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Stability of hyperbolic–parabolic mixed type equations with partial boundary condition ☆

Huashui Zhan [a], Zhaosheng Feng [b,*]

[a] School of Applied Mathematics, Xiamen University of Technology, Xiamen, Fujian 361024, China
[b] School of Mathematical and Statistical Sciences, University of Texas-Rio Grande Valley, Edinburg, TX 78539, USA

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Abstract

In this paper, we are concerned with the hyperbolic–parabolic mixed type equations with the non-homogeneous boundary condition. If it is degenerate on the boundary, the part of the boundary whose boundary value should be imposed, is determined by the entropy condition from the convection term. If there is no convection term in the equation, we show that the stability of solutions can be proved without any boundary condition. If the equation is completely degenerate, we show that the stability of solutions can be established just based on the partial boundary condition.
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1. Introduction

Consider the hyperbolic–parabolic mixed type equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(a(u,x,t)\nabla u) + \operatorname{div}(b(u)), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

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* Corresponding author.
E-mail addresses: 2012111007@xmut.edu.cn (H. Zhan), zhaosheng.feng@utrgv.edu (Z. Feng).

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with non-homogeneous boundary condition, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with the appropriately smooth boundary $\partial\Omega$, such that the distance function $d(x) = \text{dist}(x, \partial\Omega)$ is a differential function when x is near the boundary [1,2]. Throughout the paper, we assume that

$$A(u, x, t) = \int_0^u a(s, x, t)ds, \quad a(s, x, t) \geq 0,$$

and

$$a(u, x, t) \mid_{x\in\partial\Omega} = 0. \tag{2}$$

For the hyperbolic–parabolic mixed type equations, the well-posedness of the solutions has been an interesting problem. From condition (2), it implies that the equation is degenerate on the boundary. We assume that the initial condition is given by

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{3}$$

As shown in [3,4], the usual Dirichlet boundary condition

$$u(x, t) = g(x, t), \quad (x, t) \in \partial\Omega \times (0, T), \tag{4}$$

may become unnecessary. For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau \text{ and } h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|\, s \,|}{\eta}\right)_+. \tag{5}$$

For any $k \in \mathbb{R}$ and $b = (b_1, b_2, \cdots, b_N)$, we denote by

$$\Sigma_{1\eta k} = \left\{x \in \Sigma,\ S_\eta(k)[b_i(0) - b_i(k)]n_i(x) > 0\right\},$$
$$\Sigma_{2\eta k} = \left\{x \in \Sigma,\ S_\eta(k)[b_i(0) - b_i(k)]n_i(x) \leq 0\right\},$$

where $\Sigma = \partial\Omega$ is the boundary, and $\{n_i\}_{i=1}^N$ is the inner normal vector of Ω. Clearly, $\Sigma = \Sigma_{1\eta k} \bigcup \Sigma_{2\eta k}$.

Let us briefly recall the initial–boundary value problem of the equation studied in [3,4]:

$$u_t = \triangle A(u) + \text{div}(b(u)). \tag{6}$$

It was showed that instead of the usual homogeneous boundary condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T),$$

one can only use part of the boundary condition

$$u(x, t) = 0, \quad (x, t) \in \Sigma_p \times (0, T), \tag{7}$$

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to ensure the posedness of the solutions, where

$$\Sigma_p = \bigcup_{\forall \eta \geq 0,\ \forall k \in \mathbb{R}} \Sigma_{1\eta k}. \tag{8}$$

If we assume that

$$a(0) = A'(0) = 0, \tag{9}$$

by virtue of (7), the well-posedness of the solutions of equation (6) has been established in [3,4]. The pioneering work on the degenerate equation (1) only with a partial boundary condition can be traced back to Wu–Zhao [1,2], in which a very special domain Ω was considered.

In fact, since equation (1) is not only degenerate on the boundary, but also strongly degenerate in the interior of the domain, how to impose a suitable boundary condition for the problems such as the well-posedness of solutions and stability has been an important and challenging problem for a long time. It is one of our main motivations for this study. For the completely degenerate case of $a \equiv 0$, equation (1) becomes a first order hyperbolic equation, and it is well-known that a smooth solution is constant along the maximal segment of the characteristic line in Q_T. When this segment intersects both $\{0\} \times \Omega$ and $\partial\Omega$, then the problem (1)–(3)–(4) is overdetermined if the boundary condition (4) is fulfilled in the traditional sense. Thus one needs to work within a suitable framework of entropy solutions and entropy boundary conditions. In the BV setting, Bardos–Leroux–Nedelec [5] gave an interpretation of the boundary condition (4) as an "entropy" inequality on $\partial\Omega$, which is the so-called BLN condition. However, since the trace of solutions is involved in the formulation of the BLN condition, it makes no sense if the solution is merely in L^∞. Otto [6] extended the Dirichlet problem for hyperbolic equations to the L^∞ setting and proved the uniqueness of the entropy solution by introducing an integral formulation of the boundary condition. This idea had been generalized to deal with the strongly degenerate parabolic equations (1). In particular, the homogeneous boundary condition was considered by Carrillo [7], Bendahmane–Karlsen [8], Andreianov–Bendahmane–Karlsen [9] and Andreianov–Bendahmane–Karlsen–Ouaro [10], and non-homogeneous boundary condition was considered by Kobayashi–Ohwa [11], Li–Wang [12], Mascia–Porretta–Terracina [13] and Michel–Vovelle [14]. In all these works, the boundary condition is not directly shown as (4) in sense of the trace, but is implicitly contained in a family of entropy inequalities (for example, [11]), or is treated in a special weak sense such as [12].

In this study, we mainly consider how the diffusion coefficient $a(u, x, t)$ affects the boundary value condition and the solution itself. To see this, let us recall some recent results closely related to the evolutionary p-Laplacian equations. Yin–Wang [15] had studied the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) - b_i(x)D_i u + c(x,t)u = f(x,t),$$

where $D_i = \frac{\partial}{\partial x_i}$, $a(x) \in C(\Omega)$, and $a(x) \geq 0$ in Ω may be equal to zero on the boundary $\partial\Omega$. By virtue of a reasonable integral expression, the boundary was separated into three parts: the nondegenerate boundary, the weakly degenerate boundary and the strongly degenerate boundary. The boundary value condition was supplemented definitely on the nondegenerate boundary and the weakly degenerate boundary. In other words, a partial boundary condition (7) is imposed. Recently, Zhan [16] studied the equation

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$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2}\nabla u) + f(u,x,t), \quad (x,t) \in Q_T, \tag{10}$$

with $\alpha > 0$, and proved the stability of the solutions of equation (10) without any boundary condition. Meanwhile, Jiri et al. [17] investigated the equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)|u^{\alpha-1}|u, \quad (x,t) \in Q_T, \tag{11}$$

with $0 < \alpha < 1$, and showed that the uniqueness of the solution of equation (11) is invalid. So, the degeneracy of the coefficient d^α can eliminate the affection from the source term $f(u,x,t)$. It may be predicted that the degeneracy on the boundary of the diffusion coefficient $a(u,x,t)$ will change the nature of boundary condition.

Inspired by existing results in the literature mentioned above, we conjecture that a partial boundary condition similar to (7) is necessary:

$$u(x,t) = g(x,t), \quad (x,t) \in \Sigma_p \times (0,T). \tag{12}$$

To match up with the boundary degeneracy (2), we should assume that

$$a(u,x,t) = a(g(x,t),x,t) = 0, \quad x \in \partial\Omega. \tag{13}$$

However, it seems that conditions (9) and (13) strongly depend on the boundary condition $u = g(x,t)$. So, a more natural condition to reflect that the equation is degenerate on the boundary may take the form:

$$a(\cdot,x,t) = 0, \quad x \in \partial\Omega, \tag{14}$$

no matter whether $u = g(x,t)$ or not.

Clearly, (14) implies (13), but (14) is independent of the boundary value condition. A simple example satisfying (6) is

$$a(u,x,t) = d^\alpha f(u,t),$$

where $d(x)$ is the distance function.

The purpose of this paper is to give the explicit formula of Σ_p, to define a reasonable entropy solution to match up with the partial boundary condition (12), and to establish the well-posedness of the solutions. We will show that, if the equation lacks the convection term, i.e. $b_i \equiv 0$, and the coefficient a has the natural boundary degeneracy as (14) and $|a_{x_i}| \le cd(x)$, then the stability of the solutions can be proved without any boundary condition. This result is truly novel, and distinguishes from the existing results in the literature. If the equation is completely degenerate, i.e. $a(u,x,t) \equiv 0$, the equation belongs to the type of the conservation law equations, and the stability of the solutions can be analyzed based on the partial boundary condition. Unlike those works in [5–16], the weak solutions (entropy solutions) in our study is in the BV sense as the ones in [1–4], so the partial boundary condition (12) is satisfied in a traditional way.

The rest of the paper is organized as follows. In Section 2, we present the entropy solution of equation (1) and state our main results. In Section 3, some technical lemmas are introduced and the estimates of the gradient of approximate solutions are shown. In Section 4, we explore the existence of entropy solution. Sections 5 and 6 are dedicated to the stability of entropy solutions.

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2. Main results

For small $\eta > 0$, we choose $h_\eta(s)$ and $S_\eta(s)$ as described in (5). Clearly, $h_\eta(s) \in C(\mathbb{R})$, and we have

$$h_\eta(s) \geq 0, \quad |\, sh_\eta(s)\,| \leq 1, \quad |\, S_\eta(s)\,| \leq 1,$$

and

$$\lim_{\eta\to 0} S_\eta(s) = \mathrm{sgn}s, \quad \lim_{\eta\to 0} sS'_\eta(s) = 0.$$

Definition 1. A function $u \in BV(Q_T) \cap L^\infty(Q_T)$ is said to be the entropy solution of equation (1) with the initial value (3), provided that

1. there exist $g^i \in L^2(Q_T)$ $(i = 1, 2, \cdots, N)$ such that for any $\varphi(x,t) \in C_0^1(Q_T)$ there holds

$$\iint_{Q_T} \varphi(x,t) g^i(x,t) dxdt = \iint_{Q_T} \varphi(x,t) \widehat{\sqrt{a(u,x,t)}} \frac{\partial u}{\partial x_i} dxdt,$$

where

$$\widehat{\sqrt{a(u,x,t)}}(u,x,t) = \int_0^1 \sqrt{a(\tau u^+ + (1-\tau)u^-, x, t)} d\tau,$$

which is called the composite mean value of $\sqrt{a(u,x,t)}$.

2. If $\varphi \in C_0^2(Q_T)$ and $\varphi \geq 0$, for $k \in \mathbb{R}$ and for any small $\eta > 0$ there holds

$$\iint_{Q_T} \left[I_\eta(u-k)\varphi_t - B^i_\eta(u,k)\varphi_{x_i} + A_\eta(u,x,t,k)\Delta\varphi - \sum_{i=1}^N S'_\eta(u-k) \mid g^i \mid^2 \varphi \right] dxdt$$
$$+ \iint_{Q_T} \int_k^u a_{x_i}(s,x,t) S_\eta(s-k) ds \varphi_{x_i} dxdt \geq 0, \tag{15}$$

where

$$B^i_\eta(u,k) = \int_k^u b'_i(s) S_\eta(s-k) ds,$$

$$A_\eta(u,x,t,k) = \int_k^u a(s,x,t) S_\eta(s-k) ds,$$

and

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$$I_\eta(u-k)=\int_0^{u-k} S_\eta(s)ds.$$

3. The initial value is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega |\,u(x,t)-u_0(x)\,|\,dx=0. \tag{16}$$

Definition 2. A function $u\in BV(Q_T)\cap L^\infty(Q_T)$ is said to be the entropy solution of equation (1) with the initial value condition (3) and the partial boundary condition (12), if u is the weak solution in the sense of Definition 1 and partial non-homogeneous boundary condition (12) is satisfied in the sense of trace

$$\gamma u\,|_{\Sigma_p\times(0,T)}=g(x,t). \tag{17}$$

For the reasonableness of Definition 1, if equation (1) has a classical solution u, multiplying both sides of equation (1) by $\varphi S_\eta(u-k)$ and integrating it over Q_T, we are able to see that u satisfies Definition 1. On the other hand, letting $\eta\to 0$ in (15), we have

$$\iint_{Q_T}\left[|u-k|\varphi_t-\operatorname{sgn}(u-k)(b_i(u)-b_i(k))\varphi_{x_i}+|A(u,x,t)-A(k,x,t)|\Delta\varphi\right]dxdt$$
$$+\iint_{Q_T}\int_k^u a_{x_i}(s,x,t)\operatorname{sgn}(s-k)ds\varphi_{x_i}dxdt\geq 0.$$

Thus, if u is an entropy solution as described in Definition 1, then u is actually the entropy solution as defined in [1,2,18] etc.

For the choice of the partial boundary condition of (7) (similarly, (12)), we know that it agrees well with the Fichera–Oleinik theory in the previous works [3,19]. On the other hand, by the definitions of $\Sigma_{1\eta k}$ and Σ_p, we have

$$0<S_\eta(k)[b_i(0)-b_i(k)]n_i(x)=-kS_\eta(k)b_i'(\zeta)n_i(x),$$

where $\zeta\in(k,0)$. If $\eta\to 0$, then

$$b_i'(\zeta)n_i(x)<0.$$

Letting $k\to 0$ leads to

$$b_i'(0)n_i(x)<0,$$

which is in accordance with the boundary entropy condition, see [Chapter 4 of [20]].

Before stating our main results, we assume that

$$g_t(x,t)=0,\quad x\in\partial\Omega, \tag{18}$$

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and

$$a(\cdot, x, t) = 0, \quad a_{x_i}(\cdot, x, t) = 0, \quad x \in \partial\Omega, \tag{19}$$

for $i = 1, 2, \cdots, N$.

Theorem 3. *If $A(s, x, t) \in C^3$, $b_i(s) \in C^2$, $u_0(x) \in L^\infty(\Omega)$, and there is a constant $\delta > 0$ such that*

$$a(u, x, t) - \delta \sum_{s=1}^{N+1} (a_{x_s})^2 \geq 0, \tag{20}$$

then equation (1) with the initial–boundary value conditions (3) and (12) has an entropy solution in the sense of Definition 2, provided that both conditions (18) and (19) hold.

Theorem 4. *Suppose that $A(s, x, t)$ is C^2 and satisfies the boundary degeneracy (19) and $b_i(s) \equiv 0$. Suppose that there exists a constant $\delta > 0$ such that*

$$| \sqrt{a(\cdot, x, \cdot)} - \sqrt{a(\cdot, y, \cdot)} | \leq c \, | x - y |^{2+\delta}. \tag{21}$$

If u and v are solutions of equation (1) with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively, and

$$|a_{x_i}(\cdot, x, t)| \leq c d(x),$$

then there holds

$$\int_\Omega | u(x, t) - v(x, t) | \, dx \leq \int_\Omega | u_0 - v_0 | \, dx,$$

where $d(x)$ represents the distance from the boundary as usual.

Note that Theorem 4 indicates that when $b_i(s) \equiv 0$, the solutions of equation (1) is completely controlled by the initial value, and free from any limitation of the boundary condition.

Theorem 5. *Suppose that $A(s, x, t)$ is C^2 and satisfies condition (14), $b_i(s) \equiv 0$ and condition (21) holds. Let u and v be solutions of equation (1) with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively, and with the same partial non-homogeneous value condition (12). Then there holds*

$$\int_\Omega | u(x, t) - v(x, t) | \, dx \leq \int_\Omega | u_0 - v_0 | \, dx + c \cdot esssup_{\Sigma'_p \times (0,T)} |u(x, t) - v(x, t)|, \tag{22}$$

where $\Sigma'_p = \partial\Omega \setminus \Sigma_p$.

If the equation is completely degenerate, it becomes a conservation law equation, and thus we have the following result on the stability.

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Theorem 6. *Suppose that $a(s,x,t)\equiv 0$, $b_i(s)$ is a Lipchitz function, and u and v are two solutions of equation (1) with the different initial values $u_0(x)$, $v_0(x)\in L^\infty(\Omega)$ respectively. If u and v satisfy the same partial non-homogeneous value condition (12), then the stability based on the partial boundary value condition (22) is true.*

For the well-known conservation equation

$$\frac{\partial u}{\partial t}+\operatorname{div}(f(u))=0, \tag{23}$$

only if the weak solution satisfies the entropy condition, the uniqueness of the solution is true. This is another motivation for us to consider the entropy solution of equation (23) (and equation (1)). Theorem 6 makes an effort to discuss the stability of the solutions related to the conservation equations only under the partial boundary condition. Finally, we have the following more general result, which includes Theorems 4, 5 and 6 as particular cases.

Theorem 7. *Suppose that $A(s,x,t)\in C^2$, $b_i(s)$ is a Lipchitz function, and condition (21) holds. If u and v are the solutions of equation (1) with the different initial values $u_0(x)$, $v_0(x)\in L^\infty(\Omega)$ respectively, and u and v satisfy the same partial non-homogeneous value condition (12), then there holds*

$$\int_\Omega |\,u(x,t)-v(x,t)\,|\,dx\le\int_\Omega |\,u_0-v_0\,|\,dx+c\cdot esssup_{\Sigma'_p\times(0,T)}|u(x,t)-v(x,t)|.$$

3. Some lemmas

Let $\mathbb{B}(0,R)$ be a ball with the radius R in $\mathbb{R}^{N-1}$, $f\in BV(C_R^+)$, and

$$C_R^+=\mathbb{B}(0,R)\times(0,R)=\mathbb{B}_R\times(0,R).$$

Then there exists a function $f^+\in L^1(\mathbb{B}_R)$ such that for H_{N-1}-almost all $y\in\mathbb{B}_R$, there holds

$$\lim_{\rho\to 0}\rho^{-N}\int\limits_{C_\rho^+(y)}|f(z)-f^+(y)|dz=0.$$

Moreover, if $C_R=\mathbb{B}_R\times(-R,R)$, for every $g\in C_0^1(C_R;\mathbb{R}^N)$ there holds

$$\int\limits_{C_R^+} f\operatorname{div}g\,dx=-\int\limits_{C_R^+}\langle g,Df\rangle+\int\limits_{\mathbb{B}_R}f^+g\,dH_{N-1},$$

where $\mathbb{B}_\rho=\{x\in\mathbb{R}^N;\,|\,x\,|<\rho\}$. The function f^+ is called the trace of f on $\mathbb{B}_R$ and is defined by

$$f^+(y)=\lim_{\rho\to 0}\frac{1}{|C_\rho^+(y)|}\int\limits_{C_\rho^+(y)}f(z)dz.$$

This definition of trace is easy to be generalized to a smooth domain Ω.

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Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, v be the normal of Γ_u at $X = (x,t)$, and $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y - X) > 0$ and $(v, Y - X) < 0$, respectively. For the continuous functions $p(u,x,t)$ and $u \in BV(Q_T)$, we define

$$\widehat{p}(u,x,t) = \int_0^1 p(\tau u^+ + (1-\tau)u^-, x, t)d\tau,$$

which is called the composite mean value of p. If $f(s) \in C^1(\mathbb{R})$ and $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1, 2, \cdots, N, N+1,$$

where $x_{N+1} = t$.

Lemma 8. *Let u be a solution of equation (1). Then we have*

$$a(s,x,t) = 0, \quad s \in I(u^+(x,t), u^-(x,t)) \ \ a.e.\ on\ \Gamma_u, \tag{24}$$

where $I(\alpha,\beta)$ denotes the closed interval with endpoints α and β, and (24) is true in the sense of the Hausdorff measure $H_N(\Gamma_u)$.

This lemma can be proved in an analogous manner as shown in [3], so we omit the details here.

Lemma 9 *([21]). Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set and $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega)$ $(1 \le q < \infty)$. Then we have*

$$\liminf_{k\to\infty} \| f_k \|^q_{L^q(\Omega)} \ge \| f \|^q_{L^q(\Omega)}.$$

We now consider the following regularized problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right) + \varepsilon\Delta u + \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{25}$$

with the initial–boundary conditions

$$u(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{26}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \tag{27}$$

Here, $u_{0\varepsilon}(x)$ is a mollified function of u_0. We know that there exist classical solutions $u_\varepsilon \in C^2(\overline{Q_T}) \bigcap C^3(Q_T)$ to the problem (25) provided that both $a(u,x,t)$ and $b_i(u)$ satisfy the assumptions given in Theorem 3. For more details, one can refer to [1] or [Chapter 8 of [22]].

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To get estimates of the solution u_ε to the problem (25), one may apply the maximum principle to get

$$| u_\varepsilon | \leq \| u_0 \|_{L^\infty} \leq M.$$

As for the BV estimates of u_ε, one can start with the local coordinates of the boundary $\Sigma = \partial\Omega$.

Let $\delta_0 > 0$ be sufficiently small such that

$$E^{\delta_0} = \{x \in \bar{\Omega}; dist(x, \Sigma) \leq \delta_0\} \subset \bigcup_{\tau=1}^{n} V_\tau,$$

where V_τ is a region, in which one can construct local coordinates

$$y_k = F_\tau^k(x) \ \ (k = 1, 2, \cdots, N) \ \text{ and } \ y_N \mid_\Sigma = 0,$$

with F_τ^k being appropriately smooth and $F_\tau^N = F_l^N$, such that the y_N-axis coincides with the inner normal vector.

Lemma 10 *([1]). Let u_ε be the solution of equation (25) with the initial–boundary conditions (26)–(27). If the assumptions in Theorem 3 are true, then there holds*

$$\varepsilon \int_\Sigma \left| \frac{\partial u_\varepsilon}{\partial n} \right| d\sigma \leq c_1 + c_2 \left(|\nabla u_\varepsilon|_{L^1(\Omega)} + \left| \frac{\partial u_\varepsilon}{\partial t} \right|_{L^1(\Omega)} \right),$$

where the constants c_i $(i = 1, 2)$ are independent of ε.

We have the following important estimate of the solutions u_ε of the problem (25)–(27).

Theorem 11. *Let u_ε be the solution of the problem (25)–(27). If the assumptions of Theorem 3 are true, then we have*

$$|gradu_\varepsilon|_{L^1(\Omega)} \leq c,$$

where c is independent of ε, and

$$|gradu_\varepsilon|^2 = \sum_{i=1}^{N} \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 + \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2.$$

Proof of Theorem 11. For simplicity, we denote u_ε by u and let $\{n_i\}_{i=1}^N$ represent the inner normal vector of Ω as before. Differentiate both sides of equation (25) with respect to x_s $(s =$

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$1, 2, \cdots, N, N+1$). Let $x_{N+1} = t$ and sum up for s after multiplying the resulting relation by $u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}$. Integrating it over Ω yields

$$\int_\Omega \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx = \int_\Omega \frac{\partial}{\partial t} \int_0^{|\text{grad}u|} S_\eta(\tau) d\tau dx$$
$$= \frac{d}{dt} \int_\Omega I_\eta(|\text{grad}u|) dx,$$

and

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i} [a(u,x,t)u_{x_i}]_{x_s} u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\nabla u|} dx \\
&= \int_\Omega \frac{\partial}{\partial x_i} [a' u_{x_i} u_{x_s} + a_{x_s} u_{x_i} + a(u,x,t,) u_{x_i x_s}] u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx \\
&= \int_\Omega \frac{\partial}{\partial x_i} (a' u_{x_i} u_{x_s} + a_{x_s} u_{x_i}) u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx \\
&\quad + \int_\Omega \frac{\partial}{\partial x_i} (a(u,x,t) u_{x_i x_s}) u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx,
\end{aligned} \tag{28}$$

where $a' = a_u(u,x,t)$.

Let us analyze each term on the right hand side of (28) separately.

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i} (a' u_{x_i} u_{x_s}) u_{x_s} \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx \\
&= \sum_{s=1}^{N+1} \int_\Omega \frac{\partial}{\partial x_i} (a' u_{x_i}) u_{x_s}^2 \frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|} dx + \int_\Omega a' u_{x_i} \frac{\partial}{\partial x_i} I_\eta(|\text{grad}u|) dx \\
&= \int_\Omega \frac{\partial}{\partial x_i} (a' u_{x_i}) |\text{grad}u| S_\eta(|\text{grad}u|) dx \\
&\quad - \int_\Sigma a' u_{x_i} n_i I_\eta(|\text{grad}u|) d\sigma - \int_\Omega I_\eta(|\text{grad}u|) \frac{\partial}{\partial x_i} (a' u_{x_i}) dx \\
&= \int_\Omega \frac{\partial}{\partial x_i} (a' u_{x_i}) \left[|\text{grad}u| S_\eta(|\text{grad}u|) - I_\eta(|\text{grad}u|) \right] dx \\
&\quad - \int_\Sigma a' u_{x_i} n_i I_\eta(|\text{grad}u|) d\sigma,
\end{aligned} \tag{29}$$

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$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i}(u_{x_i}a_{x_s})u_{x_s}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}dx\\
&\quad=\int_\Omega \frac{\partial}{\partial x_i}(u_{x_i}a_{x_s})\frac{\partial}{\partial \xi_s}I_\eta(|\mathrm{grad}u|)dx\\
&\quad=-\int_\Sigma u_{x_i}a_{x_s}n_i\frac{\partial}{\partial \xi_s}I_\eta(|\mathrm{grad}u|)d\sigma-\int_\Omega u_{x_i}a_{x_s}\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_px_i}dx,
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i}(a(u,x,t)u_{x_ix_s})u_{x_s}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}dx\\
&\quad=\int_\Omega \frac{\partial}{\partial x_i}(a(u,x,t)u_{x_ix_s})\frac{\partial}{\partial \xi_s}I_\eta(|\mathrm{grad}u|)dx\\
&\quad=-\int_\Sigma a(g(x,t),x,t)u_{x_ix_s}n_i\frac{\partial}{\partial \xi_s}I_\eta(|\mathrm{grad}u|)d\sigma\\
&\qquad-\int_\Omega a(u)\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx,
\end{aligned}\tag{31}$$

where $\xi_s = u_{x_s}$.

Moreover, we have

$$\begin{aligned}
&\varepsilon\int_\Omega \triangle u_{x_s}u_{x_s}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}dx\\
&\quad=-\varepsilon\int_\Sigma \frac{\partial I_\eta(|\mathrm{grad}u|)}{\partial x_i}n_i d\sigma-\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx,
\end{aligned}\tag{32}$$

and

$$\begin{aligned}
&\int_\Omega \nabla(b'(u)u_{x_s})u_{x_s}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}dx\\
&\quad=\int_\Omega \frac{\partial}{\partial x_i}(b_i'(u))|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)dx+\int_\Omega b_i'(u)\frac{\partial I_\eta(|\mathrm{grad}u|)}{\partial x_i}dx\\
&\quad=\int_\Omega \frac{\partial}{\partial x_i}(b_i'(u))\left[|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)-I_\eta(|\mathrm{grad}u|)\right]dx\\
&\qquad-\int_\Sigma b_i'(g(x,t))I_\eta(|\mathrm{grad}u|)n_i d\sigma.
\end{aligned}\tag{33}$$

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From (18)–(19), we derive that

$$a_{x_s}(g(x,t),x,t)=0,\quad x\in\partial\Omega.$$

Applying this to (28)–(33), then we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega I_\eta(|\mathrm{grad}u|)dx+\varepsilon\int_\Omega u_{x_sx_i}u_{x_px_s}\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}dx\\
&\quad+\int_\Omega a(u,x,t)u_{x_sx_i}u_{x_px_s}\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}dx\\
&\quad-\int_\Omega\left[\frac{\partial}{\partial x_i}(a'(u,x,t)u_{x_i})(|\mathrm{grad}u|S_\eta(|gradu|)-I_\eta(|\mathrm{grad}u|))\right]dx\\
&\quad+\int_\Omega a_{x_s}(u,x,t)u_{x_i}\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_px_i}dx\\
&\quad-\int_\Omega\frac{\partial}{\partial x_i}(b_i'(u))\left[|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)-I_\eta(|\mathrm{grad}u|)\right)]dx\\
&\quad-\Big[\int_\Sigma a'(u,x,t)u_{x_i}n_iI_\eta(|\mathrm{grad}u|)d\sigma+\int_\Sigma a_{x_s}(u,x,t)u_{x_i}u_{x_s}n_i\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}d\sigma\\
&\quad+\int_\Sigma b_i'(u)I_\eta(|\mathrm{grad}u|)n_id\sigma+\varepsilon\int_\Sigma\frac{\partial I_\eta(|\mathrm{grad}u|)}{\partial x_i}n_id\sigma\Big]\\
&=0.
\end{aligned}\tag{34}$$

Meanwhile, if we set

$$\begin{pmatrix}v_1^i\\v_2^i\\\vdots\\v_{N+1}^i\end{pmatrix}=\begin{pmatrix}q^{11}&q^{12}&\cdots&q_{1N+1}\\q^{21}&q^{22}&\cdots&q_{1N+1}\\\vdots&&&\\q^{N+11}&q^{N+12}&\cdots&q_{N=1N+1}\end{pmatrix}\begin{pmatrix}u_{x_1x_i}\\u_{x_2x_i}\\\vdots\\u_{x_{N+1}x_i}\end{pmatrix},$$

where (q^{sp}) is the square root of $\left(\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}\right)$, then we find

$$\begin{aligned}
&\left|a_{x_s}u_{x_i}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_px_i}\right|\\
&=\left|(a_{x_1}u_{x_i},a_{x_2}u_{x_i},\cdots,a_{x_{N+1}}u_{x_i})(q^{sp})\begin{pmatrix}v_1^i\\v_2^i\\\vdots\\v_{N+1}^i\end{pmatrix}\right|
\end{aligned}$$

$$= \left| a_{x_s} u_{x_i} q^{sp} v_p^i \right|$$
$$\leq \sum_{i=1}^{N} \left[\delta \sum_{s,p=1}^{N+1} (a_{x_s} v_p^i)^2 + \frac{1}{4\delta} \sum_{s,p=1}^{N+1} (q^{sp} u_{x_i})^2 \right].$$

In view of the assumption

$$a(u,x,t) - \delta \sum_{s=1}^{N+1} (a_{x_s})^2 \geq 0,$$

we deduce that

$$\int_\Omega a(u,x,t) u_{x_s x_i} u_{x_p x_s} \frac{\partial^2 I_\eta(|\mathrm{grad} u|)}{\partial \xi_s \partial \xi_p} \varphi dx$$
$$- \int_\Omega a_{x_s}(u,x,t) u_{x_i} \frac{\partial^2 I_\eta(|\mathrm{grad} u|)}{\partial \xi_s \partial \xi_p} u_{x_p x_i} \varphi dx$$
$$\geq -\frac{1}{4\delta} \int_\Omega \sum_{s,p=1}^{N+1} \sum_{i=1}^{N} (q^{sp} u_{x_i})^2 dx$$
$$\geq -c \int_\Omega |\mathrm{grad} u| dx. \tag{35}$$

Observe that on Σ there holds

$$-b_i'(g(x,t)) \frac{\partial u}{\partial n} n_i = \varepsilon \triangle u + \frac{\partial}{\partial x_i} \left[a(g(x,t),x,t) \frac{\partial u}{\partial x_i} \right].$$

So the surface integrals in (34) can be re-written as

$$S = -\Big[\int_\Sigma b_i'(u) I_\eta(|\mathrm{grad} u|) n_i d\sigma + \varepsilon \int_\Sigma \frac{\partial I_\eta(|\mathrm{grad} u|)}{\partial x_i} n_i d\sigma$$
$$+ \int_\Sigma a' u_{x_i} n_i I_\eta(|\mathrm{grad} u|) d\sigma$$
$$+ \int_\Sigma a_{x_s}(u,x,t) u_{x_i} u_{x_s} n_i \frac{S_\eta(|\mathrm{grad} u|)}{|\mathrm{grad} u|} d\sigma \Big]$$
$$= -\varepsilon \int_\Sigma \left[\frac{\partial I_\eta(|\mathrm{grad} u|)}{\partial x_i} n_i - \triangle u \frac{I_\eta(|\mathrm{grad} u|)}{\frac{\partial u}{\partial n}} \right] d\sigma$$

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$$+\int_\Sigma a(u,x,t)\left[\frac{\partial I_\eta(|\mathrm{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\mathrm{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma$$

$$+\int_\Sigma a_{x_i}(u,x,t)u_{x_i}\frac{I_\eta(|\mathrm{grad}u|)}{\frac{\partial u}{\partial n}}d\sigma+\int_\Sigma a_{x_i}(u,x,t)u_{x_i}u_{x_s}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}n_i d\sigma$$

$$=-\varepsilon\int_\Sigma\left[\frac{\partial I_\eta(|\mathrm{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\mathrm{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma.$$

The last equality is due to (18) and (19).

In view of (18), we get

$$u_{x_{N+1}}|_\Sigma=g_t(x,t)|_\Sigma=0,$$

and then

$$\lim_{\eta\to0}S=-\varepsilon\int_\Sigma\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)(u_{x_ix_j}n_jn_i-\triangle u)d\sigma. \tag{36}$$

Using the local coordinates on V_τ $(\tau=1,2,\cdots,n)$, we have

$$y_k=F^k_\tau(x)\ \ (k=1,2,\cdots,N)\ \text{ and }\ y_m|_\Sigma=0.$$

By a straightforward calculation, on $\Sigma\bigcap V_\tau$ there holds

$$u_{x_ix_j}=\sum_{k=1}^{N}u_{y_Ny_k}F^N_{x_i}F^k_{x_j}+\sum_{k=1}^{N-1}u_{y_Ny_k}F^N_{x_i}F^k_{x_j}+u_{y_m}F^m_{x_ix_j},$$

and

$$u_{x_ix_j}n_jn_i=\frac{\sum_{k=1}^{N}u_{y_Ny_k}F^N_{x_i}F^k_{x_j}F^N_{x_j}F^N_{x_i}}{|\nabla F^N|^2}+\sum_{k=1}^{N-1}u_{y_Ny_k}F^k_{x_i}F^N_{x_j}+\frac{u_{y_m}F^m_{x_ix_j}F^N_{x_j}F^N_{x_i}}{|\nabla F^N|^2},$$

in which $F^k=F^k_\tau$. By the fact that the inner normal vector is

$$\vec{n}=-\left(\frac{\partial F^N}{\partial x_1},\cdots,\frac{\partial F^N}{\partial x_N}\right)=-\nabla F^N,$$

we find

$$u_{x_ix_j}n_jn_i-\triangle u=u_{y_m}\left(\frac{F^m_{x_ix_j}F^N_{x_j}F^N_{x_i}}{|\nabla F^N|^2}-F^m_{x_ix_i}\right).$$

It follows from Lemma 10 that $\lim_{\eta\to0}S$ can be estimated by $|\mathrm{grad}u|_{L_1(\Omega)}$.

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Letting $\eta \to 0$ in (34), using (35)–(36) and

$$\lim_{\eta\to 0}\left[|\text{grad}u|S_\eta(|\text{grad}u|) - I_\eta(|\text{grad}u|)\right] = 0,$$

we have

$$\frac{d}{dt}\int_\Omega |\text{grad}u|dx \le c_1 + c_2\int_\Omega |\text{grad}u|dx.$$

By the Gronwall inequality, we obtain

$$\int_\Omega |\text{grad}u|dxdt \le c. \tag{37}$$

By (25) and (37), since u is the short notation for u_ε as we assumed earlier, it is easy to see that

$$\iint_{Q_T} (a(u_\varepsilon, x, t) + \varepsilon) \mid \text{grad}u_\varepsilon \mid^2 dxdt \le c. \tag{38}$$

This implies that there exists a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u \in BV(Q_T) \cap L^\infty(Q_T)$ such that $u_{\varepsilon_n} \to u$ a.e. in Q_T. Consequently, the proof is completed. □

4. Existence of the solutions

In this section, we will show that the limitation function u of u_ε is a generalized solution of the problem (1)–(3)–(12).

Theorem 12. *If $A(s, x, t)$ is C^3, $b_i(s)$ is C^2, $u_0(x) \in L^\infty(\Omega)$, and there is a constant $\delta > 0$ such that condition (20) is true, then equation (1) with the initial value condition (3) has an entropy solution in the sense of Definition 1, provided that both conditions (18) and (19) are true.*

Proof of Theorem 12. Due to inequality (38), $\sqrt{a(u_\varepsilon, x, t)}\frac{\partial u_\varepsilon}{\partial x_i}$ is weakly compact in $L^2(Q_T)$. Without loss of generality, we may assume that $\sqrt{a(u_\varepsilon, x, t)}\frac{\partial u_\varepsilon}{\partial x_i}$ itself converges weakly in $L^2(Q_T)$ to a function $g^i \in L^2(Q_T)$. Then, for any $\varphi(x, t) \in C_0^1(Q_T)$ we have

$$\begin{aligned}
&\iint_{Q_T} \varphi g^i dxdt \\
&= \lim_{\varepsilon\to 0}\iint_{Q_T} \varphi\sqrt{a(u_\varepsilon, x, t)}\frac{\partial u_\varepsilon}{\partial x_i}dxdt \\
&= \lim_{\varepsilon\to 0}\iint_{Q_T} \varphi\left(\int_0^{u_\varepsilon}\sqrt{a(\tau, x, t)}d\tau\right)_{x_i} dxdt - \iint_{Q_T}\varphi\int_0^u\left(\sqrt{a(\tau, x, t)}\right)_{x_i} d\tau dxdt
\end{aligned}$$

$$= -\lim_{\varepsilon\to 0}\iint_{Q_T}\varphi_{x_i}\left(\int_0^{u_\varepsilon}\sqrt{a(\tau,x,t)}d\tau\right)dxdt - \iint_{Q_T}\varphi\int_0^{u}\left(\sqrt{a(\tau,x,t)}\right)_{x_i}d\tau dxdt$$

$$= -\iint_{Q_T}\varphi_{x_i}\left(\int_0^{u}\sqrt{a(\tau,x,t)}d\tau\right)dxdt - \iint_{Q_T}\varphi\left(\int_0^{u}(\sqrt{a(\tau,x,t)}\right)_{x_i}d\tau dxdt,$$

and

$$\iint_{Q_T}\varphi g^i dxdt = \iint_{Q_T}\varphi\widehat{\sqrt{a(u,x,t)}}\frac{\partial u}{\partial x_i}dxdt.$$

So u satisfies Part (1) of Definition 1.

Let $\varphi\in C_0^2(Q_T)$, $\varphi\geq 0$, and $\{n_i\}$ be the inner normal vector of Ω. Multiplying both sides of (25) by $\varphi S_\eta(u_\varepsilon-k)$ and integrating it over Q_T yields

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi S_\eta(u_\varepsilon-k)dxdt\\ &=\iint_{Q_T}\frac{\partial}{\partial x_i}\left(a(u_\varepsilon,x,t)\frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi S_\eta(u_\varepsilon-k)dxdt\\ &\quad+\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi S_\eta(u_\varepsilon-k)dxdt+\iint_{Q_T}\frac{\partial b_i(u_\varepsilon)}{\partial x_i}\varphi S_\eta(u_\varepsilon-k)dxdt.\end{aligned}\tag{39}$$

Considering each term on the right hand side of (39) by integration by parts gives

$$\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi S_\eta(u_\varepsilon-k)dxdt = -\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_t dxdt,\tag{40}$$

and

$$\begin{aligned}&\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi_1 S_\eta(u_\epsilon-k)dxdt\\ &=-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot[S_\eta(u_\varepsilon-k)\nabla\varphi+\varphi S'_\eta(u_\varepsilon-k)\nabla u_\varepsilon]dxdt,\\ &=-\varepsilon\iint_{Q_T}\nabla u_\varepsilon S_\eta(u_\varepsilon-k)\nabla\varphi dxdt-\varepsilon\iint_{Q_T}|\nabla u_\varepsilon|^2 S'_\eta(u_\varepsilon-k)\varphi dxdt.\end{aligned}\tag{41}$$

Denote $\nabla A=\{a(u_\varepsilon,x,t)u_{\varepsilon x_i}\}$. We further derive that

$$\begin{aligned}&\iint_{Q_T}\frac{\partial}{\partial x_i}\left(a(u_\varepsilon,x,t)\frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi S_\eta(u_\varepsilon-k)dxdt\\ &=-\iint_{Q_T}\nabla A\cdot[S_\eta(u_\varepsilon-k)\nabla\varphi+\varphi S'_\eta(u_\varepsilon-k)\nabla u_\varepsilon]dxdt\end{aligned}$$

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$$= -\iint_{Q_T} S_\eta(u_\varepsilon - k)\nabla A \cdot \nabla\varphi dxdt$$
$$- \iint_{Q_T} a(u_\varepsilon, x, t) \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt, \tag{42}$$

$$-\iint_{Q_T} S_\eta(u_\varepsilon - k)\nabla A \cdot \nabla\varphi dxdt$$
$$= \iint_{Q_T} A_\eta(u_\varepsilon, x, t, k)\triangle\varphi dxdt$$
$$+ \iint_{Q_T} \int_k^{u_\varepsilon} a_{x_i}(s, x, t) S_\eta(s - k) ds\varphi_{x_i} dxdt, \tag{43}$$

and

$$\iint_{Q_T} \frac{\partial b_i(u_\varepsilon)}{\partial x_i}\varphi S_\eta(u_\varepsilon - k)dxdt$$
$$= -\iint_{Q_T} B^i_\eta(u_\varepsilon, k)\varphi_{x_i} dxdt$$
$$= -\iint_{Q_T} [b_i(u_\varepsilon) - b_i(k)]\left[\frac{\partial\varphi}{\partial x_i} S_\eta(u_\varepsilon - k) + \varphi S'_\eta(u_\varepsilon - k)\frac{\partial u_\varepsilon}{\partial x_i}\right] dxdt. \tag{44}$$

From (39)–(42), we have

$$\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t dxdt + \iint_{Q_T} A_\eta(u_\varepsilon, x, t, k)\triangle\varphi dxdt$$
$$- \iint_{Q_T} B^i_\eta(u_\varepsilon, k)\varphi_{x_i} dxdt - \varepsilon\iint_{Q_T} \nabla u_\varepsilon \cdot \nabla\varphi S_\eta(u_\varepsilon - k)dxdt$$
$$- \varepsilon\iint_{Q_T} \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt + \iint_{Q_T} \int_k^{u_\varepsilon} a_{x_i}(s, x, t) S_\eta(s - k) ds\varphi_{x_i} dxdt$$
$$- \iint_{Q_T} a(u_\varepsilon, x, t) \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt$$
$$= 0.$$

By virtue of Lemma 9, we obtain

$$\liminf_{\varepsilon\to 0} \iint_{Q_T} S'_\eta(u_\varepsilon - k)a(u_\varepsilon, x, t)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i}\varphi dxdt$$
$$\geq \sum_{i=1}^{N} \iint_{Q_T} S'_\eta(u - k) \mid g^i \mid^2 \varphi dxdt. \tag{45}$$

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Letting $\varepsilon \to 0$ in (45), we arrive at (15). In addition, the proof of the initial value condition (3) satisfying (16) can be carried out in a similar manner as the idea shown in [Chapter 4 of [20]], so we omit the details here. □

Proof of Theorem 3. According to Theorem 11, we know that the solution u in Theorem 12 is in $BV(Q_T)$, so the trace of u on the boundary can be defined in the traditional way. This indicates that we can impose a partial boundary value condition as (17). □

5. Stability of the solutions without any boundary value condition

Proof of Theorem 4. If equation (1) lacks the convection term, i.e. $b_i \equiv 0$, let u and v be two entropy solutions of equation (1) with the initial values

$$u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x),$$

respectively, but without any boundary value condition.

By Definition 1, for any $\varphi \in C_0^2(Q_T)$, if $b_i(s) \equiv 0$ we have

$$\begin{aligned} &\iint_{Q_T} \left[I_\eta(u-k)\varphi_t + A_\eta(u,x,t,k)\Delta\varphi - \sum_{i=1}^{N} S'_\eta(u-k) \mid g^i \mid^2 \varphi \right] dxdt \\ &\quad + \iint_{Q_T} \int_k^u a_{x_i}(s,x,t) S_\eta(s-k) ds \varphi_{x_i} dxdt \\ &\geq 0, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\iint_{Q_T} \left[I_\eta(v-l)\varphi_\tau + A_\eta(v,y,\tau,l)\Delta\varphi - \sum_{i=1}^{N} S'_\eta(v-l) \mid g^i \mid^2 \varphi \right] dyd\tau \\ &\quad + \iint_{Q_T} \int_l^v a_{y_i}(s,y,\tau) S_\eta(s-l) ds \varphi_{y_i} dyd\tau \\ &\geq 0. \end{aligned} \tag{47}$$

Let

$$\psi(x,t,y,\tau) = \phi(x,t) j_h(x-y, t-\tau),$$

where $\phi(x,t) \geq 0$ and $\phi(x,t) \in C_0^\infty(Q_T)$. Take

$$j_h(x-y, t-\tau) = \omega_h(t-\tau) \Pi_{i=1}^N \omega_h(x_i - y_i),$$

where

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$$\omega(s) \in C_0^\infty(R), \quad \omega(s) \geq 0, \quad \omega_h(s) = \frac{1}{h}\omega\left(\frac{s}{h}\right),$$
$$\omega(s) = 0 \text{ if } |s| > 1,$$

and

$$\int_{-\infty}^{\infty} \omega(s)ds = 1.$$

Moreover, for any given positive constant δ there holds

$$\lim_{h\to 0} \omega_h'(s)s^{2+\delta} = 0. \tag{48}$$

We choose $k = v(y,\tau)$, $l = u(x,t)$ and $\varphi = \psi(x,t,y,\tau)$ in (46) and (47). Integrating them over Q_T yields

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\Big\{I_\eta(u-v)(\psi_t+\psi_\tau) + A_\eta(u,x,t,v)\Delta_x\psi + A_\eta(v,y,\tau,u)\Delta_y\psi \\
&\quad + \int_v^u a_{x_i}(s,x,t)S_\eta(s-v)ds\psi_{x_i} + \int_u^v a_{y_i}(s,y,\tau)S_\eta(s-u)ds\psi_{y_i} \\
&\quad - \sum_{i=1}^N S_\eta'(u-v)\left(|\, g^i(u,x,t)\,|^2 + |\, g^i(v,y,\tau)\,|^2\right)\psi\Big\}dxdtdyd\tau \\
&\geq 0.
\end{aligned} \tag{49}$$

By a straightforward calculation, we deduce that

$$\begin{aligned}
&\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \quad \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, \quad i = 1,\cdots,N, \\
&\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t} j_h, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i} j_h,
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\psi_{y_i}dxdtdyd\tau \\
&= -\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds(\phi j_h)_{x_i}dxdtdyd\tau \\
&\quad + \iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i} j_h dxdtdyd\tau
\end{aligned}$$

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$$
\begin{aligned}
&= -\iint_{Q_T}\int_{Q_T}\int_{u}^{v} a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi j_h n_i dxdtdyd\tau \\
&\quad -\iint_{Q_T}\iint_{Q_T}\int_{u}^{v} a_{y_i}(v,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau \\
&\quad +\iint_{Q_T}\iint_{Q_T}\int_{u}^{v} a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i} j_h dxdtdyd\tau \\
&= -\iint_{Q_T}\iint_{Q_T}\int_{u}^{v} a_{y_i}(v,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau \\
&\quad +\iint_{Q_T}\iint_{Q_T}\int_{u}^{v} a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i} j_h dxdtdyd\tau. \qquad (50)
\end{aligned}
$$

In view of (50), (49) becomes

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\{I_\eta(u-v)(\psi_t+\psi_\tau)+A_\eta(u,x,t,v)\psi_{x_ix_i}+A_\eta(v,y,\tau,u)\psi_{y_iy_i} \\
&\quad -\sum_{i=1}^{N} S'_\eta(u-v)\left(|\, g^i(u,x,t)\,|^2+|\, g^i(v,y,\tau)\,|^2\right)\psi \\
&\quad +\int_{v}^{u} a_{x_i}S_\eta(s-v)ds\phi_{x_i}j_h+\int_{u}^{v} a_{y_i}S_\eta(s-u)ds\phi_{x_i}j_h \\
&\quad +\int_{v}^{u} a_{x_i}S_\eta(s-v)ds\phi j_{hx_i}-\int_{u}^{v} a_{y_i}S'_\eta(s-u)dsu_{x_i}\phi j_h\}dxdtdyd\tau \\
&\geq 0. \qquad (51)
\end{aligned}
$$

For simplicity, we denote

$$
\begin{aligned}
I_3 &= \iint_{Q_T}\iint_{Q_T}[A_\eta(u,x,t,v)\psi_{x_ix_j}+A_\eta(v,y,\tau,u)\psi_{y_iy_j}]dxdtdyd\tau, \\
I_4 &= -\sum_{i=1}^{N}\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\left(|\, g^i(u,x,t)\,|^2+|\, g^i(v,y,\tau)\,|^2\right)\psi dxdtdyd\tau, \\
I_5 &= \iint_{Q_T}\iint_{Q_T}\int_{v}^{u} a_{x_i}S_\eta(s-v)ds\phi_{x_i}j_h dxdtdyd\tau,
\end{aligned}
$$

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$$I_6 = \iint\limits_{Q_T}\iint\limits_{Q_T}\int\limits_{u}^{v} a_{y_i} S_\eta(s-u)ds\phi_{x_i} j_h dxdtdyd\tau$$

$$I_7 = \iint\limits_{Q_T}\iint\limits_{Q_T}\int\limits_{v}^{u} a_{x_i} S_\eta(s-v)ds\phi j_{hx_i} dxdtdyd\tau,$$

$$I_8 = -\iint\limits_{Q_T}\iint\limits_{Q_T}\int\limits_{u}^{v} a_{y_i} S'_\eta(s-u)ds u_{x_i}\phi j_h dxdtdyd\tau.$$

Specifically, evaluating I_3 leads to

$$\begin{aligned} I_3 &= \iint\limits_{Q_T}\iint\limits_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_i x_i} j_h + 2\phi_{x_i} j_{hx_i} + \phi j_{hx_i x_i}) \\ &\quad + A_\eta(v,y,\tau,u)\phi j_{hy_i y_i}]dxdyd\tau dt \\ &= \iint\limits_{Q_T}\iint\limits_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_i x_i} j_h + \phi_{x_i} j_{hx_i}) - A_\eta(v,y,\tau,u)\phi_{x_i} j_{hx_i} \\ &\quad - \partial_{x_i} A_\eta(u,x,t,v)\phi j_{hx_i} - \partial_{x_i} A_\eta(v,y,\tau,u)\phi j_{hx_i}]dxdydtd\tau \\ &= I_{31} + I_{32} - I_7, \end{aligned}$$

where

$$I_{31} = \iint\limits_{Q_T}\iint\limits_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_i x_i} j_h + \phi_{x_i} j_{hx_i}) - A_\eta(v,y,\tau,u)\phi_{x_i} j_{hx_i}]dxdtdyd\tau$$

$$\begin{aligned} I_{32} = -\iint\limits_{Q_T}\iint\limits_{Q_T} \Bigg[&\int\limits_0^1 a(\sigma u^+ + (1-\sigma)u^-, x, t) S_\eta(\sigma u^+ + (1-\sigma)u^- - v)d\sigma \\ &+ \int\limits_0^1 \int\limits_{\sigma u^+ + (1-\sigma)u^-}^{v} a(s,y,\tau) S_\eta(s - \sigma u^+ - (1-\sigma)u^-)d\sigma ds \Bigg] \frac{\partial u}{\partial x_i}\phi j_{hx_j} dxdtdyd\tau. \end{aligned}$$

We separate $-I_4$ into two parts as

$$\begin{aligned} -I_4 &= \iint\limits_{Q_T}\iint\limits_{Q_T} S'_\eta(u-v)\left(|\, g^i(u,x,t)\,|^2 + |\, g^i(v,y,\tau)\,|^2\right)\psi dxdtdyd\tau \\ &= I_{41} + I_{42}, \end{aligned}$$

where

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$$I_{41} = \iint\limits_{Q_T} \iint\limits_{Q_T} S'_\eta(u-v)(|\, g^i(u,x,t)\,| - |\, g^i(v,y,\tau)\,|)^2 \psi dxdtdyd\tau,$$

$$I_{42} = 2\iint\limits_{Q_T} \iint\limits_{Q_T} S'_\eta(u-v) g^i(u,x,t) g^i(v,y,\tau)\psi dxdtdyd\tau.$$

By (21), a direct calculation gives

$$\begin{aligned}
&\lim_{h\to 0}\lim_{\eta\to 0}(I_{32}+I_{42})\\
&\quad = 2\lim_{h\to 0}\iint\limits_{Q_T}\iint\limits_{Q_T} \sqrt{a_{y_i}(u,y,\tau)}\,\mathrm{sgn}(v-u)\sqrt{a(u,x,t)}u_{x_i}\phi j_h dxdtdyd\tau\\
&\quad = 2\iint\limits_{Q_T} (\sqrt{a(u,x,t)})_{x_i}\sqrt{a(u,x,t)}\,\mathrm{sgn}(v-u)u_{x_i}\phi dxdt,
\end{aligned}$$

and

$$\begin{aligned}
I_8 &= -\iint\limits_{Q_T}\iint\limits_{Q_T}\int_u^v a_{y_i}(s,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau\\
&= -\iint\limits_{Q_T}\iint\limits_{Q_T}\int_u^v [a_{y_i}(s,y,\tau)-a_{y_i}(u,y,\tau)]S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau\\
&\quad -\iint\limits_{Q_T}\iint\limits_{Q_T}\int_u^v a_{y_i}(u,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau.
\end{aligned}$$

Thus, we know

$$\begin{aligned}
I_8 &\to -\iint\limits_{Q_T}\iint\limits_{Q_T}\int_u^v a_{y_i}(u,y,\tau)\,\mathrm{sgn}(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau, \text{ as } \eta\to 0\\
&= -2\iint\limits_{Q_T}\iint\limits_{Q_T}\int_u^v \sqrt{a(u,y,\tau)}(\sqrt{a(u,y,\tau)})_{y_i}\,\mathrm{sgn}(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau\\
&\to -2\iint\limits_{Q_T}\int_u^v \sqrt{a(u,x,t)}(\sqrt{a(u,x,t)})_{x_i}\,\mathrm{sgn}(v-u)dsu_{x_i}\phi dxdt, \quad \text{as } h\to 0\\
&= -\lim_{h\to 0}\lim_{\eta\to 0}(I_{32}+I_{42}).
\end{aligned}$$

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Note that

$$\lim_{\eta\to 0} A_\eta(u,x,t,v) = \lim_{\eta\to 0} A_\eta(v,y,\tau,u) = \operatorname{sgn}(u-v)(A(u,x,t)-A(v,y,\tau)).$$

So we get

$$\lim_{\eta\to 0}(A_\eta(u,x,t,v)\phi_{x_i} j_{hx_i} - A_\eta(u,y,\tau,v)\phi_{x_i} j_{hx_i}) = 0. \tag{52}$$

In view of (48) and (52), letting $\eta \to 0$ and $h \to 0$ leads to

$$\begin{aligned}\iint_{Q_T} \Big\{ &| u(x,t)-v(x,t) |\, \phi_t + \operatorname{sgn}(u-v)(A(u,x,t)-A(v,x,t))\Delta\phi \\ &+ \int_v^u a_{x_i}(s,x,t)\operatorname{sgn}(s-v)ds\phi_{x_i} + \int_u^v a_{x_i}(s,x,t)\operatorname{sgn}(s-u)ds\phi_{x_i}\Big\}dxdt \\ &\geq 0. \end{aligned} \tag{53}$$

Since $\partial\Omega$ is sufficiently smooth such that the distance function $d(x)$ is differentiable when x is near the boundary, we can choose ϕ in (53) as

$$\phi(x,t) = \omega_\lambda(x)\eta(t),$$

where $\eta(t) \in C_0^\infty(0,T)$, and $\omega_\lambda(x) \in C_0^1(\Omega)$ is defined as follows: for any given small enough $0 < \lambda$, let $0 \leq \omega_\lambda \leq 1$ and $\omega|_{\partial\Omega} = 0$. If $d(x) = \operatorname{dist}(x, \partial\Omega) \geq \lambda$, then

$$\omega_\lambda(x) = 1.$$

If $0 \leq d(x) \leq \lambda$, then

$$\omega_\lambda(d(x)) = 1 - \frac{(d(x)-\lambda)^2}{\lambda^2}.$$

By the fact that $|\nabla d| = 1$, $a \in C^1$ and $|a_{x_i}| \leq cd(x)$, it follows from (53) that

$$\iint_{Q_T} |u(x,t)-v(x,t)|\,\phi_t dxdt + c\int_0^T\int_{\Omega_\lambda} \eta(t)d\left|\frac{d-\lambda}{\lambda^2}\right| |\, u-v\,| dxdt \geq 0, \tag{54}$$

where $\Omega_\lambda = \{x \in \Omega, d(x,\partial\Omega) < \lambda\}$. In view of

$$d\left|\frac{d-\lambda}{\lambda^2}\right| = \frac{d(\lambda-d)}{\lambda^2} \leq 1,$$

letting $\lambda \to 0$ in (54) gives

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$$\int_{Q_T} |\, u(x,t) - v(x,t) \,|\, \eta_t' dxdt \geq 0.$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau - t}^{s - t} \alpha_\varepsilon(\sigma) d\sigma \quad \text{and} \quad \varepsilon < \min\{\tau, T - s\},$$

where $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$. Then we find

$$\int_0^T [\alpha_\varepsilon(t - s) - \alpha_\varepsilon(t - \tau)]\, |u - v|_{L^1(\Omega)} dt \geq 0.$$

As $\varepsilon \to 0$, one can see

$$|u(x,\tau) - v(x,\tau)|_{L^1(\Omega)} \leq |u(x,s) - v(x,s)|_{L^1(\Omega)}.$$

Consequently, the desired result follows by letting $s \to 0$. □

Proof of Theorem 5. As in the proof of Theorem 4, (46)–(53) are still valid. In (53) we choose $\phi(x,t) = \omega_\lambda(x)\eta(t)$ as before. By the fact that $|\nabla d| = 1$, $a \in C^1$ and $|a_{x_i}| \leq c$, it follows from (54) that

$$\iint_{Q_T} |u(x,t) - v(x,t)| \phi_t dxdt + \frac{c}{\lambda} \int_0^T \int_{\Omega_\lambda} \eta(t) \,|\, u - v \,|\, dxdt \geq 0. \tag{55}$$

According to the definition of the trace, we know

$$\lim_{\lambda \to 0} \int_0^T \eta(t) \frac{c}{\lambda} \int_{\Omega_\lambda} |\, u - v \,|\, dxdt = c \int_0^T \eta(t) |u - v|_\Sigma dt.$$

By (12) and (13), when $x \in \Sigma_p$ and $\gamma u = \gamma v = g(x,t)$, taking $\varepsilon \to 0$ and $\lambda \to 0$ in (55), we get

$$c \cdot \operatorname{essup}_{\Sigma_p' \times (0,T)} |u(x,t) - v(x,t)| + \int_{Q_T} |\, u(x,t) - v(x,t) \,|\, \eta_t' dxdt \geq 0.$$

Letting $0 < s < \tau < T$, we further have

$$c \cdot \operatorname{essup}_{\Sigma_p' \times (0,T)} |u(x,t) - v(x,t)| + \int_0^T [\alpha_\varepsilon(t - s) - \alpha_\varepsilon(t - \tau)]\, |u - v|_{L^1(\Omega)} dt \geq 0.$$

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As $\varepsilon \to 0$, we obtain

$$|u(x,\tau)-v(x,\tau)|_{L^1(\Omega)} \le |u(x,s)-v(x,s)|_{L^1(\Omega)} + c\cdot \operatorname{esssup}_{\Sigma'_p\times(0,T)}|u(x,t)-v(x,t)|.$$

Consequently, we arrive at the desired result by letting $s \to 0$. □

6. Completely degenerate equations

If $a(s,x,t)\equiv 0$, equation (1) is completely degenerate and becomes a conservation law equation.

Proof of Theorem 6. Let u, v be two entropy solutions of equation (1) with the initial values

$$u(x,0)=u_0(x) \ \text{ and } \ v(x,0)=v_0(x),$$

and partial non-homogeneous boundary value conditions (12).

Since $a_i(s,x,t)\equiv 0$ and $g^i=0$, according to Definition 1, if $\varphi_1 \in C_0^2(Q_T)$, we have

$$\begin{aligned} &\iint_{Q_T}\Big[I_\eta(u-k)\varphi_{1t} - B^i_\eta(u,k)\varphi_{1x_i}\Big]dxdt \ge 0,\\ &\iint_{Q_T}\Big[I_\eta(v-l)\varphi_{1\tau} - B^i_\eta(v,l)\varphi_{1y_i}\Big]dyd\tau \ge 0. \end{aligned} \tag{56}$$

Choose $\psi(x,t,y,\tau)=\phi(x,t)j_h(x-y,t-\tau)$. Take $k=v(y,\tau)$, $l=u(x,t)$ and $\varphi_1=\psi(x,t,y,\tau)$ in (56). Integrating it over Q_T yields

$$\iint_{Q_T}\iint_{Q_T}\Big[I_\eta(u-v)(\psi_t+\psi_\tau) - (B^i_\eta(u,v)\psi_{x_i} + B^i_\eta(v,u)\psi_{y_i})\Big]dxdtdyd\tau \ge 0.$$

Note that

$$\begin{aligned} \lim_{\eta\to 0} B^i_\eta(u,v) &= \lim_{\eta\to 0} B^i_\eta(v,u)\\ &= \operatorname{sgn}(u-v)(b_i(u)-b_i(v)). \end{aligned}$$

When $\eta \to 0$, we have,

$$\begin{aligned} &\iint_{Q_T}\iint_{Q_T}[B^i_\eta(u,v)\psi_{x_i} + B^i_\eta(v,u)\psi_{y_i}]dxdtdyd\tau\\ &\quad\to \iint_{Q_T}\iint_{Q_T}\operatorname{sgn}(u-v)[b_i(u)-b_i(v)]\phi_{x_i}j_h dxdtdyd\tau. \end{aligned}$$

As $h \to 0$, we have

$$\iint_{Q_T}\iint_{Q_T} \operatorname{sgn}(u-v)[b_i(u)-b_i(v)]\phi_{x_i} j_h dxdtdyd\tau \to \iint_{Q_T} \operatorname{sgn}(u-v)[b_i(u)-b_i(v)]\phi_{x_i} dxdt.$$

So as both $\eta \to 0$ and $h \to 0$, we obtain

$$\iint_{Q_T} |\, u(x,t)-v(x,t)\,|\,\phi_t - sgn(u-v)(b_i(u)-b_i(v))\phi_{x_i} \geq 0. \tag{57}$$

Choose $\phi(x,t)=\omega_\lambda(x)\eta(t)$ in (57). Since $b_i(s)$ is a Lipschitz function, using the fact that $|\nabla d|=1$, it follows from (57) that

$$\int_{Q_T} |u(x,t)-v(x,t)|\phi_t dxdt + c\int_0^T\int_{\Omega_\lambda} \eta(t)|\omega'_\lambda(d)|\,|\,u-v\,|\,dxdt \geq 0.$$

Since

$$\int_0^T\int_{\Omega_\lambda} \eta(t)|\omega'_\lambda(d)|\,|\,u-v\,|\,dxdt \leq \int_0^T \eta(t)\frac{c}{\lambda}\int_{\Omega_\lambda} |\,u-v\,|\,dxdt,$$

which is almost the same as (55), the rest of the proof is similar to that of Theorem 5. □

In the same manner, one can also prove Theorem 7, so we omit the details.

References

[1] Z. Wu, J. Zhao, The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, Chin. Ann. Math. Ser. B 4 (1) (1983) 57–76.
[2] Z. Wu, J. Zhao, Some general results on the first boundary value problem for quasilinear degenerate parabolic equations, Chin. Ann. Math. Ser. B 4 (3) (1983) 319–328.
[3] H. Zhan, Reaction diffusion equations with boundary degeneracy, Electron. J. Differential Equations 81 (2016) 1–13.
[4] H. Zhan, The solutions of a hyperbolic–parabolic mixed type equation on half-space domain, J. Differential Equations 259 (4) (2015) 1449–1481.
[5] C. Bardos, A.Y. Leroux, J.C. Nedelec, First order quasilinear equations with boundary condition, Comm. Partial Differential Equations 4 (1979) 1017–1024.
[6] F. Otto, Initial–boundary value problem for a scalar conservation law, C. R. Acad. Sci., Ser. 1 Math. 322 (1996) 729–734.
[7] J. Carrillo, Entropy solutions for nonlinear degenerate problems, Arch. Ration. Mech. Anal. 147 (1999) 269–361.
[8] M. Bendahmane, K.H. Karlsen, Uniqueness of entropy solutions for quasilinear anisotropic degenerate parabolic equation, Contemp. Math. 371 (2005) 1–27.
[9] B.P. Andreianov, M. Bendahmane, K.H. Karlsen, Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic–parabolic equations, preprint, http://www.math.ntnu.no/conservation/2009/001.html, 2009.
[10] B.P. Andreianov, M. Bendahmane, K.H. Karlsen, S. Ouaro, Well-posedness results for triply nonlinear degenerate parabolic equations, J. Differential Equations 247 (2009) 277–302.

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[11] K. Kobayasi, H. Ohwa, Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle, J. Differential Equations 252 (2012) 137–167.
[12] Y.C. Li, Q. Wang, Homogeneous Dirichlet problems for quasilinear anisotropic degenerate parabolic–hyperbolic equations, J. Differential Equations 252 (2012) 4719–4741.
[13] C. Mascia, A. Porretta, A. Terracina, Nonhomogeneous Dirichlet problems for degenerate parabolic–hyperbolic equations, Arch. Ration. Mech. Anal. 163 (2002) 87–124.
[14] A. Michel, J. Vovelle, Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods, SIAM J. Numer. Anal. 41 (2003) 2262–2293.
[15] J. Yin, C. Wang, Evolutionary weighted p-Laplacian with boundary degeneracy, J. Differential Equations 237 (2007) 421–445.
[16] H. Zhan, On a parabolic equation related to the p-Laplacian, Bound. Value Probl. 78 (2016) 1–8.
[17] B. Jiri, G. Peter, K. Lukas, T. Peter, Nonuniqueness and multi-bump solutions in parabolic problems with the p-Laplacian, J. Differential Equations 260 (2016) 991–1009.
[18] A.I. Volpert, S.I. Hudjave, Cauchy's problem for degenerate second order quasilinear parabolic equation, Math. USSR, Sb. 7 (1969) 365–387.
[19] H. Zhan, Homogeneous Dirichlet condition of an anisotropic degenerate parabolic equation, Bound. Value Probl. 22 (2015) 1–21.
[20] Z. Wu, J. Zhao, J. Yin, H. Li, Nonlinear Diffusion Equations, Word Scientific Publishing, Singapore, 2001.
[21] L.C. Evans, Weak convergence methods for nonlinear partial differential equations, in: Conference Board of the Mathematical Sciences, in: Regional Conferences Series in Mathematics, vol. 74, 1998.
[22] L.K. Gu, Second Order Parabolic Partial Differential Equations, Xiamen University Press, Xiamen, China, 2004.

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