of [46] is identical, the proof of (1.9) is the same as that of (1.6) of Theorem 1.1 in [46] and thereby omitted here.  $\square$

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## References

- [1] N.D. Alikakos,  $L^p$  bounds of solutions of reaction–diffusion equations, *Comm. Partial Differential Equations* 4 (1979) 827–868.
- [2] S. Aznavoorian, M.L. Stracke, H. Krutzsch, E. Schiffmann, L.A. Liotta, Signal transduction for chemotaxis and haptotaxis by matrix molecules in tumor cells, *J. Cell Biol.* 110 (1990) 1427–1438.
- [3] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.* 25 (2015) 1663–1763.
- [4] D. Besser, P. Verde, Y. Nagamine, F. Blasi, Signal transduction and u-PA/u-PAR system, *Fibrinolysis* 10 (1996) 215–237.
- [5] F. Blasi, J.-D. Vassalli, K. Danø, Urokinase-type plasminogen activator: proenzyme, receptor, and inhibitors, *J. Cell Biol.* 104 (1987) 801–804.
- [6] X. Cao, Boundedness in a three-dimensional chemotaxis–haptotaxis model, *arXiv:1501.05383*, 2015.
- [7] M.A.J. Chaplain, G. Lolas, Mathematical modelling of tissue invasion: dynamic heterogeneity, *Netw. Heterog. Media* 1 (2006) 399–439.
- [8] M.A.J. Chaplain, G. Lolas, Mathematical modelling of cancer invasion of tissue: the role of the urokinase plasminogen activation system, *Math. Models Methods Appl. Sci.* 15 (2005) 1685–1734.
- [9] T. Cieślak, Quasilinear nonuniformly parabolic system modelling chemotaxis, *J. Math. Anal. Appl.* 326 (2007) 1410–1426.
- [10] T. Cieślak, C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions, *J. Differential Equations* 252 (2012) 5832–5851.
- [11] T. Cieślak, M. Winkler, Finite-time blow-up in a quasilinear system of chemotaxis, *Nonlinearity* 21 (2008) 1057–1076.
- [12] A. Friedman, *Partial Differential Equations*, Holt–Rinehart–Winston, 1969.
- [13] A. Friedman, G. Lolas, Analysis of a mathematical model of tumor lymphangiogenesis, *Math. Models Methods Appl. Sci.* 15 (2005) 95–107.
- [14] B. Guo, *Viscosity Methods and the Viscosity of Difference Schemes*, Science Press, Beijing, 1993.
- [15] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 1981.
- [16] M. Herrero, J. Velázquez, A blow-up mechanism for a chemotaxis model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 24 (4) (1997) 633–683.
- [17] T. Hillen, K.J. Painter, A user’s guide to PDE models for chemotaxis, *J. Math. Biol.* 58 (2009) 183–217.
- [18] T. Hillen, K.J. Painter, M. Winkler, Convergence of a cancer invasion model to a logistic chemotaxis model, *Math. Models Methods Appl. Sci.* 23 (2013) 165–198.
- [19] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences, I, *Jahresber. Dtsch. Math.-Ver.* 105 (2003) 103–165.
- [20] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations* 215 (1) (2005) 52–107.
- [21] S. Ishida, K. Seki, T. Yokota, Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains, *J. Differential Equations* 256 (2014) 2993–3010.
- [22] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.* 26 (1970) 399–415.
- [23] D.A. Lauffenburger, A.F. Horwitz, Cell migration: a physically integrated molecular process, *Cell* 84 (1996) 359–369.
- [24] P.L. Lions, Résolution de problèmes elliptiques quasilinéaires, *Arch. Ration. Mech. Anal.* 74 (1980) 335–353.
- [25] G. Liñanu, C. Morales-Rodrigo, Asymptotic behavior of global solutions to a model of cell invasion, *Math. Models Methods Appl. Sci.* 20 (2010) 1721–1758.
- [26] A. Marciniak-Czochra, M. Ptashnyk, Boundedness of solutions of a haptotaxis model, *Math. Models Methods Appl. Sci.* 20 (2010) 449–476.

- [27] C. Morales-Rodrigo, J. Tello, Global existence and asymptotic behavior of a tumor angiogenesis model with chemotaxis and haptotaxis, *Math. Models Methods Appl. Sci.* 24 (2014) 427–464.
- [28] T. Nagai, Blow-up of nonradial solutions to parabolic–elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.* 6 (2001) 37–55.
- [29] K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, *Nonlinear Anal.* 51 (2002) 119–144.
- [30] K.J. Painter, T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, *Can. Appl. Math. Q.* 10 (2002) 501–543.
- [31] C. Stinner, C. Surulescu, M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.* 46 (2014) 1969–2007.
- [32] Z. Szymańska, C. Morales-Rodrigo, M. Lachowicz, M. Chaplain, Mathematical modelling of cancer invasion of tissue: the role and effect of nonlocal interactions, *Math. Models Methods Appl. Sci.* 19 (2009) 257–281.
- [33] Y. Tao, Boundedness in a two-dimensional chemotaxis–haptotaxis system, *arXiv:1407.7382*, 2014.
- [34] Y. Tao, Global existence of classical solutions to a combined chemotaxis–haptotaxis model with logistic source, *J. Math. Anal. Appl.* 354 (2009) 60–69.
- [35] Y. Tao, M. Wang, A combined chemotaxis–haptotaxis system: the role of logistic source, *SIAM J. Math. Anal.* 41 (2009) 1533–1558.
- [36] Y. Tao, M. Wang, Global solution for a chemotactic–haptotactic model of cancer invasion, *Nonlinearity* 21 (2008) 2221–2238.
- [37] Y. Tao, M. Winkler, Dominance of chemotaxis in a chemotaxis–haptotaxis model, *Nonlinearity* 27 (2014) 1225–1239.
- [38] Y. Tao, M. Winkler, Boundedness and stabilization in a multi-dimensional chemotaxis–aptotaxis model, *Proc. Roy. Soc. Edinburgh Sect. A* 144 (2014) 1067–1084.
- [39] Y. Tao, M. Winkler, A chemotaxis–haptotaxis model: the roles of nonlinear diffusion and logistic source, *SIAM J. Math. Anal.* 43 (2011) 685–704.
- [40] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, *J. Differential Equations* 252 (2012) 692–715.
- [41] C. Walker, G.F. Webb, Global existence of classical solutions for a haptotaxis model, *SIAM J. Math. Anal.* 38 (2007) 1694–1713.
- [42] M. Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.* 348 (2008) 708–729.
- [43] M. Winkler, Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations* 35 (2010) 1516–1537.
- [44] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, *J. Differential Equations* 248 (2010) 2889–2905.
- [45] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, *J. Math. Anal. Appl.* 384 (2011) 261–272.
- [46] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Differential Equations* 257 (2014) 1056–1077.
- [47] C. Yang, X. Cao, Z. Jiang, S. Zheng, Boundedness in a quasilinear fully parabolic Keller–Segel system of higher dimension with logistic source, *J. Math. Anal. Appl.* 430 (2015) 585–591.