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An Inhomogeneous Boundary Value Problem for Nonlinear Schrödinger Equations

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We consider a nonlinear Schrödinger equation in a domain $\Omega \subset \mathbb{R}^n$ with the inhomegeneous Dirichlet boundary condition $u = Q$ where Q is a given smooth function. The nonlinear term contributes a positive term to the energy. We prove the existence of global solutions of finite energy.

Key Words: nonlinear Schrödinger equation; inhomogenous boundary value problem; weak solution; global existence.

1. INTRODUCTION

We consider the standard inhomogeneous initial-boundary problem

$$\begin{cases} i\,\partial_t u = \Delta u - f(u) & \text{for} \quad x \in \Omega \subset \mathbb{R}^n, \\ u(x, 0) = \phi(x), & \\ u(x, t) = Q(x, t) & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{1}$$

One would think that the solution of this problem should be as easy as the corresponding homogeneous one, but it is not so. For instance, the simplest identity, conservation of the L^2 norm, takes the form

$$\partial_t \int_\Omega |u|^2\, dx = 2\,\mathrm{Im} \int_{\partial\Omega} \bar{u}\, \frac{\partial u}{\partial n}\, dS, \tag{2}$$

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where the derivative in the boundary integral is not expressible in terms of the boundary data Q. It is not obvious how to make use of such an estimate. The usual device for linear problems, that of reduction of the problem to the corresponding homogeneous one, does not work well either because the subtraction will spoil the properties of the nonlinear term. Nevertheless, we are able to prove the global existence.

The purpose of this paper is to prove the following existence theorem. Let Ω be a (bounded or unbounded) open subset of $\mathbb{R}^n$ with a C^∞ boundary. Let $f(u) = g\,|u|^{p-1}\,u$ for some constants $g > 0$ and $p > 1$. (A more general nonlinear term can be assumed but we omit such a discussion.)

THEOREM 1. *Let $\phi \in H^1(\Omega)$. Let $Q \in C^3(\partial\Omega \times (-\infty, \infty))$ have compact support and satisfy the compatibility condition $\phi(x) \equiv Q(x, 0)$ on $\partial\Omega$ in the sense of traces. Let $1 < p < \infty$. Then there exists a solution $u \in L^\infty_{loc}((-\infty, \infty); H^1(\Omega) \cap L^{p+1}(\Omega))$ to the problem (1) for $-\infty < t < \infty$. The PDE is understood in the sense of distributions while the boundary condition is understood as $u(\cdot, t) - Q(\cdot, t) \in H^1_0(\Omega)$ for a.e. t.*

In this paper we do not address the questions of uniqueness and regularity, which are quite non-trivial, except for a brief comment at the end.

There is a very large literature on nonlinear Schrödinger equations in $\mathbb{R}^n$. However, we are aware only of the following papers in a domain Ω with homogeneous boundary conditions. Y. Tsutsumi [8] and M. Tsutsumi [7] proved well-posedness for the homogeneous problem in an exterior domain with sufficiently small and smooth initial data. Large initial data in two dimensions were treated by Brezis and Gallouet [1] and M. Tsutsumi [6]. There are also some results asserting that solutions blow up under certain conditions.

For inhomogeneous boundary conditions we are aware only of certain results in one space dimension. Bu [2] proved the well-posedness of smooth solutions with arbitrarily large data for $n = 1$ and a nonlinear term of positive energy. Carroll and Bu [3] proved the same for $n = 1$ and a nonlinear cubic term of either sign. There are also some results in one dimension using inverse scattering techniques.

In order to prove Theorem 1, it is required to estimate the normal derivative $\partial u/\partial n$. We combine identity (2) with the energy identity and two other identities that involve the L^2 norm of $\partial u/\partial n$ over $\partial\Omega$ as well as many other terms. These estimates are derived in Section 2. In Section 3, we truncate the nonlinear term and combine the previous estimates for the approximate solution to obtain the required estimate for the L^2 norm of $\partial u/\partial n$. The passage to the limit in Section 4 is then standard.

2. A PRIORI ESTIMATES

Write $F(u)=\frac{2g}{p+1} f(u)\,\bar{u}=\frac{2g}{p+1}|u|^{p+1}$, $P=\nabla u|_{\partial\Omega}$, $\eta=\sum_j \partial_j\xi_j=\nabla\cdot\xi$ and $\mathbf{n}=(n_1, n_2, ..., n_n)$ standard unit outer normal vector for $\partial\Omega$. Since $\partial\Omega$ is smooth, there exists a smooth function $\xi=(\xi_1, \xi_2, ..., \xi_n)$ independent of t from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$\xi|_{\partial\Omega}=(n_1, n_2, ..., n_n)=\mathbf{n}. \tag{3}$$

In case $\partial\Omega$ is unbounded, we assume that (a) the derivatives up to third order of ξ are bounded and (b) there exists $R>0$ such that $Q(x, t)=0$ for $|x|>R$. We sometimes denote $\partial_j u=\partial u/\partial x_j$ for $j=1, 2, ..., n$.

Lemma 1. *Let u be a smooth solution to the initial-boundary value problem for the nonlinear Schrödinger equation* (1). *Then the following four identities are available. First,*

$$\partial_t \int_\Omega |u|^2\,dx=2\,\mathrm{Im}\int_{\partial\Omega} (\mathbf{n}\cdot P)\,\bar{Q}\,dS. \tag{I}$$

Second,

$$\partial_t \int_\Omega (|\nabla u|^2+F(u))\,dx=2\,\mathrm{Re}\int_{\partial\Omega} (\mathbf{n}\cdot P)\,\bar{Q}_t\,dS. \tag{II}$$

Third,

$$\begin{aligned}\partial_t \int_\Omega u(\xi\cdot\nabla\bar{u})\,dx&-\int_{\partial\Omega} Q\bar{Q}_t\,dS+\int_\Omega \eta u\bar{u}_t\,dx\\ &=-2i\int_{\partial\Omega} |\mathbf{n}\cdot P|^2\,dS-\sum_{m,j}\int_\Omega \partial_m\xi_j\,\partial_m u\,\partial_j\bar{u}\,dx+i\int_{\partial\Omega} |P|^2\,dS\\ &\quad -i\int_\Omega \eta\,|\nabla u|^2\,dx+i\int_{\partial\Omega} F(Q)\,dS-i\int_\Omega \eta F(u)\,dx.\end{aligned} \tag{III}$$

Fourth,

$$\begin{aligned} i\int_{\partial\Omega} &[2\,|\mathbf{n}\cdot P|^2-|P|^2+(\mathbf{n}\cdot\bar{P})\,Q\eta]\,dS\\ &=\int_{\partial\Omega} [Q\bar{Q}_t+iF(Q)]\,dS-\partial_t\int_\Omega u(\xi\cdot\nabla\bar{u})\,dx\\ &\quad -i\int_\Omega \{\eta[2\,|\nabla u|^2+F(u)]+(\nabla\eta\cdot\nabla\bar{u})\,u\}\,dx\\ &\quad -\int_\Omega \Big\{\sum_{j,m} \partial_m\xi_j\,\partial_m u\,\partial_j\bar{u}+\eta f(u)\,\bar{u}\Big\}\,dx.\end{aligned} \tag{IV}$$

Proof. To prove (I), we differentiate $|u|^2$ in t, substitute $u_t = -i\,\Delta u + if(u)$ and then integrate over Ω to obtain

$$\begin{aligned}
\partial_t \int_\Omega |u|^2\,dx &= 2\,\mathrm{Re}\int_\Omega \bar{u}u_t\,dx = 2\,\mathrm{Re}\int_\Omega \bar{u}(-i\,\Delta u + if(u))\,dx \\
&= 2\,\mathrm{Re}\int_\Omega \left(i\sum_j(-\partial_j(\partial_j u\cdot\bar{u}) + \partial_j u\,\partial_j\bar{u}) + ig\,|u|^{p+1}\right)dx \\
&= -2\,\mathrm{Re}\int_{\partial\Omega} i\sum_j n_j\,\partial_j u\,\bar{u}\,dS = 2\,\mathrm{Im}\int_{\partial\Omega}(\mathbf{n}\cdot P)\,\bar{Q}\,dS. \qquad (4)
\end{aligned}$$

For (II), we look at the following identity directly from the equation

$$\Delta u\bar{u}_t + \Delta\bar{u}u_t - g\,|u|^{p-1}\,(u\bar{u}_t + \bar{u}u_t) = 0 \qquad (5)$$

and obtain, after integrating over Ω, the identity

$$\begin{aligned}
0 &= 2\,\mathrm{Re}\int_\Omega \sum_j(\partial_j(\partial_j u\cdot\bar{u}_t) - \partial_j u\,\partial_j\bar{u}_t)\,dx - g\int_\Omega |u|^{p-1}\,\partial_t\,|u|^2\,dx \\
&= 2\,\mathrm{Re}\int_{\partial\Omega}\sum_j(n_j\,\partial_j u)\,\bar{u}_t\,dS - \int_\Omega \partial_t\,|\nabla u|^2\,dx - \int_\Omega \partial_t F(u)\,dx \\
&= 2\,\mathrm{Re}\int_{\partial\Omega}(\mathbf{n}\cdot P)\,\bar{Q}_t\,dS - \partial_t\left(\int_\Omega (|\nabla u|^2 + F(u))\,dx\right).
\end{aligned}$$

Thus (II) follows.

To establish (III), we write $\partial_j u = u_j$ and note that

$$u_t\bar{u}_j - \bar{u}_t u_j = \partial_t(u\bar{u}_j) - \partial_j(u\bar{u}_t). \qquad (6)$$

Multiplying (6) by ξ_j, we find

$$\xi_j(u_t\bar{u}_j - \bar{u}_t u_j) = \xi_j\,\partial_t(u\bar{u}_j) - \xi_j\,\partial_j(u\bar{u}_t) = \partial_t(u\xi_j\bar{u}_j) - \partial_j(u\bar{u}_t\xi_j) + u\bar{u}_t\,\partial_j\xi_j. \qquad (7)$$

On the other hand,

$$\begin{aligned}
\xi_j(u_t\bar{u}_j - \bar{u}_t u_j) &= \xi_j\{-2i\,\mathrm{Re}(\Delta u)\,\bar{u}_j + 2if(u)\,\bar{u}_j\} \\
&= \xi_j\left\{-2i\,\mathrm{Re}\sum_m((u_m\bar{u}_j)_m - u_m\bar{u}_{jm}) + 2if(u)\,\bar{u}_j\right\}
\end{aligned}$$

$$= -2i \operatorname{Re} \sum_m \{\partial_m(\xi_j u_m \bar{u}_j) - (\partial_m \xi_j)(u_m \bar{u}_j)\}$$
$$+ \sum_m \{i\, \partial_j(\xi_j\, |u_m|^2) - i(\partial_j \xi_j)\, |u_m|^2\}$$
$$+ i\{\partial_j(\xi_j F(u)) - (\partial_j \xi_j)\, F(u)\}. \tag{8}$$

Integrating (7) and (8) over Ω, we have the following two identities

$$\int_\Omega \xi_j(u_t \bar{u}_j - \bar{u}_t u_j)\, dx = \partial_t \int_\Omega u \xi_j \bar{u}_j\, dx - \int_{\partial\Omega} n_j u \bar{u}_t \xi_j\, dS + \int_\Omega (u\bar{u}_t)\, \partial_j \xi_j\, dx$$
$$= \partial_t \int_\Omega u \xi_j \bar{u}_j\, dx - \int_{\partial\Omega} n_j^2 Q \bar{Q}_t\, dS + \int_\Omega (u\bar{u}_t)\, \partial_j \xi_j\, dx \tag{9}$$

and

$$\int_\Omega \xi_j(u_t \bar{u}_j - \bar{u}_t u_j)\, dx$$
$$= -2i \operatorname{Re} \sum_m \int_\Omega (\partial_m(\xi_j u_m \bar{u}_j) - (\partial_m \xi_j)\, u_m \bar{u}_j)\, dx$$
$$+ i \sum_m \int_\Omega \{\partial_j(\xi_j\, |u_m|^2) - (\partial_j \xi_j)\, |u_m|^2\}\, dx$$
$$+ i \int_\Omega \{\partial_j(\xi_j F(u)) - (\partial_j \xi_j)\, F(u)\}\, dx$$
$$= -2i \operatorname{Re} \sum_m \int_{\partial\Omega} n_m n_j P_m \bar{P}_j\, dS + 2i \operatorname{Re} \sum_m \int_\Omega \partial_m \xi_j\, \partial_m u\, \partial_j \bar{u}\, dx$$
$$+ i \sum_m \int_{\partial\Omega} n_j^2\, |P_m|^2\, dS - i \sum_m \int_\Omega (\partial_j \xi_j)\, |u_m|^2\, dx$$
$$+ i \int_{\partial\Omega} n_j^2 F(Q)\, dS - i \int_\Omega (\partial_j \xi_j)\, F(u)\, dx. \tag{10}$$

Combining (9), (10) and adding $j = 1, 2, ..., n$, we obtain

$$\partial_t \int_\Omega u(\xi \cdot \nabla \bar{u})\, dx - \int_{\partial\Omega} Q \bar{Q}_t\, dS + \int_\Omega u \bar{u}_t \eta\, dx$$
$$= -2i \int_{\partial\Omega} |\mathbf{n} \cdot P|^2\, dS - \sum_{m,\, j} \int_\Omega \partial_m \xi_j\, \partial_m u\, \partial_j \bar{u}\, dx + i \int_{\partial\Omega} |P|^2\, dS$$
$$- i \int_\Omega \eta\, |\nabla u|^2\, dx + i \int_{\partial\Omega} F(Q)\, dS - i \int_\Omega \eta F(u)\, dx. \tag{11}$$

Therefore (III) follows. In (III) we must still transform the term $\int_\Omega \eta u \bar{u}_t \, dx$ into a usable form. This is the purpose of (IV).

To prove (IV), we multiply our Schrödinger equation (1) by $\eta \bar{u}$ and integrate over Ω to obtain

$$\begin{aligned} 0 &= \int_\Omega (iu_t - \Delta u + f(u)) \, \eta \bar{u} \, dx \\ &= \int_\Omega u_t \bar{u} \eta \, dx - \sum_m \int_{\partial\Omega} n_m P_m \eta \bar{Q} \, dS \\ &\quad + \sum_m \int_\Omega u_m \eta_m \bar{u} \, dx + \int_\Omega \eta \, |\nabla u|^2 \, dx + \int_\Omega f(u) \, \eta \bar{u} \, dx. \end{aligned} \tag{12}$$

We divide (12) by i and take the complex conjugate to obtain

$$\begin{aligned} \int_\Omega \eta u \bar{u}_t \, dx &= -i \int_{\partial\Omega} \mathbf{n} \cdot \bar{P} Q \eta \, dS + i \int_\Omega (\nabla \eta \cdot \nabla \bar{u}) \, u \, dx \\ &\quad + i \int_\Omega \eta \, |\nabla u|^2 \, dx + \int_\Omega \eta f(u) \, \bar{u} \, dx. \end{aligned} \tag{13}$$

Finally, we replace $\int_\Omega \eta u \bar{u}_t \, dx$ in (III) by (13) to obtain

$$\begin{aligned} &\partial_t \int_\Omega u(\xi \cdot \nabla u) \, dx - \int_{\partial\Omega} Q \bar{Q}_t \, dS - i \int_{\partial\Omega} \mathbf{n} \cdot \bar{P} Q \eta \, dS \\ &\quad + i \int_\Omega (\nabla \eta \cdot \nabla \bar{u}) \, u \, dx + 2i \int_\Omega \eta \, |\nabla u|^2 \, dx + \int_\Omega \eta f(u) \, \bar{u} \, dx \\ &= -2i \int_{\partial\Omega} |n \cdot P|^2 \, dS - \sum_{m,j} \int_\Omega \partial_m \xi_j \, \partial_m u \, \partial_j \bar{u} \, dx + i \int_{\partial\Omega} |P|^2 \, dS \\ &\quad + i \int_{\partial\Omega} F(Q) \, dS - i \int_\Omega \eta F(u) \, dx. \end{aligned} \tag{14}$$

Rearranging (14), we get (IV). ∎

We will prove the global existence theorem by using the above estimates.

3. THE APPROXIMATE EQUATION

We will approximate the original equation by truncating the nonlinear term as follows. Let q_0 be an upper bound for $|Q|$. For any $k > q_0$, we define

$$f_k(u) = \begin{cases} g\,|u|^{p-1}\,u & |u| < k \\ gk^{p-1}u & |u| \geqslant k. \end{cases} \tag{15}$$

First, we construct the local solution of the truncated problem (15). For convenience we only consider $t \geqslant 0$.

LEMMA 2. *For any $k > q_0$ and $c_0 > 0$, there exists $T_0 > 0$ such that if $\|\phi\|_{H^1} \leqslant c_0$ then there exists a unique solution $u^{(k)} \in C([0, T_0]; H^1(\Omega))$ which solves*

$$\begin{cases} i\,\partial_t u^{(k)} = \Delta u^{(k)} - f_k(u^{(k)}), & x \in \Omega \subset \mathbb{R}^n, \quad t > 0 \\ u^{(k)}(x, 0) = \phi(x) & \\ u^{(k)}(x, t) = Q(x, t) & \text{for} \quad (x, t) \in \partial\Omega \times [0, \infty). \end{cases} \tag{16}$$

Proof. Notice that f_k is globally Lipschitz for each $k > 0$. By the compact support assumption, $Q = 0$ when $|x| > R$. We shall always take $k > q_0$, so that $f_k(Q) = f(Q)$. For convenience we drop the superscript k. We first convert the problem into a problem with a homogeneous boundary condition, by writing $v = u - \tilde{Q}$. Here we choose $\tilde{Q}(x, t)$ to be any C^3 function on $\bar{\Omega} \times [0, \infty)$ with compact support in x such that

$$\Delta \tilde{Q} = f(Q) - iQ_t \qquad \text{on } \partial\Omega, \tag{17}$$

$$\tilde{Q} = Q \qquad \text{on } \partial\Omega. \tag{18}$$

(In fact, any finite number of derivatives can be specified on $\partial\Omega$; see for instance Lemma 13.1 in [4]). Then v satisfies the following problem, equivalent to (1),

$$\begin{cases} iv_t - \Delta v = h_k \\ v(0) = \phi - \tilde{Q}(0) \\ v|_{\partial\Omega} = 0 \end{cases} \tag{19}$$

where $h_k = f_k(v + \tilde{Q}) - i\tilde{Q}_t + \Delta\tilde{Q}$. Notice that by (17), (18) and (19), h_k formally vanishes on $\partial\Omega$. Let $e^{i\Delta t}$ be the evolution operator for the linear Schrödinger equation with homogeneous boundary condition. Then $e^{i\Delta t}$ is a group of unitary operators on $H_0^1(\Omega)$ to itself. Now, the problem (19) can be written as the integral equation

$$v(t) = e^{i\Delta t}v(0) + \int_0^t e^{i\Delta(t-\tau)}h_k(\tau)\,d\tau = \mathcal{N}v(t) \tag{20}$$

where $v(t) \in H_0^1(\Omega)$ and $\mathcal{N}$ is defined by the right side of (20).

Taking the $H_0^1(\Omega)$ norm for each $T > 0$, there exists a constant $\tilde{c}_{k,T}$ such that

$$\begin{aligned}\|\mathcal{N}v(t)\|_{H_0^1} &\leqslant 1 \cdot \|v(0)\|_{H_0^1} + \int_0^t 1 \cdot \|f_k(\tilde{Q}+v) - i\tilde{Q}_t + \Delta\tilde{Q})(\tau)\|_{H_0^1}\,d\tau,\\ &\leqslant \|v(0)\|_{H_0^1} + c_k \int_0^t \|v(\tau)\|_{H_0^1}\,d\tau + \tilde{c}_{k,T}\end{aligned} \tag{21}$$

for $0 \leqslant t \leqslant T$. Since f_k is Lipschitz,

$$\begin{aligned}\|\mathcal{N}v(t) - \mathcal{N}w(t)\|_{H_0^1} &\leqslant \|v(0) - w(0)\|_{H_0^1} + \int_0^t \|f_k(\tilde{Q}+v) - f_k(\tilde{Q}+w)\|_{H_0^1}\,d\tau\\ &\leqslant \|v(0) - w(0)\|_{H_0^1} + c_k \int_0^t \|v(\tau) - w(\tau)\|_{H_0^1}\,d\tau.\end{aligned} \tag{22}$$

Let $|||\cdot|||$ denote the norm in $C([0, T_0]; H_0^1(\Omega))$. Let $v(0)$ be given in $H_0^1(\Omega)$ and $\|v(0)\|_{H_0^1} \leqslant c_0'$. Let

$$B = \{v \in C([0, T_0]; H_0^1(\Omega)) : |||v||| \leqslant c^*, v(0) = \psi\} \tag{23}$$

where $c^* = 2(c_0' + \tilde{c}_{k,T})$. If $T_0 \leqslant \frac{1}{2c_k}$, then (21) and (22) imply that $\mathcal{N}$ is a contraction on B. Hence for any $k > q_0$ and any $c_0' > 0$, there exists $T_0 > 0$ such that if $\|v(0)\|_{H_0^1} \leqslant c_0'$, then there is a unique solution $v^{(k)} \in B$ to (20). Here T_0 depends on k and c_0'. If we define $\psi = \phi - \tilde{Q}(0)$, then $u^{(k)} = v^{(k)} + \tilde{Q}$ is the unique solution to (16) in $[0, T_0]$. ∎

The following lemma establishes an a priori bound for the solution $u^{(k)}$ above.

LEMMA 3. *Let $T > 0$ and $k > q_0$ be fixed. Let $u^{(k)}$ be a solution of* (16) *in the space $C([0, T]; H^1(\Omega))$. Then there exists a constant $C_T > 0$ independent of k such that $\|u^{(k)}(t)\|_{H^1} \leqslant C_T$ for all $0 \leqslant t \leqslant T$.*

Proof. Define $F_k(u) = G_k(|u|)$, $G_k' = g_k$ and $f_k(u) = g_k(|u|) \cdot \frac{u}{|u|}$. Then $g_k(0) = 0$, $g_k \geqslant 0$ and $G_k \geqslant 0$. Further, $\bar{u}f_k(u) = \frac{g_k(|u|)}{|u|}u\bar{u} = g_k(|u|)\,|u| \geqslant 0$. We check each of the identities (I)–(IV) in the case that f is replaced by f_k.

Because of the numerous integrations by parts, these identities are justifiable only for solutions of sufficient regularity. If we approximate ϕ, Q and f_k by sufficient smooth functions, then the unique solution $u^{(k)}$ of Lemma 2 is also smooth, by the same kind of contraction argument. Then the identities are derived for the approximations and a subsequent passage to the limit leads to the following identities for $u^{(k)}$. First,

$$\partial_t \int_\Omega |u|^2\, dx = 2\,\mathrm{Im} \int_{\partial\Omega} (\mathbf{n}\cdot P)\,\bar{Q}\, dS, \tag{I_k}$$

where $u = u^{(k)}$ is a solution to (16). Second,

$$\partial_t \int_\Omega (|\nabla u|^2 + F_k(u))\, dx = 2\,\mathrm{Re} \int_{\partial\Omega} (\mathbf{n}\cdot P)\,\bar{Q}_t\, dS. \tag{II_k}$$

Third,

$$\begin{aligned} &\partial_t \int_\Omega u(\xi\cdot\nabla u)\, dx - \int_{\partial\Omega} Q\bar{Q}_t\, dS + \int_\Omega \eta u \bar{u}_t\, dx \\ &\quad = -2i \int_{\partial\Omega} |\mathbf{n}\cdot P|^2\, dS - \sum_{m,\,j} \int_\Omega \partial_m \xi_j\, \partial_m u\, \partial_j \bar{u}\, dx + i \int_{\partial\Omega} |P|^2\, dS \\ &\quad\quad - i \int_\Omega \eta\, |\nabla u|^2\, dx + i \int_{\partial\Omega} F_k(Q)\, dS - i \int_\Omega \eta F_k(u)\, dx. \end{aligned} \tag{III_k}$$

Fourth,

$$\begin{aligned} &i \int_{\partial\Omega} (2\,|\mathbf{n}\cdot P|^2 - |P|^2 + (\mathbf{n}\cdot\bar{P})\, Q\eta)\, dS \\ &\quad = \int_{\partial\Omega} [Q\bar{Q}_t + iF_k(Q)]\, dS - \partial_t \int_\Omega u(\xi\cdot\nabla u)\, dx \\ &\quad\quad - i \int_\Omega (\eta[2\,|\nabla u|^2 + F_k(u)] + (\nabla\eta\cdot\nabla\bar{u})\, u)\, dx \\ &\quad\quad + \int_\Omega \left(-\sum_{j,\,m} \partial_m \xi_j\, \partial_m u\, \partial_j \bar{u} + \eta f_k(u)\, \bar{u} \right) dx. \end{aligned} \tag{IV_k}$$

Our next goal is to obtain a bound on the integral of $|\mathbf{n}\cdot P|^2$ where $P = \nabla u|_{\partial\Omega}$ and $u = u^{(k)}$ is a smooth solution of the approximate problem. Then at each point we can write

$$|P|^2 = |\mathbf{n}\cdot P|^2 + |A\cdot P|^2 = |\mathbf{n}\cdot P|^2 + |A\cdot\nabla\tilde{Q}|^2 \tag{24}$$

where $A \cdot P$ denotes the tangential component of P. Substituting (24) into (IV_k), integrating over $\Omega \times [0, t]$ and using the assumption that up to three derivatives of ξ are bounded, we obtain

$$\begin{aligned}
\int_0^t \int_{\partial\Omega} |\mathbf{n} \cdot P|^2 \, dS \, d\tau \leqslant{}& \left| \int_\Omega u(\xi \cdot \nabla u) \, dx \right| + \left| \int_\Omega \phi(\xi \cdot \nabla \phi) \, dx \right| \\
&+ \int_0^t \int_{\partial\Omega} |A \cdot \nabla \tilde{Q}|^2 \, dS \, d\tau \\
&+ \int_0^t \int_{\partial\Omega} |Q \bar{Q}_t| \, dS \, d\tau + c \int_0^t \int_{\partial\Omega} |(\mathbf{n} \cdot \bar{P}) \, Q| \, dS \, d\tau \\
&+ c \int_0^t \int_\Omega |\nabla u| \, |u| \, dx \, d\tau + c \int_0^t \int_\Omega |\nabla u|^2 \, dx \, d\tau \\
&+ \int_0^t \int_{\partial\Omega} F_k(Q) \, dS \, d\tau + c \int_0^t \int_\Omega F_k(u) \, dx \, d\tau. \qquad (25)
\end{aligned}$$

Here we note that for $k > q_0$, $F_k(Q) = F(Q)$ and Im $\eta f_k(u) \, \bar{u} = 0$. Since $\phi \in H^1(\Omega)$, Q is C^3 with compact support in x, each term in (25) involving ϕ and Q is bounded. Therefore (25) is estimated as

$$\begin{aligned}
\int_0^t \int_{\partial\Omega} |\mathbf{n} \cdot P|^2 \, dS \, d\tau \leqslant{}& c' + c' \left(\int_0^t \int_{\partial\Omega} |\mathbf{n} \cdot P|^2 \, dS \, d\tau \right)^{1/2} \\
&+ c' \int_\Omega (|u|^2 + |\nabla u|^2) \, dx + c' \int_0^t \int_\Omega (|u|^2 + |\nabla u|^2) \, dx \, d\tau \\
&+ c' \int_0^t \int_\Omega F_k(u) \, dx \, d\tau. \qquad (26)
\end{aligned}$$

It is important to note that all the constants denoted by c and c' only depend on n, p, Q, T, ϕ and $\partial\Omega$, but not on k or u. We write

$$J^2 = \int_0^t \int_{\partial\Omega} |\mathbf{n} \cdot P|^2 \, dS \, d\tau. \qquad (27)$$

Then (26) is equivalent to $J^2 \leqslant \alpha^2 + 2\beta J$, where $2\beta = c'$ and α^2 is the sum of all the other terms in the right side of (26). By completing the square, we have $(J - \beta)^2 \leqslant \alpha^2 + \beta^2$. Taking the square root of both sides, we obtain

$$J \leqslant 2\beta + \alpha = c' + \alpha. \qquad (28)$$

Denoting

$$\gamma(t)=\int_{\Omega}(|u|^2+|\nabla u|^2+F_k(u))\,dx, \tag{29}$$

we have

$$\alpha^2\leqslant c_1+c_2\gamma+c_3\int_0^t\gamma(\tau)\,d\tau. \tag{30}$$

Now we use (I_k), (II_k) to estimate $\gamma(t)$ as follows. From (I_k)

$$\|u\|_2^2\leqslant c'+\tilde{c}\left(\int_0^t\int_{\partial\Omega}|\mathbf{n}\cdot P|^2\,dS\,d\tau\right)^{1/2}. \tag{31}$$

From (II_k)

$$\|\nabla u\|_2^2+\int_{\Omega}F_k(u)\,dx\leqslant c_0+\hat{c}\left(\int_0^t\int_{\partial\Omega}|\mathbf{n}\cdot P|^2\,dS\,d\tau\right)^{1/2}. \tag{32}$$

Also we will continue to use c_0, c', $\hat{c}$, $\tilde{c}$ etc as generic constants depending on T. By summing (31) and (32), we have

$$\gamma(t)\leqslant m+m'J. \tag{33}$$

But (28) and (30) imply that

$$J\leqslant c_1'+c_2'\sqrt{\gamma}+c_3'\int_0^t\gamma(\tau)\,d\tau. \tag{34}$$

By the last two inequalities and Gronwall's lemma, $\gamma(t)$ is bounded on $[0, T]$ for any T. Since $F_k>0$, we deduce that $\|u\|_{H^1}$ is bounded for bounded T. This proves Lemma 3. ∎

4. GLOBAL EXISTENCE THEOREM FOR THE H^1 SOLUTION

Now that we have the bound on the energy, the existence theorem follows by a well-known argument. For more details of the following proof, see [5] for instance.

Proof of Theorem 1. Let $u^{(k)}$ be the solution in Lemma 2. It follows from Lemma 3 that it has a unique extension (we still call it $u^{(k)}$) to

$0 \leqslant t < \infty$ such that for all T, $u^{(k)} \in C([0, T]; H^1(\Omega))$ and there exists a constant C_T such that

$$\sup_{0 \leqslant t \leqslant T} \|u^{(k)}(t)\|_{H^1} \leqslant C_T. \tag{35}$$

For $T=1$, there exists a subsequence $u_1^{(k)}$ converging weakly* in $L^\infty([0, 1]; H^1(\Omega))$. Similarly, for $T=2$, there exists a subsequence of $u_1^{(k)}$, denoted by $u_2^{(k)}$, converging weakly* in $L^\infty([0, 2]; H^1(\Omega))$. We repeat the same process for $T=1, 2, ...$ and choose the diagonal sequence $u_k^{(k)}$, $k=1, 2, ...$. Then there exists $u \in L^\infty_{loc}([0, \infty); H^1(\Omega))$ such that $u_k^{(k)}$ converges weakly* to u in $L^\infty([0, T]; H^1(\Omega))$ for any $T>0$. By the fact that $\int_\Omega F_k(u^{(k)})\,dx$ is bounded, we know that $f_k(u^{(k)})$ is also bounded in $L^\infty([0, T]; L^1 + L^2)$. By (16),

$$\partial_t u^{(k)} = -i\,\Delta u^{(k)} + i f_k(u^{(k)}) \tag{36}$$

is bounded in $L^\infty([0, T]; L^1 + H^{-1})$. By Aubin's Compactness Theorem and Cantor diagonalization, there exists a subsequence of $u^{(k)}$ (again called $u^{(k)}$) which converges to u a.e. in $\Omega \times [0, \infty)$. Therefore, $f_k(u^{(k)})$ also converges a.e. to $f(u)$. Since the integral of $F_k(u^{(k)})$ is bounded (for bounded t), it follows from Egoroff's lemma that $f_k(u^{(k)}) \to f(u)$ strongly in $L^1(\Omega')$ for any bounded set $\Omega' \subset \Omega \times [0, \infty)$. Therefore, it follows easily that u is a solution of (1) for $0 \leqslant t < \infty$. The case of $-\infty < t \leqslant 0$ is proven in the same way. ∎

Uniqueness is an open problem for large p, even in free space. In our case we can prove uniqueness for small p provided that we impose the condition

$$\|e^{i\Delta t}\|_{\mathscr{L}(L^1(\Omega), L^\infty(\Omega))} \leqslant \frac{c}{t^{n/2}}, \tag{37}$$

where $e^{i\Delta t}$ denotes the evolution operator for the free Schrödinger equation with homogeneous boundary condition on $\partial\Omega$, as above. We note that the linear estimate (37) is true in many cases. It is true if $\Omega = \mathbb{R}^n$. It is also true in the case of a half space, as it is easily proven by explicit formula using the method of even extensions.

THEOREM 2 (Uniqueness). *If $1 < p < 1 + \frac{4}{n-2}$ and if (37) is true, then the solution in Theorem 1 is unique.*

Proof. The estimate (37) is combined with the fact that the L^2 norm is preserved. An interpolation between these two estimates yields the $L^{(p+1)'} \to L^{p+1}$ estimate

$$\|e^{i\Delta t}u_0\|_{p+1} \leqslant ct^{-(n/2)(p+1)/(p-1)} \|u_0\|_{(p+1)'} \tag{38}$$

for $1 \leqslant p < \infty$. Uniqueness follows from exactly the same argument as on p. 19 of [5]. ∎

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