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Strong convergence of the solutions of the linear elasticity and uniformity of asymptotic expansions in the presence of small inclusions ☆

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ABSTRACT

We consider the Lamé system of linear elasticity when the inclusion has the extreme elastic constants. We show that the solutions to the Lamé system converge in appropriate H^1-norms when the shear modulus tends to infinity (the other modulus, the compressional modulus is fixed), and when the bulk modulus and the shear modulus tend to zero. Using this result, we show that the asymptotic expansion of the displacement vector in the presence of small inclusion is uniform with respect to Lamé parameters.

1. Introduction

Recently there is growing interest in partial differential equations with high contrast coefficients in various contexts. Among them are the photonic and phononic band gap problems where the electromagnetic parameter or the bulk modulus tend to infinity, biomedical imaging where anomalous tissues have large material parameters, and the stress concentration in between two inclusions with extreme material properties to name a few. See [10,9,3] and references therein. The purpose of this paper is to prove two basic theorems in relation to the PDEs with high contrast coefficients. The first

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one is to show that when the material parameters tend to the extreme, the corresponding solutions converge strongly in appropriate norms. The other one is to show that the asymptotic expansion of the solution in the presence of small inclusions holds uniformly with respect to material parameters. We prove these facts in the context of the system of linear elasticity. Corresponding results for the conductivity equation (a scalar equation) have been obtained in [22,13,29].

We consider a linear isotropic elastic body containing an inclusion with different elastic parameters. When the bulk and shear moduli of the inclusion are finite, the solution satisfies the transmission condition along the interface (the boundary of the inclusion). If the shear modulus of the inclusion is infinity, then the interface transmission condition is replaced by a null condition of the displacement (see Section 2). If the bulk and shear moduli are zero, then it is replaced by the traction zero condition on the boundary of the inclusion. The first objective of this paper is to prove the convergence in an appropriate H^1 space of the solution to the Lamé system as the bulk and shear moduli tend to the extreme (zero or infinity) (see Theorem 4.2).

The second objective of this paper is to prove a closely related problem of uniformity of the asymptotic expansion. In imaging small inclusions from boundary measurements, it is of fundamental importance to catch the boundary signature of the presence of anomalies. In this respect, an asymptotic expansion of the boundary perturbations of the solutions due to the presence of the inclusion, as the diameter of the inclusion tends to zero, has been derived. The asymptotic expansion is derived in [22,18,6] for the conductivity (scalar) equation, and in [12,28] for the Lamé system of linear isotropic elasticity. The asymptotic expansions have been effectively used for imaging diametrically small inclusions. See for example [18,25,7,8,26,1]. We also mention the topological derivative based shape optimization where the asymptotic expansion is an essential ingredient (see for example [17,23,5,2]). In [2] topological derivative based detection algorithms for the localization of an elastic inclusion of vanishing characteristic size have been developed and their resolution and stability with respect to measurement and medium noises analyzed.

In these applications, it is important to know that the asymptotic expansion holds uniformly with respect to the pair of Lamé parameters. We prove this in the second half of this paper under the assumption that the compressional modulus is bounded, which is necessary (see Theorem 6.1 for precise statements). It is worthwhile to mention that this result may have a relation with the cloaking as discussed in [29,14].

The methods of this paper are different from those of [29], where uniform validity of the asymptotic expansion for the conductivity (scalar) equations is proved, in that they are based on the layer potential techniques. The solutions to the Lamé system can be expressed as a single layer potential on the boundary of inclusion. We show that $H^{-1/2}$-norms of the potentials are bounded uniformly with respect to Lamé parameters, and the main results follow from this fact.

This paper is organized as follows. In Section 2, we set up the problems for finite and extreme moduli, and review the representation of solutions using layer potential techniques. In Section 3, we prove that the energy functional is uniformly bounded. As a consequence, we obtain that the potentials on the boundary of the inclusion are uniformly bounded. In Section 4 we show that these potentials converge as the bulk and shear moduli tend to extreme values and prove Theorem 4.2. In Section 5, we briefly discuss that similar boundedness and convergence result hold to be true for the boundary value problem. Section 6 is to prove Theorem 6.1 which asserts that the small volume expansion holds independently of Lamé parameters. The results and methods hold to be true even if there are multiple inclusions. We make a brief remark on this in the last section.

2. Problem setting and representation of solutions

Let D be an elastic inclusion which is a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary. Let (λ, μ) be the pair of Lamé (shear and compressional) parameters of D while (λ_0, μ_0) is that of the background $\mathbb{R}^d \setminus D$. Then the elasticity tensors for the inclusions and the background can be written respectively as $\mathbb{C}^1 = (C^1_{ijk\ell})$ and $\mathbb{C}^0 = (C^0_{ijk\ell})$ where

$$C^1_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

$$C^0_{ijk\ell} = \lambda_0 \delta_{ij}\delta_{k\ell} + \mu_0(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

and the elasticity tensor for $\mathbb{R}^d$ in the presence of the inclusion D is given by

$$\mathbb{C} := 1_D \mathbb{C}^1 + (1 - 1_D)\mathbb{C}^0, \tag{2.1}$$

where 1_D is the indicator function of D. We assume that the strong convexity condition holds, *i.e.*,

$$\mu > 0, \qquad d\lambda + 2\mu > 0, \qquad \mu_0 > 0, \quad \text{and} \quad d\lambda_0 + 2\mu_0 > 0. \tag{2.2}$$

We also assume that

$$(\lambda - \lambda_0)(\mu - \mu_0) > 0, \tag{2.3}$$

which is required to have the representation of the displacement vectors by the single layer potential in the following. We also denote the bulk modulus by κ which is given by $\kappa = \lambda + 2\mu/d$.

We consider the problem of the Lamé system of the linear elasticity: For a given function $\mathbf{h}$ satisfying $\nabla \cdot \mathbb{C}^0 \nabla^s \mathbf{h} = 0$ in $\mathbb{R}^d$,

$$\begin{cases} \nabla \cdot \mathbb{C} \nabla^s \mathbf{u} = 0 & \text{in } \mathbb{R}^d, \\ \mathbf{u}(\boldsymbol{x}) - \mathbf{h}(\boldsymbol{x}) = O\big(|\boldsymbol{x}|^{1-d}\big) & \text{as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{2.4}$$

where $\nabla^s \mathbf{u}$ is the symmetric gradient (or the strain tensor), *i.e.*,

$$\nabla^s \mathbf{u} := \frac{1}{2}\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\big) \quad (T \text{ for transpose}).$$

Let

$$\mathcal{L}_{\lambda,\mu}\mathbf{u} := \nabla \cdot \mathbb{C}^1 \nabla^s \mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}$$

and define the corresponding conormal derivative $\partial \mathbf{u}/\partial \nu$ on ∂D by

$$\frac{\partial \mathbf{u}}{\partial \nu} := \mathbb{C}^1\big(\nabla^s \mathbf{u}\big)\mathbf{n} = \lambda(\nabla \cdot \mathbf{u})\mathbf{n} + \mu\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\big)\mathbf{n} \quad \text{on } \partial D, \tag{2.5}$$

where $\mathbf{n}$ is the outward unit normal to ∂D. Let $D^c := \mathbb{R}^d \setminus \overline{D}$. Let $\mathcal{L}_{\lambda_0,\mu_0}$ and $\frac{\partial}{\partial \nu_0}$ be those corresponding to (λ_0, μ_0). Then (2.4) is equivalent to the following problem:

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{u} = 0 & \text{in } D^c, \\ \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 & \text{in } D, \\ \mathbf{u}|_- = \mathbf{u}|_+ & \text{on } \partial D, \\ \dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_- = \dfrac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ & \text{on } \partial D, \\ \mathbf{u}(\boldsymbol{x}) - \mathbf{h}(\boldsymbol{x}) = O\big(|\boldsymbol{x}|^{1-d}\big) & \text{as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{2.6}$$

where the subscripts $+$ and $-$ indicate the limits from outside and inside D, respectively.

We also consider the two limiting cases of (2.6): when both κ and μ tend to 0, and when $\mu \to \infty$ while λ is fixed. In relation to the latter case it is worth mentioning that if $\lambda \to \infty$ and μ is fixed, (2.6) approaches to a different problem. Roughly speaking, if $\lambda \to \infty$, then $\nabla \cdot \mathbf{u}$ is approaching to 0 while $\lambda\nabla\cdot\mathbf{u}$ stays bounded. So (2.6) approaches to the modified Stokes' problem $\mu\Delta\mathbf{u}+\nabla p=0$ with $p=\lambda\nabla\cdot\mathbf{u}$. (See [4].) Hence we assume that λ is bounded throughout this paper.

If $\kappa=\mu=0$ (or $\lambda=\mu=0$), one can easily see what the limiting problem should be. Since $\frac{\partial\mathbf{u}}{\partial\nu}|_-=0$, we have from the fourth line of (2.6) that $\frac{\partial\mathbf{u}}{\partial\nu_0}|_+=0$. So the elasticity equation in this case is

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}=0 & \text{in } D^c,\\ \dfrac{\partial\mathbf{u}}{\partial\nu_0}\Big|_+=0 & \text{on } \partial D,\\ \mathbf{u}(\boldsymbol{x})-\mathbf{h}(\boldsymbol{x})=O\big(|\boldsymbol{x}|^{1-d}\big) & \text{as } |\boldsymbol{x}|\to\infty. \end{cases} \tag{2.7}$$

To describe the equation when $\mu=\infty$ while λ remains bounded, we need to introduce the following functional space: Let Ψ be the $d(d+1)/2$-dimensional vector space defined by

$$\Psi:=\big\{\psi=\big(\psi^{(1)},\dots,\psi^{(d)}\big)^T\colon\ \partial_i\psi^{(j)}+\partial_j\psi^{(i)}=0,\ 1\leqslant i,j\leqslant d\big\}. \tag{2.8}$$

We emphasize that Ψ is the space of solutions $\mathcal{L}_{\lambda,\mu}\mathbf{u}=0$ in D and $\partial\mathbf{u}/\partial\nu=0$ on ∂D for any (λ,μ). Let ψ_j, $j=1,\dots,d(d+1)/2$, be a basis of Ψ. If $\mu\to\infty$, then from the second and fourth equations in (2.6) we have

$$\Delta\mathbf{u}+\nabla\nabla\cdot\mathbf{u}=0\quad\text{in } D,\qquad \big(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\big)\mathbf{n}=0\quad\text{on } \partial D,$$

which is another elasticity equation (with $\mu=1$ and $\lambda=0$) with zero traction on the boundary. Thus there are constants α_j such that

$$\mathbf{u}(\boldsymbol{x})=\sum_{j=1}^{d(d+1)/2}\alpha_j\psi_j(\boldsymbol{x}),\quad \boldsymbol{x}\in D.$$

So, the elasticity problem when $\mu=\infty$ is

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}=0 & \text{in } D^c,\\ \mathbf{u}=\displaystyle\sum_{j=1}^{d(d+1)/2}\alpha_j\psi_j & \text{on } \partial D,\\ \mathbf{u}(\boldsymbol{x})-\mathbf{h}(\boldsymbol{x})=O\big(|\boldsymbol{x}|^{1-d}\big) & \text{as } |\boldsymbol{x}|\to\infty. \end{cases} \tag{2.9}$$

We need extra conditions to determine the coefficients α_j. Note that the solution $\mathbf{u}$ to (2.6) satisfies

$$\int_{\partial D}\frac{\partial\mathbf{u}}{\partial\nu_0}\Big|_+\cdot\psi_l\,d\sigma=0,\quad l=1,\dots,\frac{d(d+1)}{2}. \tag{2.10}$$

So, by taking a (formal) limit, one can expect that the solution $\mathbf{u}$ to (2.9) should satisfy the same condition, and the constants α_j in (2.9) are determined by this orthogonality condition.

We now review the representation of the solution to (2.6) using the single layer potential for the Lamé system following [27,20,21,9]. The Kelvin matrix of the fundamental solution $\boldsymbol{\Gamma}=(\Gamma_{ij})_{i,j=1}^d$ to the Lamé system $\mathcal{L}_{\lambda,\mu}$ is given by

$$\Gamma_{ij}(\boldsymbol{x}) := \begin{cases} \frac{\alpha}{2\pi}\delta_{ij}\ln|\boldsymbol{x}| - \frac{\beta}{2\pi}\frac{x_i x_j}{|\boldsymbol{x}|^2} & \text{if } d=2, \\ -\frac{\alpha}{4\pi}\frac{\delta_{ij}}{|\boldsymbol{x}|} - \frac{\beta}{2\pi}\frac{x_i x_j}{|\boldsymbol{x}|^3} & \text{if } d=3, \end{cases} \quad \boldsymbol{x} \neq 0$$

where

$$\alpha = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \beta = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right).$$

When D is a simply connected domain, the single layer potentials of the density function $\varphi = [\varphi_1 \ \cdots \ \varphi_d]^T$ on ∂D associated with the Lamé parameters (λ, μ) are defined by

$$\mathcal{S}_D[\varphi](\boldsymbol{x}) := \int_{\partial D} \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{2.11}$$

We denote by $\Gamma^{\mathbf{0}}$, $\mathcal{S}_D^0[\varphi]$ the fundamental solution and the single layer potential associate with the Lamé parameter (λ_0, μ_0) respectively. The conormal derivative of $\mathcal{S}_D[\varphi]$ enjoys the jump relation on ∂D:

$$\frac{\partial}{\partial \nu}\mathcal{S}_D[\varphi]\big|_{\pm} = \left(\pm\frac{1}{2}I + \mathcal{K}_D^*\right)[\varphi] \quad \text{a.e. on } \partial D, \tag{2.12}$$

where $\mathcal{K}_D^*$ is defined by

$$\mathcal{K}_D^*[\varphi](\boldsymbol{x}) = \text{p.v.}\int_{\partial D} \frac{\partial}{\partial \nu_{\boldsymbol{x}}}\Gamma(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,d\sigma(\boldsymbol{y}) \quad \text{a.e. } \boldsymbol{x} \in \partial D, \tag{2.13}$$

where p.v. stands for the Cauchy principal value. We denote by $\mathcal{K}_D^{0*}$ the operator corresponding to (λ_0, μ_0).

We introduce a weighted norm, $\|\mathbf{u}\|_{H^1_w(\Omega)}$, in two dimensions: let Ω be either $\mathbb{R}^d$ or D^c, and let

$$\|\mathbf{u}\|^2_{H^1_w(\Omega)} := \int_{\Omega} \frac{|\mathbf{u}(\boldsymbol{x})|^2}{\sqrt{1+|\boldsymbol{x}|^2}}\,d\boldsymbol{x} + \int_{\Omega} \left|\nabla \mathbf{u}(\boldsymbol{x})\right|^2 d\boldsymbol{x}.$$

This weighted norm is introduced because the solutions $\mathbf{u}$ satisfies only $\mathbf{u}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1})$ in two dimensions as $|\boldsymbol{x}| \to \infty$. For convenience in presenting results of this paper, we put $W(\Omega) := H^1_w(\Omega)$ in two dimensions, and $W(\Omega) := H^1(\Omega)$, the usual Sobolev space, in three dimensions. Let Ψ be the space introduced in (2.8), and define

$$H_{\Psi}^{-1/2}(\partial D) := \left\{\varphi \in H^{-1/2}(\partial D)^d\colon\ \langle \varphi, \psi\rangle = 0 \text{ for all } \psi \in \Psi\right\}. \tag{2.14}$$

Here $\langle \varphi, \psi\rangle$ denotes the $H^{-1/2}$–$H^{1/2}$ product. Then $\pm\frac{1}{2}I + \mathcal{K}_D^*$ is invertible on $H_{\Psi}^{-1/2}(\partial D)$. We also have

$$\left\|\mathcal{S}_D[\varphi]\right\|_{W(\mathbb{R}^d)} \leqslant C\|\varphi\|_{H^{-1/2}(\partial D)} \tag{2.15}$$

for all $\varphi \in H^{-1/2}(\partial D)^d$.

It is proved in [21] that the solution $\mathbf{u}$ to (2.6) is represented as

$$\mathbf{u}(\boldsymbol{x}) = \begin{cases} \mathbf{h}(\boldsymbol{x}) + \mathcal{S}_D^0[\varphi](\boldsymbol{x}), & \boldsymbol{x} \in D^c, \\ \mathcal{S}_D[\psi](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases} \tag{2.16}$$

where the pair $(\varphi, \psi) \in H_\Psi^{-1/2}(\partial D) \times H^{-1/2}(\partial D)$ is the solutions to

$$\begin{cases} \mathcal{S}_D[\psi](\boldsymbol{x}) - \mathcal{S}_D^0[\varphi](\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x}) \\ \left.\dfrac{\partial \mathcal{S}_D[\psi]}{\partial \nu}\right|_-(\boldsymbol{x}) - \left.\dfrac{\partial \mathcal{S}_D^0[\varphi]}{\partial \nu_0}\right|_+(\boldsymbol{x}) = \dfrac{\partial \mathbf{h}}{\partial \nu_0}(\boldsymbol{x}) \end{cases} \quad \text{for } \boldsymbol{x} \in \partial D. \tag{2.17}$$

Even if $\kappa = \mu = 0$ or $\mu = \infty$, we have a similar representation:

$$\mathbf{u}_\kappa(\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x}) + \mathcal{S}_D^0[\varphi_\kappa](\boldsymbol{x}), \quad \boldsymbol{x} \in D^c, \ \kappa = 0, \infty. \tag{2.18}$$

When $\kappa = \mu = 0$, φ_0 satisfies

$$\left(\frac{1}{2} I + \left(\mathcal{K}_D^0\right)^*\right)[\varphi_0] = -\frac{\partial \mathbf{h}}{\partial \nu_0} \quad \text{on } \partial D, \tag{2.19}$$

and if $\mu = \infty$, then φ_∞ satisfies

$$\left(-\frac{1}{2} I + \left(\mathcal{K}_D^0\right)^*\right)[\varphi_\infty] = -\frac{\partial \mathbf{h}}{\partial \nu_0} \quad \text{on } \partial D. \tag{2.20}$$

We emphasize that $\varphi_\kappa \in H_\Psi^{-1/2}(\partial D)$. See, for example, [9] for details of the above mentioned representation of the solutions.

A similar representation formula holds for the solutions to the boundary value problems. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ containing D, which is also Lipschitz. Let $\mathbf{u}$ be the solution to

$$\nabla \cdot \mathbb{C} \nabla^s \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.21}$$

with either the Dirichlet boundary condition $\mathbf{u} = \mathbf{f}$ or the Neumann boundary condition $\frac{\partial \mathbf{u}}{\partial \nu_0} = \mathbf{g}$ on $\partial\Omega$. Let

$$\mathbf{h}(\boldsymbol{x}) := -\int_{\partial\Omega} \boldsymbol{\Gamma}^{\mathbf{0}}(\boldsymbol{x} - \boldsymbol{y}) \left.\frac{\partial \mathbf{u}}{\partial \nu_0}\right|_-(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) + \int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^{\mathbf{0}}(\boldsymbol{x} - \boldsymbol{y})}{\partial \nu_0(\boldsymbol{y})} \mathbf{u}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega. \tag{2.22}$$

Then $\mathbf{u}$ is represented as

$$\mathbf{u}(\boldsymbol{x}) = \begin{cases} \mathbf{h}(\boldsymbol{x}) + \mathcal{S}_D^0[\varphi](\boldsymbol{x}), & \boldsymbol{x} \in \Omega \setminus \overline{D}, \\ \mathcal{S}_D[\psi](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases} \tag{2.23}$$

where the pair $(\varphi, \psi) \in H_\Psi^{-1/2}(\partial D) \times H^{-1/2}(\partial D)$ is the solutions to (2.17).

3. Energy estimates

Let

$$J[\mathbf{u}] := \frac{1}{2}\int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s \mathbf{u} + \frac{1}{2}\int_{D^c} \mathbb{C}^0 \nabla^s (\mathbf{u}-\mathbf{h}) : \nabla^s (\mathbf{u}-\mathbf{h}). \tag{3.1}$$

Here and throughout this paper $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^d a_{ij} b_{ij}$ for $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. For the solution $\mathbf{u}$ to (2.6), we prove that $J[\mathbf{u}]$ is bounded regardless of κ and μ. More precisely we prove the following lemma.

Lemma 3.1. *Let $\mathbf{u}$ be the solution to (2.6). If $\lambda \leqslant \Lambda$ for some constant Λ, then there is a constant C depending on Λ, but otherwise independent of μ and κ, such that*

$$J[\mathbf{u}] \leqslant C \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|^2_{H^{-1/2}(\partial D)}. \tag{3.2}$$

As a consequence of Lemma 3.1 we have

Lemma 3.2. *Let φ be the potential defined in (2.16). If $\lambda \leqslant \Lambda$ for some constant Λ, then there is a constant C depending on Λ, but otherwise independent of μ and κ, such that*

$$\|\varphi\|_{H^{-1/2}(\partial D)} \leqslant C \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|_{H^{-1/2}(\partial D)}. \tag{3.3}$$

Proof. Let $\mathbf{v} := \mathbf{u} - \mathbf{h}$. Then (2.16) yields $\mathbf{v}(\boldsymbol{x}) = \mathcal{S}_D^0[\varphi](\boldsymbol{x})$ for $\boldsymbol{x} \in D^c$. Thus, we have from (2.12)

$$\left.\frac{\partial \mathbf{v}}{\partial \nu_0}\right|_+ = \left(\frac{1}{2} I + \left(\mathcal{K}_D^0\right)^*\right)[\varphi] \quad \text{on } \partial D.$$

Since $\frac{1}{2} I + (\mathcal{K}_D^0)^*$ is invertible on $H_\Psi^{-1/2}(\partial D)$, we have

$$\|\varphi\|_{H^{-1/2}(\partial D)} \leqslant C \left\| \left.\frac{\partial \mathbf{v}}{\partial \nu_0}\right|_+ \right\|_{H^{-1/2}(\partial D)}.$$

Let η be a function in $H^{1/2}(\partial D)$ satisfying $\int_{\partial D} \eta = 0$ and let $\mathbf{w}$ be the solution to $\Delta \mathbf{w} = 0$ in D^c with $\mathbf{w}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d})$ and $\mathbf{w} = \eta$ on ∂D, so that the following estimate holds:

$$\left\|\nabla^s \mathbf{w}\right\|_{L^2(D^c)} \leqslant C \|\eta\|_{H^{1/2}(\partial D)}.$$

Since

$$\int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{w} \, d\boldsymbol{x} = -\int_{\partial D} \left.\frac{\partial \mathbf{v}}{\partial \nu_0}\right|_+ \cdot \eta \, d\sigma(\boldsymbol{x}), \tag{3.4}$$

we have

$$\left| \int_{\partial D} \frac{\partial \mathbf{v}}{\partial \nu_0}\bigg|_+ \cdot \eta \, d\sigma(\boldsymbol{x}) \right| \leqslant C \left\| \nabla^s \mathbf{v} \right\|_{L^2(D^c)} \left\| \nabla^s \mathbf{w} \right\|_{L^2(D^c)} \leqslant C \left\| \nabla^s \mathbf{v} \right\|_{L^2(D^c)} \|\eta\|_{H^{1/2}(\partial D)}.$$

Since $\eta \in H^{1/2}(\partial D)$ is arbitrary, we have from (3.2)

$$\left\| \frac{\partial \mathbf{v}}{\partial \nu_0}\bigg|_+ \right\|_{H^{-1/2}(\partial D)} \leqslant C \left\| \nabla^s \mathbf{v} \right\|_{L^2(D^c)} \leqslant C \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|_{H^{-1/2}(\partial D)},$$

and so follows (3.3). $\square$

Proof of Lemma 3.1. Let $\mathbf{v} := \mathbf{u} - \mathbf{h}$. It is known (see for example [16]) that $\mathbf{v}$ is the minimizer in $W(\mathbb{R}^d)$ of the functional

$$I[\mathbf{v}] := \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{C}\big(\nabla^s \mathbf{v} + 1_D \mathbb{G} \nabla^s \mathbf{h}\big) : \big(\nabla^s \mathbf{v} + 1_D \mathbb{G} \nabla^s \mathbf{h}\big), \tag{3.5}$$

where $\mathbb{G} = \mathbf{I}_4 - (\mathbb{C}^1)^{-1}\mathbb{C}^0$ and $\mathbf{I}_4$ is the identity 4-tensor. Note that

$$\begin{aligned} I[\mathbf{v}] &= \frac{1}{2} \int_D \big(\mathbb{C}^1 \nabla^s(\mathbf{v} + \mathbf{h}) - \mathbb{C}^0 \nabla^s \mathbf{h}\big) : \big(\nabla^s(\mathbf{v} + \mathbf{h}) - \big(\mathbb{C}^1\big)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h}\big) + \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{v} \\ &= \frac{1}{2} \int_D \mathbb{C}^1 \nabla^s \mathbf{v} : \nabla^s \mathbf{v} + \int_D \big(\mathbb{C}^1 - \mathbb{C}^0\big) \nabla^s \mathbf{v} : \nabla^s \mathbf{h} \\ &\quad + \frac{1}{2} \int_D \big(\mathbb{C}^1 - 2\mathbb{C}^0 + \mathbb{C}^0 \big(\mathbb{C}^1\big)^{-1} \mathbb{C}^0\big) \nabla^s \mathbf{h} : \nabla^s \mathbf{h} + \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{v}. \end{aligned} \tag{3.6}$$

Let

$$\mathbf{v}_\infty := \mathbf{u}_\infty - \mathbf{h}. \tag{3.7}$$

Then $\mathbf{v}_\infty \in W(\mathbb{R}^d)$, and $(\mathbf{v}_\infty + \mathbf{h})|_D \in \Psi$ which implies that $\nabla^s(\mathbf{v}_\infty + \mathbf{h}) = 0$ in D. So, we have from the first line in (3.6) that

$$I[\mathbf{v}_\infty] = \frac{1}{2} \int_D \mathbb{C}^0 \big(\mathbb{C}^1\big)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} + \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v}_\infty : \nabla^s \mathbf{v}_\infty. \tag{3.8}$$

We then have

$$\begin{aligned} J[\mathbf{u}] &= J[\mathbf{v} + \mathbf{h}] \\ &= \frac{1}{2} \int_D \mathbb{C}^1 \nabla^s \mathbf{v} : \nabla^s \mathbf{v} + \int_D \mathbb{C}^1 \nabla^s \mathbf{v} : \nabla^s \mathbf{h} + \frac{1}{2} \int_D \mathbb{C}^1 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} + \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{v} \\ &= I[\mathbf{v}] + \int_D \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{h} + \int_D \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} - \frac{1}{2} \int_D \mathbb{C}^0 \big(\mathbb{C}^1\big)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} \\ &= I[\mathbf{v}] + \int_D \mathbb{C}^0 \nabla^s \mathbf{u} : \nabla^s \mathbf{h} - \frac{1}{2} \int_D \mathbb{C}^0 \big(\mathbb{C}^1\big)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h}. \end{aligned}$$

Since $I[\mathbf{v}] \leqslant I[\mathbf{v}_\infty]$, it follows from (3.8) that

$$J[\mathbf{u}] \leqslant I[\mathbf{v}_\infty] + \int_D \mathbb{C}^0 \nabla^s \mathbf{u} : \nabla^s \mathbf{h} - \frac{1}{2} \int_D \mathbb{C}^0 (\mathbb{C}^1)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h}$$

$$= \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v}_\infty : \nabla^s \mathbf{v}_\infty + \int_D \mathbb{C}^0 \nabla^s \mathbf{u} : \nabla^s \mathbf{h}. \tag{3.9}$$

Note that since λ is bounded, we have

$$\begin{aligned} \int_D \mathbb{C}^0 \nabla^s \mathbf{u} : \nabla^s \mathbf{h} &= C\left(\int_D |\nabla^s \mathbf{u}|^2 + \int_D |\nabla^s \mathbf{h}|^2 \right) \\ &\leqslant C\left(\frac{1}{\mu} \int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s \mathbf{u} + \int_D |\nabla^s \mathbf{h}|^2 \right) \\ &\leqslant C\left(\frac{1}{\mu} J[\mathbf{u}] + \int_D |\nabla^s \mathbf{h}|^2 \right). \end{aligned}$$

So, if μ is sufficiently large, then we have from (3.9) that

$$J[\mathbf{u}] \leqslant C\left(\|\nabla^s \mathbf{v}_\infty\|^2_{L^2(D^c)} + \|\nabla^s \mathbf{h}\|^2_{L^2(D)} \right)$$

for some constant C. Since

$$\|\nabla^s \mathbf{v}_\infty\|_{L^2(D^c)} \leqslant C\|\varphi_\infty\|_{H^{-1/2}(\partial D)} \leqslant C \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|_{H^{-1/2}(\partial D)},$$

we have (3.2) when μ is large.

When κ and μ are bounded, we need a function which plays the role of $\mathbf{v}_\infty$ in the above. For that we use φ_0 in (2.18): define

$$\mathbf{v}_0(\boldsymbol{x}) := \mathcal{S}_D^0[\varphi_0](\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d. \tag{3.10}$$

It is worth emphasizing that $\mathbf{v}_0$ is defined not only on D^c but on $\mathbb{R}^d$. Then one can show as above that

$$J[\mathbf{u}] \leqslant I[\mathbf{v}_0] + \int_D \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{h} + \int_D \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} - \frac{1}{2} \int_D \mathbb{C}^0 (\mathbb{C}^1)^{-1} \mathbb{C}^0 \nabla^s \mathbf{h} : \nabla^s \mathbf{h}.$$

Using (3.6), one can see that

$$\begin{aligned} J[\mathbf{u}] \leqslant{} & \frac{1}{2} \int_D \mathbb{C}^1 \nabla^s \mathbf{v}_0 : \nabla^s \mathbf{v}_0 + \int_D (\mathbb{C}^1 - \mathbb{C}^0) \nabla^s \mathbf{v}_0 : \nabla^s \mathbf{h} + \frac{1}{2} \int_D \mathbb{C}^1 \nabla^s \mathbf{h} : \nabla^s \mathbf{h} \\ & + \frac{1}{2} \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{v}_0 : \nabla^s \mathbf{v}_0 + \int_D \mathbb{C}^0 \nabla^s \mathbf{v} : \nabla^s \mathbf{h}. \end{aligned} \tag{3.11}$$

Since $\frac{\partial(\mathbf{v}_0+\mathbf{h})}{\partial\nu_0}|_+ = 0$ on ∂B, we have

$$\int_D \mathbb{C}^0\nabla^s\mathbf{v} : \nabla^s\mathbf{h} = \int_{\partial D} \frac{\partial\mathbf{h}}{\partial\nu_0}\cdot\mathbf{v} = -\int_{\partial D} \frac{\partial\mathbf{v}_0}{\partial\nu_0}\bigg|_+ \cdot\mathbf{v} = \int_{D^c} \mathbb{C}^0\nabla^s\mathbf{v}_0 : \nabla^s\mathbf{v}$$
$$\leqslant \frac{C}{\epsilon}\int_{D^c} \mathbb{C}^0\nabla^s\mathbf{v}_0 : \nabla^s\mathbf{v}_0 + C\epsilon\int_{D^c} \mathbb{C}^0\nabla^s\mathbf{v} : \nabla^s\mathbf{v}$$

for a (small) constant ϵ. If ϵ is sufficiently small, then we obtain by combining this with (3.11)

$$J[\mathbf{u}] \leqslant C\left(\|\nabla^s\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^s\mathbf{h}\|^2_{L^2(D)}\right) \tag{3.12}$$

for some constant C independent of κ and μ. Since

$$\|\nabla^s\mathbf{v}_0\|_{L^2(D^c)} \leqslant C\|\varphi_0\|_{H^{-1/2}(\partial D)} \leqslant C\left\|\frac{\partial\mathbf{h}}{\partial\nu_0}\right\|_{H^{-1/2}(\partial D)},$$

we have (3.2). This completes the proof. □

4. Convergence of potentials and solutions

Lemma 3.2 shows that the potential defined in (2.16) is uniformly bounded with respect to μ and λ as long as λ is bounded. We now prove the following lemma.

Lemma 4.1. *Let φ, φ_0 and φ_∞ be potentials defined by* (2.16), (2.19) *and* (2.20), *respectively.*

(i) *Suppose that $\lambda \leqslant \Lambda$ for some constant Λ. There are constants μ_1 and C such that*

$$\|\varphi - \varphi_\infty\|_{H^{-1/2}(\partial D)} \leqslant \frac{C}{\sqrt{\mu}}\left\|\frac{\partial\mathbf{h}}{\partial\nu_0}\right\|_{H^{-1/2}(\partial D)} \tag{4.1}$$

for all $\mu \geqslant \mu_1$.

(ii) *There are constants δ and C such that*

$$\|\varphi - \varphi_0\|_{H^{-1/2}(\partial D)} \leqslant C(\kappa+\mu)^{1/4}\left\|\frac{\partial\mathbf{h}}{\partial\nu_0}\right\|_{H^{-1/2}(\partial D)} \tag{4.2}$$

for all $\kappa, \mu \leqslant \delta$.

Proof. We may assume that $\|\frac{\partial\mathbf{h}}{\partial\nu_0}\|_{H^{-1/2}(\partial D)} = 1$. Let $\mathbf{w} = \mathbf{u} - \mathbf{u}_\infty$ so that $\mathbf{w}$ satisfies

$$\begin{cases} \nabla\cdot\mathbb{C}^0\nabla^s\mathbf{w} = 0 & \text{in } D^c, \\ \mathbf{w} - \mathbf{u}|_{\overline{D}} \in \Psi & \text{in } \overline{D}, \\ \mathbf{w}(\boldsymbol{x}) = O\left(|\boldsymbol{x}|^{1-d}\right) & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{4.3}$$

Since $\nabla^s(\mathbf{w}-\mathbf{u}) = 0$ in D, we have from Lemma 3.1 that

$$\int_D |\nabla^s\mathbf{w}|^2\,d\boldsymbol{x} \leqslant \frac{1}{2\mu}\int_D \mathbb{C}^1\nabla^s\mathbf{w} : \nabla^s\mathbf{w}\,d\boldsymbol{x} = \frac{1}{2\mu}\int_D \mathbb{C}^1\nabla^s\mathbf{u} : \nabla^s\mathbf{u}\,d\boldsymbol{x} \leqslant \frac{C}{\mu}. \tag{4.4}$$

Let ψ_j, $j=1,\dots,d(d+1)/2$, be a basis of Ψ as before, and let

$$\mathbf{e}=\sum_{j=1}^{d(d+1)/2}\beta_j\psi_j,$$

where β_j are chosen so that

$$\int_D\big[(\mathbf{w}-\mathbf{e})\cdot\psi_j+\nabla(\mathbf{w}-\mathbf{e}):\nabla\psi_j\big]=0,\quad j=1,\dots,d(d+1)/2. \tag{4.5}$$

We then apply Korn's inequality (see for example [19]) to $\mathbf{w}-\mathbf{e}$ to have

$$\int_D\big[|\mathbf{w}-\mathbf{e}|^2+\big|\nabla(\mathbf{w}-\mathbf{e})\big|^2\big]\leqslant C\int_D\big|\nabla^s(\mathbf{w}-\mathbf{e})\big|^2\leqslant C\int_D\big|\nabla^s\mathbf{w}\big|^2 \tag{4.6}$$

for some constant C independent of μ. It then follows from (4.4) that

$$\|\mathbf{w}-\mathbf{e}\|_{H^1(D)}\leqslant\frac{C}{\sqrt{\mu}}, \tag{4.7}$$

and from the trace theorem that

$$\|\mathbf{w}-\mathbf{e}\|_{H^{1/2}(\partial D)}\leqslant\frac{C}{\sqrt{\mu}} \tag{4.8}$$

for some constant C independent of μ. By the strong convexity (2.2) of $\mathbb{C}^0$, there is a constant C such that

$$\begin{aligned}\big\|\nabla^s\mathbf{w}\big\|^2_{L^2(D^c)}&\leqslant C\int_{D^c}\mathbb{C}^0\nabla^s\mathbf{w}:\nabla^s\mathbf{w}\,d\boldsymbol{x}=-C\int_{\partial D}\frac{\partial\mathbf{w}}{\partial\nu_0}\bigg|_+\cdot\mathbf{w}\,d\sigma(\boldsymbol{x})\\&=-C\int_{\partial D}\frac{\partial\mathbf{w}}{\partial\nu_0}\bigg|_+\cdot(\mathbf{w}-\mathbf{e})\,d\sigma(\boldsymbol{x})\leqslant C\|\mathbf{w}-\mathbf{e}\|_{H^{1/2}(\partial D)}\bigg\|\frac{\partial\mathbf{w}}{\partial\nu_0}\bigg|_+\bigg\|_{H^{-1/2}(\partial D)},\end{aligned}$$

where the second equality holds because of the orthogonality property (2.10). It then follows from (4.8) that

$$\big\|\nabla^s\mathbf{w}\big\|^2_{L^2(D^c)}\leqslant\frac{C}{\sqrt{\mu}}\bigg\|\frac{\partial\mathbf{w}}{\partial\nu_0}\bigg|_+\bigg\|_{H^{-1/2}(\partial D)}. \tag{4.9}$$

We then obtain using the $H^{-1/2}$–$H^{1/2}$ duality, divergence theorem on D^c, Korn's inequality, and the trace theorem that

$$\bigg\|\frac{\partial\mathbf{w}}{\partial\nu_0}\bigg|_+\bigg\|^2_{H^{-1/2}(\partial D)}\leqslant C\big\|\nabla^s\mathbf{w}\big\|^2_{L^2(D^c)},$$

and from (4.9) that

$$\left\| \frac{\partial \mathbf{w}}{\partial \nu_0}\Big|_+ \right\|_{H^{-1/2}(\partial D)} \leqslant \frac{C}{\sqrt{\mu}}. \tag{4.10}$$

Using the representations (2.16) and (2.18), we have

$$\mathbf{w}(\boldsymbol{x}) = \mathbf{u}(\boldsymbol{x}) - \mathbf{u}_\infty(\boldsymbol{x}) = \mathcal{S}_D^0[\varphi - \varphi_\infty](\boldsymbol{x}), \quad \boldsymbol{x} \in D^c. \tag{4.11}$$

Thus, (2.12) yields

$$\frac{\partial \mathbf{w}}{\partial \nu_0}\Big|_+ = \left(\frac{1}{2}I + \left(\mathcal{K}_D^0\right)^*\right)[\varphi - \varphi_\infty] \quad \text{on } \partial D. \tag{4.12}$$

So, (4.1) follows from (4.10).

To prove (4.2), let $\mathbf{v}_0$ be as defined in (3.10) and let $\mathbf{z} := \mathbf{v} - \mathbf{v}_0$ in $\mathbb{R}^d$. Then $\mathbf{z} = \mathbf{u} - \mathbf{u}_0$ in D^c and the following holds

$$\begin{aligned}\int_{D^c} \left|\nabla^s \mathbf{z}\right|^2 d\boldsymbol{x} &\leqslant C \int_{D^c} \mathbb{C}^0 \nabla^s \mathbf{z} : \nabla^s \mathbf{z}\, d\boldsymbol{x} = -C \int_{\partial D} \frac{\partial \mathbf{z}}{\partial \nu_0}\Big|_+ \cdot \mathbf{z} = -C \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot \mathbf{z} \\ &= -C\left(\int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot \mathbf{u} - \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot \mathbf{u}_0\right)\end{aligned}$$

for some constant $C > 0$. Since

$$\int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot \mathbf{u} = \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu}\Big|_- \cdot \mathbf{u} = \int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s \mathbf{u} \geqslant 0,$$

we have

$$\begin{aligned}\int_{D^c} \left|\nabla^s \mathbf{z}\right|^2 d\boldsymbol{x} &\leqslant C \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot \mathbf{u}_0 = C \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu_0}\Big|_+ \cdot (\mathbf{v}_0 + \mathbf{h}) \\ &= C \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \nu}\Big|_- \cdot (\mathbf{v}_0 + \mathbf{h}) = C \int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s (\mathbf{v}_0 + \mathbf{h}).\end{aligned}$$

By Cauchy's inequality, we obtain that

$$\int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s (\mathbf{v}_0 + \mathbf{h}) \leqslant C\left(\int_D \mathbb{C}^1 \nabla^s \mathbf{u} : \nabla^s \mathbf{u}\right)^{1/2} \left(\int_D \mathbb{C}^1 \nabla^s (\mathbf{v}_0 + \mathbf{h}) : \nabla^s (\mathbf{v}_0 + \mathbf{h})\right)^{1/2}.$$

Thus, from (3.2) it follows that

$$\begin{aligned}\int_{D^c} \left|\nabla^s \mathbf{z}\right|^2 d\boldsymbol{x} &\leqslant C\left(\int_D \mathbb{C}^1 \nabla^s (\mathbf{v}_0 + \mathbf{h}) : \nabla^s (\mathbf{v}_0 + \mathbf{h})\right)^{1/2} \\ &\leqslant C\sqrt{\kappa + \mu}\left\|\nabla^s (\mathbf{v}_0 + \mathbf{h})\right\|_{L^2(D)} \leqslant C\sqrt{\kappa + \mu}\end{aligned}$$

for a constant C independent of κ. Therefore, we arrive at

$$\left\| \frac{\partial \mathbf{z}}{\partial \nu_0}\bigg|_+ \right\|_{H^{-1/2}(\partial D)} \leqslant C \left\| \nabla^s \mathbf{z} \right\|_{L^2(D^c)} \leqslant C(\kappa+\mu)^{1/4}. \tag{4.13}$$

Note that $\mathbf{z} = \mathbf{u} - \mathbf{u}_0 = \mathcal{S}_D^0[\varphi - \varphi_0]$ in D^c. So by the same reasoning as above we have (4.2), and the proof is complete. □

As a consequence of Lemma 4.1, we obtain the first main result of this paper.

Theorem 4.2. *Suppose that* (2.2) *and* (2.3) *hold. Let* $\mathbf{u}$, $\mathbf{u}_\infty$ *and* $\mathbf{u}_0$ *be the solutions to* (2.6), (2.9) *and* (2.7), *respectively.*

(i) *Suppose that $\lambda \leqslant \Lambda$ for some constant Λ. There are constants μ_1 and C such that*

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{W(\mathbb{R}^d)} \leqslant \frac{C}{\sqrt{\mu}} \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|_{H^{-1/2}(\partial D)} \tag{4.14}$$

for all $\mu \geqslant \mu_1$.

(ii) *There are constants δ and C such that*

$$\|\mathbf{u} - \mathbf{u}_0\|_{W(D^c)} \leqslant C(\kappa+\mu)^{1/4} \left\| \frac{\partial \mathbf{h}}{\partial \nu_0} \right\|_{H^{-1/2}(\partial D)} \tag{4.15}$$

for all $\kappa, \mu \leqslant \delta$.

It is not clear if the convergence rate, $\mu^{-1/2}$ and $(\kappa+\mu)^{1/4}$, are optimal or not.

Proof of Theorem 4.2. Assume that $\|\frac{\partial \mathbf{h}}{\partial \nu_0}\|_{H^{-1/2}(\partial D)} = 1$. Since $\mathbf{u} - \mathbf{u}_0 = \mathcal{S}_D^0[\varphi - \varphi_0]$ on D^c, (4.15) follows from (4.2).

Since $\mathbf{u} - \mathbf{u}_\infty = \mathcal{S}_D^0[\varphi - \varphi_\infty]$ on D^c, we have

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{W(D^c)} \leqslant \frac{C}{\sqrt{\mu}}. \tag{4.16}$$

Moreover, we have

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{H^{1/2}(\partial D)} = \left\| \mathcal{S}_D^0[\varphi - \varphi_\infty] \right\|_{H^{1/2}(\partial D)} \leqslant C\|\varphi - \varphi_\infty\|_{H^{-1/2}(\partial D)} \leqslant \frac{C}{\sqrt{\mu}},$$

and hence

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{H^1(D)} \leqslant \frac{C}{\sqrt{\mu}}. \tag{4.17}$$

This completes the proof. □

5. Boundary value problems

We now show that the results on the boundedness of the energy functional and on the convergence of solutions similar to the previous ones hold for the boundary value problems.

Let Ω be a bounded domain in $\mathbb{R}^d$ and let D be an open subset in Ω. We assume that Ω and D have Lipschitz boundaries and satisfy

$$\operatorname{dist}(D, \partial\Omega) \geqslant c$$

for some $c > 0$. We consider

$$\begin{cases} \nabla \cdot \mathbb{C}\nabla^s \mathbf{u} = 0 & \text{in } \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu_0} = \mathbf{g} & \text{on } \partial\Omega, \\ \mathbf{u}|_{\partial\Omega} \in L^2_\Psi(\partial\Omega), \end{cases} \tag{5.1}$$

where $\mathbb{C}$ is given by

$$\mathbb{C} = 1_D \mathbb{C}^1 + 1_{\Omega\setminus\overline{D}} \mathbb{C}^0. \tag{5.2}$$

The Dirichlet problem can be treated in the exactly same way.

The relevant energy functional for the boundary value problem is

$$J_\Omega[\mathbf{u}] := \frac{1}{2} \int_\Omega \mathbb{C}\nabla^s \mathbf{u} : \nabla^s \mathbf{u}. \tag{5.3}$$

Then the solution $\mathbf{u}$ to (5.1) is the minimizer of J_Ω over $H^1(\Omega)$ with the given boundary condition. Let $\mathbf{u}_\infty$ be the solution when $\mu = \infty$ (λ is bounded). Then we have

$$J_\Omega[\mathbf{u}] \leqslant J_\Omega[\mathbf{u}_\infty] = \frac{1}{2} \int_\Omega \mathbb{C}\nabla^s \mathbf{u}_\infty : \nabla^s \mathbf{u}_\infty.$$

Since $\mathbf{u}_\infty|_D \in \Psi$, we have $\mathbb{C}^1 \nabla^s \mathbf{u}_\infty = 0$ in D, and so,

$$J_\Omega[\mathbf{u}] \leqslant \frac{1}{2} \int_{\Omega\setminus D} \mathbb{C}^0 \nabla^s \mathbf{u}_\infty : \nabla^s \mathbf{u}_\infty \leqslant C\|\mathbf{g}\|_{H^{-1/2}(\partial\Omega)}, \tag{5.4}$$

where C is independent of μ, λ.

Using (5.4) one can show as before that

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{H^1(\Omega)} \leqslant \frac{C}{\sqrt{\mu}} \tag{5.5}$$

for all $\mu \geqslant \mu_1$ when λ is bounded, and

$$\|\mathbf{u} - \mathbf{u}_0\|_{H^1(\Omega\setminus D)} \leqslant C(\kappa + \mu)^{1/4} \tag{5.6}$$

for all $\kappa, \mu \leqslant \delta$. Here $\mathbf{u}_0$ is the solution when $\kappa = \mu = 0$. We also note that (5.4) together with Korn's inequality implies that $\|\mathbf{u}\|_{H^{1/2}(\partial\Omega)} \leqslant C$ independently of μ, λ. It then follows from (2.22) that

$$\|\mathbf{h}\|_{H^1(\Omega)} \leqslant C \tag{5.7}$$

independently of μ, λ.

6. Uniformity of asymptotic expansions

Let us first recall the notion of elastic moment tensors (EMTs) associated to the inclusion D with the Lamé constants (μ, λ) when the background Lamé constants are (λ_0, μ_0). Let $\alpha, \beta \in \mathbb{N}^d$ be multi-indices, and let $\{\mathbf{e}_k\}_{k=1}^d$ be the standard basis of $\mathbb{R}^d$. For $\alpha \in \mathbb{N}^d$, $j = 1, \ldots, d$, let $(\varphi_\alpha^j, \psi_\alpha^j)$ be the solution to (2.17) with $\mathbf{h}$ replaced by $\boldsymbol{x}^\alpha \mathbf{e}_j$. Here $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The EMT associated with D is defined by

$$M_{\alpha\beta}^j(D) = \int_{\partial D} \boldsymbol{x}^\beta \varphi_\alpha^j \, d\sigma \tag{6.1}$$

for $\alpha, \beta \in \mathbb{N}^d$ and $j = 1, \ldots, d$. It is worth mentioning that $M_{\alpha\beta}^j$ is a vector, *i.e.*, $M_{\alpha\beta}^j = (m_{\alpha\beta}^{j1}, m_{\alpha\beta}^{j2}, m_{\alpha\beta}^{j3})$ where

$$m_{\alpha\beta}^{jp}(D) = \int_{\partial D} \boldsymbol{x}^\beta \mathbf{e}_p \cdot \varphi_\alpha^j \, d\sigma. \tag{6.2}$$

If $|\alpha| = |\beta| = 1$, we may write $m_{\alpha\beta}^{jp}$ as m_{ijpq} for $i, j, p, q = 1, \ldots, d$. It is known that (m_{ijpq}) is an (anisotropic) elasticity tensor. See [12]. We emphasize that

$$\left\|\varphi_\alpha^j\right\|_{H^{-1/2}(\partial D)} \leqslant C \tag{6.3}$$

for some C independent of μ and κ as long as λ is bounded.

Suppose that D is diametrically small and it is given by

$$D = \boldsymbol{z}_0 + \epsilon B, \tag{6.4}$$

where ϵ is a small parameter representing the diameter of D, B is a reference domain containing 0, and $\boldsymbol{z}_0$ represents the location of D. Let $\mathbf{u}$ be the solution to (5.1) and $\mathbf{U}$ be the background solution, *i.e.*, the solution to

$$\begin{cases} \nabla \cdot \mathbb{C}^0 \nabla^s \mathbf{U} = 0 & \text{in } \Omega, \\ \dfrac{\partial \mathbf{U}}{\partial \nu_0} = \mathbf{g} & \text{on } \partial\Omega, \\ \mathbf{U}|_{\partial\Omega} \in L^2_\Psi(\partial\Omega). \end{cases} \tag{6.5}$$

Let $\mathbf{N}$ be the Neumann function which is the solution to

$$\begin{cases} \nabla \cdot \mathbb{C}^0 \nabla^s \mathbf{N}(\boldsymbol{x}, \boldsymbol{y}) = -\delta_{\boldsymbol{y}}(\boldsymbol{x})\mathbf{I} & \text{in } \Omega, \\ \dfrac{\partial \mathbf{N}(\cdot, \boldsymbol{y})}{\partial \nu_0} = -\dfrac{1}{|\partial\Omega|}\mathbf{I} & \text{on } \partial\Omega, \\ \mathbf{N}(\cdot, \boldsymbol{y}) \in L^2_\Psi(\partial\Omega) & \text{for each } \boldsymbol{y} \in \Omega, \end{cases} \tag{6.6}$$

where $\mathbf{I}$ is the identity matrix. It is proved in [12] (see also [7]) that the following asymptotic expansion holds on $\partial\Omega$: for $\boldsymbol{x} \in \partial\Omega$

$$\mathbf{u}(\boldsymbol{x}) = \mathbf{U}(\boldsymbol{x}) - \sum_{j=1}^{d} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d+1-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} \partial^\alpha U_j(\boldsymbol{z}_0) \partial_z^\beta \mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0) M_{\alpha\beta}^j(B) + O\left(\epsilon^{2d}\right),$$

where $\mathbf{U} = (U_1, \ldots, U_d)$. Our goal in this section is to show that this asymptotic formula holds uniformly in λ and μ. More precisely we have the following result.

Theorem 6.1. *Suppose that (2.2) and (2.3) hold. We have for $\boldsymbol{x} \in \partial\Omega$*

$$\mathbf{u}(\boldsymbol{x}) = \mathbf{U}(\boldsymbol{x}) - \sum_{j=1}^{d} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d+1-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} \partial^\alpha U_j(\boldsymbol{z}_0) \partial_z^\beta \mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0) M_{\alpha\beta}^j(B) + \mathbf{E}(\boldsymbol{x}), \tag{6.7}$$

where the error term satisfies

$$\|\mathbf{E}\|_{L^\infty(\partial\Omega)} \leqslant C\epsilon^{2d} \tag{6.8}$$

for some constant C independent of μ and λ as long as $\lambda \leqslant \Lambda$ for some constant Λ.

We emphasize that (6.7) contains not only the leading order (ϵ^d) term but also higher order terms up to ϵ^{2d-1}. The terms higher than ϵ^{2d} are expressed in terms of not only EMTs but also interactions between the boundary $\partial\Omega$ and the inclusion(s), and become much more complicated.

Proof of Theorem 6.1. We closely follow the proof in [12]. By (2.23), the solution $\mathbf{u}$ can be written in the form:

$$\mathbf{u}(\boldsymbol{x}) = \begin{cases} \mathbf{h}(\boldsymbol{x}) + \mathcal{S}_D^0[\varphi](\boldsymbol{x}), & \boldsymbol{x} \in \Omega \setminus \overline{D}, \\ \mathcal{S}_D[\psi](\boldsymbol{x}), & \boldsymbol{x} \in D, \end{cases} \tag{6.9}$$

where $\mathbf{h}$ is the function given by (2.22) and $\varphi, \psi \in H^{-1/2}(\partial D)$ are the solutions to

$$\begin{cases} \mathcal{S}_D[\psi](\boldsymbol{x}) - \mathcal{S}_D^0[\varphi](\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x}) \\ \dfrac{\partial \mathcal{S}_D[\psi]}{\partial \nu}\bigg|_-(\boldsymbol{x}) - \dfrac{\partial \mathcal{S}_D^0[\varphi]}{\partial \nu_0}\bigg|_+(\boldsymbol{x}) = \dfrac{\partial \mathbf{h}}{\partial \nu_0}(\boldsymbol{x}) \end{cases} \quad \text{for } \boldsymbol{x} \in \partial D. \tag{6.10}$$

Let $\tilde{\varphi}(\boldsymbol{x}) := \epsilon\varphi(\mathbf{z}_0 + \epsilon\boldsymbol{x})$ and $\tilde{\psi}(\boldsymbol{x}) := \epsilon\psi(\mathbf{z}_0 + \epsilon\boldsymbol{x})$. By a change of variables, (6.10) can be scaled into, for $\boldsymbol{x} \in \partial B$,

$$\begin{cases} \mathcal{S}_B[\tilde{\psi}](\boldsymbol{x}) - \mathcal{S}_B^0[\tilde{\varphi}](\boldsymbol{x}) = \mathbf{h}(\mathbf{z}_0 + \epsilon\boldsymbol{x}) - \delta_{d2} \dfrac{\alpha \ln \epsilon}{2\pi} \displaystyle\int_{\partial B} \tilde{\psi}, \\ \dfrac{\partial \mathcal{S}_B[\tilde{\psi}]}{\partial \nu}\bigg|_-(\boldsymbol{x}) - \dfrac{\partial \mathcal{S}_B^0[\tilde{\varphi}]}{\partial \nu_0}\bigg|_+(\boldsymbol{x}) = \dfrac{\partial}{\partial \nu_0}\mathbf{h}(\mathbf{z}_0 + \epsilon\boldsymbol{x}), \end{cases} \tag{6.11}$$

where δ_{d2} is the Kronecker delta function. By the Taylor expansion of $\mathbf{h} = (h_1, \ldots, h_d)$, we have

$$\mathbf{h}(\mathbf{z}_0 + \epsilon\boldsymbol{x}) = \sum_{j=1}^{d} \sum_{|\alpha|=0}^{d} \frac{\epsilon^{|\alpha|}}{\alpha!} \partial^\alpha h_j(\mathbf{z}_0) \boldsymbol{x}^\alpha \mathbf{e}_j + O\left(\epsilon^{d+1}\right). \tag{6.12}$$

Here the error term is independent of λ, μ, and $\partial^\alpha \mathbf{h}(\boldsymbol{z}_0)$ is bounded independently of λ, μ as long as λ is bounded because of (5.7). By (6.12) and the linearity of (6.11) we have

$$\tilde{\varphi}(\boldsymbol{x}) = \sum_{j=1}^{d} \sum_{|\alpha|=1}^{d} \frac{\epsilon^{|\alpha|}}{\alpha!} \partial^\alpha h_j(\boldsymbol{z}_0) \varphi_\alpha^j + O\big(\epsilon^{d+1}\big), \tag{6.13}$$

where φ_α^j is given in the definition of EMT. Here we see from Lemma 3.2 that the error term in (6.13) is uniform with respect to $\lambda, \ \mu$ as long as λ is bounded.

It is known that

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{U}(\boldsymbol{x}) - \int_{\partial D} \mathbf{N}(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \partial\Omega \tag{6.14}$$

(see [12]). Using (6.13) and the Taylor expansion of $\mathbf{N}(\boldsymbol{x}, \boldsymbol{y})$, we have

$$\mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0 + \epsilon \boldsymbol{y}) = \sum_{|\beta|=0}^{\infty} \frac{1}{\beta!} \epsilon^{|\beta|} \partial_{\boldsymbol{z}}^\beta \mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0) \boldsymbol{y}^\beta, \quad \boldsymbol{x} \in \partial\Omega,$$

and so, for $\boldsymbol{x} \in \partial\Omega$,

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) &= \mathbf{U}(\boldsymbol{x}) - \int_{\partial B} \mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0 + \epsilon \boldsymbol{y}) \tilde{\varphi}(\boldsymbol{y}) \epsilon^{d-2} d\sigma(\boldsymbol{y}) \\ &= \mathbf{U}(\boldsymbol{x}) - \sum_{j=1}^{d} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d+1-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} \partial^\alpha h_j(\boldsymbol{z}_0) \partial_z^\beta \mathbf{N}(\boldsymbol{x}, \boldsymbol{z}_0) M_{\alpha\beta}^j + O\big(\epsilon^{2d}\big). \end{aligned} \tag{6.15}$$

The formula (6.15) implies in particular that

$$\|\mathbf{u} - \mathbf{U}\|_{H^{1/2}(\partial\Omega)} = O\big(\epsilon^d\big),$$

where $O(\epsilon^d)$ is uniform with respect to λ and μ. Since

$$\begin{aligned} \mathbf{h}(\boldsymbol{x}) &= \int_{\partial\Omega} \frac{\partial \Gamma^{\mathbf{0}}(\boldsymbol{x} - \boldsymbol{y})}{\partial \nu_0(\boldsymbol{y})} \mathbf{u}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) - \int_{\partial\Omega} \Gamma^{\mathbf{0}}(\boldsymbol{x} - \boldsymbol{y}) \mathbf{g}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\ &= \mathbf{U}(\boldsymbol{x}) + \int_{\partial\Omega} \frac{\partial \Gamma^{\mathbf{0}}(\boldsymbol{x} - \boldsymbol{y})}{\partial \nu_0(\boldsymbol{y})} (\mathbf{u} - \mathbf{U})(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \end{aligned}$$

we have

$$\partial^\alpha \mathbf{h}(\boldsymbol{z}_0) = \partial^\alpha \mathbf{U}(\boldsymbol{z}_0) + O\big(\epsilon^d\big), \tag{6.16}$$

independently of λ and μ. By substituting this into (6.15), we have (6.1). □

7. The case of multiple inclusions

So far we deal with the case where the inclusion D is a simply connected inclusion, but the methods and results of this paper work even when there are multiple simply connected inclusions.

Let us make a brief remark on the case when D has n disjoint simply connected components, say $D_1, \dots, D_n$. In this case, the solution $\mathbf{u}$ to (2.6) is represented as

$$\mathbf{u}(\boldsymbol{x}) = \begin{cases} \mathbf{h}(\boldsymbol{x}) + \sum_{j=1}^{n} \mathcal{S}_j^0[\varphi^{(j)}](\boldsymbol{x}), & \boldsymbol{x} \in D^c, \\ \mathcal{S}_j[\psi^{(j)}](\boldsymbol{x}), & \boldsymbol{x} \in D_j, \ j = 1, \dots, n, \end{cases} \tag{7.1}$$

where $\mathcal{S}_j^0$ and $\mathcal{S}_j$ denote single layer potentials on ∂D_j, and $\varphi^{(j)}, \psi^{(j)}$ are the solutions to

$$\begin{cases} \mathcal{S}_j\big[\psi^{(j)}\big](\boldsymbol{x}) - \displaystyle\sum_{j=1}^{n} \mathcal{S}_j^0\big[\varphi^{(j)}\big](\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x}) \\ \dfrac{\partial \mathcal{S}_j[\psi^{(j)}]}{\partial \nu}\bigg|_{-}(\boldsymbol{x}) - \displaystyle\sum_{j=1}^{n} \dfrac{\partial \mathcal{S}_j^0[\varphi^{(j)}]}{\partial \nu_0}\bigg|_{+}(\boldsymbol{x}) = \dfrac{\partial \mathbf{h}}{\partial \nu_0}(\boldsymbol{x}) \end{cases} \quad \text{for } \boldsymbol{x} \in \partial D_j, \ j = 1, \dots, n. \tag{7.2}$$

If κ is either 0 or ∞, we have a similar representation:

$$\mathbf{u}_\kappa(\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x}) + \sum_{j=1}^{n} \mathcal{S}_j^0\big[\varphi_\kappa^{(j)}\big](\boldsymbol{x}), \quad \boldsymbol{x} \in D^c, \ \kappa = 0, \infty, \tag{7.3}$$

where $\varphi_\kappa^{(j)}$ satisfies appropriate integral equations. We can show in a similar way that

$$\sum_{j=1}^{n} \big\|\varphi^{(j)} - \varphi_\infty^{(j)}\big\|_{H^{-1/2}(\partial D_j)} \leqslant \frac{C}{\sqrt{\mu}} \sum_{j=1}^{n} \bigg\| \frac{\partial \mathbf{h}}{\partial \nu_0} \bigg\|_{H^{-1/2}(\partial D_j)} \tag{7.4}$$

and

$$\sum_{j=1}^{n} \big\|\varphi^{(j)} - \varphi_0^{(j)}\big\|_{H^{-1/2}(\partial D_j)} \leqslant C(\kappa + \mu)^{1/4} \sum_{j=1}^{n} \bigg\| \frac{\partial \mathbf{h}}{\partial \nu_0} \bigg\|_{H^{-1/2}(\partial D_j)}. \tag{7.5}$$

So, Theorems 4.2 and 6.1 are valid even if D has several components.

It is worth mentioning that if there are multiple inclusions, the convergence depends on the distance between inclusions since $|\nabla \mathbf{u}|$ may be arbitrarily large as the distance between inclusions tends to zero. This fact was proved in, for example, [11,30,13] for the conductivity problem. (See [3] for an extensive list of recent papers on this problem.) For the elasticity problem, it is shown in [24] by numerical computations that $|\nabla \mathbf{u}|$ may blow up as the distance between inclusions tends to zero.

8. Conclusion

In this paper we have used layer potential techniques to prove uniform convergence in an appropriate function spaces of solutions to the Lamé system as the bulk and shear moduli tend to extreme values (zero or ∞) provided that the compressional modulus is bounded. Making use of this result, we have shown that the asymptotic expansion of the solution due to the presence of diametrically small inclusions is uniform with respect to the bulk and shear moduli. These results are obtained under the assumption that the Lamé parameters of the background and inclusions are constant. We expect that the same results hold even if the background Lamé parameters are not constants, but variables, even though the methods of this paper do not apply to that case. Another interesting case is when the inclusion is thin, and the thickness tends to zero [15]. In this case we expect that the asymptotic expansion is not uniform.

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