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On center singularity for compressible spherically symmetric nematic liquid crystal flows

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Abstract

The paper is concerned with a simplified system, proposed by Ericksen [6] and Leslie [20], modeling the flow of nematic liquid crystals. In the first part, we give a new Serrin's continuation principle for strong solutions of general compressible liquid crystal flows. Based on new observations, we establish a localized Serrin's regularity criterion for the 3D compressible spherically symmetric flows. It is proved that the classical solution loses its regularity in finite time if and only if, either the concentration or vanishing of mass forms or the norm inflammation of gradient of orientation field occurs around the center.

MSC: 35Q35; 35B65; 76N10

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1. Introduction

In this paper, we consider the motion of the compressible nematic liquid crystal flows, which is described by the following simplified Ericksen–Leslie system,

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$$\begin{cases} \rho_t + \mathrm{div}(\rho U) = 0, \\ \rho U_t + \rho(U\cdot\nabla)U - \mu\Delta U - (\mu+\lambda)\nabla(\mathrm{div}U) + \nabla P = -\Delta d\cdot\nabla d, \\ d_t + (U\cdot\nabla)d = \Delta d + |\nabla d|^2 d. \end{cases} \tag{1.1}$$

Here $\rho \geq 0$ is the density of the fluid and U is the velocity field. d denotes the macroscopic average of the nematic liquid crystal orientation field, which conforms to $|d| = 1$. $P = P(\rho)$ is the pressure of the fluid, which is a function of the density. The equation of state is given by

$$P(\rho) = a\rho^\gamma, \quad a > 0,\ \gamma > 1. \tag{1.2}$$

The constants μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid respectively, and they satisfy the following physical conditions,

$$\mu > 0, \qquad 3\lambda + 2\mu \geq 0. \tag{1.3}$$

The three equations in (1.1) are the equations for conservation of mass, linear momentum and angular momentum respectively.

The domain Ω is a bounded ball with radius R, namely,

$$\Omega = B_R = \{x \in \mathbb{R}^3;\ |x| \leq R < \infty\}.$$

We study an initial boundary value problem for (1.1) with the initial condition

$$(\rho, U, d)(0, x) = (\rho_0, U_0, d_0)(x), \quad x \in \Omega, \tag{1.4}$$

and the boundary condition

$$U(t, x) = 0, \quad \frac{\partial d}{\partial \vec{n}}(t, x) = 0, \quad t \geq 0,\ x \in \partial\Omega. \tag{1.5}$$

And we are looking for the smooth spherically symmetric solution (ρ, U, d) of the problem (1.1)–(1.5) which enjoys the form

$$\rho(t, x) = \rho(t, |x|), \quad U(t, x) = u(t, |x|)\frac{x}{|x|}, \quad d(t, x) = d(t, |x|). \tag{1.6}$$

Then, for the initial data to be consistent with the form (1.7), we assume the initial data (ρ_0, U_0, d_0) also takes the form

$$\rho_0(x) = \rho_0(|x|), \quad U_0(x) = u_0(|x|)\frac{x}{|x|}, \quad d_0(x) = d_0(|x|). \tag{1.7}$$

In this paper, we further assume the initial density is uniformly positive, that is,

$$\rho_0(x) = \rho_0(|x|) \geq \underline{\rho} > 0, \quad x \in \Omega, \tag{1.8}$$

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for a positive constant $\underline{\rho}$. Then it is noted that as long as the classical solution of (1.1)–(1.5) exists the density ρ is positive, that is, the vacuum never occurs. It is also noted that the assumption (1.6) implies

$$U(t,x) + U(t,-x) = 0, \quad x \in \Omega, \tag{1.9}$$

and ∇d follows the same principle. We necessarily have $U(t,0) = 0$ and $\nabla d(t,0) = 0$.

In the spherical coordinates, the original system (1.1) under the assumption (1.6) takes the form

$$\begin{cases} \rho_t + (\rho u)_r + m\dfrac{\rho u}{r} = 0, \\ (\rho u)_t + \left[\rho u^2 + P(\rho)\right]_r + m\dfrac{\rho u^2}{r} = \kappa\left(u_r + m\dfrac{u}{r}\right)_r - \dfrac{1}{2}(|d_r|^2)_r - \dfrac{m}{r}|d_r|^2, \\ d_t + ud_r = d_{rr} + |d_r|^2 d + \dfrac{m}{r} d_r, \end{cases} \tag{1.10}$$

where $m = N - 1$ and $\kappa = 2\mu + \lambda$.

In this paper, we will focus on the following three-dimensional ($N = 3$) initial boundary value problem for the system (1.10). The initial condition is given as

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \tag{1.11}$$

and the boundary condition is

$$u(t,r) = 0, \; d_r(t,r) = 0 \text{ for } r \in \{0, R\}. \tag{1.12}$$

Now, we consider the following Lagrangian transformation:

$$t = t, \quad y = \int_0^r \rho(t,\tau)\,\tau^m\, d\tau. \tag{1.13}$$

Then, it follows from (1.10) and (1.13) that

$$y_t = \int_0^r \frac{\partial \rho}{\partial t}(t,\tau)\cdot \tau^m\, d\tau = -\int_0^r \left[\frac{\partial(\rho u)}{\partial \tau} + m\frac{\rho u}{\tau}\right]\cdot \tau^m\, d\tau = -\rho u r^m, \tag{1.14}$$

and $y_r = \rho(t,r) r^m$. Consequently,

$$r_t = u, \qquad r_y = (\rho r^m)^{-1}. \tag{1.15}$$

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Hence, the system (1.10) can be further reduced to

$$\begin{cases} \rho_t + \rho^2 (r^m u)_y = 0, \\ r^{-m} u_t + p_y = \kappa \left[\rho (r^m u)_y \right]_y - \frac{1}{2} \left(r^{2m} \rho^2 |d_y|^2 \right)_y - m r^{m-1} \rho |d_y|^2, \\ d_t = r^m \rho \left(r^m \rho d_y \right)_y + m r^{m-1} \rho d_y + r^{2m} \rho^2 |d_y|^2 d, \end{cases} \tag{1.16}$$

where $t \geq 0$, $y \in [0, M_0]$ and M_0 is defined by

$$M_0 = \int_0^R \rho_0(r) r^{N-1} dr = \int_0^R \rho(t, r) r^{N-1} dr, \tag{1.17}$$

according to the conservation of mass. We denote by E_0 the initial energy

$$E_0 = \int_0^R r^{N-1} \left(\rho_0 \frac{u_0^2}{2} + \frac{a \rho_0^\gamma}{\gamma - 1} + \frac{|(d_0)_r|^2}{2} \right) dr, \tag{1.18}$$

and define a rectangle $Q_{t,y}$, for $t \geq 0$ and $y \in [0, M_0]$, as

$$Q_{t,y} = [0, t] \times [y, M_0]. \tag{1.19}$$

There are some special related cases. When the density ρ is initially constant and the fluid is incompressible, then the density remains constant and consequently the whole system (1.1) becomes the incompressible version. For this case, Lin [21], Lin–Liu [23,24] initiated the rigorous mathematical analysis. For some more recent results, please refer to [9,22,25,31] and references therein.

When d is a constant vector field, the system (1.1) becomes a compressible Navier–Stokes one for the isentropic flow, which is one of the most important models in fluid dynamics. There are a huge amount of literatures about that. For an overview, see, for example, [8,26] and references therein. The local strong solution was proved to exist uniquely ([2,3,28,29]). Whether the local strong solution blows up or not is still an open problem. Some blowup criteria have been contributed to study the mechanism for possible breakdown. The earlier ones rely on the high-order bound of density or velocity, see [4,7]. Later, efforts were made toward the criteria in terms of low-order bound of density only or velocity only. In this direction, Huang [14] and Huang–Xin [18] first established a blowup criterion, analogous to the Beale–Kato–Majda criterion [1] for the ideal incompressible flows, which can be stated as follows: suppose $T^* < \infty$ is the maximal existence time for a local strong solution and

$$\lambda < 7\mu, \tag{1.20}$$

then

$$\lim_{T\to T^*}\int_0^T \|\nabla u\|_{L^\infty}\,dt=\infty. \tag{1.21}$$

It is improved later in [17] to

$$\lim_{T\to T^*}\int_0^T \|\mathcal{D}u\|_{L^\infty}\,dt=\infty, \tag{1.22}$$

where $\mathcal{D}u$ is the deformation tensor, $\mathcal{D}u=\frac{\nabla u+(\nabla u)^T}{2}$. More recently, one Serrin type's criterion was given by Huang–Li–Xin [16] (without the assumption (1.20)), which is

$$\lim_{T\to T^*}\left(\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}+\|\sqrt{\rho}u\|_{L^s(0,T;L^r)}\right)=\infty,\quad \text{with } 2/s+3/r\le 1. \tag{1.23}$$

(1.23) is called as Serrin type, since it is almost the same as the well-known Serrin's criterion for the 3D incompressible Navier–Stokes equations. In the same paper, Huang–Li–Xin [16] gave another criterion in terms of density, but with the artificial condition (1.20), which is

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)}=\infty. \tag{1.24}$$

The same result was also obtained by Sun–Wang–Zhang [30] independently.

Now, let's return to the system (1.1). There are not so many results. It is not clear whether a global weak solution to (1.1)–(1.5) exists in dimensions greater than one. For the one dimensional case, such an existence has been derived by Ding–Wang–Wen [5]. On the other hand, strong solutions are proved to exist, refer to [11]. The strong solution to the system (1.1)–(1.5) is defined as follows.

Definition 1.1 *(Strong solutions).* (ρ,U,d) is called a strong solution to (1.1) in $\Omega\times(0,T)$, if for some $3<q<\infty$,

$$\begin{cases}\rho\ge 0,\quad \rho\in C([0,T];W^{1,q}),\quad \rho_t\in C([0,T];L^2\cap L^q),\\ U\in C([0,T];H_0^1\cap H^2)\cap L^2(0,T;W^{2,q}),\quad U_t\in L^2(0,T;H_0^1)\ \text{and}\ \sqrt{\rho}U_t\in L^\infty(0,T;L^2),\\ \nabla d\in C([0,T];H^2)\cap L^2(0,T;H^3),\quad d_t\in C([0,T];H^1)\cap L^2(0,T;H^2),\ |d|=1,\end{cases}$$

and (ρ,U,d) satisfies (1.1) a.e. in $\Omega\times(0,T)$.

It is also an open problem whether the strong solution blows up in finite time. One step toward this problem is to make clear the mechanism for the possible breakdown. After the recent blowup results in the study for the compressible Navier–Stokes system, some similar blowup criteria

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for the compressible nematic liquid crystal flow were derived, see [11,12,19,27] and references therein. In particular, it was shown by Huang–Wang–Wen [11,12] that

$$\lim_{T\to T^*}\left(\|\mathcal{D}U\|_{L^1(0,T;L^\infty)}+\|\nabla d\|_{L^2(0,T;L^\infty)}\right)=\infty, \tag{1.25}$$

or (with the crucial artificial assumption $7\mu>9\lambda$)

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\nabla d\|_{L^3(0,T;L^\infty)}\right)=\infty, \tag{1.26}$$

where T^* is the maximal time of existence of strong solutions.

In [19], the authors established a Serrin type criterion,

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|U\|_{L^{s_1}(0,T;L^{r_1})}+\|\nabla d\|_{L^{s_2}(0,T;L^{r_2})}\right)=\infty, \tag{1.27}$$

with r_i and s_i satisfying

$$\frac{2}{s_i}+\frac{3}{r_i}\le 1,\quad 3<r_i\le\infty,\quad i=1,2.$$

Our first result gives a new blowup criterion for further use. It can be stated as follows.

Theorem 1.1. *For $\tilde{q}\in(3,6]$, assume that the initial data (ρ_0,U_0,d_0) satisfy*

$$\rho_0\in W^{1,\tilde{q}},\ \inf\rho_0>0,\ \ U_0\in H^2\cap H_0^1,\ \ \nabla d_0\in H^2,\ |d_0|=1.$$

Let (ρ,U,d) be a strong solution to the initial boundary value problem (1.1)–(1.5). If $T^*<\infty$ is the maximal time of existence, then

$$\limsup_{t\to T^*}\left(\left\|(\rho(t),\frac{1}{\rho}(t))\right\|_{L^\infty}+\|\nabla d\|_{L^2(0,t;L^\infty)}\right)=\infty \tag{1.28}$$

provided

$$\lambda<2\mu. \tag{1.29}$$

Remark 1.1. As in [12], Criterion (1.28) restates the important role of liquid crystal orientation field. To prevent breakdown of classical solutions, we may only need to track the behavior of ∇d rather than the velocity U. In fact, we will give the localized version of (1.28), which is our main result in this paper.

Let us mention that the global radially symmetric strong solutions exist in annular domains, which was proved in [10]. So we guess the singularity will happen around the center. We try to capture the possible singular behavior at the center. In fact we have the following localized Serrin's continuation principle.

Theorem 1.2. *Assume that the initial data* (ρ_0, U_0, d_0) *satisfy* (1.7)–(1.8) *and*

$$(\rho_0, U_0, d_0) \in H^3, \quad |d_0| = 1. \tag{1.30}$$

Let (ρ, U, d) *be a classical spherically symmetric solution to the initial boundary value problem* (1.1)–(1.5) *in* $[0, T] \times \Omega$*, and* T^* *be the upper limit of* T*, that is, the maximal time of existence of the classical solution. Then, if* $T^* < \infty$*, the following continuation principle holds*

$$\limsup_{(t,r)\to(T^*,0)} \left(\rho(t,r) + \frac{1}{\rho}(t,r) + \|\nabla d\|_{L^2(0,t;L^\infty(0,r))}\right) = \infty. \tag{1.31}$$

Remark 1.2. Theorem 1.2 characterizes a local behavior of blowup phenomenon for compressible liquid crystal flows. (1.31) can be viewed as a localization of Serrin's continuation principle. It asserts that the concentration or vanishing of the density, the unbounded Serrin norm inflammation of the gradient liquid crystal field around the center should be responsible for the formation of singularities.

Remark 1.3. Theorem 1.2 is consistent with the result in [10], where it is proved that the global radially symmetric strong solutions exist in annular domain.

Remark 1.4. The method we adopt here is in the same spirit as that in [13,15], where compressible Navier–Stokes equations were considered.

This paper is organized as follows. Section 2 is devoted to some preliminaries. We will illustrate the main ideas of Theorem 1.1 in Section 3 and concentrate on the proof of Theorem 1.2 in Section 4.

2. Preliminaries

First, let us explain some notations and conventions used throughout this paper.

Notations. For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces in Ω are denoted by:

$$L^p = L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2},$$

and $H_0^1 = \{v \in H^1 \mid v|_{\partial\Omega} = 0\}$. We denote

$$\dot{f} = f_t + U \cdot \nabla f, \qquad \int f dx = \int_\Omega f dx.$$

Next, one proposition is presented for the Lamé system, which comes from the momentum equation $(1.1)_2$. Assume that $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain. Suppose $V \in H_0^1$ is a weak solution to the Lamé system,

$$\begin{cases} \mu\Delta V + (\mu+\lambda)\nabla \mathrm{div} V = F, & \text{in } \Omega, \\ V(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

In what follows, we denote $\mathcal{L}V = \mu\Delta V + (\mu+\lambda)\nabla \mathrm{div} V$. Owing to the uniqueness of solution, we denote $V = \mathcal{L}^{-1}F$.

The system is an elliptic system under the assumption (1.3), hence some regularity estimates can be derived. For a proof, refer to [30].

Proposition 2.1. *Let $q \in (1,\infty)$. Then there exists some constant C depending only on λ, μ, q and Ω such that*

- *if $F \in L^q$, then*

$$\|V\|_{W^{2,q}} \le C\|F\|_{L^q}; \tag{2.1}$$

- *if $F \in W^{-1,q}$ (i.e., $F = \mathrm{div} f$ with $f = (f_{ij})_{3\times 3}$, $f_{ij} \in L^q$), then*

$$\|V\|_{W^{1,q}} \le C\|f\|_{L^q}.$$

3. Proof of Theorem 1.1

Suppose the conclusion of Theorem 1.1 fails, then there exists some $M_1 \in (0,\infty)$ such that for any $t < T^*$, the following inequality holds true

$$\|\rho(t)\|_{L^\infty} + \left\|\frac{1}{\rho}(t))\right\|_{L^\infty} + \|\nabla d\|_{L^2(0,t;L^\infty)} \le M_1. \tag{3.1}$$

We then prove the following inequality

$$\limsup_{t\to T^*}\left(\|\rho(t)\|_{L^\infty} + \|\nabla d\|_{L^3(0,t;L^\infty)}\right) \le C. \tag{3.2}$$

It gives a contradiction to the definition of T^*, according to (1.26).

The first step of the proof is the derivation of basic energy inequality. Multiplying $(1.1)_2$ and $(1.1)_3$ by U and $(\triangle d + |\nabla d|^2 d)$ respectively, adding the resulting equations together, we obtain the following energy estimate:

Proposition 3.1. *It holds that for every* $0 < t < T^*$,

$$\begin{aligned}
&\int \left(\frac{1}{2}\rho|U|^2 + \frac{1}{2}|\nabla d|^2 + \frac{a\rho^\gamma}{\gamma-1}\right)dx \\
&\qquad + \int_0^t \int \left(\mu|\nabla U|^2 + (\mu+\lambda)|\mathrm{div} U|^2 + |\Delta d + |\nabla d|^2 d|^2\right)dxds \\
&= \int \left[\frac{1}{2}\rho_0|U_0|^2 + \frac{1}{2}|\nabla d_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1}\right]dx.
\end{aligned}$$

The key estimate of Theorem 1.1 lies on the following enhanced energy estimates.

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Lemma 3.1. *Under the condition (3.1), as long as $\lambda < 2\mu$, it holds that for every $0 < t < T^*$,*

$$\int \left(\rho|U|^4 + |\nabla d|^4\right) dx + \int_0^t \int \left(|\nabla U|^2|U|^2 + |\nabla d|^2|\nabla^2 d|^2\right) dxds \le C. \tag{3.3}$$

Proof. The momentum equation $(1.1)_2$ can be written as follows,

$$\rho U_t + (\rho U \cdot \nabla)U - \mu\Delta U - (\mu+\lambda)\nabla \mathrm{div} U + \nabla P = -\mathrm{div}(M(d)), \tag{3.4}$$

where

$$M(d) \triangleq \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_3. \tag{3.5}$$

Multiplying (3.4) by $q|U|^{q-2}U$, and integrating over Ω yields

$$\begin{aligned}
&\frac{d}{dt}\int \rho|U|^q\, dx + \int \Big\{q|U|^{q-2}\Big[\mu|\nabla U|^2 + (\lambda+\mu)(\mathrm{div}U)^2 + \mu(q-2)|\nabla|U||^2\Big] \\
&\qquad + q(\lambda+\mu)(\nabla|U|^{q-2})\cdot U \mathrm{div}U\Big\}\, dx \\
&\le q\int P\mathrm{div}(|U|^{q-2}U)\, dx + C\int |\nabla d|^2|U|^{q-2}|\nabla U|\, dx \\
&\le C\int P|U|^{q-2}|\nabla U|\, dx + C\int |\nabla d|^2|U|^{q-2}|\nabla U|\, dx.
\end{aligned} \tag{3.6}$$

Here,

$$\begin{aligned}
&q|U|^{q-2}\Big[\mu|\nabla U|^2 + (\lambda+\mu)(\mathrm{div}U)^2 + \mu(q-2)|\nabla|U||^2\Big] \\
&\quad + q(\lambda+\mu)(\nabla|U|^{q-2})\cdot U\mathrm{div}U \\
&\ge q|U|^{q-2}\big[\mu|\nabla U|^2 + (\lambda+\mu)(\mathrm{div}U)^2 + \mu(q-2)\,|\nabla|U||^2 \\
&\qquad - (\lambda+\mu)(q-2)|\nabla|U||\cdot|\mathrm{div}U|\big] \\
&= q|U|^{q-2}\left[\mu|\nabla U|^2 + (\lambda+\mu)(\mathrm{div}U - \frac{1}{2}|\nabla|U||)^2\right] \\
&\quad + q|U|^{q-2}\left[\mu(q-2) - \frac{1}{4}(\lambda+\mu)(q-2)^2\right]|\nabla|U||^2.
\end{aligned} \tag{3.7}$$

Note that $|\nabla|U|| \le |\nabla U|$, taking $q = 4$ and $\lambda < 2\mu$, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|U|^4\,dx + \int |\nabla U|^2|U|^2\,dx \\
&\le C\int P|U|^2|\nabla U|\,dx + C\int |\nabla d|^2|U|^2|\nabla U|\,dx \\
&\le C\int \rho|U|^4\,dx + C\int |\nabla U|^2\,dx + \frac{1}{4}\int |\nabla U|^2|U|^2\,dx + C\int |\nabla d|^4|U|^2\,dx \\
&\le C\int \rho|U|^4\,dx + C\int |\nabla U|^2\,dx + \frac{1}{4}\int |\nabla U|^2|U|^2\,dx + C\int |\nabla d|^2\left(|U|^4 + |\nabla d|^4\right)dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\int \rho|U|^4\,dx + \int |\nabla U|^2|U|^2\,dx \\
&\le C\int \rho|U|^4\,dx + C\int |\nabla U|^2\,dx + C\|\nabla d\|_{L^\infty}^2\left(\int \rho|U|^4\,dx + \int |\nabla d|^4\,dx\right),
\end{aligned} \tag{3.8}$$

where we used the lower bound for density.

Applying the gradient operator to (1.1)$_3$, then one gets that

$$\nabla d_t - \nabla\triangle d = -\nabla(U\cdot\nabla d) + \nabla(|\nabla d|^2 d). \tag{3.9}$$

Multiplying (3.9) by $4|\nabla d|^2\nabla d$ and integrating the resulting equation over Ω lead to

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^4\,dx + 4\int \left(|\nabla d|^2|\nabla^2 d|^2 + 2|\nabla d|^2|\nabla(|\nabla d|)|^2\right)dx \\
&= 4\int |\nabla d|^2\nabla d\cdot\left[-\nabla(U\cdot\nabla d) + \nabla(|\nabla d|^2 d)\right]dx \\
&\le C\int |\nabla d|^3|\nabla^2 d||U|\,dx + C\int |\nabla^2 d||\nabla d|^4\,dx \\
&\le 2\int |\nabla^2 d|^2|\nabla d|^2\,dx + C\int |U|^2|\nabla d|^4\,dx + C\int |\nabla d|^6\,dx \\
&\le 2\int |\nabla^2 d|^2|\nabla d|^2\,dx + C\|\nabla d\|_{L^\infty}^2\cdot\left(\int \rho|U|^4\,dx + \int |\nabla d|^4\,dx\right).
\end{aligned} \tag{3.10}$$

Adding the two equations, (3.8) and (3.10) together, employing the Gronwall's inequality, we have

$$\int \left(\rho|U|^4 + |\nabla d|^4\right)dx + \int_0^t \int \left(|U|^2|\nabla U|^2 + |\nabla d|^2|\nabla^2 d|^2\right)dxds \le C, \tag{3.11}$$

which completes the proof for Lemma 3.1. □

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Lemma 3.2. *Under the condition (3.1), as long as $\lambda < 2\mu$, it holds that for every $0 < t < T^*$,*

$$\int \left(|\nabla U|^2 + |\nabla^2 d|^2\right) dx + \int_0^t \int \left(|\nabla d_t|^2 + |\nabla^3 d|^2\right) dxds \le C.$$

Proof. Let us take the L^2-norm of both sides of the equation (3.9). Using Young's inequality, we get that

$$\begin{aligned} &\frac{d}{dt}\int |\nabla^2 d|^2\, dx + \int (|\nabla \Delta d|^2 + |\nabla d_t|^2)\, dx \\ &\le C\int \left(|\nabla U|^2|\nabla d|^2 + |U|^2|\nabla^2 d|^2 + |\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2\right) dx. \end{aligned} \tag{3.12}$$

Here the second term on the right hand can be estimated as follows,

$$\begin{aligned} &\int |U|^2|\nabla^2 d|^2 dx \\ &\le \int \left(|U||\nabla U||\nabla d||\nabla^2 d| + |U|^2|\nabla d||\nabla \Delta d|\right) dx \\ &\le C\int \left(|U|^2|\nabla U|^2 + |\nabla d|^2|\nabla^2 d|^2\right) dx + \frac{1}{2}\|\nabla \Delta d\|_{L^2}^2 + \int |U|^4|\nabla d|^2\, dx \\ &\le \frac{1}{2}\|\nabla \Delta d\|_{L^2}^2 + C\int |U|^2|\nabla U|^2\, dx + C\|\nabla d\|_{L^\infty}^2 \left(\int |\nabla^2 d|^2 dx + \int \rho U^4 dx\right), \end{aligned}$$

where for the last inequality we used the assumption (3.1),

$$\rho \ge M_1^{-1}.$$

Hence,

$$\begin{aligned} &\frac{d}{dt}\int |\nabla^2 d|^2\, dx + \int (|\nabla \Delta d|^2 + |\nabla d_t|^2)\, dx \\ &\le C\|\nabla d\|_{L^\infty}^2 \int (|\nabla U|^2 + |\nabla^2 d|^2 + \rho|U|^4 + |\nabla d|^4)\, dx + C\int |U|^2|\nabla U|^2\, dx. \end{aligned} \tag{3.13}$$

Multiplying the momentum equation (3.4) by U_t and then integrating over Ω, then one gets that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(\mu|\nabla U|^2+(\mu+\lambda)(\mathrm{div}U)^2\right)dx+\int \rho|\dot U|^2\,dx\\
&=\int \rho\dot U\cdot(U\cdot\nabla)U\,dx+\int P\mathrm{div}U_t\,dx-\int \mathrm{div}(M(d))\cdot U_t\,dx\\
&=\int \rho\dot U\cdot(U\cdot\nabla)U\,dx+\frac{d}{dt}\int P\mathrm{div}U\,dx-\int P_t\mathrm{div}U\,dx\\
&\quad+\frac{d}{dt}\int M(d)\cdot\nabla U\,dx-\int M(d)_t\cdot\nabla U\,dx.
\end{aligned}
\tag{3.14}
$$

To deal with the term $\int (P_t\cdot\mathrm{div}U+M(d)_t\cdot\nabla U)\,dx$, we split U into two parts, v and w. Let

$$
v=\mathcal{L}^{-1}\left(\nabla P+\mathrm{div}M(d)\right),\quad \text{and}\quad w=U-v.
$$

Now,

$$
\begin{aligned}
&\int [P_t\cdot\mathrm{div}v+M(d)_t\cdot\nabla v]\,dx\\
&=-\int \mathrm{div}\,[P_t\mathbb{I}_3+M(d)_t]\cdot v\,dx\\
&=-\int (\mathcal{L}v)_t\cdot v\,dx\\
&=\frac{1}{2}\frac{d}{dt}\int \left|(-\mathcal{L})^{\frac{1}{2}}v\right|^2dx.
\end{aligned}
\tag{3.15}
$$

Note that the equation for P is

$$
P_t+\mathrm{div}(UP)+(\gamma-1)P\mathrm{div}U=0,
\tag{3.16}
$$

hence it follows from Proposition 2.1,

$$
\begin{aligned}
&\left|\int P_t\cdot\mathrm{div}w\,dx\right|\\
&\le C\int P|U|\cdot|\nabla^2 w|\,dx+C\int P|\nabla U|\cdot|\nabla w|\,dx\\
&\le C\|U\|_{L^2}\cdot\|\nabla^2 w\|_{L^2}+C\|\nabla U\|_{L^2}\cdot\|\nabla w\|_{L^2}\\
&\le C\|\nabla U\|_{L^2}\|\rho\dot U\|_{L^2}\\
&\le \frac{1}{4}\|\sqrt{\rho}\dot U\|_{L^2}^2+C\|\nabla U\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
&\left|\int M(d)_t \cdot \nabla w\, dx\right| \\
&\le \left|\int M(d)_t \cdot \nabla U\, dx\right| + \left|\int M(d)_t \cdot \nabla v\, dx\right| \\
&\le C\int |\nabla d_t|\cdot|\nabla d|\cdot|\nabla U|\, dx + C\int |\nabla d_t|\cdot|\nabla d|\cdot|\nabla v|\, dx \\
&\le \frac{1}{4}\|\nabla d_t\|_{L^2}^2 + C\int |\nabla d|^2|\nabla U|^2\, dx + C\int |\nabla d|^2|\nabla v|^2\, dx \\
&\le \frac{1}{4}\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\left(\int |\nabla U|^2\, dx + \int (|P|^2 + |M(d)|^2)\, dx\right) \\
&\le \frac{1}{4}\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\left(1 + \int (|\nabla U|^2 + |\nabla d|^4)\, dx\right).
\end{aligned}
\tag{3.18}
$$

Taking the estimates (3.15)–(3.18) into (3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\left[\int \left(\mu|\nabla U|^2 + (\mu+\lambda)(\mathrm{div}U)^2\right] dx + 2J(t)\right) + \int \rho|\dot{U}|^2\, dx \\
&\le \frac{1}{2}\|\nabla d_t\|_{L^2}^2 + C\int |U|^2|\nabla U|^2\, dx + C\int |\nabla U|^2\, dx \\
&\qquad + C\|\nabla d\|_{L^\infty}^2\left[1 + \int (|\nabla U|^2 + |\nabla d|^4)\, dx\right],
\end{aligned}
\tag{3.19}
$$

where

$$
J(t) = \frac{1}{2}\int \left|(-\mathcal{L})^{\frac{1}{2}} v\right|^2 dx - \int P\mathrm{div}U\, dx - \int M(d)\cdot\nabla U\, dx.
$$

Adding the two equations (3.12) and (3.19) together, we have

$$
\begin{aligned}
&\frac{d}{dt}\left[\int \left(\mu|\nabla U|^2 + (\mu+\lambda)(\mathrm{div}U)^2 + |\nabla^2 d|^2\right] dx + 2J(t)\right) + \int \left(\rho|\dot{U}|^2 + |\nabla\Delta d|^2\right) dx \\
&\le C\|\nabla d\|_{L^\infty}^2 \int (|\nabla U|^2 + |\nabla^2 d|^2)\, dx + C\|\nabla d\|_{L^\infty}^2\left[1 + \int (|\nabla d|^4 + \rho|U|^4)\, dx\right] \\
&\quad + C\int |U|^2|\nabla U|^2\, dx + C\int |\nabla U|^2\, dx.
\end{aligned}
$$

Employing Gronwall's inequality and taking (3.11) into account, we get that

$$
\int \mu|\nabla U|^2 + (\mu+\lambda)(\mathrm{div}U)^2 + |\nabla^2 d|^2\, dx + \int_0^t \int \rho|\dot{U}|^2 + |\nabla d_t|^2 + |\nabla\Delta d|^2\, dx ds \le C. \qquad \square
$$

Hence, it follows from Lemma 3.2 and energy equality that

$$\|\nabla d\|_{L^3(0,t;L^\infty)} \le C,$$

which completes the proof of Theorem 1.1 with the help of the criterion (1.26).

4. Proof of Theorem 1.2

Suppose the conclusion of Theorem 1.2 fails, then there exists a $r_0 \in (0, R)$, $M_2 \in (1, \infty)$ such that for any $r \in (0, r_0)$ and $t < T^*$, the following inequality holds true

$$\rho(t,r) + \frac{1}{\rho}(t,r) + \|d_r\|^2_{L^2(0,t;L^\infty(0,r))} \le M_2. \tag{4.1}$$

Note that for spherically symmetric solutions,

$$\triangle U = \nabla \mathrm{div} U = \left(u_r + m\frac{u}{r}\right)_r \cdot \frac{x}{r}, \tag{4.2}$$

we rewrite the equation of momentum as

$$(\rho U)_t + \mathrm{div}(\rho U \otimes U) + \nabla P = \frac{1}{3}\kappa \Delta U + \frac{2}{3}\kappa \nabla \mathrm{div} U - \Delta d \cdot \nabla d, \tag{4.3}$$

where $\kappa = 2\mu + \lambda$. The new equation satisfies the assumption (1.29). According to Theorem 1.1, it remains to show that in fact there exists a constant $C > 0$ depending only on $M_2, \rho_0, u_0, d_0, r_0, R, a$ and T^* such that for any $r \in (r_0/2, R)$ and $t < T^*$

$$\rho(t,r) + \frac{1}{\rho}(t,r) + \|d_r\|^2_{L^2(0,t;L^\infty(r,R))} \le C. \tag{4.4}$$

First, we give the basic energy equality in the spherical coordinate system.

Proposition 4.1. *It holds that for every* $0 < t < T^*$,

$$\begin{aligned} &\int_0^R r^m \left(\frac{\rho u^2}{2} + \frac{a\rho^\gamma}{\gamma - 1} + \frac{|d_r|^2}{2}\right) dr \\ &\quad + \int_0^t \int_0^R r^m \left(\kappa u_r^2 + m\kappa \frac{u^2}{r^2} + |d_{rr} + |d_r|^2 d|^2 + m\frac{|d_r|^2}{r^2}\right) dr ds \\ &\quad = \int_0^R r^m \left(\frac{\rho_0 u_0^2}{2} + \frac{a\rho_0^\gamma}{\gamma - 1} + \frac{|(d_0)_r|^2}{2}\right) dr. \end{aligned} \tag{4.5}$$

With the help of (4.1) and Proposition 4.1, the regularity of d is at hand.

Lemma 4.1. *Let $r_1 = \frac{r_0}{2}$. Under the condition (4.1), for every $0 \le t \le T < T^*$, it holds that*

$$\int_0^t \int_{r_1}^R r^m |d_{rr}|^2 \, dr ds \le C(E_0 + E_0^3 T), \tag{4.6}$$

and then

$$\int_0^t \int_0^R r^m |d_{rr}|^2 \, dr ds \le C(M_2 E_0 + E_0^3 T). \tag{4.7}$$

Consequently,

$$\|d_r\|^2_{L^2(0,t;L^\infty(0,R))} \le C(M_2 E_0 + E_0^3 T + M_2). \tag{4.8}$$

Proof. By Gagliardo–Nirenberg inequality and Poincaré's inequality,

$$\|d_r\|^4_{L^4(r_1,R)} \le C(r_1, R)\|d_r\|^3_{L^2(r_1,R)} \cdot \|d_{rr}\|_{L^2(r_1,R)}. \tag{4.9}$$

As $|d| = 1$, one obtains

$$\begin{aligned}
&\int_{r_1}^R r^m |d_{rr}|^2 \, dr \\
&\le 2\int_{r_1}^R r^m |d_{rr} + |d_r|^2 d|^2 \, dr + 2\int_{r_1}^R r^m |d_r|^4 \, dr \\
&\le 2\int_{r_1}^R r^m |d_{rr} + |d_r|^2 d|^2 \, dr + 2R^m \int_{r_1}^R |d_r|^4 \, dr \\
&\le 2\int_{r_1}^R r^m |d_{rr} + |d_r|^2 d|^2 \, dr + C\|d_r\|^3_{L^2(r_1,R)} \|d_{rr}\|_{L^2(r_1,R)} \\
&\le 2\int_{r_1}^R r^m |d_{rr} + |d_r|^2 d|^2 \, dr + C\|r^{\frac{m}{2}} d_r\|^3_{L^2(r_1,R)} \|r^{\frac{m}{2}} d_{rr}\|_{L^2(r_1,R)} \\
&\le C\int_{r_1}^R r^m |d_{rr} + |d_r|^2 d|^2 \, dr + \frac{1}{2}\int_{r_1}^R r^m |d_{rr}|^2 \, dr + C\|r^{\frac{m}{2}} d_r\|^6_{L^2(r_1,R)}.
\end{aligned} \tag{4.10}$$

Hence, (4.6) follows immediately from energy equality (4.5) and (4.10).

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To prove (4.7), it amounts to show

$$\begin{aligned}
&\int_0^t\int_0^{r_1} r^m|d_{rr}|^2\,drds\\
&\le 2\int_0^t\int_0^{r_1}\left(r^m|d_{rr}+|d_r|^2d|^2\right)drds+2\int_0^t\int_0^{r_1} r^m|d_r|^4\,drds\\
&\le 2\int_0^t\int_0^{r_1} r^m|d_{rr}+|d_r|^2d|^2\,drds+2\int_0^t\left(\|d_r\|^2_{L^\infty(0,r_1)}\int_0^{r_1} r^m|d_r|^2\,drds\right)\\
&\le CE_0+CE_0M_2.
\end{aligned}\tag{4.11}$$

We then use the following Sobolev inequalities for radially symmetric functions with the boundary condition $d_r(t,0)=0$ and conclude

$$\begin{aligned}
|d_r|r^m=\left|\int_0^r (d_r\tau^m)_\tau\,d\tau\right|&=\left|\int_0^r\left((d_r)_\tau+m\frac{d_r}{\tau}\right)\tau^m\,d\tau\right|\\
&\le\left[\int_0^R\left(d_{rr}+m\frac{d_r}{\tau}\right)^2\tau^m\,d\tau\right]^{\frac12}\left(\frac{r^{m+1}}{m+1}\right)^{\frac12}.
\end{aligned}\tag{4.12}$$

Therefore,

$$\|d_r\|^2_{L^2(0,t;L^\infty(r_1,R))}\le C(M_2E_0+E_0^3T).\tag{4.13}$$

Combining with the assumption (4.1), we have

$$\|d_r\|^2_{L^2(0,t;L^\infty(0,R))}\le C(M_2E_0+E_0^3T+M_2).\qquad\square\tag{4.14}$$

It remains to establish the upper and lower bounds of the density away from the center. We then work in **Lagrangian coordinates**.

The following lemma gives a relationship between r and y.

Lemma 4.2. *There exists a positive constant C_1 depending only on a,γ, such that*

$$r(t,y)\ge C_1^{-1}y^{\frac{\gamma}{(m+1)(\gamma-1)}}E_0^{-\frac{1}{(m+1)(\gamma-1)}},\tag{4.15}$$

and

$$R^{m+1}-r(t,y)^{m+1}\ge C_1^{-1}(M_0-y)^{\frac{\gamma}{\gamma-1}}E_0^{-\frac{1}{\gamma-1}},\tag{4.16}$$

for all $(t,y)\in Q_{T,0}=[0,T]\times[0,M_0]$.

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Proof. By the energy equality (4.5), one has

$$\begin{aligned} y &= \int_0^r \rho\tau^m \, d\tau = \int_0^r \rho\tau^{\frac{m}{\gamma}} \tau^{m\left(1-\frac{1}{\gamma}\right)} \, d\tau \\ &\le \left(\int_0^r \rho^\gamma \tau^m \, d\tau\right)^{\frac{1}{\gamma}} \left(\int_0^r \tau^m \, d\tau\right)^{1-\frac{1}{\gamma}} \\ &\le C r^{(m+1)(1-\frac{1}{\gamma})} E_0^{\frac{1}{\gamma}}. \end{aligned} \tag{4.17}$$

Consequently,

$$r(t, y) \ge C_1^{-1} y^{\frac{\gamma}{(m+1)(\gamma-1)}} E_0^{-\frac{1}{(m+1)(\gamma-1)}}. \tag{4.18}$$

Similarly,

$$\begin{aligned} M_0 - y &= \int_r^R \rho\tau^m \, d\tau = \int_r^R \rho\tau^{\frac{m}{\gamma}} \tau^{m(1-\frac{1}{\gamma})} \, d\tau \\ &\le \left(\int_r^R \rho^\gamma \tau^m \, d\tau\right)^{\frac{1}{\gamma}} \left(\int_r^R \tau^m \, d\tau\right)^{1-\frac{1}{\gamma}} \\ &\le C \left[R^{m+1} - r(t, y)^{m+1}\right]^{1-\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}}. \qquad \square \end{aligned} \tag{4.19}$$

We are now in a position to establish the pointwise estimates of the density away from the center. We need to fix one point $y_0 \in (0, M_0)$, such that

$$r(t, y_0) \le \frac{r_0}{2}, \quad \text{for every } 0 \le t < T^*. \tag{4.20}$$

Choose some point y_0 satisfying

$$y_0 \le \left(\frac{r_0}{2}\right)^{m+1} \frac{1}{M_2 \cdot (m+1)}.$$

It can be verified such a point satisfies (4.20), since

$$y = \int_0^r \rho(t, \tau)\tau^m \, d\tau \ge M_2^{-1} \cdot \frac{1}{m+1} r^{m+1}, \quad \text{for every } 0 \le t < T^*.$$

Lemma 4.3. *Under the condition (4.1), there exists a constant C such that*

$$\rho(t,y)+\frac{1}{\rho}(t,y)\le C, \quad (t,y)\in[0,T]\times[y_0,M_0]. \tag{4.21}$$

Proof. In view of (1.15) and (1.16), it holds

$$\begin{aligned}\kappa(\log\rho)_{ty}&=\kappa\left(\frac{\rho_t}{\rho}\right)_y=-\kappa\left[\rho(r^m u)_y\right]_y\\&=-r^{-m}u_t-p_y-\frac{1}{2}\left(r^{2m}\rho^2|d_y|^2\right)_y-mr^{m-1}\rho|d_y|^2\\&=-(r^{-m}u)_t-m\frac{u^2}{r^{m+1}}-p_y-\frac{1}{2}\left(r^{2m}\rho^2|d_y|^2\right)_y-mr^{m-1}\rho|d_y|^2.\end{aligned}\tag{4.22}$$

Thus, for $y>y_0>0$, integrating (4.22) over $(0,t)\times(y_0,y)$, we deduce that

$$\begin{aligned}\kappa\log\frac{\rho(t,y)}{\rho(t,y_0)}&=\kappa\log\frac{\rho_0(y)}{\rho_0(y_0)}+\int_{y_0}^{y}\left[(r^{-m}u)(0,z)-(r^{-m}u)(t,z)\right]dz\\&\quad+\int_0^t[p(s,y_0)-p(s,y)]\,ds-\int_0^t\int_{y_0}^{y}m\frac{u^2(s,z)}{r^{m+1}}\,dzds\\&\quad-\int_0^t\int_{y_0}^{y}\left[\frac{1}{2}(r^{2m}\rho^2|d_y|^2)_y+mr^{m-1}\rho|d_y|^2\right]dzds\end{aligned}\tag{4.23}$$

which is equivalent to

$$\begin{aligned}\frac{\rho(t,y)}{\rho(t,y_0)}&=\frac{\rho_0(y)}{\rho_0(y_0)}\exp\left(\kappa^{-1}\int_{y_0}^{y}\left[(r^{-m}u)(0,z)-(r^{-m}u)(t,z)\right]dz\right)\\&\quad\cdot\exp\left(\kappa^{-1}\int_0^t[p(s,y_0)-p(s,y)]\,ds\right)\\&\quad\cdot\exp\left(-\kappa^{-1}\int_0^t\int_{y_0}^{y}m\frac{u^2(s,z)}{r^{m+1}}\,dzds\right)\\&\quad\cdot\exp\left(-\kappa^{-1}\int_0^t\int_{y_0}^{y}\left[\frac{1}{2}(r^{2m}\rho^2|d_y|^2)_y+mr^{m-1}\rho|d_y|^2\right]dzds\right).\end{aligned}\tag{4.24}$$

We can rewrite (4.24) as

$$\rho(t,y)=\mathcal{P}(t)\mathcal{U}(t,y)\exp\left(-\kappa^{-1}\int_0^t p(s,y)\,ds\right) \tag{4.25}$$

where

$$\mathcal{P}(t)=\frac{\rho(t,y_0)}{\rho_0(y_0)}\exp\left(\kappa^{-1}\int_0^t p(s,y_0)\,ds\right) \tag{4.26}$$

and

$$\begin{aligned}\mathcal{U}(t,y)&=\rho_0(y)\exp\left(\kappa^{-1}\int_{y_0}^{y}\left[(r^{-m}u)(0,z)-(r^{-m}u)(t,z)\right]dz\right)\\&\quad\cdot\exp\left(-\kappa^{-1}\int_0^t\int_{y_0}^{y} m\frac{u^2(s,z)}{r^{m+1}}\,dzds\right)\\&\quad\cdot\exp\left(-\kappa^{-1}\int_0^t\int_{y_0}^{y}\left[\frac{1}{2}(r^{2m}\rho^2|d_y|^2)_y+mr^{m-1}\rho|d_y|^2\right]dzds\right)\\&=\rho_0(y)\exp\left(\kappa^{-1}\sum_{i=1}^{3}\chi_i\right).\end{aligned} \tag{4.27}$$

For every $t\in[0,T]$, $y\in[y_0,M_0]$, it follows from the equation of mass, the energy inequality (4.5) and Lemma 4.2 that

$$\begin{aligned}\left|\int_{y_0}^{y} r^{-m}u(t,z)\,dz\right|&\le\int_{y_0}^{M_0} r^{-m}|u|\,dy\\&=\int_{r(y_0)}^{R}\rho|u|\,dr\\&\le Cr(y_0)^{-m}\int_{r(y_0)}^{R}\rho|u|r^m\,dr\\&\le Cr(y_0)^{-m}\left(\int_0^R\rho r^m\,dr\right)^{\frac{1}{2}}\left(\int_0^R\rho u^2r^m\,dr\right)^{\frac{1}{2}}\\&\le Cr(y_0)^{-m}M_0^{\frac{1}{2}}E_0^{\frac{1}{2}}\\&\le Cy_0^{-\frac{m\gamma}{(m+1)(\gamma-1)}}M_0^{\frac{1}{2}}E_0^{\frac{(m+1)\gamma+m-1}{2(m+1)(\gamma-1)}},\end{aligned} \tag{4.28}$$

which implies

$$|\chi_1| \le C y_0^{-\frac{m\gamma}{(m+1)(\gamma-1)}} M_0^{\frac{1}{2}} E_0^{\frac{(m+1)\gamma+m-1}{2(m+1)(\gamma-1)}}. \tag{4.29}$$

Meanwhile, using the energy inequality (4.5) and Lemma 4.2 again,

$$\begin{aligned}
|\chi_2| &\le \int_0^T \int_{y_0}^{M_0} m r^{-m} \frac{|u|^2}{r}\, dy ds \\
&= \int_0^T \int_{r(y_0)}^{R} m \frac{\rho|u|^2}{r}\, dr ds \\
&\le C r(y_0)^{-m-1} \int_0^T \int_0^R \rho|u|^2 r^m\, dr ds \\
&\le C r(y_0)^{-m-1} E_0 T \\
&\le C y_0^{-\frac{\gamma}{\gamma-1}} E_0^{\frac{\gamma}{\gamma-1}} T.
\end{aligned} \tag{4.30}$$

Employing Lemma 4.1 and Proposition 4.1, we get that

$$\begin{aligned}
|\chi_3| &= \left| \int_0^t \int_{y_0}^{y} \left[\frac{1}{2} (r^{2m}\rho^2 |d_y|^2)_y + m r^{m-1} \rho |d_y|^2 \right] dz ds \right| \\
&= \left| \int_0^T \int_{r(y_0)}^{r(y)} \left[\frac{1}{2} \left| (|d_r|^2)_r \right| + \frac{m}{r} |d_r|^2 \right] dr ds \right| \\
&\le \int_0^T \int_{r(y_0)}^{R} \left(|d_r \cdot d_{rr}| + \frac{m}{r} |d_r|^2 \right) dr ds \\
&\le \int_0^T \int_{r(y_0)}^{R} \left(r|d_{rr}|^2 + \frac{|d_r|^2}{r} + \frac{m}{r} |d_r|^2 \right) dr ds \\
&\le C r(y_0)^{-1} \int_0^T \int_{r(y_0)}^{R} \left[r^2 |d_{rr}|^2 + (m+1)|d_r|^2 \right] dr ds \\
&\le C y_0^{-\frac{\gamma}{(m+1)(\gamma-1)}} E_0^{\frac{1}{(m+1)(\gamma-1)}} (M_2 E_0 + E_0^3 T).
\end{aligned} \tag{4.31}$$

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Let us set

$$H = y_0^{-\frac{m\gamma}{(m+1)(\gamma-1)}} M_0^{\frac{1}{2}} E_0^{\frac{(m+1)\gamma+m-1}{2(m+1)(\gamma-1)}} + y_0^{-\frac{\gamma}{\gamma-1}} E_0^{\frac{\gamma}{\gamma-1}} T + y_0^{-\frac{\gamma}{(m+1)(\gamma-1)}} E_0^{\frac{1}{(m+1)(\gamma-1)}} (M_2 E_0 + E_0^3 T). \tag{4.32}$$

The above estimates for $\mathcal{U}(t, y)$ can be simply written as

$$\begin{aligned} \mathcal{U}(t, y) &\le C \sup_{x\in\Omega} \rho_0(x) \cdot \exp(C\kappa^{-1} H), \\ \mathcal{U}^{-1}(t, y) &\le C \left(\inf_{x\in\Omega} \rho_0(x)\right)^{-1} \cdot \exp(C\kappa^{-1} H). \end{aligned} \tag{4.33}$$

Next, we are in a position to estimate $\mathcal{P}(t)$. It follows from (4.25) that

$$\begin{aligned} \frac{d}{dt}\left[\exp\left(\frac{\gamma}{\kappa}\int_0^t p(s, y)\, ds\right)\right] &= \frac{a\gamma}{\kappa}\rho(t, y)^\gamma \exp\left(\frac{\gamma}{\kappa}\int_0^t p(s, y)\, ds\right) \\ &= \frac{a\gamma}{\kappa}\left[\mathcal{P}(t)\mathcal{U}(t, y)\right]^\gamma, \end{aligned} \tag{4.34}$$

which implies

$$\exp\left(\frac{1}{\kappa}\int_0^t p(s, y)ds\right) = \left(1 + \frac{a\gamma}{\kappa}\int_0^t \left[\mathcal{P}(s)\mathcal{U}(s, y)\right]^\gamma ds\right)^{1/\gamma}. \tag{4.35}$$

In view of (4.25) and (4.35), we have

$$\rho(t, y) = \frac{\mathcal{P}(t)\mathcal{U}(t, y)}{\left(1 + \frac{a\gamma}{\kappa}\int_0^t (\mathcal{P}(s)\mathcal{U}(s, y))^\gamma ds\right)^{1/\gamma}}. \tag{4.36}$$

Observe that

$$\int_{y_0}^{M_0} \frac{dy}{\rho(t, y)} = \int_{r(t,y_0)}^{R} r^m dr = \frac{R^{m+1} - [r(t, y_0)]^{m+1}}{m+1}.$$

Then, $\mathcal{P}(t)$ can be estimated as

$$
\begin{aligned}
&\frac{R^{m+1}-(r(t,y_0))^{m+1}}{m+1}\mathcal{P}(t)\\
&=\int_{y_0}^{M_0}\frac{\mathcal{P}(t)}{\rho(t,y)}\,dy\\
&=\int_{y_0}^{M_0}\frac{\left(1+\frac{a\gamma}{\kappa}\int_0^t[\mathcal{P}(s)\mathcal{U}(s,y)]^\gamma\,ds\right)^{1/\gamma}}{\mathcal{U}(t,y)}\,dy\\
&\le\int_{y_0}^{M_0}\frac{1}{\mathcal{U}(t,y)}\,dy+\left(\frac{a\gamma}{\kappa}\right)^{1/\gamma}\left[\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right]\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]\int_{y_0}^{M_0}\left[\int_0^t\mathcal{P}(s)^\gamma\,ds\right]^{1/\gamma}dy\\
&\le(M_0-y_0)\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]\cdot\left[1+C\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\cdot\left(\int_0^t\mathcal{P}(s)^\gamma\,ds\right)^{\frac{1}{\gamma}}\right]
\end{aligned}
\tag{4.37}
$$

Taking γ-th power of both sides of (4.37), it follows from Lemma 4.2 that we have

$$
\begin{aligned}
\left(\frac{M_0-y_0}{E_0}\right)^{\frac{\gamma}{\gamma-1}}\mathcal{P}(t)^\gamma\le{}&C\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]^\gamma\\
&+C\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]^\gamma\cdot\left[\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right]^\gamma\cdot\left[\int_0^t\mathcal{P}(s)^\gamma\,ds\right].
\end{aligned}
\tag{4.38}
$$

Therefore, by Gronwall's inequality, we deduce from (4.38) that

$$
\begin{aligned}
\mathcal{P}(t)\le{}&C\left(\frac{E_0}{M_0-y_0}\right)^{\frac{1}{\gamma-1}}\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]\\
&\cdot\exp\left\{CT\left(\frac{E_0}{M_0-y_0}\right)^{\frac{\gamma}{\gamma-1}}\cdot\left[\sup_{Q_{T,y_0}}\mathcal{U}^{-1}(t,y)\right]^\gamma\cdot\left[\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right]^\gamma\right\}.
\end{aligned}
\tag{4.39}
$$

Finally, recalling (4.26) and (4.36), we have

$$
\rho(t,y)\le\mathcal{P}(t)\cdot\left[\sup_{Q_{T,y_0}}\mathcal{U}(t,y)\right],
\tag{4.40}
$$

and

$$
\begin{aligned}
\rho(t,y)^{-1} &\le \mathcal{P}^{-1}(t)\cdot\left[\sup_{Q_{T,y_0}} \mathcal{U}^{-1}(t,y)\right]\cdot\left(1+\frac{a\gamma}{\kappa}\int_0^t [\mathcal{P}(s)\mathcal{U}(s,y)]^\gamma\, ds\right)^{\frac{1}{\gamma}} \\
&\le \frac{\rho_0(y_0)}{\rho(t,y_0)}\cdot\left[\sup_{Q_{T,y_0}} \mathcal{U}^{-1}(t,y)\right]\cdot\left(1+\frac{a\gamma}{\kappa}\int_0^t [\mathcal{P}(s)\mathcal{U}(s,y)]^\gamma\, ds\right)^{\frac{1}{\gamma}} \qquad (4.41) \\
&\le CM_2^2\left[\sup_{Q_{T,y_0}} \mathcal{U}^{-1}(t,y)\right]\cdot\left(1+\frac{a\gamma}{\kappa}\int_0^t [\mathcal{P}(s)\mathcal{U}(s,y)]^\gamma\, ds\right)^{\frac{1}{\gamma}},
\end{aligned}
$$

where the last inequality is due to the fact $r(t,y_0)\le \frac{r_0}{2}$. Plugging the estimates (4.33), (4.39) into (4.40) and (4.41), we can deduce the desired pointwise estimate for ρ. Thus the proof of Lemma 4.3, therefore, Theorem 1.2, is completed. □

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References

[1] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.
[2] Y. Cho, H.J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 83 (2004) 243–275.
[3] H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for isentropic compressible fluids, J. Differential Equations 190 (2003) 504–523.
[4] H.J. Choe, B.J. Jin, Regularity of weak solutions of the compressible Navier–Stokes equations, J. Korean Math. Soc. 40 (2003) 1031–1050.
[5] S.J. Ding, C.Y. Wang, H.Y. Wen, Weak solution to compressible hydrodynamic flow of liquid crystals in dimension one, Discrete Contin. Dyn. Syst. Ser. B (2011) 357–371.
[6] J.L. Ericksen, Hydrostatic theory of liquid crystal, Arch. Ration. Mech. Anal. 9 (1962) 371–378.
[7] J.S. Fan, S. Jiang, Blow-up criteria for the Navier–Stokes equations of compressible fluids, J. Hyperbolic Differ. Equ. 5 (2008) 167–185.
[8] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford Science Publication, Oxford, 2004.
[9] M.C. Hong, J.K. Li, Z.P. Xin, Blow-up criteria of strong solutions to the Ericksen–Leslie system in $\mathbb{R}^3$, Comm. Partial Differential Equations 39 (7) (2014) 1284–1328.
[10] J.R. Huang, S.J. Ding, Spherically symmetric solutions to compressible hydrodynamic flow of liquid crystals in N dimensions, Chin. Ann. Math. Ser. B 33 (2012) 453–478.
[11] T. Huang, C.Y. Wang, H.Y. Wen, Strong solutions of the compressible nematic liquid crystal flow, J. Differential Equations 252 (3) (2012) 2222–2265.
[12] T. Huang, C.Y. Wang, H.Y. Wen, Blow up criterion for compressible nematic liquid crystal flows in dimension three, Arch. Ration. Mech. Anal. 204 (1) (2012) 285–311.
[13] X.D. Huang, On formation of singularity of spherically symmetric nonbarotropic flows, Kyoto J. Math. 55 (1) (2015) 1–15.

[14] X.D. Huang, Some Results on Blowup of Solutions to the Compressible Navier–Stokes Equations, PhD Thesis, The Chinese University of Hong Kong, 2009.
[15] X.D. Huang, A. Matsumura, A characterization on breakdown of smooth spherically symmetric solutions of the isentropic system of compressible Navier–Stokes equations, Osaka J. Math. 52 (1) (2015) 271–285.
[16] X.D. Huang, J. Li, Z.P. Xin, Serrin type criterion for the three-dimensional viscous compressible flows, SIAM J. Math. Anal. 43 (2011) 1872–1886.
[17] X.D. Huang, J. Li, Z.P. Xin, Blowup criterion for the three-dimensional viscous barotropic flows with vacuum states, Comm. Math. Phys. 301 (2011) 23–35.
[18] X.D. Huang, Z.P. Xin, A blow-up criterion for classical solutions to the compressible Navier–Stokes equations, Sci. China Math. 53 (3) (2010) 671–686.
[19] X.D. Huang, Y. Wang, A Serrin criterion for compressible nematic liquid crystal flows, Math. Methods Appl. Sci. 36 (11) (2013) 1363–1375.
[20] F.M. Leslie, Some constitutive equations for liquid crystals, Arch. Ration. Mech. Anal. 28 (1968) 265–283.
[21] F.H. Lin, Nonlinear theory of defects in nematic liquid crystal: phase transition and flow phenomena, Comm. Pure Appl. Math. 42 (1989) 789–814.
[22] F.H. Lin, J.Y. Lin, C.Y. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal. 197 (2010) 297–336.
[23] F.H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Comm. Pure Appl. Math. 48 (1995) 501–537.
[24] F.H. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst. 2 (1) (1996) 1–22.
[25] F.H. Lin, C. Liu, Existence of solutions for the Ericksen–Leslie system, Arch. Ration. Mech. Anal. 154 (2000) 135–156.
[26] P.L. Lions, Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models, Oxford University Press, New York, 1998.
[27] X.G. Liu, L.M. Liu, A blow-up criterion for the compressible liquid crystal system, arXiv:1011.4399v2.
[28] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, Bull. Soc. Math. France 90 (1962) 487–497.
[29] J. Serrin, On the uniqueness of compressible fluid motion, Arch. Ration. Mech. Anal. 3 (1959) 271–288.
[30] Y.Z. Sun, C. Wang, Z.F. Zhang, A Beale–Kato–Majda blow-up criterion for the compressible Navier–Stokes equations, J. Math. Pures Appl. (9) 95 (2011) 36–47.
[31] H. Wu, X. Xu, C. Liu, On the general Ericksen–Leslie system: Parodi's relation, well-posedness and stability, Arch. Ration. Mech. Anal. 208 (2013) 59–107.

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