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Ground state solutions for an indefinite Kirchhoff type problem with steep potential well ☆

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Abstract

In this paper we study an indefinite Kirchhoff type equation with steep potential well. Under some suitable conditions, the existence and the non-existence of nontrivial solutions are obtained by using variational methods. Furthermore, the phenomenon of concentration of solutions is also explored.

Keywords: Kirchhoff type problem; Steep potential well; Ground state; Nehari manifold; Mountain pass

1. Introduction

In this paper, we investigate the existence and concentration of solutions to a class of Kirchhoff type problems:

$$\begin{cases} -\left(a\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\,dx + b\right)\Delta u + \lambda V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$K_{\lambda,a}$}$$

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where $N \geqslant 3$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, the parameters $a, b, \lambda > 0$ and the potential V satisfies the following conditions:

$(V1)$ $V \in C(\mathbb{R}^N)$ and $V \geqslant 0$ on $\mathbb{R}^N$;
$(V2)$ there exists $c > 0$ such that the set $\{V < c\} = \{x \in \mathbb{R}^N \mid V(x) < c\}$ is nonempty and has finite measure;
$(V3)$ $\Omega = \mathit{int}\, V^{-1}(0)$ is nonempty and has smooth boundary with $\overline{\Omega} = V^{-1}(0)$.

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(a \int_{\Omega} |\nabla u|^2 \, dx + b\right) \Delta u = f(x, u), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, u denotes the displacement, $f(x, u)$ is the external force and b is the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. The solvability of the Kirchhoff type equation (1.1) has been well-studied in general dimensions and domains by various authors (see, for examples, [10,11,14,16,18,20,22,24] and the references therein).

Nonlocal effect also finds its applications in biological systems. A parabolic version of Eq. (1.2) can, in theory, be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed dependent on the "energy" of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type $u_t - a(\int_{\Omega} u \, dx)\Delta u = f$. Chipot and Lovat [8] and Corrêa et al. [9], for examples, studied the existence of solutions and their uniqueness for such nonlocal problems as well as their corresponding elliptic problems.

More recently, the stationary problem of Eq. (1.1):

$$\begin{cases} -\left(a \displaystyle\int_{\Omega} |\nabla u|^2 \, dx + b\right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has been studied by many authors; for example, Alves et al. [1] and many others [5,7,19] using variational methods, proved the existence of positive solutions while Zhang and Perera [25] obtained sign changing solutions via invariant sets of descent flow. Bensedik and Bouchekif [5] studied the asymptotically linear case and obtained the existence of positive solutions for Eq. (1.2) when the assumptions about the asymptotic behaviors of f are the following

(f_1) $t \longmapsto \frac{f(x,t)}{t}$ is a nondecreasing function for any fixed $x \in \overline{\Omega}$;
(f_2) $\lim_{t \to 0} \frac{f(x,t)}{t} = \overline{p}(x)$ and $\lim_{t \to \infty} \frac{f(x,t)}{t} = \overline{q}(x)$ uniformly in $x \in \Omega$, where $0 \leqslant \overline{p}(x)$, $\overline{q}(x) \in L^{\infty}(\Omega)$ and $\sup_{x \in \Omega} \overline{p}(x) < m_0 \lambda_1$, λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

On the other hand, the conditions $(V1)$–$(V3)$ imply that λV represents a potential well whose depth is controlled by λ. λV is called a steep potential well if λ is sufficiently large and one expects to find solutions which localize near its bottom Ω. This problem has found much interest after being first introduced by Bartsch and Wang [4] in the study of the existence of positive solutions for nonlinear Schrödinger equations and has been attracting much attention, see [2,3,21,23]. Very recently, Jiang and Zhou [15] first applied the steep potential well into Schrödinger–Poisson system, and obtained the existence and concentration results by combining domains approximation with priori estimates. Later, Zhao et al. [26] studied another Schrödinger–Poisson system with V satisfying the conditions $(V1)$–$(V3)$, namely,

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$, $K \geqslant 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3)$ (or $L^\infty(\mathbb{R}^3)$). They obtained the existence and concentration results for $p \in (3,6)$ via variational methods. In particular, the potential V is allowed to be sign-changing for the case $p \in (4,6)$.

Inspired by the above facts, the aim of this paper is to consider the Kirchhoff type equations with steep potential well. To the author's knowledge, this case seems to be considered by few authors. We mainly study the existence of ground state solution for Eq. $(K_{\lambda,a})$ with the indefinite nonlinear term $f(x,u)$. Furthermore, the non-existence and concentration of nontrivial solutions are also discussed.

Before stating our results we need to introduce some notations and definitions.

Notation 1.1. Throughout this paper, we denote by $|\cdot|_r$ the L^r-norm, $1 \leqslant r \leqslant \infty$, and we have to use the notations $p^\pm = \sup\{\pm p, 0\}$ and the critical exponent $2^* = \frac{2N}{N-2}$ for $N \geqslant 3$. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$.

Definition 1.1. u is a ground state of Eq. $(K_{\lambda,a})$ we mean that u is such a solution of Eq. $(K_{\lambda,a})$ which has the least energy among all nontrivial solutions of Eq. $(K_{\lambda,a})$.

We need the following the minimum problem: for each positive integer $k = 1, 3, 4$,

$$\lambda_1^{(k)} = \inf\left\{ \left(\int_\Omega |\nabla u|^2\,dx \right)^{\frac{k+1}{2}} \;\middle|\; u \in H_0^1(\Omega),\ \int_\Omega q|u|^{k+1}\,dx = 1 \right\}, \tag{1.3}$$

where q is a bounded function on $\overline{\Omega}$ with $q^+ \not\equiv 0$. Then $\lambda_1^{(k)} > 0$, which is achieved by some $\phi_k \in H_0^1(\Omega)$ which $\int_\Omega q|\phi_k|^{k+1}\,dx = 1$ and $\phi_k > 0$ a.e. in Ω, by the compactness of Sobolev embedding from $H_0^1(\Omega)$ into $L^{k+1}(\Omega)$ and Fatou's lemma (see Figueiredo [13]). In particular,

$$\lambda_1^{(k)} \int_\Omega q|u|^{k+1}\,dx \leqslant \left(\int_\Omega |\nabla u|^2\,dx \right)^{\frac{k+1}{2}} \quad \text{for all } u \in H_0^1(\Omega). \tag{1.4}$$

Now, we give our main results.

Theorem 1.2. *Suppose that the conditions* $(V1)$–$(V3)$ *hold. In addition, for each* $k = 1, 3, 4$, *we assume that the function* f *satisfies the following conditions*:

$(D1)$ $f(x,s)$ *is a continuous function on* $\mathbb{R}^N \times \mathbb{R}$ *such that* $f(x,s) \equiv 0$ *for all* $s < 0$ *and* $x \in \mathbb{R}^N$. *Moreover, there exists* $p \in L^\infty(\mathbb{R}^N)$ *with*

$$\left|p^+\right|_\infty < \Theta_0 := \frac{\overline{S}^2 \min\{b,1\}}{|\{V < c\}|^{\frac{2^*-2}{2^*}}}$$

such that

$$\lim_{s \to 0^+} \frac{f(x,s)}{s^k} = p(x) \quad \textit{uniformly in } x \in \mathbb{R}^N$$

and

$$\frac{f(x,s)}{s^k} \geqslant p(x) \quad \textit{for all } s > 0 \textit{ and } x \in \overline{\Omega},$$

where $\overline{S}$ *is the best constant for the embedding of* $D^{1,2}(\mathbb{R}^N)$ *in* $L^{2^*}(\mathbb{R}^N)$, *and* $|\cdot|$ *is the Lebesgue measure*;

$(D2)$ *there exists* $q \in L^\infty(\mathbb{R}^N)$ *with* $q^+ \not\equiv 0$ *on* $\overline{\Omega}$ *such that*

$$\lim_{s \to \infty} \frac{f(x,s)}{s^k} = q(x) \quad \textit{uniformly in } x \in \mathbb{R}^N;$$

$(D3)$ *there exists a constant* d_0 *satisfying* $0 \leqslant d_0 < \frac{\overline{S}^2 \min\{b,1\}}{4} |\{V < c\}|^{\frac{2-2^*}{2^*}}$ *such that*

$$F(x,s) - \frac{1}{4} f(x,s)s \leqslant d_0 s^2 \quad \textit{for all } s > 0 \textit{ and } x \in \mathbb{R}^N.$$

Then we have the following results:

(i) *If* $k = 1$, $N \geqslant 3$ *and* $b\lambda_1^{(1)} < 1$, *then there exists a positive number* a^* *such that for every* $a \in (0, a^*)$, *there exists* $\Lambda^* > 0$ *such that Eq.* $(K_{\lambda,a})$ *has at least one nontrivial solution for all* $\lambda > \Lambda^*$.

(ii) *If* $k = 3$ *and* $N = 3$, *then for each* $a \in (0, 1/\lambda_1^{(3)})$ *there exists* $\Lambda^* > 0$ *such that Eq.* $(K_{\lambda,a})$ *has at least one nontrivial solution for all* $\lambda > \Lambda^*$.

(iii) *If* $k = 4$ *and* $N = 3$, *then for each* $a > 0$ *there exists* $\Lambda^* > 0$ *such that Eq.* $(K_{\lambda,a})$ *has at least one nontrivial solution for all* $\lambda > \Lambda^*$.

Theorem 1.3. *Suppose that the conditions* $(V1)$–$(V3)$ *hold. In addition, for each* $k = 1, 3, 4$, *we assume that the function* f *satisfies the conditions* $(D1)$–$(D3)$ *and the following condition*:

$(D4)$ $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *and* $s \longmapsto \frac{f(x,s)}{s^k}$ *is non-decreasing function for any fixed* $x \in \mathbb{R}$.

Let $\Lambda^* > 0$ be as in Theorem 1.2. Then we have the following results:

(i) *If $k = 1$, $N \geqslant 3$ and $b\lambda_1^{(1)} < 1$, then there exists a positive number a^* such that for every $a \in (0, a^*)$, there exists $\Lambda^{**} \geqslant \Lambda^*$ such that Eq. $(K_{\lambda,a})$ has a ground state solution for all $\lambda > \Lambda^*$.*
(ii) *If $k = 3$ and $N = 3$, then for each $a \in (0, 1/\lambda_1^{(3)})$ and $\lambda > \Lambda^*$, Eq. $(K_{\lambda,a})$ has a ground state solution.*
(iii) *If $k = 4$ and $N = 3$, then for each $a > 0$ and $\lambda > \Lambda^*$, Eq. $(K_{\lambda,a})$ has a ground state solution.*

Now, we consider the following the minimum problem:

$$\widehat{\lambda}_1^{(3)} = \inf\left\{\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 \,\Big|\, u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} q|u|^4\,dx = 1\right\} \geqslant 0. \tag{1.5}$$

Clearly, $\widehat{\lambda}_1^{(3)} \leqslant \lambda_1^{(3)}$. Then we have the following result.

Theorem 1.4. *Suppose that the conditions $(V1)$–$(V3)$ hold. In addition, for each $k = 1, 3$, we assume that the function f satisfies the conditions $(D2)$ and $(D4)$. Then we have the following results*:

(i) *If $k = 1$, $N \geqslant 3$ and $b > |q|_\infty \bar{S}^{-2}|\Omega|^{\frac{2^*-2}{2^*}}$, then there exists positive number Λ_* such that for every $a > 0$ and $\lambda > \Lambda_*$, Eq. $(K_{\lambda,a})$ does not admit any nontrivial solution.*
(ii) *If $k = 3$ and $N = 3$ and $\widehat{\lambda}_1^{(3)} > 0$, then for every $a \geqslant 1/\widehat{\lambda}_1^{(3)}$ and $\lambda > 0$, Eq. $(K_{\lambda,a})$ does not admit any nontrivial solution.*

On the concentration of solutions we have the following result.

Theorem 1.5. *Let u_λ be the solution obtained by Theorem 1.2. Then for every $r \in [2, 2^*)$, $u_\lambda \to u_0$ strongly in $L^r(\mathbb{R}^N)$ as $\lambda \to \infty$, where $u_0 \in H_0^1(\Omega)$ is the nontrivial solution of equation*:

$$\begin{cases} -\left(a\displaystyle\int_\Omega |\nabla u|^2\,dx + b\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{K_∞}$$

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Sections 3–6, we give the proofs of our main results.

2. Variational setting and preliminaries

In this section, we give the variational setting for Eq. $(K_{\lambda,a})$ following [11] and establish compactness conditions. Let

$$X = \left\{u \in H^1(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty\right\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V u v)\, dx, \qquad \|u\| = \langle u, u\rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V u v)\, dx, \qquad \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\| \leqslant \|u\|_\lambda$ for $\lambda \geqslant 1$.

Set $X_\lambda = (X, \|u\|_\lambda)$. It follows from the conditions $(V1)$–$(V2)$ and the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx \\
&\quad= \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\{V<c\}} u^2\, dx + \int_{\{V\geqslant c\}} u^2\, dx \\
&\quad\leqslant \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \left(\int_{\{V<c\}} 1\, dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{\{V<c\}} |u|^{2^*}\, dx\right)^{\frac{2}{2^*}} + \frac{1}{c}\int_{\{V\geqslant c\}} V(x) u^2\, dx \\
&\quad\leqslant \left(1 + \left|\{V<c\}\right|^{\frac{2^*-2}{2^*}} \overline{S}^{-2}\right) \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{c}\int_{\mathbb{R}^N} V(x) u^2\, dx \\
&\quad\leqslant \max\left\{1 + \left|\{V<c\}\right|^{\frac{2^*-2}{2^*}} \overline{S}^{-2}, \frac{1}{c}\right\} \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} V(x) u^2\, dx\right),
\end{aligned}$$

which implies that the imbedding $X \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Moreover, using the same conditions and techniques, for any $r \in [2, 2^*]$, we also have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^r\, dx &\leqslant \left(\int_{\{V\geqslant c\}} u^2\, dx + \int_{\{V<c\}} u^2\, dx\right)^{\frac{2^*-r}{2^*-2}} \left(\overline{S}^{-2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{\frac{2^*}{2}}\right)^{\frac{r-2}{2^*-2}} \\
&\leqslant \left(\frac{1}{\lambda c}\int_{\{V\geqslant c\}} \lambda V(x) u^2\, dx + \left(\int_{\{V<c\}} 1\, dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{\{V<c\}} |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}\right)^{\frac{2^*-r}{2^*-2}} \\
&\quad\cdot \left(\overline{S}^{-2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda V(x) u^2\, dx\right)^{\frac{2^*}{2}}\right)^{\frac{r-2}{2^*-2}}
\end{aligned}$$

$$\leqslant \left(\frac{1}{\lambda c} \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx + \left| \{V < c\} \right|^{\frac{2^*-2}{2^*}} \overline{S}^{-2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{2^*-r}{2^*-2}}$$

$$\cdot \left(\overline{S}^{-2^*} \|u\|_\lambda^{2^*} \right)^{\frac{r-2}{2^*-2}}$$

$$\leqslant \left(\max\left\{ \frac{1}{\lambda c}, \frac{|\{V < c\}|^{\frac{2^*-2}{2^*}}}{\overline{S}^2} \right\} \|u\|_\lambda^2 \right)^{\frac{2^*-r}{2^*-2}} \overline{S}^{\frac{-2^*(r-2)}{2^*-2}} \|u\|_\lambda^{\frac{2^*(r-2)}{2^*-2}}$$

$$= \left(\max\left\{ \frac{\overline{S}^2}{\lambda c}, \left| \{V < c\} \right|^{\frac{2^*-2}{2^*}} \right\} \right)^{\frac{2^*-r}{2^*-2}} \overline{S}^{-r} \|u\|_\lambda^r \quad \text{for all } \lambda > 0, \tag{2.1}$$

this implies that

$$\int_{\mathbb{R}^N} |u|^r \, dx \leqslant \left| \{V < c\} \right|^{\frac{2^*-r}{2^*}} \overline{S}^{-r} \|u\|_\lambda^r \quad \text{for all } \lambda \geqslant \frac{\overline{S}^2}{c} \left| \{V < c\} \right|^{\frac{2-2^*}{2^*}}. \tag{2.2}$$

It is well known that Eq. $(K_{\lambda,a})$ is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^N)$ by

$$J_{\lambda,a}(u) = \frac{1}{2} \left(b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \lambda V u^2 \, dx \right) + \frac{a}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^N} F(x,u) \, dx,$$

where $F(x,u) = \int_0^u f(x,s) \, ds$. Furthermore, it is easy to prove that the functional $J_{\lambda,a}$ is of class C^1 in $H^1(\mathbb{R}^N)$, and that

$$\left\langle J'_{\lambda,a}(u), v \right\rangle = \left(b + a \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \int_{\mathbb{R}^N} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \lambda V u v \, dx - \int_{\mathbb{R}^N} f(x,u) v \, dx.$$

Hence, if $u \in H^1(\mathbb{R}^N)$ is a critical point of $J_{\lambda,a}$, then u is a solution of Eq. $(K_{\lambda,a})$. Furthermore, we have the following result.

Lemma 2.1. *Suppose that the conditions* $(V1)$–$(V3)$ *hold. In addition, for each* $k = 1, 3, 4$, *we assume that the function* f *satisfies the condition* $(D3)$. *Then for each nontrivial solution* u_λ *of Eq.* $(K_{\lambda,a})$, *we have* $J_{\lambda,a}(u_\lambda) > 0$.

Proof. If u_λ is a nontrivial solution of Eq. $(K_{\lambda,a})$, then

$$\left(b \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 \, dx + \int_{\mathbb{R}^N} \lambda V u_\lambda^2 \, dx \right) + a \left(\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 \, dx \right)^2 = \int_{\mathbb{R}^N} f(x, u_\lambda) u_\lambda \, dx.$$

Moreover, by the condition $(D3)$,

$$\int_{\mathbb{R}^N}\left[F(x,u_\lambda)-\frac{1}{4}f(x,u_\lambda)u_\lambda\right]dx \leqslant \int_{\mathbb{R}^N} d_0u_\lambda^2\,dx,$$

this implies that

$$\begin{aligned}
J_{\lambda,a}(u_\lambda) &= \frac{1}{4}\left(b\int_{\mathbb{R}^N}|\nabla u_\lambda|^2\,dx+\int_{\mathbb{R}^N}\lambda V u_\lambda^2\,dx\right)-\int_{\mathbb{R}^N}\left[F(x,u_\lambda)-\frac{1}{4}f(x,u_\lambda)u_\lambda\right]dx\\
&\geqslant \frac{\min\{b,1\}}{4}\|u_\lambda\|_\lambda^2-d_0\int_{\mathbb{R}^N}u_\lambda^2\,dx\\
&\geqslant \left(\frac{\min\{b,1\}}{4}-d_0\big|\{V<c\}\big|^{\frac{2^*-2}{2^*}}\overline{S}^{-2}\right)\|u_\lambda\|_\lambda^2\\
&> 0.
\end{aligned}$$

This completes the proof. □

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence.

Theorem 2.2. *(See [12], Mountain Pass Theorem.) Let E be a real Banach space with its dual space E^*, and suppose that $I\in C^1(E,\mathbb{R})$ satisfies*

$$\max\big\{I(0),I(e)\big\}\leqslant\mu<\eta\leqslant\inf_{\|u\|=\rho}I(u),$$

for some $\mu<\eta$, $\rho>0$ and $e\in E$ with $\|e\|>\rho$. Let $\hat{c}\geqslant\eta$ be characterized by

$$\hat{c}=\inf_{\gamma\in\Gamma}\max_{0\leqslant\tau\leqslant1}I\big(\gamma(\tau)\big),$$

where $\Gamma=\{\gamma\in C([0,1],E)\colon\ \gamma(0)=0,\ \gamma(1)=e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\}\subset E$ such that

$$I(u_n)\to\hat{c}\geqslant\eta\quad\text{and}\quad\big(1+\|u_n\|\big)\big\|I'(u_n)\big\|_{E^*}\to0,\quad\text{as } n\to\infty.$$

In what follows, we give the following lemmas which ensure that the functional $J_{\lambda,a}$ has the mountain pass geometry.

Lemma 2.3. *Suppose that the conditions $(V1)$–$(V2)$ hold. In addition, for each $k=1,3,4$, we assume that the function f satisfies the conditions $(D1)$–$(D2)$. Then there exist $\rho>0$ and $\eta>0$ such that $\inf\{J_{\lambda,a}(u)\colon\ u\in X_\lambda$ with $\|u\|_\lambda=\rho\}>\eta$ for all $\lambda>\frac{\overline{S}^2}{c}|\{V<c\}|^{\frac{2-2^*}{2^*}}$.*

Proof. For any $\epsilon > 0$, it follows from the conditions $(D1)$ and $(D2)$ that there exists $C_\epsilon > 0$ such that

$$F(x,s) \leqslant \frac{|p^+|_\infty + \epsilon}{2} s^2 + \frac{C_\epsilon}{r}|s|^r, \quad \text{for all } s \in \mathbb{R}, \tag{2.3}$$

where $\max\{2,k\} < r < 2^*$. Then, by (2.2), (2.3) and the Sobolev inequality, for every $u \in X_\lambda$ and $\lambda \geqslant \frac{\bar{S}^2}{c}|\{V<c\}|^{\frac{2-2^*}{2^*}}$,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\,dx &\leqslant \frac{|p^+|_\infty + \epsilon}{2}\int_{\mathbb{R}^N} u^2\,dx + \frac{C_\epsilon}{r}\int_{\mathbb{R}^N}|u|^r\,dx \\ &\leqslant \frac{(|p^+|_\infty+\epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}}{2\bar{S}^2}\|u\|_\lambda^2 + \frac{C_\epsilon|\{V<c\}|^{\frac{2^*-r}{2^*}}}{r\bar{S}^r}\|u\|_\lambda^r,\end{aligned}$$

this implies that

$$\begin{aligned}J_{\lambda,a}(u) &= \frac{1}{2}\left(b\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \int_{\mathbb{R}^N}\lambda V u^2\,dx\right) + \frac{a}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^N}F(x,u)\,dx \\ &\geqslant \frac{1}{2}\left(b\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \int_{\mathbb{R}^N}\lambda V u^2\,dx\right) - \frac{(|p^+|_\infty+\epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}}{2\bar{S}^2}\|u\|_\lambda^2 \\ &\quad - \frac{C_\epsilon|\{V<c\}|^{\frac{2^*-r}{2^*}}}{r\bar{S}^r}\|u\|_\lambda^r \\ &\geqslant \frac{1}{2}\left(\min\{b,1\} - \frac{(|p^+|_\infty+\epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}}{\bar{S}^2}\right)\|u\|_\lambda^2 - \frac{C_\epsilon|\{V<c\}|^{\frac{2^*-r}{2^*}}}{r\bar{S}^r}\|u\|_\lambda^r.\end{aligned}$$

Therefore, by the condition $(D1)$, fixing $\epsilon \in (0, \Theta_0 - |p^+|_\infty)$ and letting $\|u\|_\lambda = \rho > 0$ small enough, it is easy to see that there is $\eta > 0$ such that this lemma holds. □

Lemma 2.4. *Suppose that the conditions $(V1)$–$(V2)$ hold. In addition, for each $k = 1,3,4$, we assume that the function f satisfies the conditions $(D1)$–$(D2)$. Let $\rho > 0$ be as in Lemma 2.3. Then we have the following results*:

(i) *If $k = 1$, $N \geqslant 3$ and $\lambda_1^{(1)} < \frac{1}{b}$, then there exist $a^* > 0$ and $e \in H^1(\mathbb{R}^N)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $a \in (0,a^*)$ and $\lambda > 0$.*

(ii) *If $k = 3$ and $N = 3$, then there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $0 < a < 1/\lambda_1^{(3)}$ and $\lambda > 0$.*

(iii) *If $k = 4$ and $N = 3$, then there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $a > 0$ and $\lambda > 0$.*

Proof. (i) By $b\lambda_1^{(1)} < 1$, the condition $(D2)$ and Fatou's lemma, we have

$$\lim_{t\to+\infty}\frac{J_{\lambda,0}(t\phi_1)}{t^2} = \frac{1}{2}\left(b\int_{\mathbb{R}^N}|\nabla\phi_1|^2\,dx + \int_{\mathbb{R}^N}\lambda V\phi_1^2\,dx\right) - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\phi_1)}{t^2\phi_1}\phi_1\,dx$$
$$\leqslant \frac{b}{2}\int_{\Omega}|\nabla\phi_1|^2\,dx - \frac{1}{2}\int_{\Omega}q\phi_1^2\,dx \leqslant \frac{1}{2}\left(b - \frac{1}{\lambda_1^{(1)}}\right)\int_{\Omega}|\nabla\phi_1|^2\,dx$$
$$< 0,$$

where $J_{\lambda,0}(u) = J_{\lambda,a}(u)$ with $a = 0$. Thus, if $J_{\lambda,0}(t\phi_1) \to -\infty$ as $t \to +\infty$, then there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,0}(e) < 0$. Since $J_{\lambda,a}(e) \to J_{\lambda,0}(e)$ as $a \to 0^+$, we see that there exists $a^* > 0$ such that $J_{\lambda,a}(e) < 0$ for all $a \in (0, a^*)$.

(ii) and (iii) By (1.3), we define

$$\psi_k = \begin{cases} \phi_3, & \text{if } k = 3,\\ \phi_4, & \text{if } k = 4.\end{cases}$$

Then, by $(D1)$, $(D2)$ and Fatou's lemma, one has

$$\lim_{t\to+\infty}\frac{J_{\lambda,a}(t\psi_k)}{t^{k+1}} = \begin{cases} \frac{a}{4}(\int_{\mathbb{R}^N}|\nabla\phi_3|^2\,dx)^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\phi_3)}{t^4\phi_3^4}\phi_3^4\,dx, & \text{if } k = 3,\\ -\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\phi_4)}{t^5\phi_4^5}\phi_4^5\,dx, & \text{if } k = 4\end{cases}$$
$$\leqslant \begin{cases} \frac{1}{4}[a(\int_{\Omega}|\nabla\phi_3|^2\,dx)^2 - \int_{\Omega}q\phi_3^4\,dx], & \text{if } k = 3,\\ -\frac{1}{5}\int_{\Omega}q\phi_4^5\,dx, & \text{if } k = 4\end{cases}$$
$$= \begin{cases} \frac{1}{4}(a\lambda_1^{(3)} - 1), & \text{if } k = 3,\\ -\frac{1}{5}, & \text{if } k = 4\end{cases}$$
$$< 0,$$

this implies that $J_{\lambda,a}(t\psi_k) \to -\infty$ as $t \to +\infty$. Therefore, there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ and the lemma is proved. □

3. Proof of Theorem 1.2

First we define

$$\alpha_{\lambda,a} = \inf_{\gamma\in\Gamma_\lambda}\max_{0\leqslant t\leqslant 1} J_{\lambda,a}\big(\gamma(t)\big),$$

and

$$\alpha_{0,a}(\Omega) = \inf_{\gamma\in\bar{\Gamma}_\lambda(\Omega)}\max_{0\leqslant t\leqslant 1} J_{\lambda,a}|_{H_0^1(\Omega)}\big(\gamma(t)\big),$$

where $J_{\lambda,a}|_{H_0^1(\Omega)}$ is a restriction of $J_{\lambda,a}$ on $H_0^1(\Omega)$,

$$\Gamma_\lambda = \{\gamma \in C([0,1], X_\lambda) \colon \gamma(0) = 0,\ \gamma(1) = e\}$$

and

$$\overline{\Gamma}_\lambda(\Omega) = \{\gamma \in C([0,1], H_0^1(\Omega)) \colon \gamma(0) = 0,\ \gamma(1) = e\}.$$

Note that

$$J_{\lambda,a}|_{H_0^1(\Omega)}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{a}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)\,dx \quad \text{for } u \in H_0^1(\Omega),$$

and $\alpha_{0,a}(\Omega)$ independent of λ. Moreover, if the conditions $(D1)$–$(D3)$ hold, similar to the proofs of Lemmas 2.3 and 2.4, we can conclude that $J_{\lambda,a}|_{H_0^1(\Omega)}$ also satisfies the mountain pass hypothesis in Theorem 2.2. Note that $H_0^1(\Omega) \subset X_\lambda$ for all $\lambda > 0$, then $0 < \eta \leqslant \alpha_{\lambda,a} \leqslant \alpha_{0,a}(\Omega)$ for all $\lambda \geqslant \frac{\overline{S}^2}{c}|\{V < c\}|^{\frac{2-2^*}{2^*}}$. Define

$$m(k) = \begin{cases} a^*, & \text{if } k = 1, \\ 1/\lambda_1^{(3)}, & \text{if } k = 3, \\ \infty, & \text{if } k = 4. \end{cases}$$

Then for each $k \in \{1,3,4\}$ and $a \in (0, m(k))$, we can take a positive number D_a such that

$$0 < \eta \leqslant \alpha_{\lambda,a} \leqslant \alpha_{0,a}(\Omega) < D_a \quad \text{for all } \lambda \geqslant \frac{\overline{S}^2}{c}\left|\{V < c\}\right|^{\frac{2-2^*}{2^*}}.$$

Thus, by Lemmas 2.3 and 2.4 and Theorem 2.2, we obtain that for each $a \in (0, m(k))$ and $\lambda \geqslant \frac{\overline{S}^2}{c}|\{V < c\}|^{\frac{2-2^*}{2^*}}$, there exists $\{u_n\} \subset X_\lambda$ such that

$$J_{\lambda,a}(u_n) \to \alpha_{\lambda,a} > 0 \quad \text{and} \quad \left(1 + \|u_n\|_\lambda\right)\left\|J'_{\lambda,a}(u_n)\right\|_{X_\lambda^{-1}} \to 0, \quad \text{as } n \to \infty, \tag{3.1}$$

where $0 < \eta \leqslant \alpha_{\lambda,a} \leqslant \alpha_{0,a}(\Omega) < D_a$. Furthermore, we have the following result.

Lemma 3.1. *Suppose that the conditions $(V1)$–$(V3)$ hold. In addition, for each $k = 1,3,4$, we assume that the function f satisfies the conditions $(D1)$–$(D3)$. Then for each $\lambda \geqslant \frac{\overline{S}^2}{c}|\{V < c\}|^{\frac{2-2^*}{2^*}}$ and $\{u_n\}$ defined in (3.1), we have $\{u_n\}$ is bounded in X_λ.*

Proof. For n large enough, by $(D3)$ and (2.2), we have

$$\begin{aligned} \alpha_{\lambda,a} + 1 &\geqslant J_{\lambda,a}(u_n) - \frac{1}{4}\langle J'_{\lambda,a}(u_n), u_n\rangle \\ &= \frac{1}{4}\left(b\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V u_n^2\,dx\right) + \int_{\mathbb{R}^N} \left[\frac{1}{4} f(x,u_n)u_n - F(x,u_n)\right]dx \end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{\min\{b,1\}}{4}\|u_n\|_\lambda^2 - \int\limits_{\mathbb{R}^N}\left[F(x,u_n) - \frac{1}{4}f(x,u_n)u_n\right]dx \\
&\geqslant \frac{\min\{b,1\}}{4}\|u_n\|_\lambda^2 - d_0\int\limits_{\mathbb{R}^N} u_n^2\,dx \\
&\geqslant \left(\frac{\min\{b,1\}}{4} - d_0\big|\{V<c\}\big|^{\frac{2^*-2}{2^*}}\overline{S}^{-2}\right)\|u_n\|_\lambda^2,
\end{aligned}$$

this implies that

$$\|u_n\|_\lambda \leqslant \left(\frac{4\overline{S}^2(\alpha_{\lambda,a}+1)}{\overline{S}^2\min\{b,1\} - 4d_0|\{V<c\}|^{\frac{2^*-2}{2^*}}}\right)^{1/2}.$$

Therefore, $\{u_n\}$ is bounded in X_λ. □

Next, we investigate the compactness conditions for the functional $J_{\lambda,a}$. Recall that a C^1 functional I satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $(1+\|u_n\|)\|I'(u_n)\|_{E^*} \to 0$ has a convergent subsequence, and such sequence is called a $(C)_c$-sequence.

Proposition 3.2. *Suppose that the conditions $(V1)$–$(V3)$ hold. In addition, for each $k = 1,3,4$, we assume that the function f satisfies the conditions $(D1)$–$(D3)$. Then for each $D > 0$ there exists $\overline{\Lambda}_0 = \Lambda(D) \geqslant \frac{4d_0}{c} > 0$ such that $J_{\lambda,a}$ satisfies the $(C)_\alpha$-condition in X_λ for all $\alpha < D$ and $\lambda > \overline{\Lambda}_0$.*

Proof. Let $\{u_n\}$ be a $(C)_\alpha$-sequence with $\alpha < D$. Then, by Lemma 3.1, $\{u_n\}$ is bounded in X_λ. Therefore, there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{aligned}
&u_n \rightharpoonup u_0 \quad \text{weakly in } X_\lambda; \\
&u_n \to u_0 \quad \text{strongly in } L^r_{loc}\big(\mathbb{R}^N\big),\ \text{for } 2 \leqslant r < 2^*.
\end{aligned}$$

Moreover, $J'_{\lambda,a}(u_0) = 0$. Now we prove that $u_n \to u_0$ strongly in X_λ. Let $v_n = u_n - u_0$. It follows from the condition $(V2)$ that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} v_n^2\,dx &= \int\limits_{\{V\geqslant c\}} v_n^2\,dx + \int\limits_{\{V<c\}} v_n^2\,dx \\
&\leqslant \frac{1}{\lambda c}\int\limits_{\{V\geqslant c\}} \lambda c v_n^2\,dx + \int\limits_{\{V<c\}} v_n^2\,dx \\
&\leqslant \frac{1}{\lambda c}\left(b\int\limits_{\mathbb{R}^N}|\nabla v_n|^2 + \int\limits_{\mathbb{R}^N}\lambda V v_n^2\right) + o(1).
\end{aligned} \tag{3.2}$$

Then, by the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^r\,dx &\leqslant \left(\int_{\mathbb{R}^N} |v_n|^2\,dx\right)^{\frac{2^*-r}{2^*-2}} \left(\int_{\mathbb{R}^N} |v_n|^{2^*}\,dx\right)^{\frac{r-2}{2^*-2}} \\
&\leqslant \left(\frac{1}{\lambda c}\left(b\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} \lambda V v_n^2\right)\right)^{\frac{2^*-r}{2^*-2}} \left(\overline{S}^{-2^*} b^{\frac{2}{2^*}}\left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\right)^{\frac{2^*}{2}}\right)^{\frac{r-2}{2^*-2}} \\
&\leqslant \left(\frac{1}{\lambda c}\right)^{\frac{2^*-r}{2^*-2}} \overline{S}^{-\frac{2^*(r-2)}{2^*-2}} b^{\frac{2(r-2)}{2^*(2^*-2)}} \left(b\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} \lambda V v_n^2\right)^{r/2} + o(1). \qquad (3.3)
\end{aligned}$$

Moreover, by the conditions $(D1)$–$(D2)$ and Brezis–Lieb Lemma [6], we have

$$J_{\lambda,a}(v_n) = J_{\lambda,a}(u_n) - J_{\lambda,a}(u_0) + o(1) \quad \text{and} \quad J'_{\lambda,a}(v_n) = o(1).$$

Consequently, this together with the condition $(D3)$, (3.2) and Lemma 2.1, we obtain

$$\begin{aligned}
D &\geqslant \alpha - J_{\lambda,a}(u_0) \\
&\geqslant J_{\lambda,a}(v_n) - \frac{1}{4}\langle J'_{\lambda,a}(v_n), v_n\rangle + o(1) \\
&= \frac{1}{4}\left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V v_n^2\,dx\right) - \int_{\mathbb{R}^N}\left[F(x,v_n) - \frac{1}{4}f(x,v_n)v_n\right]dx + o(1) \\
&\geqslant \frac{1}{4}\left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V v_n^2\,dx\right) - d_0\int_{\mathbb{R}^N} v_n^2\,dx + o(1) \\
&\geqslant \frac{\lambda c - 4d_0}{4\lambda c}\left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V v_n^2\,dx\right) + o(1),
\end{aligned}$$

which implies that for every $\lambda > \frac{4d_0}{c}$,

$$\min\{b,1\}\|v_n\|_\lambda^2 \leqslant \left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V v_n^2\,dx\right) \leqslant \frac{4\lambda c D}{\lambda c - 4d_0} + o(1).$$

Moreover, by (2.2), one has

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^r\,dx &\leqslant \big|\{V<c\}\big|^{\frac{2^*-r}{2^*}} \overline{S}^{-r}\|u\|_\lambda^r \\
&\leqslant \big|\{V<c\}\big|^{\frac{2^*-r}{2^*}} \overline{S}^{-r}\left(\frac{4\lambda c D}{\min\{b,1\}(\lambda c - 4d_0)}\right)^{\frac{r}{2}} + o(1). \qquad (3.4)
\end{aligned}$$

Since $\langle J'_{\lambda,a}(v_n), v_n\rangle = o(1)$ and

$$\int_{\mathbb{R}^N} f(x, v_n)v_n\,dx \leqslant \left(\left|p^+\right|_\infty + \epsilon\right)\int_{\mathbb{R}^N} v_n^2\,dx + C_\epsilon \int_{\mathbb{R}^N} |v_n|^r\,dx,$$

it follows from (3.3) and (3.4) that

$$\begin{aligned}
o(1) &= \left(b\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} \lambda V v_n^2\,dx\right) + a\left(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\right)^2 \\
&\quad - \left(\left|p^+\right|_\infty + \epsilon\right)\int_{\mathbb{R}^N} v_n^2\,dx - C_\epsilon \int_{\mathbb{R}^N} |v_n|^r\,dx \\
&\geqslant \min\{b, 1\}\|v_n\|_\lambda^2 - \frac{(|p^+|_\infty + \epsilon)}{\lambda c}\min\{b, 1\}\|v_n\|_\lambda^2 \\
&\quad - C_\epsilon\left(\int_{\mathbb{R}^N} |v_n|^r\,dx\right)^{(r-2)/r}\left(\int_{\mathbb{R}^N} |v_n|^r\,dx\right)^{2/r} \\
&\geqslant \min\{b, 1\}\left(1 - \frac{(|p^+|_\infty + \epsilon)}{\lambda c}\right)\|v_n\|_\lambda^2 \\
&\quad - \left(\left|\{V < c\}\right|^{\frac{2^*-r}{2^*}}\overline{S}^{-r}\right)^{(r-2)/r}\left(\frac{4\lambda c D}{\min\{b, 1\}(\lambda c - 4d_0)}\right)^{(r-2)/2} \\
&\quad \cdot \left[\left(\frac{1}{\lambda c}\right)^{\frac{2^*-r}{2^*-2}}\overline{S}^{-\frac{2^*(r-2)}{2^*-2}} b^{\frac{2(r-2)}{2^*(2^*-2)}}\right]^{2/r}\|v_n\|_\lambda^2 \\
&\geqslant \|v_n\|_\lambda^2 \cdot \min\{b, 1\} \cdot \left[1 - \frac{(|p^+|_\infty + \epsilon)}{\lambda c}\right. \\
&\quad \left. - \left(\frac{4\lambda c D|\{V < c\}|^{\frac{2^*-r}{2^*}}}{\min(b, 1)(\lambda c - 4d_0)\overline{S}^r}\right)^{\frac{r-2}{r}}\left(\left(\frac{1}{\lambda c}\right)^{\frac{2^*-r}{2^*-2}}\overline{S}^{-\frac{2^*(p-2)}{2^*-2}} b^{\frac{2(r-2)}{2^*(2^*-2)}}\right)^{2/r}\right] + o(1).
\end{aligned}$$

Therefore, there exists $\overline{\Lambda}_0 = \overline{\Lambda}_0(D) \geqslant \frac{4d_0}{c} > 0$ such that $v_n \to 0$ strongly in X_λ for $\lambda > \overline{\Lambda}_0$. This completes the proof. □

Now we give the proof of Theorem 1.2: By Proposition 3.2 and $0 < \eta \leqslant \alpha_{\lambda,a} \leqslant \alpha_{0,a}(\Omega)$ for all $\lambda \geqslant \frac{\overline{S}^2}{c}|\{V < c\}|^{\frac{2-2^*}{2^*}}$, for each $a \in (0, m(k))$ and $D_a > \alpha_{0,a}(\Omega)$ there exists

$$\Lambda^* \geqslant \max\left\{\frac{\overline{S}^2}{c|\{V < c\}|^{\frac{2^*-2}{2^*}}}, \frac{4d_0}{c}\right\} > 0$$

such that for every $\lambda > \Lambda^*$ and $(C)_{\alpha_{\lambda,a}}$-sequence $\{u_n\}$ for $J_{\lambda,a}$ on X_λ there exist a subsequence $\{u_n\}$ and $u_\lambda \in X_\lambda$ such that $u_n \to u_\lambda$ strongly in X_λ. Moreover, $J_{\lambda,a}(u_\lambda) = \alpha_{\lambda,a}$ and u_λ is a nontrivial solution of Eq. $(K_{\lambda,a})$.

4. Proof of Theorem 1.3

First we define the Nehari manifold

$$\mathcal{N}_{\lambda,a} := \left\{u \in X_\lambda \setminus \{0\}: \left\langle J'_{\lambda,a}(u), u\right\rangle = 0\right\} \quad \text{for } \lambda > \Lambda^*.$$

Then by Proposition 3.2, $\mathcal{N}_{\lambda,a}$ is nonempty. Define

$$\begin{aligned}\Psi_{\lambda,a}(u) &= \left\langle J'_{\lambda,a}(u), u\right\rangle \\ &= b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx + a\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^N} f(x,u)u\,dx.\end{aligned}$$

Then for $u \in \mathcal{N}_{\lambda,a}$,

$$\begin{aligned}\left\langle \Psi'_{\lambda,a}(u), u\right\rangle &= 2\left(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx\right) + 4a\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} f_u(x,u)u^2\,dx - \int_{\mathbb{R}^N} f(x,u)u\,dx \\ &= -2\left(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx\right) + 3\int_{\mathbb{R}^N} f(x,u)u\,dx \\ &\quad - \int_{\mathbb{R}^N} f_u(x,u)u^2\,dx. \end{aligned} \tag{4.1}$$

Moreover, we have the following result.

Proposition 4.1. *Suppose that the conditions $(V1)$–$(V3)$ hold. In addition, for each $k = 1, 3, 4$, we assume that the function f satisfies the conditions $(D1)$–$(D4)$. Then we have the following results*:

- (i) *Let $a^* > 0$ be as in Lemma 2.4. If $k = 1$ and $b\lambda_1^{(1)} < 1$, then for each $a \in (0, a^*)$, there exists $\Lambda^{**} > 0$ such that $\mathcal{N}_{\lambda,a}$ is a natural constraint for all $\lambda > \Lambda^{**}$.*
- (ii) *Let $\Lambda^* > 0$ be as in Theorem 1.2. If $k = 3$, then for each $a \in (0, 1/\lambda_1^{(3)})$ and $\lambda > \Lambda^*$, $\mathcal{N}_{\lambda,a}$ is a natural constraint.*
- (iii) *Let $\Lambda^* > 0$ be as in Theorem 1.2. If $k = 4$, then for each $a > 0$ and $\lambda > \Lambda^*$, $\mathcal{N}_{\lambda,a}$ is a natural constraint.*

Proof. (i) By (4.1) and the condition $(D4)$, for $u \in \mathcal{N}_{\lambda,a}$ one has

$$\left\langle \Psi'_{\lambda,a}(u), u\right\rangle \leqslant -2\left(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx - \int_{\mathbb{R}^N} f(x,u)u\,dx\right)$$

$$\leqslant -2\bigg(\int_{\{V>0\}} \big(b|\nabla u|^2 + \lambda V u^2\big)\,dx - \int_{\{V>0\}} qu^2\,dx + \bigg(b - \frac{1}{\lambda_1^{(1)}}\bigg)\int_{\Omega} |\nabla u|^2\,dx\bigg)$$

$$\leqslant -2\bigg(\int_{\{V>0\}} \big(b|\nabla u|^2 + \lambda V u^2\big)\,dx - \int_{\{V>0\}} qu^2\,dx\bigg). \tag{4.2}$$

Moreover, by the condition $(V2)$, there exists $c_0 > 0$ such that

$$|q|_\infty b^{-1}\overline{S}^{-2}\big|\{0 < V \leqslant c_0\}\big| \leqslant 1,$$

which implies that

$$\begin{aligned}\int_{\{V>0\}} qu^2\,dx &\leqslant |q|_\infty \int_{\{0<V\leqslant c_0\}} u^2\,dx + |q|_\infty \int_{\{V>c_0\}} u^2\,dx \\ &\leqslant |q|_\infty b^{-1}\overline{S}^{-2}\big|\{0 < V \leqslant c_0\}\big| \int_{\{0<V\leqslant c_0\}} b|\nabla u|^2\,dx + \frac{|q|_\infty}{\lambda c_0}\int_{\{V>c_0\}} \lambda V u^2\,dx \\ &\leqslant \int_{\{0<V\leqslant c_0\}} b|\nabla u|^2\,dx + \frac{|q|_\infty}{\lambda c_0}\int_{\{V>c_0\}} \lambda V u^2\,dx. \end{aligned} \tag{4.3}$$

Therefore, by (4.2) and (4.3), for every $\lambda > \frac{c_0}{|q|_\infty}$, we have

$$\begin{aligned}\big\langle \Psi'_{\lambda,a}(u), u\big\rangle &\leqslant -2b\bigg(\int_{\{V>0\}} |\nabla u|^2\,dx - \int_{\{0<V\leqslant c_0\}} |\nabla u|^2\,dx\bigg) \\ &\quad - 2\bigg(\int_{\{V>0\}} \lambda V u^2\,dx - \frac{|q|_\infty}{\lambda c_0}\int_{\{V>c_0\}} \lambda V u^2\,dx\bigg) \\ &< 0.\end{aligned}$$

Take $\Lambda^{**} = \max\{\frac{c_0}{|q|_\infty}, \Lambda^*\}$, where $\Lambda^* > 0$ is as in Theorem 1.2. Then $\mathcal{N}_{\lambda,a}$ is a natural constraint for all $\lambda > \Lambda^{**}$.

(ii) and (iii) By (4.1) and the condition $(D4)$, for $u \in \mathcal{N}_{\lambda,a}$ we have

$$\begin{aligned}\big\langle \Psi'_{\lambda,a}(u), u\big\rangle &= 2\bigg(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx\bigg) + 4a\bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)^2 \\ &\quad - \int_{\mathbb{R}^N} f_u(x,u)u^2\,dx - \int_{\mathbb{R}^N} f(x,u)u\,dx \\ &\leqslant 2\bigg(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx\bigg) + 4a\bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)^2\end{aligned}$$

$$
\begin{aligned}
&- (k+1)\int_{\mathbb{R}^N} f(x,u)u\,dx \\
&= -2\left(b\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} \lambda V u^2\,dx\right) - (k-3)\int_{\mathbb{R}^N} f(x,u)u\,dx < 0.
\end{aligned}
$$

Therefore, $\mathcal{N}_{\lambda,a}$ is a natural constraint. This completes the proof. $\square$

Now we give the proof of Theorem 1.3: For any $u \in \mathcal{N}_{\lambda,a}$, we have

$$
0 = \left\langle J'_{\lambda,a}(u), u\right\rangle \geqslant \min\{b,1\}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} f(x,u)u\,dx.
$$

Now, choose $\epsilon \in (0, \Theta_0 - |p^+|_\infty)$ as in the proof of Lemma 2.3 and use the conditions $(D1)$ and $(D2)$ to get

$$
\left|\int_{\mathbb{R}^N} f(x,u)u\,dx\right| \leqslant \int_{\mathbb{R}^N} \left[\left(\left|p^+\right|_\infty + \epsilon\right)u^2 + C_\epsilon |u|^r\right] dx. \tag{4.4}
$$

Therefore, for every $u \in \mathcal{N}_{\lambda,a}$ we have

$$
0 \geqslant \min\{b,1\}\|u\|_\lambda^2 - \frac{(|p^+|_\infty + \epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}}{\bar{S}^2}\|u\|_\lambda^2 - \frac{C_\epsilon|\{V<c\}|^{\frac{2^*-r}{2^*}}}{\bar{S}^r}\|u\|_\lambda^r. \tag{4.5}
$$

We recall that $u \neq 0$ whenever $u \in \mathcal{N}_{\lambda,a}$ and (4.5) implies that

$$
\|u\|_\lambda \geqslant \left(\frac{\min\{b,1\} - (|p^+|_\infty + \epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}\bar{S}^{-2}}{C_\epsilon|\{V<c\}|^{\frac{2^*-2}{2^*}}\bar{S}^{-r}}\right)^{1/(r-2)} > 0, \quad \text{for all } u \in \mathcal{N}_{\lambda,a}. \tag{4.6}
$$

Hence any limit point of a sequence in the Nehari manifold is different from zero. Use a similar argument to the proof in Lemma 2.1, we can claim that there exists $d_0 > 0$ such that $J_{\lambda,a}(u) > d_0$ for all $u \in \mathcal{N}_{\lambda,a}$, i.e., $J_{\lambda,a}$ is bounded from below on $\mathcal{N}_{\lambda,a}$. So, we may define

$$
\beta_{\lambda,a} = \inf\left\{J_{\lambda,a}(u):\ u \in \mathcal{N}_{\lambda,a}\right\},
$$

and $\beta_{\lambda,a} > 0$. Let $\{\bar{u}_n\} \subset \mathcal{N}_{\lambda,a}$ be such that $J_{\lambda,a}(\bar{u}_n) \to \beta_{\lambda,a}$ as $n \to \infty$. Following almost the same procedures as the proofs of Lemma 3.1 and Proposition 3.2, we can show that $\{\bar{u}_n\}$ is bounded in X_λ and it has a convergent subsequence, strongly converging to $v_0 \in \mathcal{N}_{\lambda,a}$. Thus, $J_{\lambda,a}(v_0) = \beta_{\lambda,a}$. Moreover, by Proposition 4.1, $J'_{\lambda,a}(v_0) = 0$. Therefore, $v_0 \in X_\lambda$ is a ground state solution of Eq. $(K_{\lambda,a})$.

5. Proof of Theorem 1.4

In this section, we give the proof of Theorem 1.4.

Proof. Suppose that u is a nontrivial solution of Eq. $(K_{\lambda,a})$. Then

$$\langle J'_{\lambda,a}(u), u\rangle = \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx + a\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^N} f(x,u)u\, dx = 0.$$

(i) By the conditions $(V1)$–$(V3)$ and $b > |q|_\infty \overline{S}^{-2}|\Omega|^{\frac{2^*-2}{2^*}}$, there exists $c_1 > 0$ such that

$$b > |q|_\infty \overline{S}^{-2}\left|\{V < c_1\}\right|^{\frac{2^*-2}{2^*}},$$

which implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} qu^2\, dx &\leqslant |q|_\infty \int_{\{V<c_1\}} u^2\, dx + |q|_\infty \int_{\{V\geqslant c_1\}} u^2\, dx \\
&\leqslant |q|_\infty \left|\{V < c_1\}\right|^{\frac{2^*-2}{2^*}} \overline{S}^{-2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{|q|_\infty}{\lambda c_1} \int_{\{V\geqslant c_1\}} \lambda V u^2\, dx \\
&< b\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{|q|_\infty}{\lambda c_1} \int_{\{V\geqslant c_1\}} \lambda V u^2\, dx.
\end{aligned}$$

Then, by the conditions $(D2)$, $(D4)$ and (2.2), for $\lambda > \Lambda_* := \frac{|q|_\infty}{c_1}$ we have

$$\begin{aligned}
0 = \langle J'_{\lambda,a}(u), u\rangle &\geqslant \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx - \int_{\mathbb{R}^N} qu^2\, dx \\
&> \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx - \left(b\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{|q|_\infty}{\lambda c_1} \int_{\{V\geqslant c_1\}} \lambda V u^2\, dx\right) \\
&\geqslant \left(1 - \frac{|q|_\infty}{\lambda c_1}\right) \int_{\{V\geqslant c_1\}} \lambda V u^2\, dx \geqslant 0,
\end{aligned}$$

which is a contradiction. Therefore, Eq. $(K_{\lambda,a})$ does not admit any nontrivial solution.

(ii) We consider the proof in two cases:

Case I ($\int_{\mathbb{R}^N} qu^4\, dx = 0$): By (2.2), we have

$$\begin{aligned}
0 &= \langle J'_{\lambda,a}(u), u\rangle \\
&= b\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} \lambda V u^2\, dx + a\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^N} f(x,u)u\, dx
\end{aligned}$$

$$\geqslant \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx + a\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^N} q u^4\, dx$$

$$= \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx + a\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 > 0,$$

which is a contradiction.

Case II ($\int_{\mathbb{R}^N} q u^4\, dx > 0$): Set

$$v = \frac{u}{(\int_{\mathbb{R}^N} q u^4\, dx)^{1/4}}.$$

Clearly, $\int_{\mathbb{R}^N} q v^4\, dx = 1$. Then, by the conditions $(D2)$, $(D4)$ and (1.5), we have

$$\begin{aligned}
0 &= \langle J'_{\lambda,a}(u), u\rangle \\
&\geqslant \int_{\mathbb{R}^N} \left(b|\nabla u|^2 + \lambda V u^2\right) dx + \frac{1}{\widehat{\lambda}_1^{(3)}}\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^N} q u^4\, dx \\
&= \left(\int_{\mathbb{R}^N} q u^4\, dx\right)^{1/2} \int_{\mathbb{R}^N} \left(b|\nabla v|^2 + \lambda V v^2\right) dx \\
&\quad + \frac{1}{\widehat{\lambda}_1^{(3)}}\left(\int_{\mathbb{R}^N} q u^4\, dx\right)\left(\int_{\mathbb{R}^N} |\nabla v|^2\, dx\right)^2 - \int_{\mathbb{R}^N} q u^4\, dx \\
&= \left(\int_{\mathbb{R}^N} q u^4\, dx\right)^{1/2} \int_{\mathbb{R}^N} \left(b|\nabla v|^2 + \lambda V v^2\right) dx \\
&\quad + \frac{1}{\widehat{\lambda}_1^{(3)}} \int_{\mathbb{R}^N} q u^4\, dx \left(\left(\int_{\mathbb{R}^N} |\nabla v|^2\, dx\right)^2 - \widehat{\lambda}_1^{(3)}\right) > 0,
\end{aligned}$$

which is a contradiction. Therefore, Eq. $(K_{\lambda,a})$ does not admit any nontrivial solution. This completes the proof. □

6. Concentration for solutions

In this section, we investigate the concentration for solutions and give the proof of Theorem 1.5.

Proof of Theorem 1.5. We follows the argument in [2] (or see [26]). For any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the critical points of $J_{\lambda_n,a}$ obtained in Theorem 1.2. Since

$$D \geqslant \alpha_{\lambda_n,a} = J_{\lambda_n,a}(u_n) \geqslant \left(\frac{\min\{b,1\}}{4} - d_0\left|\{V < c\}\right|^{\frac{2^*-2}{2^*}} \overline{S}^{-2}\right)\|u_n\|_{\lambda_n}^2,$$

one has

$$\|u_n\|_{\lambda_n} \leqslant C_0, \tag{6.1}$$

where the constant C_0 is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup u_0$ weakly in X and $u_n \to u_0$ strongly in $L^r_{loc}(\mathbb{R}^N)$ for $2 \leqslant r < 2^*$. By Fatou's lemma, we have

$$\int_{\mathbb{R}^N} V u_0^2\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} V u_n^2\,dx \leqslant \liminf_{n\to\infty} \frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,$$

this implies that $u_0 = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$, and $u_0 \in H^1_0(\Omega)$ by the condition $(V3)$. Now for any $\varphi \in C_0^\infty(\Omega)$, since $\langle J'_{\lambda_n,a}(u_n), \varphi\rangle = 0$, it is easy to check that

$$\left(b + a\int_{\mathbb{R}^N} |\nabla u_0|^2\,dx\right)\int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla\varphi\,dx = \int_{\mathbb{R}^N} f(x,u_0)\varphi\,dx,$$

that is, u_0 is a weak solution of (K_∞) by the density of $C_0^\infty(\Omega)$ in $H^1_0(\Omega)$. Now we show that $u_n \to u_0$ in $L^r(\mathbb{R}^N)$ for $2 \leqslant r < 2^*$. Otherwise, by Lions vanishing lemma [17] there exist $\delta > 0$, $R_0 > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\int_{B^N(x_n,R_0)} (u_n - u_0)^2\,dx \geqslant \delta.$$

Since $|B(x_n, R_0) \cap \{x \in \mathbb{R}^N\colon\ V < c\}| \to 0$ as $x_n \to \infty$, by Hölder inequality, we have

$$\int_{B(x_n,R_0)\cap\{V<c\}} (u_n - u_0)^2\,dx \to 0.$$

Consequently,

$$\begin{aligned}
\|u_n\|^2_{\lambda_n} &\geqslant \lambda_n c \int_{B(x_n,R_0)\cap\{V\geqslant c\}} (u_n)^2\,dx = \lambda_n c \int_{B(x_n,R_0)\cap\{V\geqslant c\}} (u_n - u_0)^2\,dx \\
&= \lambda_n c\left(\int_{B(x_n,R_0)} (u_n - u_0)^2\,dx - \int_{B(x_n,R_0)\cap\{V<c\}} (u_n - u_0)^2\,dx + o(1)\right) \\
&\to \infty,
\end{aligned}$$

which contradicts (6.1). Therefore, $u_n \to u_0$ in $L^r(\mathbb{R}^N)$ for $2 \leqslant r < 2^*$. Then, by the conditions $(D1)$, $(D2)$ and $u_n \to u_0$ in $L^r(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \to \int_{\mathbb{R}^N} f(x,u_0)u_0\,dx. \tag{6.2}$$

Now, choose $\epsilon \in (0, \Theta_0 - |p^+|_\infty)$ as in the proof of Lemma 2.3. Since $\langle J'_{\lambda_n,a}(u_n), u_n\rangle = 0$, by (2.2), (4.4) and the fact that $u_n \neq 0$, for n large we have

$$\begin{aligned}\min\{b,1\}\|u_n\|^2_{\lambda_n} &\leqslant \left(b\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\,dx + \int\limits_{\mathbb{R}^N}\lambda_n V u_n^2\,dx\right)\\ &\leqslant \frac{(|p^+|_\infty+\epsilon)|\{V<c\}|^{\frac{2^*-2}{2^*}}}{\bar{S}^2}\|u_n\|^2_{\lambda_n} + \frac{C_\epsilon|\{V<c\}|^{\frac{2^*-r}{2^*}}}{\bar{S}^r}\|u_n\|^r_{\lambda_n},\end{aligned}$$

which implies that

$$\|u_n\|_{\lambda_n} \geqslant \left(\frac{\bar{S}^r(\Theta_0 - |p^+|_\infty - \epsilon)\min\{b,1\}}{C_\epsilon\Theta_0|\{V<c\}|^{\frac{2^*-r}{2^*}}}\right)^{1/(r-2)} > 0. \tag{6.3}$$

Moreover,

$$\begin{aligned}\int\limits_{\mathbb{R}^N} f(x,u_n)u_n\,dx &= \left(b\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\,dx + \int\limits_{\mathbb{R}^N}\lambda_n V u_n^2\,dx\right) + a\left(\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\,dx\right)^2\\ &\geqslant \min\{1,b\}\|u_n\|^2_{\lambda_n}.\end{aligned} \tag{6.4}$$

Therefore, by (6.2)–(6.4), we have

$$\int\limits_{\mathbb{R}^N} f(x,u_0)u_0\,dx \geqslant \min\{b,1\}\left(\frac{\bar{S}^r(\Theta_0 - |p^+|_\infty - \epsilon)\min\{b,1\}}{C_\epsilon\Theta_0|\{V<c\}|^{\frac{2^*-r}{2^*}}}\right)^{2/(r-2)} > 0,$$

this shows that $u_0 \neq 0$. This completes the proof. □

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