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Inverse Jacobian multipliers and Hopf bifurcation on center manifolds

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Abstract

In this paper we consider a class of higher dimensional differential systems in $\mathbb{R}^n$ which have a two dimensional center manifold at the origin with a pair of pure imaginary eigenvalues. First we characterize the existence of either analytic or C^∞ inverse Jacobian multipliers of the systems around the origin, which is either a center or a focus on the center manifold. Later we study the cyclicity of the system at the origin through Hopf bifurcation by using the vanishing multiplicity of the inverse Jacobian multiplier.

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1. Background and statement of the main results

For real planar differential systems, the problems on center–focus and Hopf bifurcation are classical and related. They are important subjects in the bifurcation theory and also in the study of Hilbert's 16th problem [6,7,15,17].

For planar non-degenerate center, Poincaré provided an equivalent characterization.

Poincaré center Theorem. *For a real planar analytic differential system with the origin as a singularity having a pair of pure imaginary eigenvalues, the origin is a center if and only if the*

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system has a local analytic first integral, and if and only if the system is analytically equivalent to

$$\dot{u} = -\mathbf{i}u\big(1 + g(uv)\big), \qquad \dot{v} = \mathbf{i}v\big(1 + g(uv)\big),$$

with $g(uv)$ without constant terms, where we have used the conjugate complex coordinates instead of the two real ones.

This result has a higher dimensional version, see for instance [13,18,20], which characterizes the equivalence between the analytic integrability and the existence of analytic normalization of analytic differential systems to its Poincaré–Dulac normal form of a special type.

Reeb [14] in 1952 provided another characterization on planar centers via inverse integrating factor. Recall that a function V is an *inverse integrating factor* of a planar differential system if $1/V$ is an integrating factor of the system. From [8,9,11,12] we know that inverse integrating factors have better properties than integrating factors.

Reeb center Theorem. *Real planar analytic differential system*

$$\dot{x} = -y + f_1(x, y), \qquad \dot{y} = x + f_2(x, y),$$

has the origin as a center if and only if it admits a real analytic local inverse integrating factor with non-vanishing constant part.

Poincaré center Theorem was extended to higher dimensional differential systems which have a two dimensional center manifold by Lyapunov. Consider analytic differential systems in $\mathbb{R}^n$

$$\begin{aligned} \dot{x} &= -y + f_1(x, y, z) = F_1(x, y, z), \\ \dot{y} &= x + f_2(x, y, z) = F_2(x, y, z), \\ \dot{z} &= Az + f(x, y, z) = F(x, y, z), \end{aligned} \tag{1.1}$$

with $z = (z_3, \dots, z_n)^{tr}$, A is a real square matrix of order $n-2$, and $f = (f_3, \dots, f_n)^{tr}$ and $F = (F_3, \dots, F_n)^{tr}$. Hereafter we use *tr* to denote the transpose of a matrix. Moreover we assume that $\mathbf{f} := (f_1, f_2, f) = O(|(x, y, z)|^2)$ are n dimensional vector valued analytic functions. We denote by

$$\mathcal{X} = F_1(x, y, z)\frac{\partial}{\partial x} + F_2(x, y, z)\frac{\partial}{\partial y} + \sum_{j=3}^{n} F_j(x, y, z)\frac{\partial}{\partial z_j}$$

the vector field associated to systems (1.1).

Assume that the eigenvalues of A all have non-zero real parts. Then from the Center Manifold Theorem we get that system (1.1) has a center manifold tangent to the (x, y) plane at the origin (of course center manifolds are not necessary unique, and may not be analytic even not C^∞). Moreover this center manifold can be represented as

$$\mathcal{M}^c = \bigcap_{j=3}^{n} \big\{z_j = h_j(x, y)\big\}. \tag{1.2}$$

Lyapunov center Theorem. *Assume that A has no eigenvalues with vanishing real parts. The following statements hold.*

(a) *System (1.1) restricted to the center manifold has the origin as a center if and only if it admits a real analytic local first integral of the form $\Phi(x, y, z) = x^2 + y^2 + \text{higher order term}$ in a neighborhood of the origin in $\mathbb{R}^n$.*
(b) *If the condition in statement* (a) *holds, then the center manifold is unique and analytic.*

For a proof of the Lyapunov center Theorem, we refer to [16] and [2, Theorems 3.1, 3.2 and §5].

Reeb center Theorem via inverse integrating factor was extended to differential systems in $\mathbb{R}^3$ by Buică, García and Maza [4]. A smooth function $J(x)$ is an *inverse Jacobian multiplier* of system (1.1) if

$$\mathcal{X}(J) = J \operatorname{div} \mathcal{X}.$$

In fact, if $J(x)$ is an inverse Jacobian multiplier of system (1.1) then $1/J$ is a Jacobian multiplier of the system, i.e.

$$\partial_x\left(\frac{F_1}{J}\right) + \partial_y\left(\frac{F_2}{J}\right) + \partial_{z_3}\left(\frac{F_3}{J}\right) + \cdots + \partial_{z_n}\left(\frac{F_n}{J}\right) = 0,$$

where ∂_x denotes the partial derivative with respect to x.

Buică et al.'s main results in [4] can be summarized as follows.

Buică, García and Maza center–focus Theorem. *Assume that system (1.1) is defined in $\mathbb{R}^3$ and A is a non-zero real number. The following statements hold.*

(a) *System (1.1) restricted to the center manifold has the origin as a center if and only if it admits an analytic local inverse Jacobian multiplier of the form $J(x, y, z) = z + \text{higher order term}$ in a neighborhood of the origin in $\mathbb{R}^3$. Moreover, if such an inverse Jacobian multiplier exists, then the analytic center manifold $\mathcal{M}^c \subset J^{-1}(0)$.*
(b) *If system (1.1) restricted to the center manifold has the origin as a focus, then there exists a local C^∞ and non-flat inverse Jacobian multiplier of the form $J(x, y, z) = z(x^2 + y^2)^k + \text{higher order term}$ with $k \geqslant 2$, in a neighborhood of the origin in $\mathbb{R}^3$. Moreover, there exists a local C^∞ center manifold $\mathcal{M}$ such that $\mathcal{M} \subset J^{-1}(0)$.*

In this paper we will extend Buică et al.'s results to any finite dimensional differential system (1.1). We should say that this extension is not trivial, because for higher dimensional differential systems we need new ideas and techniques than those in [4,5]. Parts of the methods in [4,5] are only suitable for three dimensional differential systems but not for higher dimensional ones.

Let $\lambda_3, \ldots, \lambda_n$ be the eigenvalues of the matrix A. Then system (1.1) at the origin has the eigenvalues $\lambda = (\mathbf{i}, -\mathbf{i}, \lambda_3, \ldots, \lambda_n)$, where $\mathbf{i} = \sqrt{-1}$. Let

$$\mathcal{R} = \left\{k \in \mathbb{Z}^n\colon\ \langle k, \lambda\rangle = 0,\ k + e_j \in \mathbb{Z}^n_+,\ j = 3, \ldots, n\right\},$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers, e_j is the unit vector with its jth component equal to 1 and the others all vanishing, and $\langle k, \lambda\rangle = k_1\mathbf{i} - k_2\mathbf{i} + \sum_{j=3}^n k_j\lambda_j$. We remark that in the definition $\mathcal{R}$ we choose $k \in \mathbb{Z}^n$ but not $k \in \mathbb{Z}_+^n$, because we will also discuss the case $\langle k, \lambda\rangle = \lambda_j$ for $k \in \mathbb{Z}_+^n$ and $j \in \{1, \dots, n\}$.

In this paper we have a basic assumption.

(H) $\mathcal{R}$ is one dimensional and A can be diagonalizable in $\mathbb{C}$.

Clearly if A has its eigenvalues either all having positive real parts or all having negative real parts, then $\mathcal{R}$ has only one linearly independent element with generator $(1, 1, 0)$. For three dimensional differential systems of the form (1.1), this condition always holds provided that A is a nonzero real number.

By the assumption (H) we get easily that $\operatorname{Re}\lambda_j \neq 0$ for $j = 3, \dots, n$. So from the Center Manifold Theorem we get that system (1.1) has a center manifold tangent to the (x, y) plane at the origin, and it can be represented in (1.2).

In the case that A has complex eigenvalues, we assume without loss of generality that there exists an $m \in \mathbb{Z}_+$ with $2m \leqslant n - 2$ such that λ_{3+2j} and λ_{3+2j+1}, $j = 0, 1, \dots, m - 1$, are conjugate complex eigenvalues of A. Of course if $m = 0$ then all the eigenvalues are real.

Our first result provides an equivalent characterization on the center manifold $\mathcal{M}^c$ at the origin via inverse Jacobian multipliers.

Theorem 1.1. *Assume that the analytic differential system (1.1) satisfies (H) and the eigenvalues of A either all having positive real parts or all having negative real parts. The following statements hold.*

(a) *System (1.1) restricted to $\mathcal{M}^c$ has the origin as a center if and only if the system has a local analytic inverse Jacobian multiplier of the form*

$$J(x, y, z) = \prod_{j=0}^{m-1} \left[\left(z_{3+2j} - p_{3+2j}(x, y, z)\right)^2 + \left(z_{3+2j+1} - p_{3+2j+1}(x, y, z)\right)^2\right] \times \prod_{l=3+2m}^{n} \left(z_l - p_l(x, y, z)\right) V(x, y, z), \tag{1.3}$$

in a neighborhood of the origin in $\mathbb{R}^n$, where $p_j = O(|(x, y, z)|^2)$ for $j = 3, \dots, n$, and $V(0, 0, 0) = 1$. For $m = 0$ the first product does not appear.

(b) *If system (1.1) has the inverse Jacobian multiplier as in statement* (a)*, then the center manifold $\mathcal{M}^c$ is unique and analytic, and $\mathcal{M}^c \subset J^{-1}(0)$.*

We note that the set of matrices satisfying (H) is a full Lebesgue measure subset in the set of real matrices of order n.

The second result shows the existence of C^∞ smooth local inverse Jacobian multiplier provided that the origin on the center manifold is a focus.

Theorem 1.2. *Assume that the differential system (1.1) satisfies (H). The following statements hold.*

(a) *If system (1.1) restricted to $\mathcal{M}^c$ has the origin as a focus, then the system has a local C^∞ inverse Jacobian multiplier of the form*

$$J(x,y,z)=\prod_{j=0}^{m-1}\left[\left(z_{3+2j}-p_{3+2j}(x,y,z)\right)^2+\left(z_{3+2j+1}-p_{3+2j+1}(x,y,z)\right)^2\right]$$
$$\times\prod_{s=3+2m}^{n}\left(z_s-p_s(x,y,z)\right)\left[\left(x-q_1(x,y,z)\right)^2+\left(y-q_2(x,y,z)\right)^2\right]^l$$
$$\times h\left(\left(x-q_1(x,y,z)\right)^2+\left(y-q_2(x,y,z)\right)^2\right)V(x,y,z), \tag{1.4}$$

in a neighborhood of the origin in $\mathbb{R}^n$, where $l\geqslant 2$, $p_j,q_i=O(|(x,y,z)|^2)$, and $h(0)=V(0,0,0)=1$.

(b) *There exists a local C^∞ center manifold $\mathcal{M}$ such that $\mathcal{M}\subset J^{-1}(0)$.*

We call l *vanishing multiplicity* of the inverse Jacobian multiplier.

Next we will study the Hopf bifurcation of system (1.1) under small perturbations through inverse Jacobian multipliers. In this direction the first study is due to Buică, García and Maza [5] for a three dimensional differential system.

Consider an analytic perturbation of system (1.1) in the following form

$$\begin{aligned}\dot{x}&=-y+g_1(x,y,z,\varepsilon)=G_1(x,y,z,\varepsilon),\\ \dot{y}&=x+g_2(x,y,z,\varepsilon)=G_2(x,y,z,\varepsilon),\\ \dot{z}&=Az+g(x,y,z,\varepsilon)=G(x,y,z,\varepsilon),\end{aligned}\tag{1.5}$$

where $\varepsilon\in\mathbb{R}^m$ is an m dimensional parameter and $\|\varepsilon\|\ll 1$, $\mathbf{g}:=(g_1,g_2,g)=O(|(x,y,z)|)$ are analytic in a neighborhood of the origin, and $\mathbf{g}(x,y,z,0)=\mathbf{f}(x,y,z)$ with $\mathbf{f}$ defined in (1.1). These conditions make sure that the origin is always a singularity of system (1.5) for all $\|\varepsilon\|\ll 1$. In addition, in order to keep the monotone property of the origin, we assume that the determinant of the Jacobian matrix of $\mathbf{G}=(G_1,G_2,G)$ with respect to (x,y,z) at the origin has the eigenvalues

$$\alpha(\varepsilon)\pm\mathbf{i},\qquad \lambda_j+\mu_j(\varepsilon),\quad j=3,\dots,n,$$

satisfying $\alpha(0)=\mu_j(0)=0$. For convenience we denote by $\mathcal{X}_\varepsilon$ the vector field associated to (1.5). Then $\mathcal{X}_0=\mathcal{X}$.

Next we shall study the Hopf bifurcation of system (1.5) at the origin when the parameters ε vary near $0\in\mathbb{R}^m$. That is, when the values of ε change, the stability of the origin of system (1.5) will probably change, and so there bring appearance or disappearance of small amplitude limit cycles of system (1.5) which are bifurcated from the origin, i.e. if ε tend to 0 these limit cycles will approach to the origin. The maximal number of limit cycles which can be bifurcated from the Hopf at the origin of systems (1.5) is called *cyclicity* of system (1.1) at the origin under the perturbation (1.5). Denote this number by $\mathrm{Cycl}(\mathcal{X}_\varepsilon,0)$.

Now we can state our third result on the Hopf bifurcation.

Theorem 1.3. *Assume that the analytic differential system (1.1) satisfies (H). If system (1.1) restricted to $\mathcal{M}^c$ has the origin as a focus, then* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = l - 1$, *where l is the vanishing multiplicity of the inverse Jacobian multiplier defined in Theorem 1.2.*

This result is an extension of the main result of [5] to any finite dimensional differential systems.

For the real differential system (1.1) there always exists an invertible linear transformation which sends A to its Jordan normal form. So in what follows we assume without loss of generality that A in system (1.1) is in the real Jordan normal form.

In the rest of this paper we will prove our main results. In the next section we will prove Theorems 1.1 and 1.2. The proof of Theorem 1.3 will be given in Section 3.

2. Proofs of Theorems 1.1 and 1.2

2.1. Preparation to the proof

For simplifying notations we will use conjugate complex coordinates instead of the real ones which correspond to conjugate complex eigenvalues of the linear part of system (1.1) at the origin.

Set $\xi = x + \mathbf{i}y$, $\eta = x - \mathbf{i}y$. Since A is real, if it has complex eigenvalues, they should appear in pair. Corresponding to a pair of conjugate complex eigenvalues of A, the associated coordinates are z_j and z_{j+1} by assumption. Then instead of this pair of real coordinates we choose a pair of conjugate complex coordinates $\zeta_j = z_j + \mathbf{i}z_{j+1}$ and $\zeta_{j+1} = z_j - \mathbf{i}z_{j+1}$. Under these new coordinates system (1.1) can be written in

$$\begin{aligned}
\dot{\xi} &= -\mathbf{i}\xi + \widetilde{f}_1(\xi, \eta, \zeta) = \widetilde{F}_1(\xi, \eta, \zeta), \\
\dot{\eta} &= \mathbf{i}\eta + \widetilde{f}_2(\xi, \eta, \zeta) = \widetilde{F}_2(\xi, \eta, \zeta), \\
\dot{\zeta} &= B\zeta + \widetilde{f}(\xi, \eta, \zeta) = \widetilde{F}(\xi, \eta, \zeta),
\end{aligned} \tag{2.1}$$

with $B = \mathrm{diag}(\lambda_3, \ldots, \lambda_n)$, where we have used the assumption (H) and the fact that A is in the real Jordan normal form. Denote by $\widetilde{\mathcal{X}}$ the vector field associated to system (2.1). We note that system (2.1) is different from system (1.1) only in a rotation. But using the coordinates (ξ, η, ζ), some expressions will be simpler than in the coordinates (x, y, z). This idea was first introduced in [19].

First we recall a basic fact on inverse Jacobian multipliers of vector fields under transformations, which will be used in the full paper.

Lemma 2.1. *Let $\mathcal{X}$ be the vector field associated to system (1.1) and J be an inverse Jacobian multiplier of $\mathcal{X}$. Under an invertible smooth transformation of coordinates $(x, y, z) = \Phi(u, v, w)$, the vector field $\mathcal{X}$ becomes*

$$\dot{\mathbf{w}} = \big(D\Phi(\mathbf{w})\big)^{-1}\mathbf{F} \circ \Phi(\mathbf{w}),$$

where $\mathbf{F} = (F_1, F_2, F)^{tr}$ and $\mathbf{w} = (u, v, w)^{tr}$. Then this last system has an inverse Jacobian multiplier $\widetilde{J}(\mathbf{w}) = \frac{J(\Phi(\mathbf{w}))}{D\Phi(\mathbf{w})}$.

Recall that hereafter we use $D\Phi$ to denote the determinant of the Jacobian matrix of Φ with respect to its variables.

In the proof of our main results we need the Poincaré–Dulac normal form theorem. For an analytic or formal differential system in $\mathbb{R}^n$ or $\mathbb{C}^n$

$$\dot{x} = Cx + f(x), \tag{2.2}$$

with C in the Jordan normal form, and $f(x)$ has no constant and linear part, the Poincaré–Dulac normal formal theorem shows that system (2.2) can always be transformed to a system of the form

$$\dot{y} = Cy + g(y),$$

through a near identity transformation $x = y + \psi(y)$ with $\psi(0) = 0$ and $\partial\psi(0) = 0$, where $g(y)$ contains resonant terms only, and $\partial\psi(y)$ denotes the Jacobian matrix of ψ with respect to y. Recall that a monomial $y^k e_j$ in the jth component of $g(y)$ is *resonant* if $\mu_j = \langle k, \mu\rangle$, where $\mu = (\mu_1, \ldots, \mu_n)$ are the eigenvalues of C. The transformation from (2.2) to its normal form is called *normalization*. Usually the normalization is not unique. If a normalization contains only non-resonant terms, then it is called *distinguished normalization*. Distinguished normalization is unique. A monomial x^k in a normalization or in a function is *resonant* if $\langle k, \mu\rangle = 0$.

In our case, by the Poincaré–Dulac normal form theorem we have the following result.

Lemma 2.2. *Under the assumption* (H) *system* (2.1) *is formally equivalent to*

$$\begin{aligned} \dot{u} &= -u\big(\mathbf{i} + g_1(uv)\big), \\ \dot{v} &= v\big(\mathbf{i} + g_2(uv)\big), \\ \dot{w}_j &= w_j\big(\lambda_j + g_j(uv)\big), \quad j = 3, \ldots, n, \end{aligned} \tag{2.3}$$

through a distinguished normalization of the form $(x, y, z) = (u, v, w) + \cdots$, *where dots denote the higher order terms.*

About the smoothness of the transformation in Lemma 2.2 we have the following results.

Lemma 2.3. *Under the assumption* (H), *for system* (2.1) *to its Poincaré–Dulac normal form* (2.3) *the following statements hold.*

(a) *If system* (2.1) *restricted to the center manifold* $\mathcal{M}^c$ *has the origin as a focus, then the distinguished normalization is* C^∞.
(b) *If system* (2.1) *restricted to* $\mathcal{M}^c$ *has the origin as a center, and the eigenvalues of* A *have either all positive real parts or all negative real parts, then the distinguished normalization is analytic.*

Proof. (a) We note that u and v are conjugate in (2.3), we have $g_2 = \bar{g}_1$. Since the origin of system (2.3) on $w = 0$ is a focus, it follows that $\operatorname{Re} g_1 \neq 0$. So our vector fields (2.3) are outside the exception set which was defined on page 254 of [1]. Hence we get from Theorem 1 of Belitskii [1] that the distinguished normalization from systems (2.1) to (2.3) is C^∞.

(b) Since the eigenvalues of A have non-vanishing real parts, we have

$$\lambda_j \neq k_1(-\mathbf{i}) + k_2\mathbf{i} = (k_2 - k_1)\mathbf{i}, \quad k_1, k_2 \in \mathbb{Z}_+ \text{ for } j = 3, \dots, n.$$

So by Theorem 10.1 of [2], system (2.1) is formally equivalent to

$$\begin{aligned} \dot{u} &= -u\big(\mathbf{i} + g_1(uv)\big), \\ \dot{v} &= v\big(\mathbf{i} + g_2(uv)\big), \\ \dot{\rho}_j &= \lambda_j \rho_j + h_j(u, v, \rho), \quad j = 3, \dots, n, \end{aligned} \tag{2.4}$$

with $g_1, g_2 = o(1)$, $h_j = O(|(u, v, \rho)|^2)$ and $h_j(u, v, 0) = 0$ for $j = 3, \dots, n$, through a distinguished normalization of the form

$$\xi = u + \psi_1(u, v, \rho), \qquad \eta = v + \psi_2(u, v, \rho), \qquad \zeta = \rho + \psi(u, v),$$

where $\rho = (\rho_3, \dots, \rho_n)$ and $\psi = (\psi_3, \dots, \psi_n)$ with $\psi_1, \psi_2, \psi = O(|(u, v, \rho)|^2)$. System (2.4) is called a *quasi-normal form* of system (2.1), see [2].

By the assumption system (2.4) has the origin as a center on the center manifold $w = 0$ and so has a formal first integral. By Zhang [18] we get that $g_1(uv) = g_2(uv)$ in (2.4). Applying Theorems 10.2, 3.2 and §5 of [2] to our case, we get that the distinguished normalization from system (2.1) to (2.4) is convergent. This means that systems (2.1) and (2.4) are analytically equivalent through a near identity change of variables.

Next we prove that system (2.4) is analytically equivalent to system (2.3). Take the change of variables

$$u = u, \qquad v = v, \qquad \rho = w + \varphi(u, v, w),$$

for which system (2.4) is transformed to (2.3). Then we have

$$\begin{aligned} \frac{\partial \varphi}{\partial w} Bw - \mathbf{i}\frac{\partial \varphi}{\partial u} u + \mathbf{i}\frac{\partial \varphi}{\partial v} v - B\varphi = {} & Bh\big(u, v, w + \varphi(u, v, w)\big) \\ & - \frac{\partial \varphi}{\partial w} wg + ug_1 \frac{\partial \varphi}{\partial u} - vg_2 \frac{\partial \varphi}{\partial v}, \end{aligned} \tag{2.5}$$

where $wg = (w_3 g_3, \dots, w_n g_n)^{tr}$, and we look φ as a column vector and $\frac{\partial \varphi}{\partial w}$ is the Jacobian matrix of φ with respect to w. The linear operator

$$L = \frac{\partial}{\partial w} Bw - \mathbf{i}\frac{\partial}{\partial u} u + \mathbf{i}\frac{\partial}{\partial v} v - B,$$

has the spectrum

$$\big\{\langle k, \lambda\rangle - p\mathbf{i} + 1\mathbf{i} - \lambda_j \colon k \in \mathbb{Z}_+^{n-2},\ |k| = l,\ p, q \in \mathbb{Z}_+,\ j = 3, \dots, n\big\},$$

in the linear space $\mathcal{H}^{l+p+q}$ which consists of $n - 2$ dimensional vector valued homogeneous polynomials of degree l in w and of degrees p and q in u and v, respectively.

Expanding φ, h, g_1, g_2 and g in the Taylor series, and equating the homogeneous terms in (2.5) which have the same degree, we get from induction and the assumption (H) that Eqs. (2.5) have a formal series solution φ with its monomials all nonresonant. Moreover, by the assumption that A has its eigenvalues either all having positive real parts or all having negative real parts, there exists a number $\sigma > 0$ such that if $\langle k, \lambda\rangle - p\mathbf{i} + l\mathbf{i} - \lambda_j \neq 0$ for $(p, l, k) \in \mathbb{Z}_+^n$ we have

$$\left\| \langle k, \lambda\rangle - p\mathbf{i} + l\mathbf{i} - \lambda_j \right\| \geqslant \sigma.$$

This shows that φ in the transformation does not contain small denominators. Then similar to the proof of the classical Poincaré–Dulac normal form theorem we can prove that φ is convergent, see for instance [2,18], where similar proofs on convergence of φ were provided. This proves statement (b), and consequently the lemma. □

Next result shows the existence of analytic integrating factor on the center manifold provided the existence of analytic inverse Jacobian multiplier of system (2.1) in a neighborhood of the origin.

Lemma 2.4. *Assume that system (2.1) has an analytic inverse Jacobian multiplier of the form*

$$J(\xi, \eta, \zeta) = \big(\zeta_3 - \phi_3(\xi, \eta, \zeta)\big) \ldots \big(\zeta_n - \phi_n(\xi, \eta, \zeta)\big) V(\xi, \eta, \zeta),$$

with $\phi_j = O(|(\xi, \eta, \zeta)|^2)$ for $j \in \{3, \ldots, n\}$ and V analytic, and $V(0,0,0) \neq 0$. Then:

(a) *$\mathcal{M} = \bigcap_{j=3}^n \{\zeta_j = \phi_j(\xi, \eta, \zeta)\}$ is an invariant analytic center manifold of $\widetilde{\mathcal{X}}$ in a neighborhood of the origin.*
(b) *$V|_{\mathcal{M}}$ is an analytic inverse integrating factor of $\widetilde{\mathcal{X}}|_{\mathcal{M}}$.*

Proof. (a) By the expression of J we get from $\widetilde{\mathcal{X}}(J) = J \operatorname{div} \widetilde{\mathcal{X}}$ that

$$\sum_{j=3}^n \widetilde{\mathcal{X}}(\zeta_j - \phi_j)(\zeta_3 - \phi_3) \ldots (\widehat{\zeta_j - \phi_j}) \ldots (\zeta_n - \phi_n) V(\xi, \eta, \zeta) + (\zeta_3 - \phi_3) \ldots (\zeta_n - \phi_n) \widetilde{\mathcal{X}}(V)$$
$$= (\zeta_3 - \phi_3) \ldots (\zeta_n - \phi_n) V \operatorname{div} \widetilde{\mathcal{X}},$$

where $(\widehat{\zeta_j - \phi_j})$ denotes its absence in the product. Since $\zeta_3 - \phi_3, \ldots, \zeta_n - \phi_n$ are relatively pairwise coprime in the algebra of analytic functions which are defined in a neighborhood of the origin, so there exist analytic functions

$$L_0(\xi, \eta, \zeta), \quad L_3(\xi, \eta, \zeta), \quad \ldots, \quad L_n(\xi, \eta, \zeta),$$

such that

$$\begin{aligned} \widetilde{\mathcal{X}}\big(V(\xi, \eta, \zeta)\big) &= L_0(\xi, \eta, \zeta) V(\xi, \eta, \zeta), \\ \widetilde{\mathcal{X}}\big(\zeta_j - \phi_j(\xi, \eta, \zeta)\big) &= L_j(\xi, \eta, \zeta)\big(\zeta_j - \phi_j(\xi, \eta, \zeta)\big), \end{aligned} \tag{2.6}$$

for $j = 3, \dots, n$. This shows that $\zeta_j = \phi_j(\xi, \eta, \zeta)$, $j = 3, \dots, n$, are invariant under the flow of $\widetilde{\mathcal{X}}$.

Applying the Implicit Function Theorem to the equations

$$\zeta_j - \phi_j(\xi, \eta, \zeta) = 0, \quad j = 3, \dots, n,$$

we get a unique solution $\zeta = k(\xi, \eta)$, i.e.

$$\zeta_j = k_j(\xi, \eta), \quad j = 3, \dots, n,$$

in a neighborhood of the origin, which is analytic. Hence

$$\mathcal{M} = \bigcap_{j=3}^{n} \big\{\zeta_j = k_j(\xi, \eta)\big\},$$

in a neighborhood of the origin. Again the Implicit Function Theorem shows that $k_j(0, 0) = 0$ and $\partial_\xi k_j(0, 0) = \partial_\eta k_j(0, 0) = 0$ for $j = 3, \dots, n$. These imply that $\mathcal{M}$ is an analytic center manifold of $\widetilde{\mathcal{X}}$ in a neighborhood of the origin which is tangent to the (ξ, η) plane.

(b) Since

$$\widetilde{\mathcal{X}}\big(\zeta_j - \phi_j(\xi, \eta, \zeta)\big) = 0 \quad \text{on } \mathcal{M},$$

we have

$$\begin{aligned} &\widetilde{F}_j\big(\xi, \eta, k(\xi, \eta)\big) - \widetilde{F}_1\big(\xi, \eta, k(\xi, \eta)\big)\frac{\partial \phi_j}{\partial \xi} - \widetilde{F}_2\big(\xi, \eta, k(\xi, \eta)\big)\frac{\partial \phi_j}{\partial \eta} \\ &\quad - \partial_\zeta \phi_j \widetilde{F}\big(\xi, \eta, k(\xi, \eta)\big) = 0, \quad j = 3, \dots, n. \end{aligned}$$

Here we have used the conventions $\partial_\zeta \phi_j = (\partial_{\zeta_3}\phi_j, \dots, \partial_{\zeta_n}\phi_j)$ and $\widetilde{F} = (\widetilde{F}_3, \dots, \widetilde{F}_n)^{tr}$. Write these equations in a unified vector form, we have

$$(E - \partial_\zeta \phi)\widetilde{F} = \widetilde{F}_1 \partial_\xi \phi + \widetilde{F}_2 \partial_\eta \phi \quad \text{on } \mathcal{M}, \tag{2.7}$$

where $\partial_s \phi = (\partial_s \phi_3, \dots, \partial_s \phi_n)^{tr}$, $s \in \{\xi, \eta\}$.

In addition, since

$$k(\xi, \eta) = \phi\big(\xi, \eta, k(\xi, \eta)\big),$$

we have

$$(E - \partial_\zeta \phi)\partial_\xi k = \partial_\xi \phi, \qquad (E - \partial_\zeta \phi)\partial_\eta k = \partial_\eta \phi, \tag{2.8}$$

where $\partial_s k = (\partial_s k_3, \dots, \partial_s k_n)^{tr}$, $s \in \{\xi, \eta\}$.

Set $C(\xi,\eta)=V(\xi,\eta,k(\xi,\eta))$. Some calculations show that

$$
\begin{aligned}
\widetilde{\mathcal{X}}|_{\mathcal{M}}\big(C(\xi,\eta)\big) &= \widetilde{F}_1\big(\xi,\eta,k(\xi,\eta)\big)\frac{\partial C}{\partial \xi}+\widetilde{F}_2\big(\xi,\eta,k(\xi,\eta)\big)\frac{\partial C}{\partial \eta}\\
&= \widetilde{F}_1[w]\left(\frac{\partial V}{\partial \xi}+\frac{\partial V}{\partial \zeta_3}\frac{\partial k_3}{\partial \xi}+\cdots+\frac{\partial V}{\partial \zeta_n}\frac{\partial k_n}{\partial \xi}\right)\\
&\quad+\widetilde{F}_2[w]\left(\frac{\partial V}{\partial \eta}+\frac{\partial V}{\partial \zeta_3}\frac{\partial k_3}{\partial \eta}+\cdots+\frac{\partial V}{\partial \zeta_n}\frac{\partial k_n}{\partial \eta}\right)\\
&= \widetilde{F}_1[w]\frac{\partial V}{\partial \xi}+\widetilde{F}_1[w]\partial_\zeta V(E-\partial_\zeta\phi)^{-1}\partial_\xi\phi\\
&\quad+\widetilde{F}_2[w]\frac{\partial V}{\partial \eta}+\widetilde{F}_2[w]\partial_\zeta V(E-\partial_\zeta\phi)^{-1}\partial_\eta\phi\Big|_{\mathcal{M}}\\
&= \widetilde{F}_1[w]\frac{\partial V}{\partial \xi}+\widetilde{F}_2[w]\frac{\partial V}{\partial \eta}+\partial_\zeta V\widetilde{F}[w]\Big|_{\mathcal{M}}\\
&= \widetilde{\mathcal{X}}(V)|_{\mathcal{M}}=L_0V|_{\mathcal{M}}=L_0|_{\mathcal{M}}C,
\end{aligned}
\tag{2.9}
$$

where $[w]=(\xi,\eta,k(\xi,\eta))$, and in the third and fourth equalities we have used respectively (2.8) and (2.7). Recall that $\partial_\zeta V=(\partial_{\zeta_3}V,\dots,\partial_{\zeta_n}V)$.

Next we shall prove that $L_0|_{\mathcal{M}}=\operatorname{div}(\widetilde{\mathcal{X}}|_{\mathcal{M}})$. From the definition of inverse Jacobian multipliers and (2.6), we get that

$$J\operatorname{div}\widetilde{\mathcal{X}}=\widetilde{\mathcal{X}}(J)=(L_0+L_3+\cdots+L_n)J.$$

This reduces to

$$L_0=\operatorname{div}\widetilde{\mathcal{X}}-L_3-\cdots-L_n. \tag{2.10}$$

Note that for $j=3,\dots,n$

$$
\begin{aligned}
L_j\big(\zeta_j-\phi_j(\xi,\eta,\zeta)\big)&=\widetilde{\mathcal{X}}\big(\zeta_j-\phi_j(\xi,\eta,\zeta)\big)\\
&=\widetilde{F}_j-\widetilde{F}_1\partial_\xi\phi_j-\widetilde{F}_2\partial_\eta\phi_j-\partial_\zeta\phi_j\widetilde{F}.
\end{aligned}
$$

Writing these equations in vector form gives

$$\operatorname{diag}(L_3,\dots,L_n)\big(\zeta-\phi(\xi,\eta,\zeta)\big)=(E-\partial_\zeta\phi)\widetilde{F}-\partial_\xi\phi\,\widetilde{F}_1-\partial_\eta\phi\,\widetilde{F}_2. \tag{2.11}$$

Recall that $\widetilde{F}$, $\partial_\xi\phi$, $\partial_\eta\phi$ are $n-2$ dimensional column vectors.

On the center manifold $\mathcal{M}$ we have

$$\phi_j(\xi,\eta,\zeta)=\zeta_j,\qquad j=3,\dots,n.$$

So from these we get that

$$\partial_\xi\partial_{\zeta_s}\phi_j=\partial_\eta\partial_{\zeta_s}\phi_j=\partial_{\zeta_s}\partial_{\zeta_l}\phi_j=0\quad\text{on }\mathcal{M},\ \text{for all }3\leqslant s,j,l\leqslant n.$$

Differentiating (2.11) with respect to ζ, together with these last equalities, yield

$$\operatorname{diag}(L_3,\dots,L_n)(E-\partial_\zeta\phi)=(E-\partial_\zeta\phi)\partial_\zeta\widetilde{F}-\partial_\xi\phi\,\partial_\zeta\widetilde{F}_1-\partial_\eta\phi\,\partial_\zeta\widetilde{F}_2\quad\text{on } \mathcal{M}.$$

We note that $\partial_\zeta\widetilde{F}$ is a matrix of order $n-2$, and $\partial_\zeta\widetilde{F}_s$ for $s=1,2$ are $n-2$ dimensional horizontal vectors. Rewrite this last equation in the following form

$$\begin{aligned}\operatorname{diag}(L_3,\dots,L_n)&=(E-\partial_\zeta\phi)\partial_\zeta\widetilde{F}(E-\partial_\zeta\phi)^{-1}\\&\quad-\partial_\xi\phi\,\partial_\zeta\widetilde{F}_1(E-\partial_\zeta\phi)^{-1}-\partial_\eta\phi\,\partial_\zeta\widetilde{F}_2(E-\partial_\zeta\phi)^{-1}.\end{aligned}\tag{2.12}$$

Since similar matrices have the same trace, we have

$$\operatorname{trace}\big((E-\partial_\zeta\phi)\partial_\zeta\widetilde{F}(E-\partial_\zeta\phi)^{-1}\big)=\operatorname{trace}(\partial_\zeta\widetilde{F})=\sum_{j=3}^{n}\partial_{\zeta_j}\widetilde{F}_j.\tag{2.13}$$

Moreover some calculations show that

$$\begin{aligned}\operatorname{trace}\big(\partial_\xi\phi\,\partial_\zeta\widetilde{F}_1(E-\partial_\zeta\phi)^{-1}\big)&=\operatorname{trace}\big((E-\partial_\zeta\phi)^{-1}\partial_\xi\phi\,\partial_\zeta\widetilde{F}_1\big)\\&=\partial_\zeta\widetilde{F}_1(E-\partial_\zeta\phi)^{-1}\partial_\xi\phi,\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}\operatorname{trace}\big(\partial_\eta\phi\,\partial_\zeta\widetilde{F}_2(E-\partial_\zeta\phi)^{-1}\big)&=\operatorname{trace}\big((E-\partial_\zeta\phi)^{-1}\partial_\eta\phi\,\partial_\zeta\widetilde{F}_2\big)\\&=\partial_\zeta\widetilde{F}_2(E-\partial_\zeta\phi)^{-1}\partial_\eta\phi.\end{aligned}\tag{2.15}$$

Combining (2.12), (2.13), (2.14) and (2.15) gives

$$L_3+\dots+L_n=\sum_{j=3}^{n}\partial_{\zeta_j}\widetilde{F}_j-\partial_\zeta\widetilde{F}_1(E-\partial_\zeta\phi)^{-1}\partial_\xi\phi-\partial_\zeta\widetilde{F}_2(E-\partial_\zeta\phi)^{-1}\partial_\eta\phi.$$

This together with (2.10) show that

$$\begin{aligned}L_0|_{\mathcal{M}}&=\partial_\xi\widetilde{F}_1+\partial_\eta\widetilde{F}_2+\partial_\zeta\widetilde{F}_1(E-\partial_\zeta\phi)^{-1}\partial_\xi\phi+\partial_\zeta\widetilde{F}_2(E-\partial_\zeta\phi)^{-1}\partial_\eta\phi\big|_{\mathcal{M}}\\&=\partial_\xi\widetilde{F}_1+\partial_\eta\widetilde{F}_2+\partial_\zeta\widetilde{F}_1\,\partial_\xi k+\partial_\zeta\widetilde{F}_2\,\partial_\eta k\big|_{\mathcal{M}}\\&=\partial_\xi\widetilde{F}_1\big(\xi,\eta,k(\xi,\eta)\big)+\partial_\eta\widetilde{F}_2\big(\xi,\eta,k(\xi,\eta)\big)=\operatorname{div}(\widetilde{\mathcal{X}}|_{\mathcal{M}}),\end{aligned}\tag{2.16}$$

where in the second equality we have used (2.8).

Now the equalities (2.9) and (2.16) verify that $C(\xi,\eta)$ is an analytic inverse integrating factor of the vector field $\widetilde{\mathcal{X}}|_{\mathcal{M}}$.

We complete the proof of the lemma. □

Remark 2.5. Replacing analyticity by C^∞ smoothness Lemma 2.4 holds, too.

We now study the properties of C^∞ inverse Jacobian multiplier restricted to center manifolds.

Lemma 2.6. *Assume that system (1.1) satisfies (H) and has a C^∞ inverse Jacobian multiplier, written in conjugate complex coordinates as*

$$J(\xi,\eta,\zeta)=\prod_{j=3}^{n}\big(\zeta_j-\psi_j(\xi,\eta,\zeta)\big)V(\xi,\eta,\zeta),$$

where $\psi_j=O(|(\xi,\eta,\zeta)|^2)$ and V has no factor $\zeta_l-\psi_l(\xi,\eta,\zeta)$ for any $l\in\{3,\dots,n\}$. Then the following statements hold.

(a) *$\mathcal{M}^*=\bigcap_{j=3}^{n}\{\zeta_j=\psi_j(\xi,\eta,\zeta)\}$ is a center manifold of system (1.1) at the origin.*
(b) *For any smooth center manifold $\mathcal{M}$ of system (1.1) at the origin, if $\mathcal{X}|_{\mathcal{M}}$ has the origin as a center, then $J|_{\mathcal{M}}=0$.*

Proof. (a) As in the proof of Lemma 2.4 there exist C^∞ smooth functions $L_3,\dots,L_n$ such that

$$\widetilde{\mathcal{X}}\big(\zeta_j-\psi_j(\xi,\eta,\zeta)\big)=L_j(\xi,\eta,\zeta)\big(\zeta_j-\psi_j(\xi,\eta,\zeta)\big),\quad j=3,\dots,n,$$

where $\widetilde{\mathcal{X}}$ is $\mathcal{X}$ written in the conjugate complex coordinates as those did in (2.1). Note that each surface $\zeta_j-\psi_j(\xi,\eta,\zeta)$ is invariant under the flow of $\widetilde{\mathcal{X}}$. By the Implicit Function Theorem the equations

$$\zeta_j-\psi_j(\xi,\eta,\zeta)=0,\quad j=3,\dots,n,$$

have a unique solution $\zeta=k(\xi,\eta)$, which is C^∞. Representing $\zeta=k(\xi,\eta)$ in the cartesian coordinates gives

$$z=h(x,y),\quad \text{i.e.}\quad z_j=h_j(x,y),\quad j=3,\dots,n.$$

Clearly $\partial_x h_j(0,0)=\partial_y h_j(0,0)=0$ for $j=3,\dots,n$. Then

$$\mathcal{M}^*=\bigcap_{j=3}^{n}\big\{\zeta_j-k_j(\xi,\eta)\big\}=\bigcap_{j=3}^{n}\big\{z_j-h_j(x,y)\big\},$$

is a center manifold of system (1.1) at the origin.

(b) Let $P_0=(x_0,y_0,z_0)$ be any point on $\mathcal{M}$ in a sufficiently small neighborhood of the origin, and let φ_t be the orbit of (1.1) passing through P_0. Then we have

$$\frac{dJ(\varphi_t)}{dt}=\mathcal{X}(J)|_{\varphi_t}=J\operatorname{div}\mathcal{X}|_{\varphi_t}.$$

This equation has the solution

$$J\big(\varphi_t(x_0,y_0,z_0)\big)=J(x_0,y_0,z_0)\exp\left(\int_0^t\operatorname{div}\mathcal{X}|_{\varphi_s}\,ds\right). \tag{2.17}$$

By the assumption φ_t is a periodic orbit. Denote its period by T_0. This last equation can be simplified to

$$J(x_0, y_0, z_0) = J(x_0, y_0, z_0) \exp\left(\int_0^{T_0} \operatorname{div} \mathcal{X}|_{\varphi_s} \, ds\right). \tag{2.18}$$

Restricted to the center manifold $\mathcal{M}$ system (1.1) becomes

$$\dot{x} = -y + f_1\big(x, y, h(x, y)\big), \qquad \dot{y} = x + f_2\big(x, y, h(x, y)\big). \tag{2.19}$$

Written in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, we get from this two dimensional system

$$\frac{d\theta}{1 + O(r)} = dt.$$

Integrating along the periodic orbit gives

$$T_0 = 2\pi + O(r).$$

So we have

$$\int_0^{T_0} \operatorname{div} \mathcal{X}|_{\varphi_s} \, ds = \int_0^{T_0} \big(\Lambda + O\big(|P_0|\big)\big) \, ds = 2\pi\Lambda + O\big(|P_0|\big).$$

This together with (2.18) yield that in a sufficiently small neighborhood of the origin

$$J(x_0, y_0, z_0) = 0.$$

By the arbitrariness of $P_0 \in \mathcal{M}$ we get that $J|_{\mathcal{M}} \equiv 0$. This proves statement (b).

We complete the proof of the lemma. □

Having the above preparations we can prove Theorems 1.1 and 1.2.

2.2. *Proof of Theorem 1.1*

(a) *Sufficiency*. If the matrix A has conjugate complex eigenvalues, we write system (1.1) in (2.1). By Lemma 2.4 and its proof we get that system (2.1) has an analytic center manifold $\mathcal{M} = \bigcap_{j=3}^{n} \{\zeta_j = k_j(\xi, \eta)\}$. Again by Lemma 2.4, system (2.1) restricted to $\mathcal{M}$, i.e. (2.19), has an analytic inverse integrating factor $C(\xi, \eta) = \widetilde{V}(\xi, \eta, k(\xi, \eta))$, where $\widetilde{V}$ is $V(x, y, z)$ written in (ξ, η, ζ).

We note that either $\zeta_j = z_j$ is a real coordinate or $\zeta_j = z_j + \mathbf{i}z_{j+1}$ and $\zeta_{j+1} = z_j - \mathbf{i}z_{j+1}$ for some j are conjugate complex coordinates. In the latter write $k_j(\xi, \eta) = h_j(x, y) + \mathbf{i}h_{j+1}(x, y)$, we have $z_j = h_j(x, y)$. In the former write $h_j = k_j$. Then we have

$$\mathcal{M} = \bigcap_{j=3}^{n} \big\{z_j = k_j(x, y)\big\}.$$

Since $C(0,0) = V(0,0,0) \neq 0$, integrating the one-form

$$\frac{x + f_2(x,y,h(x,y))}{V(x,y,h(x,y))}\,dx + \frac{y - f_1(x,y,h(x,y))}{V(x,y,h(x,y))}\,dy,$$

provides an analytic first integral $H(x,y)$ of (2.19) and it has the form $H(x,y) = (x^2+y^2)/C(0,0) +$ higher order term. So we get from the Poincaré center Theorem that the vector field $\mathcal{X}$ has the origin as a center on the center manifold $\mathcal{M}$.

The vector field $\mathcal{X}$ restricted to the center manifold $\mathcal{M}$ has the origin as a center and it has an analytic first integral. These facts together with Theorems 6.3 and 7.1 of Sijbrand [16] show that the center manifold at the origin is unique and analytic. So we have $\mathcal{M}^c = \mathcal{M}$. Hence system (1.1) restricted to $\mathcal{M}^c$ has the origin as a center.

Necessity. First we write system (1.1) in (2.1) with the conjugate complex coordinates. Lemmas 2.2 and 2.3 show that system (2.1) is analytically equivalent to its distinguished normal form, i.e. system (2.3), through a distinguished normalization.

For the analytic differential system (2.3) we have $g_1 = g_2$ by the proof of Lemma 2.3. We can check easily that $\widetilde{J} = w_3 \ldots w_n$ is an inverse Jacobian multiplier of system (2.3) and is clearly analytic. Hence using the near identity analytic transformation from (2.1) to (2.3) we get that system (2.1) has an analytic inverse Jacobian multiplier

$$J^* = \big(\zeta_3 - \phi_3(\xi,\eta,\zeta)\big)\ldots\big(\zeta_n - \phi_n(\xi,\eta,\zeta)\big)/D(\xi,\eta,\zeta),$$

where $D(\xi,\eta,\zeta)$ is the determinant of the Jacobian matrix of the transformation from (2.1) to (2.3), and satisfies $D(0,0,0) = 1$.

Going back to the (x,y,z) coordinates we get that system (1.1) has an analytic inverse Jacobian multiplier of the form (1.3).

(b) The analyticity and uniqueness of the center manifolds were proved in the sufficient part of statement (a). $\mathcal{M}^c \subset J^{-1}(0)$ follows from Lemma 2.6 (b) and the first assertion.

We complete the proof of the theorem. □

2.3. *Proof of Theorem 1.2*

(a) Under the assumption of the theorem, we get from Lemma 2.3 (a) that system (2.1) is locally C^∞ equivalent to its Poincaré–Dulac normal form (2.3) with $g_1 \neq g_2$. Direct calculations show that system (2.3) has the C^∞ inverse Jacobian multiplier

$$\widetilde{J} = w_3 \ldots w_n uv\big(g_2(uv) - g_1(uv)\big),$$

where $g_1(s)$, $g_2(s)$ are C^∞ functions and $g_2 - g_1$ is non-flat at $s = 0$. This shows that $\widetilde{J} = w_3 \ldots w_n (uv)^l h(uv)$ with $l \geqslant 2$ and $h(0) \neq 0$. Without loss of generality we can assume $h(0) = 1$. By the inverse transformations from (2.1) to (2.3) we get that system (2.3) has a C^∞ inverse Jacobian multiplier of the form

$$J(\xi,\eta,\zeta) = \prod_{j=3}^{n}\big(\zeta_j - \phi_j(\xi,\eta,\zeta)\big)\big((\xi-\phi_1)(\eta-\phi_2)\big)^l h\big((\xi-\phi_1)(\eta-\phi_2)\big)/D(\xi,\eta,\zeta),$$

where the C^∞ smoothness follows from the facts that $\widetilde{J}$ and the near identity transformation from (2.1) to (2.3) are both C^∞ smooth, $D(\xi,\eta,\zeta)$ is the determinant of the Jacobian matrix of the transformation satisfying $D(0,0,0)=1$.

Note that ϕ_1 and ϕ_2 are conjugate. And for $j=3,\dots,n$, either ϕ_j is real if ζ_j is real, or ϕ_j and ϕ_k are conjugate if some ζ_j and ζ_k are conjugate. So written the conjugate complex coordinates (ξ,η,ζ) (if exist) in the real ones (x,y,z) we get that system (1.1) has the inverse Jacobian multiplier in the prescribed form (1.4).

(b) The proof follows from statement (a) and Lemma 2.6.

We complete the proof of the theorem. □

3. Proof of Theorem 1.3

3.1. Preparation to the proof

Under the assumption of Theorem 1.3 we get from Theorem 1.2 that system (1.1) has a C^∞ inverse Jacobian multiplier of the form

$$J(x,y,z)=\prod_{j=0}^{m-1}\left[\left(z_{3+2j}-\psi_{3+2j}(x,y,z)\right)^2+\left(z_{3+2j+1}-\psi_{3+2j+1}(x,y,z)\right)^2\right]$$
$$\times\prod_{s=3+2m}^{n}\left(z_s-\psi_s(x,y,z)\right)\left(x^2+y^2\right)^l V(x,y,z),$$

where $V(0,0,0)=1$. Moreover, it follows from the proofs of Lemmas 2.4 and 2.6 that system (1.1) has a C^∞ center manifold $\mathcal{M}^c$ at the origin, which is defined by the intersection of the invariant surfaces

$$z_j=\psi_j(x,y,z),\qquad j=3,\dots,n. \tag{3.1}$$

Furthermore the center manifold can be represented as $\mathcal{M}^c=\bigcap_{j=3}^n\{z_j=h_j(x,y)\}$, where $z=h(x,y)$ is the unique solution of (3.1) defined in a neighborhood of the origin, which is obtained from the Implicit Function Theorem. Recall that $z=(z_3,\dots,z_n)$ and $h=(h_3,\dots,h_n)$.

If $m>0$, set for $j=0,\dots,m-1$

$$\zeta_{3+2j}=z_{3+2j}+\mathbf{i}z_{3+2j+1},\qquad \psi^*_{3+2j}(x,y,\zeta)=\psi_{3+2j}+\mathbf{i}\psi_{3+2j+1},$$
$$\zeta_{3+2j+1}=z_{3+2j}-\mathbf{i}z_{3+2j+1},\qquad \psi^*_{3+2j+1}(x,y,\zeta)=\psi_{3+2j}-\mathbf{i}\psi_{3+2j+1}.$$

Note that the determinant of the Jacobian matrix of the transformation from (x,y,z) to (x,y,ζ) is a nonzero constant. If $\psi_j(x,y,z)\neq 0$, we take the change of variables

$$(u,v,w)=\Phi(x,y,z)=\left(x,y,\zeta-\psi^*(x,y,\zeta)\right), \tag{3.2}$$

where $\zeta=(\zeta_3,\dots,\zeta_{2m+2},z_{2m+3},\dots,z_n)$ and $\psi^*=(\psi^*_3,\dots,\psi^*_{2m+2},\psi_{2m+3},\dots,\psi_n)$. Then system (1.1) is transformed to

$$\begin{aligned}
\dot{u} &= -v + g_1(u, v, w),\\
\dot{v} &= u + g_2(u, v, w),\\
\dot{w}_j &= w_j(\lambda_j + g_j(u, v, w)), \quad j = 3, \ldots, n
\end{aligned} \tag{3.3}$$

where $g_1, g_2 = O(|(u, v, w)|^2)$ and $g_j = O(|(u, v, w)|)$, $j = 3, \ldots, n$. Correspondingly system (3.3) has the center manifold $w = 0$. Moreover system (3.3) has the associated inverse Jacobian multiplier

$$\frac{J \circ \Phi^{-1}(u, v, w)}{D\Phi^{-1}(u, v, w)} = w_3 \ldots w_n \big(u^2 + v^2\big)^l \widetilde{V}(u, v, w)$$

where $\widetilde{V}(0, 0, 0) \neq 0$, and $D\Phi^{-1}$ is the determinant of the Jacobian matrix of Φ^{-1} with respect to its variables and $D\Phi(0, 0, 0) = 1$.

Since systems (1.1) and (3.3) are C^∞ equivalent in a neighborhood of the origin and the corresponding inverse Jacobian multipliers have the same forms, so in what follows we assume without loss of generality that system (1.1) has the center manifold $z = 0$ and the coordinate hyperplane $z_j = 0$ is invariant for $j = 3, \ldots, n$.

Taking the cylindrical coordinate changes

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = rs,$$

with $r \geqslant 0$, system (1.5) is transformed to

$$\dot{\theta} = 1 + \Theta(\theta, r, s, \varepsilon), \qquad \dot{r} = R(\theta, r, s, \varepsilon) \qquad \dot{s} = As + S(\theta, r, s, \varepsilon), \tag{3.4}$$

where

$$\begin{aligned}
\Theta(\theta, r, s, \varepsilon) &= \frac{\cos\theta g_2(r\cos\theta, r\sin\theta, rs, \varepsilon) - \sin\theta g_1(r\cos\theta, r\sin\theta, rs, \varepsilon)}{r},\\
R(\theta, r, s, \varepsilon) &= \cos\theta g_1(r\cos\theta, r\sin\theta, rs, \varepsilon) + \sin\theta g_2(r\cos\theta, r\sin\theta, rs, \varepsilon),\\
S(\theta, r, s, \varepsilon) &= \frac{g(r\cos\theta, r\sin\theta, rs, \varepsilon) - sR(\theta, r, s, \varepsilon)}{r},
\end{aligned}$$

where $g = (g_3, \ldots, g_n)^{tr}$ with $g_3, \ldots, g_n$ given in (3.3). Notice that

$$\begin{aligned}
&R(\theta, 0, s, \varepsilon) = 0, \qquad R(\theta, r, s, 0) = O\big(r^2\big),\\
&\Theta(\theta, r, s, 0) = O(r), \qquad S(\theta, r, s, 0) = O(r).
\end{aligned}$$

Corresponding to the inverse Jacobian multiplier $J(x, y, z)$ of system (1.1), system (3.4) with $\varepsilon = 0$ has the inverse Jacobian multiplier

$$J(r\cos\theta, r\sin\theta, rs)/r^{n-1} = s_3 \ldots s_n r^{2l-1} k(\theta, r, s), \tag{3.5}$$

with $k(\theta, 0, 0) = \text{constant} \neq 0$.

For $|\varepsilon| \ll 1$ and $|r|$ suitably small, we always have $\dot{\theta} > 0$. So system (3.4) can be equivalently written in

$$\frac{dr}{d\theta} = \frac{R(\theta, r, s, \varepsilon)}{1 + \Theta(\theta, r, s, \varepsilon)} =: p(\theta, r, s, \varepsilon),$$
$$\frac{ds}{d\theta} = \frac{As + S(\theta, r, s, \varepsilon)}{1 + \Theta(\theta, r, s, \varepsilon)} =: As + q(\theta, r, s, \varepsilon). \tag{3.6}$$

Furthermore, we have

$$p(\theta, 0, s, \varepsilon) = 0, \qquad p(\theta, r, s, 0) = O\big(r^2\big), \qquad q(\theta, r, s, 0) = O(r). \tag{3.7}$$

And $q = (q_3, \ldots, q_n)^{tr}$ with q_j having the factor s_j when $\varepsilon = 0$ for $j = 3, \ldots, n$.

Associated to system (3.6) we have a vector field

$$\mathcal{Y}_\varepsilon = \partial_\theta + p(\theta, r, s, \varepsilon)\partial_r + \big\langle As + q(\theta, r, s, \varepsilon), \partial_s\big\rangle,$$

where $\partial_s = (\partial_{s_3}, \ldots, \partial_{s_n})$. Related to the inverse Jacobian multiplier (3.5) of system (3.4), the vector field $\mathcal{Y}_0$ has the inverse Jacobian multiplier

$$J_c(\theta, r, s) = \frac{J(r\cos\theta, r\sin\theta, rs)}{r^{n-1}(1 + \Theta(\theta, r, s, 0))} = s_3 \ldots s_n r^{2l-1} K(\theta, r, s), \tag{3.8}$$

where $K = 1 + O(r)$.

Clear p, q are periodic in θ with period 2π, and they are well defined on the cylinder $\mathcal{C} = \{(\theta, r, s, \varepsilon) \in \mathbb{R}/(2\pi\mathbb{R}) \times \mathbb{R}^{n-1} \times \mathbb{R}^m\colon\ |r|, |\varepsilon| \ll 1\}$. Furthermore we note that each periodic orbit of system (1.5) corresponds to a unique periodic orbit of system (3.6) on $\mathcal{C}$. So, to study the periodic orbits of system (1.5) is equivalent to study the periodic orbits of system (3.6).

Denote by $\psi_\theta(r_0, s_0, \varepsilon)$ the solution of system (3.6) with the initial point $\psi_0(r_0, s_0, \varepsilon) = (r_0, s_0) \in \mathcal{C}$. We have

$$\psi_\theta(r_0, s_0, \varepsilon) = \big(r_\theta(r_0, s_0, \varepsilon), s_\theta(r_0, s_0, \varepsilon)\big).$$

On the cylinder $\mathcal{C}$, $\theta = 2\pi$ coincides with $\theta = 0$. We define the Poincaré map on the transversal section $\theta = 0$ of the flow of (3.6) by

$$\mathcal{P}(r_0, s_0; \varepsilon) = \psi_{2\pi}(r_0, s_0, \varepsilon).$$

Since system (3.6) is analytic, and so is the Poincaré map $\mathcal{P}$. Set

$$\mathcal{P}(r_0, s_0, \varepsilon) = \big(\mathcal{P}_r(r_0, s_0, \varepsilon), \mathcal{P}_s(r_0, s_0, \varepsilon)\big),$$

with

$$\mathcal{P}_r(r_0, s_0, \varepsilon) = r_{2\pi}(r_0, s_0, \varepsilon) \quad \text{and} \quad \mathcal{P}_s(r_0, s_0, \varepsilon) = s_{2\pi}(r_0, s_0, \varepsilon).$$

Then

$$\mathcal{P}_r(r_0, s_0, \varepsilon) = r_0 + \int_0^{2\pi} p\bigl(v, r_v(r_0, s_0, \varepsilon), s_v(r_0, s_0, \varepsilon), \varepsilon\bigr)\, dv,$$

$$\mathcal{P}_s(r_0, s_0, \varepsilon) = e^{A2\pi}\left(Es_0 + \int_0^{2\pi} e^{-Av} q\bigl(v, r_v(r_0, s_0, \varepsilon), s_v(r_0, s_0, \varepsilon), \varepsilon\bigr)\, dv\right),$$

where E is the unit matrix of order $n-2$.

Define the displacement function by

$$\mathcal{D}(r_0, s_0, \varepsilon) = \mathcal{P}(r_0, s_0, \varepsilon) - (r_0, s_0).$$

Then the periodic orbit of system (3.6) is uniquely determined by the zero of the displacement function $\mathcal{D}$. Set

$$\mathcal{D}_r = \mathcal{P}_r - r_0, \qquad \mathcal{D}_s = \mathcal{P}_s - s_0.$$

Then $\mathcal{D} = (\mathcal{D}_r, \mathcal{D}_s)$.

In order to study the zeros of $\mathcal{D}(r_0, s_0, \varepsilon)$ on (r_0, s_0) for any fixed ε sufficiently small, we will solve $\mathcal{D}_s(r_0, s_0, \varepsilon) = 0$ in s_0 as a function of (r_0, ε) in a small neighborhood of $(r_0, \varepsilon) = (0, 0)$. In fact, by (3.7) we get easily that

$$\mathcal{D}_s(0, 0, 0) = 0, \qquad \frac{\partial \mathcal{D}_s}{\partial s}(0, 0, 0) = e^{2\pi A} - E.$$

These together with the assumption on A show that the matrix $e^{2\pi A} - E$ is invertible. So the Implicit Function Theorem yields that $D_s(r_0, s_0, \varepsilon) = 0$ has a unique solution $s_0 = s^*(r_0, \varepsilon)$ in a neighborhood of $(r_0, \varepsilon) = (0, 0)$, which is analytic. Substituting s^* into $\mathcal{D}_r$ gives

$$d(r_0, \varepsilon) := \mathcal{D}_r\bigl(r_0, s^*(r_0, \varepsilon), \varepsilon\bigr).$$

Note that $d(r_0, \varepsilon)$ is analytic. Thus the number of periodic orbits of system (3.6) is equal to the number of positive roots r_0 of $d(r_0, \varepsilon) = 0$.

Having the above preparation we can prove Theorem 1.3.

3.2. *Proof of Theorem 1.3*

As we discussed in Subsection 3.1, for proving Theorem 1.3 we only need to study the number of zeros of $d(r_0, \varepsilon)$ in r_0.

From the expression of the inverse Jacobian multiplier J_c it follows that J_c is periodic in θ with period 2π. The inverse Jacobian multiplier J_c and the Poincaré map $\mathcal{P}(r_0, s_0, 0)$ of system (3.6) with $\varepsilon = 0$ has the relation

$$J_c\bigl(0, \mathcal{P}(r_0, s_0, 0)\bigr) = J_c(0, r_0, s_0) D\mathcal{P}(r_0, s_0, 0), \tag{3.9}$$

where $D\mathcal{P}$ denotes the determinant of the Jacobian matrix of $\mathcal{P}$ with respect to (r_0, s_0). For a proof, see [3]. Here for completeness we provide a proof. From (2.17) we have

$$J_c\big(0, \varphi_\theta(r_0, s_0, 0)\big) = J_c\big(0, \varphi_0(r_0, s_0, 0)\big) \exp\left(\int_0^\theta \operatorname{div} \mathcal{Y}_0 \circ \varphi_s(r_0, s_0, 0)\, ds\right), \tag{3.10}$$

where $\varphi_\theta(r_0, s_0, \varepsilon)$ is the flow of the vector field $\mathcal{Y}_\varepsilon$ or of system (3.6) satisfying $\varphi_0(r_0, s_0, \varepsilon) = (r_0, s_0)$. Restricted (3.10) to $\theta = 2\pi$ and by the definition of the Poincaré map, we have

$$J_c\big(0, \mathcal{P}(r_0, s_0, 0)\big) = J_c(0, r_0, s_0) \exp\left(\int_0^{2\pi} \operatorname{div} \mathcal{Y}_0 \circ \varphi_s(r_0, s_0, 0)\, ds\right). \tag{3.11}$$

Since the Jacobian matrix $\frac{\partial \varphi_\theta(r_0, s_0, \varepsilon)}{\partial(r_0, s_0)}$ satisfies the variational equations of system (3.6) along the solution $(r, s) = \varphi_\theta(r_0, s_0, \varepsilon)$,

$$\frac{dZ}{d\theta} = \frac{\partial(p, As + q)}{\partial(r, s)} \circ \varphi(r_0, s_0, \varepsilon) Z.$$

By the Liouvellian formula we have

$$\det \frac{\partial \varphi_\theta(r_0, s_0, \varepsilon)}{\partial(r_0, s_0)} = \det \frac{\partial \varphi_0(r_0, s_0, \varepsilon)}{\partial(r_0, s_0)} \exp\left(\int_0^\theta \operatorname{div} \mathcal{Y}_\varepsilon \circ \varphi_s(r_0, s_0, \varepsilon)\, ds\right).$$

Taking $\varepsilon = 0$ and $\theta = 2\pi$, this last equation can be written in

$$\det \frac{\partial \varphi_{2\pi}(r_0, s_0, 0)}{\partial(r_0, s_0)} = \exp\left(\int_0^{2\pi} \operatorname{div} \mathcal{Y}_0 \circ \varphi_s(r_0, s_0, 0)\, ds\right).$$

This together with (3.11) verify (3.9).

Writing (3.9) in components and using (3.8), we have

$$\mathcal{P}_{s3} \ldots \mathcal{P}_{sn} \mathcal{P}_r^{2l-1} K(0, \mathcal{P}_r, \mathcal{P}_s) = s_{03} \ldots s_{0n} r_0^{2l-1} K(0, r_0, s_0) D\mathcal{P}(r_0, s_0, 0), \tag{3.12}$$

where $\mathcal{P}_s = (\mathcal{P}_{s3}, \ldots, \mathcal{P}_{sn})$.

Since the hyperplane $s_j = 0$ is invariant under the flow of (3.6) with $\varepsilon = 0$ for $j = 3, \ldots, n$, we get that

$$\mathcal{P}_s(r_0, s_0, 0) = \big(s_{03} \mathcal{P}_{s3}^*(r_0, s_0), \ldots, s_{0n} \mathcal{P}_{sn}^*(r_0, s_0)\big) =: \big\langle s_0, \mathcal{P}_s^*(r_0, s_0)\big\rangle, \tag{3.13}$$

where $s_0 = (s_{03}, \ldots, s_{0n})$. These together with (3.12) show that

$$P_{s3}^* \ldots \mathcal{P}_{sn}^* \mathcal{P}_r^{2l-1} K(0, \mathcal{P}_r, \mathcal{P}_s) = r_0^{2l-1} K(0, r_0, s_0) D\mathcal{P}(r_0, s_0, 0). \tag{3.14}$$

Direct calculations show that $D\mathcal{P}(r_0, s_0, 0)|_{s_0=0} = \mathcal{P}^*_{s3} \dots \mathcal{P}^*_{sn} \partial_r \mathcal{P}_r|_{s_0=0}$. So (3.14) is simplified to

$$\mathcal{P}_r^{2l-1} K(0, \mathcal{P}_r, 0) = r_0^{2l-1} K(0, r_0, 0) \partial_r \mathcal{P}_r \quad \text{for } s_0 = 0. \tag{3.15}$$

From (3.13) it follows that the solution $s^*(r_0, 0)$ of $\mathcal{D}_s(r_0, s_0, \varepsilon) = 0$ with $\varepsilon = 0$ satisfies $s^*(r_0, 0) \equiv 0$. So we have $d(r_0, 0) = \mathcal{D}(r_0, 0, 0)$. Expanding $d(r_0, 0)$ in the Taylor series gives

$$d(r_0, 0) = \delta_k r_0^k + O\big(r_0^{k+1}\big),$$

with $\delta_k \neq 0$ a constant. Then

$$\mathcal{P}_r(r_0, 0, 0) = r_0 + \delta_k r_0^k + O\big(r_0^{k+1}\big).$$

Consequently we have $K(0, \mathcal{P}_r, 0) = K(0, r_0, 0) + O(r_0^k)$. Substituting these expressions in (3.15), with some simple calculations, gives

$$K(0, r_0, 0)\big[(2l-1)\delta_k r_0^{2l-2+k} + O\big(r^{2l-1+k}\big)\big] + O\big(r_0^{2l-1+k}\big) = K(0, r_0, 0) k \delta_k r_0^{2l-2+k}.$$

Since $K(0, 0, 0) = 1$, equating the coefficients of r_0^{2l-2+k} in the last equation we get

$$k = 2l - 1.$$

Note that $l \geqslant 2$ by Theorem 1.2, it follows that $k \geqslant 3$.

From the expression of $d(r_0, 0)$ and the Weierstrass Preparation Theorem we get that $d(r_0, \varepsilon)$ has at most $2l - 1$ zeros. Since system (3.4) is invariant under the symmetric change of variables $(\theta, r, s) \to (\theta + \pi, -r, -s)$, and $r_0 = 0$ is always a solution of $d(r_0, \varepsilon) = 0$, these verify that $d(r_0, \varepsilon) = 0$ has at most $l - 1$ positive roots.

We note that the 2π periodic solutions of (3.6) one to one correspond to periodic orbits of (1.5) in a neighborhood of the origin. While each 2π periodic solution of (3.6) in a neighborhood of the origin is uniquely determined by a positive zero of $d(r_0, \varepsilon)$. So system (1.5) has at most $l - 1$ small amplitude limit cycles which are bifurcated from the Hopf on the two dimensional center manifold.

Finally we provide an example showing that there exist systems of form (1.5) which do have $l - 1$ limit cycles under sufficient small perturbation. Consider a special perturbation to system (3.3)

$$\begin{aligned}
\dot{u} &= -v + g_1(u, v, w) + u h(u, v, \varepsilon), \\
\dot{v} &= u + g_2(u, v, w) + v h(u, v, \varepsilon), \\
\dot{w}_j &= w_j\big(\lambda_j + g_j(u, v)\big) + w_j h(u, v, \varepsilon), \quad j = 3, \dots, n
\end{aligned} \tag{3.16}$$

with $h(u, v, \varepsilon) = \sum_{s=1}^{l-1} \varepsilon^{l-s} a_s (u^2 + v^2)^s$ and ε a single parameter. Recall that if λ_j is complex with nonvanishing imaginary part, it must have a conjugate one, saying λ_{j+1}, then the variables w_j and w_{j+1} are conjugate complex ones. Write system (3.16) in cylindrical coordinates (θ, r, s), we get a system as in the form (3.4) with $\Theta(\theta, r, s, \varepsilon)$ and $S(\theta, r, s, \varepsilon)$ independent of ε, and

$R(\theta, r, s, \varepsilon) = R(\theta, r, s, 0) + \sum_{s=1}^{l-1} \varepsilon^{l-s} a_s r^{2s+1}$. Then similar to [5,10] we get that for $|\varepsilon| \ll 1$ and suitable choices of $a_1, \ldots, a_{l-1}$ system (3.16) can have $l-1$ small amplitude limit cycles in a neighborhood of the origin.

We complete the proof of the theorem. □

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