Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 4055–4072

Journal of Differential Equations

www.elsevier.com/locate/jde

# Splitting scheme for the stability of strong shock profile

Linglong Du

*Department of Applied Mathematics, Donghua University, Shanghai, China*

Received 20 January 2016; revised 10 June 2016

**Abstract**

In this paper, we use the Laplace transform and Dirichlet–Neumann map to give a systematical scheme to study the small wave perturbation of general shock profile with general amplitude. Here we use certain non-classical shock waves for a rotationally invariant system of viscous conservation laws to demonstrate this scheme. We derive an explicit solution and show that it converges pointwise to another over-compressive profile exponentially, when the perturbations of the initial data to a given over-compressive shock profile are sufficiently small.


*MSC:* 35L65; 35L67

*Keywords:* Splitting scheme; Strong shock profile; Dirichlet–Neumann map; Conservation law; Nonlinear stability

## 1. Introduction

Consider the following simple rotationally invariant system originated from the study of MHD and nonlinear elasticity by Freistühler [3],

$$\begin{cases} \tilde{u}_t + (\tilde{u}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{u}_{xx}, \\ \tilde{v}_t + (\tilde{v}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{v}_{xx}. \end{cases} \tag{1.1}$$

The characteristics are

*E-mail address:* matdl@dhu.edu.cn.

http://dx.doi.org/10.1016/j.jde.2016.06.018

$$r_1(\tilde{u},\tilde{v})=(\tilde{v},-\tilde{u}),\quad r_2(\tilde{u},\tilde{v})=(\tilde{u},\tilde{v}),\quad \lambda_1(\tilde{u},\tilde{v})=\tilde{u}^2+\tilde{v}^2,\quad \lambda_2(\tilde{u},\tilde{v})=3\lambda_1(\tilde{u},\tilde{v}). \tag{1.2}$$

The 1-characteristic is linearly degenerate and the 2-characteristic is genuinely nonlinear except at the origin

$$\nabla\lambda_1\cdot r_1(\tilde{u},\tilde{v})=0,\quad \nabla\lambda_2\cdot r_2(\tilde{u},\tilde{v})=6(\tilde{u}^2+\tilde{v}^2). \tag{1.3}$$

A viscous shock wave has end states along the same radial direction through the origin, i.e., in the direction of $r_2(\tilde{u},\tilde{v})$. The system is rotational invariant and so, without loss of generality, consider the end states to have the second component zero $(\tilde{u}_\pm,0)$. When $\tilde{u}_\pm$ are of the same sign, the shock is classical, and there is only one connecting orbit along the $\tilde{u}_-$ axis. When $\tilde{u}_+<0$, the shock may cross the non-strictly hyperbolic point $(\tilde{u},\tilde{v})=0$ and becomes over-compressive.

Here we are interested in over-compressive shock, which is characterized by

$$\lambda_2(\tilde{u}_-,\tilde{v}_-)>\lambda_1(\tilde{u}_-,\tilde{v}_-)>b>\lambda_2(\tilde{u}_+,\tilde{v}_+)>\lambda_1(\tilde{u}_+,\tilde{v}_+), \tag{1.4}$$

$b$ is the speed of over-compressive shock. (1.4) implies that the over-compressive shock is a node–node connection. Thus when exists, there is a 1-parameter family of viscous profiles. More details about the over-compressive shock will be given in the Section 2.

The goal of this paper is to prove that the over-compressive shock profiles for (1.1) can be stable against small perturbations using our new approach: given the profile $\Phi=(\phi,\psi)$ of an (appropriate) over-compressive shock profile, and a function $(\tilde{u}_0(x),\tilde{v}_0(x))$ (of appropriate type) such that

$$\begin{pmatrix} u_0(x)\\ v_0(x)\end{pmatrix}\equiv\begin{pmatrix}\tilde{u}_0(x)\\ \tilde{v}_0(x)\end{pmatrix}-\Phi \tag{1.5}$$

is small (in an appropriate sense), then the solution $(\tilde{u},\tilde{v})$ of (1.1) with initial data $(\tilde{u}_0(x),\tilde{v}_0(x))$ converges time-asymptotically to another profile $\Phi^*=(\phi^*,\psi^*)$. We give the pointwise convergence rate to the new profile.

Since there is a 1-parameter family of viscous shocks with given end states, the stability of the shock profile is to be understood in the following way: the perturbation of a stable shock profile would converge to the same or another profile in the 1-parameter family. Thus, in addition to the phase shift, the perturbation also changes the time-asymptotic profile of the solution. Therefore, instead of using the conservation laws to identify the phase shift and diffusion waves as for the Laxian shocks, we use the two conservation laws to identify the phase shift and the new profile. This should make the situation well-posed as we have two conservation laws and the same number of parameters to determine. Here and thereafter, we set $\mu=1$ without loss of generality.

The main theorem is given as follows:

**Theorem 1.1** *(Main Theorem). Give an over-compressive shock profile $\Phi=(\phi,0)$ with the shock speed $b$. Let $C$ be a universal positive constant, and let the initial perturbation defined by (1.5) satisfy $|u_0(x),v_0(x)|\le O(1)\varepsilon e^{-|x|/C}$. Then for $\varepsilon$ sufficiently small, there exists a unique profile $\Phi^*$ with $\Phi(\pm\infty)=\Phi^*(\pm\infty)$, such that the solution of (1.1) satisfies:*

$$\left|\begin{Bmatrix} \tilde{u}(x,t) \\ \tilde{v}(x,t) \end{Bmatrix} - \Phi^*(x-bt-x_0)\right| \le O(1)\varepsilon e^{-(|x-bt|+t)/C}.$$

*Here the constant $x_0$ and profile $\Phi^*$ are determined by*

$$\int_R (u_0, v_0)(x)dx = \int_R ((\phi^*, \psi^*)(x-bt-x_0) - (\phi,\psi)(x-bt))dx.$$

The stability of viscous shock profile has been studied widely, as an issue of physical importance and interest, for example [3,4,9–11,15,17]. However, many existing stability results for systems depend much on the assumptions of weak shock strength, see [11,13,16]. The purpose of this paper is to present a unified approach of studying small wave perturbation around the general shock profile with general amplitude. Our method involves an interesting blend of the Laplace transformation and Dirichlet–Neumann map introduced by Liu and Yu [12], described in Section 2. Although the shock we considered is over-compressive (no transversal waves in the solution) and the technical difficulty in our model is similar to those in [6,7,15], one could apply this scheme to other type of shocks because it doesn't require much restrictive assumptions. The advantage of our splitting scheme is that it can give the sharp pointwise analysis of each part by separating the hyperbolic wave structure in the far fields and local structure around shock front. We will use this method to study the Lax shock wave of discrete Boltzmann equation in the near future. For another interesting functional analytic approach to the stability of general viscous shock waves, one could see series of results by Zumbrun and various collaborators, e.g. [14,17]. That method is based on the framework of the Evans-function criterion method including Gap Lemma [5], functional analysis and semigroup method, generalizing the earlier work by [15].

Now we outline the ingredients of splitting scheme to elaborate the ideas of our new approach. It is used to handle the linearized system around the general amplitude profile $\phi$:

$$\begin{cases} u_t + (f(\phi)u)_x = u_{xx}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.6}$$

with

$$\begin{cases} \phi(\pm\infty) = M_\pm, \\ |\phi'(x)| = O(1)e^{-|x|/C}, \\ |u_0(x)| = O(1)\varepsilon e^{-|x|/C}. \end{cases}$$

The scheme is carried out in three steps and we study the structure of the linear wave propagation around the profile.

In the first step, we obtain a non-decaying structure which is caused by initial data through the standard procedure. This is done in Section 3.1 with approximate manipulations. The non-decaying component is defined as $\beta(t)\Psi(x)$, has the property that

$$\begin{cases} |\beta'(t)| = O(1)\varepsilon \frac{e^{-C_1 t}}{\sqrt{t+1}}, \\ |\Psi(x)| = O(1)e^{-|x|/C}. \end{cases}$$

$\Psi(x)$ is the stationary solution of (1.6) with $\Psi(\pm\infty) = 0$. With this observation, we extract the non-decaying part precisely. Otherwise, one would fail to get the nonlinear stability.

The remainder $z(x,t) \equiv u(x,t) - \beta(t)\Psi(x)$ satisfies the following

$$\begin{cases} z_t + (f(\phi)z)_x = z_{xx} + S(x,t), \\ |z(x,0)| = 0, |S(x,t)| = O(1)\varepsilon e^{-(|x|+t)/C}. \end{cases} \tag{1.7}$$

In the second step, we construct a new function to approximate the remainder, which satisfies a modified error equation:

$$\begin{cases} r_t + (f(\phi_L)r)_x = r_{xx} + S(x,t), \\ r(x,0) = z(x,0), \end{cases} \tag{1.8}$$

where

$$\phi_L(x) = \begin{cases} \phi(\infty) & \text{if } x \geq L, \\ \phi(x) & \text{if } |x| < L, \\ \phi(-\infty) & \text{if } x \leq -L, \end{cases} \tag{1.9}$$

choosing $L$ sufficiently large, $L = O(1)|\ln\varepsilon|$. Here we only modify the values of the shock profile at far fields. Due to this modification, one can split the whole space domain into three parts: the far field $(-\infty, -L]$, $[L, \infty)$, and finite domain region $[-L, L]$. This is similar to the work by Kreiss [6,7]. Convergence rate of $r(x,t)$ at each boundary can be obtained through the wave interactions in these three domains.

When $|x| > L$, (1.8) turns to be a constant coefficient problem. The solvability condition for the constant coefficient problem in each far field gives a Dirichlet–Neumann map:

$$\mathbb{L}_s[r]_x(-L-, s) = \mathbf{D}_1\mathbb{L}_s[r](-L-, s) + I_1(-L-, s), \tag{1.10}$$

$$\mathbb{L}_s[r]_x(L+, s) = \mathbf{D}_2\mathbb{L}_s[r](L+, s) + I_2(L+, s). \tag{1.11}$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are Dirichlet–Neumann kernels defined by (2.5), $I_1$ and $I_2$ are terms due to initial datum and source term $S(x,t)$.

In the finite domain, we introduce a transition function $G(x,s)$ in the Laplace space which satisfies:

$$\begin{cases} sG - f(\phi_L)G_x - G_{xx} = 0, \\ G(-L,s) = 1, \\ G_x(-L,s) = -\mathbf{D}_1 - f(\phi_L)(-\infty). \end{cases} \tag{1.12}$$

Applying the Laplace transformation to the first equation in (1.8) with respect to time variable $t$, we multiply the transformed equation with $G(x,s)$, integrate in the domain $[-L, L]$,

$$\int_{-L}^{L} G(x,s)\,(s\mathbb{L}_s[r] + (f(\phi_L)\mathbb{L}_s[r])_x - \mathbb{L}_s[r]_{xx} - z(x,0) - \mathbb{L}_s[S](x,s))\,dx = 0.$$

Integrating by parts, with the help of $C^1$ continuity of $r$ at $x = -L$ and the special boundary datum in (1.12), we have

$$\begin{aligned} & I_1(-L,s) + \int_{-L}^{L} G(x,s)\mathbb{L}_s[S](x,s)dx \\ & = (Gf(\phi_L)\mathbb{L}_s[r] - G\mathbb{L}_s[r]_x + G_x\mathbb{L}_s[r])|_L, \end{aligned} \tag{1.13}$$

which gives another relationship for $\mathbb{L}_s[r]_x(L,s)$ and $\mathbb{L}_s[r](L,s)$.

Combining (1.11), (1.13) and $C^1$ continuity of $r$ at $x = L$ together, we identify $\mathbb{L}_s[r]_x(L,s)$ and $\mathbb{L}_s[r](L,s)$. Similarly one can get the boundary information at $x = -L$.

Once the boundary information is clear, one could solve the problem (1.8) in each domain respectively. In the far fields, we only need to consider the constant-coefficient initial boundary problem. The behavior of solution in the finite domain can be obtained through the standard PDE method.

The truncation error of the above approximation $e(x,t) \equiv z(x,t) - r(x,t)$ satisfies:

$$\begin{cases} e_t + (f(\phi)e)_x = e_{xx} + ((f(\phi_L) - f(\phi))r)_x, \\ e(x,0) = 0. \end{cases} \tag{1.14}$$

This is similar to system (1.7) and hence we can construct it's solution by iteration.

In the third step, we introduce an iterative scheme:

$$\begin{cases} r^0 = r, \\ e^0 = z, \\ e^k = e^{k-1} - r^{k-1}, k \geq 1. \end{cases}$$

$r^k$ is the approximate solution of $e^k$. The smallness and pointwise localization property of $r^k$ will assure that the series of errors $\sum_{j=0}^{\infty} r^j$ obtained in each iterative step converges and establish the wave propagation patterns for system (1.7).

The plan of this paper is the following: some preliminary preparations are given in Section 2; in Section 3, we apply the splitting scheme to solve the linear decoupled system and give the pointwise estimate of the Green's function; the nonlinear stability of perturbation for the general shock profile is proved in Section 4.

## 2. Preliminaries

### 2.1. *Profiles of over-compressive shocks*

Profile $(\phi, \psi)$ of any viscous shock wave is a heteroclinic orbit of the O.D.E. system

$$\begin{aligned} \phi' &= (\phi^2 + \psi^2 - b)\phi - b_1, \\ \psi' &= (\phi^2 + \psi^2 - b)\psi - b_2, \end{aligned} \tag{2.1}$$

the speed $b$ and the relative flux $b_1, b_2$ are given by the Rankine–Hugoniot conditions:

$$(\tilde{u}_+^2 + \tilde{v}_+^2 - b)\tilde{u}_+ = (\tilde{u}_-^2 + \tilde{v}_-^2 - b)\tilde{u}_- = b_1,$$
$$(\tilde{u}_+^2 + \tilde{v}_+^2 - b)\tilde{v}_+ = (\tilde{u}_-^2 + \tilde{v}_-^2 - b)\tilde{v}_- = b_2, \tag{2.2}$$

with $\tilde{u}_\pm = \phi(\pm\infty)$, $\tilde{v}_\pm = \psi(\pm\infty)$.

Since the system (1.1) has a non-strictly hyperbolic point $(\tilde{u}, \tilde{v}) = 0$, there may exist shock waves which are more or less compressive than the classical Lax shocks. These are called the overcompressive or undercompressive shocks, that have distinct behaviors. When the shock is classical, there are three characteristics impinging on the shock and one leaving the shock. The shock is called overcompressive if more than three characteristics are impinging on the shock i.e. (1.4), and undercompressive if less than three characteristics are impinging on the shock. Here we focus attention on the case of overcompressive shocks. We quote a lemma in [3] which characterizes the overcompressive shock waves.

**Lemma 2.1.**

(i) $\tilde{u}_\pm$, $\tilde{v}_\pm$ *satisfy the inequality* (1.4) *with* $b_1$ *and* $b_2$ *from* (2.2) *if and only if* $(\tilde{u}_-, \tilde{v}_-) \neq (0, 0)$ *and* $(\tilde{u}_+, \tilde{v}_+) = \alpha(\tilde{u}_-, \tilde{v}_-)$, $\alpha \in (-\frac{1}{2}, 0)$.
(ii) *In this case, there is a 1-parameter family of viscous profiles that satisfy* (2.1) *and* (2.2).

*2.2. Dirichlet–Neumann map*

We list the procedure of constructing the Dirichlet–Neumann map we will need in the following sections, please refer to [12] for more details.

**Definition 2.2.** For a function $y(t)$ defined for $t \geq 0$, its Laplace transform and inverse Laplace transform are defined as follows:

$$Y(s) = \mathbb{L}[y](s) \equiv \int_0^\infty e^{-st} y(t)dt, \quad \text{for } s \in \{z \in \mathbb{C} | Re(s) \geq 0\};$$

$$y(t) = \mathbb{L}^{-1}[Y](t) \equiv \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} Y(s)ds,$$

$$Re(s) = \gamma, \quad \text{s.t. } \gamma \text{ is greater than the real part of all singularities of } Y(s).$$

Consider the following half space problem:

$$\begin{cases} V_t + AV_x = V_{xx}, & \text{for } x > 0, \quad t > 0, \\ V(0,t) = V_b(0,t), \\ V(x,0) = V_0(x), \end{cases} \tag{2.3}$$

$A$ is a constant. A functional property on given boundary datum $V_b$ is imposed for the consistency with zero initial datum case and for an application condition for the Bromwich integral:

$$V_b \in \{g | \mathbb{L}[g](s) \text{ exists for } Re(s) > 0, g^{[n]}(0) = 0 \text{ for } n \in \mathbb{N} \cup \{0\}\}.$$

For convenience, we define $\mathbb{L}_s$ as the Laplace transform with respect to time variable $t$, $\mathbb{L}_\xi$ as the Laplace transform with respect to space variable $x$, $\mathbb{L}^{-1}$ as the inverse Laplace transform.

Case 1: $V_0(x) = 0$.

Applying the Laplace–Laplace transform to the equation of $V$ in (2.3), one has:

$$(s + A\xi - \xi^2)\mathbb{J}[V] = (A - \xi)\mathbb{L}_s[V_b](0, s) - \partial_x \mathbb{L}_s[V](0, s),$$

$\mathbb{J}[V] \equiv \mathbb{L}_\xi[\mathbb{L}_s[V]]$.

By purely algebraic manipulations one obtains the solution in terms of variables $\xi$, $s$, $\mathbb{L}_s[V_b]$ and $\partial_x \mathbb{L}_s[V]$:

$$\begin{aligned}\mathbb{J}[V] = &\frac{-\partial_x \mathbb{L}_s[V](0, s) + \frac{1}{2}\mathbb{L}_s[V_b](0, s)(A - \sqrt{A^2 + 4s})}{\sqrt{A^2 + 4s}(\xi - \frac{1}{2}(A + \sqrt{A^2 + 4s}))} \\ &+ \frac{-\partial_x \mathbb{L}_s[V](0, s) + \frac{1}{2}\mathbb{L}_s[V_b](0, s)(A + \sqrt{A^2 + 4s})}{\sqrt{A^2 + 4s}(\xi - \frac{1}{2}(A - \sqrt{A^2 + 4s}))}.\end{aligned} \tag{2.4}$$

Note that the characteristic polynomial $s + A\xi - \xi^2$ has only one positive root $\xi = \frac{1}{2}(A + \sqrt{A^2 + 4s})$ for $s > 0$, the boundedness of the solution $V$ requires that $|V| < \infty$ as $x \to \infty$. This results in:

$$\frac{-\partial_x \mathbb{L}_s[V](0, s) + \frac{1}{2}\mathbb{L}_s[V_b](0, s)(A - \sqrt{A^2 + 4s})}{\sqrt{A^2 + 4s}} = 0,$$

which gives the Dirichlet–Neumann map:

$$\partial_x \mathbb{L}_s[V](0, s) = -\frac{2s}{A + \sqrt{A^2 + 4s}}\mathbb{L}_s[V_b](0, s) \equiv \mathbf{D}\mathbb{L}_s[V_b](0, s). \tag{2.5}$$

Case 2: $V_0(x) \neq 0$.

One needs to shift the initial datum to the boundary by introducing:

$$\begin{cases} \bar{V} = V - I(x, t), \\ I(x, t) = \int_0^\infty \mathbb{G}^A(x - y, t)V_0(y)dy. \end{cases} \tag{2.6}$$

$\mathbb{G}^A$ is the fundamental solution for

$$\begin{cases} {\mathbb{G}^A}_t + A{\mathbb{G}^A}_x = {\mathbb{G}^A}_{xx}, \quad \text{for } x \in R, \quad t > 0, \\ \mathbb{G}^A(x, 0) = \delta(x). \end{cases} \tag{2.7}$$

Therefore, $\bar{V}$ has the initial value and satisfies:

$$\begin{cases} \bar{V}_t + A\bar{V}_x = \bar{V}_{xx}, \quad \text{for } x > 0, \quad t > 0, \\ \bar{V}(0, t) = V_b(0, t) - I(0, t), \\ \bar{V}(x, 0) = 0, \end{cases} \tag{2.8}$$

which is the Case 1.

## 3. Pointwise estimate of solution for the linearized rotationally invariant system

Linearizing system (1.1) around the given profile $\Phi = (\phi, 0)$ gives the decoupled system:

$$\begin{cases} u_t + ((3\phi^2 - b)u)_{\bar{x}} = u_{\bar{x}\bar{x}}, \\ v_t + ((\phi^2 - b)v)_{\bar{x}} = v_{\bar{x}\bar{x}}, \\ u_0(\bar{x}), v_0(\bar{x}) = O(1)\varepsilon e^{-|\bar{x}|/C}, \end{cases} \tag{3.1}$$

here we have changed the coordinate system to make the presentation of this problem convenient:

$$\begin{cases} \bar{x} = x - bt, \\ t = t. \end{cases}$$

Set $f_1(\phi) \equiv 3\phi^2 - b$, $f_2(\phi) \equiv \phi^2 - b$. Without loss of generality, we assume the end states $(\tilde{u}_-, \tilde{u}_+) = (1, \alpha)$ because of Lemma 2.1.

### 3.1. *Extract the non-decaying structure*

Before extracting the non-decaying structure, we lay down a standard procedure to catch the non-decaying structure and give the pointwise estimates.

Approximate the over-compressive shock profile with a discontinuous function, which takes the form:

$$H(\bar{x}, t) = \begin{cases} \phi(-\infty) = 1 & \bar{x} < 0, \\ \phi(\infty) = \alpha & \bar{x} > 0. \end{cases}$$

Consider the approximate linearized equations:

$$\begin{cases} \bar{u}_t + (f_1(H)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \\ \bar{v}_t + (f_2(H)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-|\bar{x}|/C}. \end{cases} \tag{3.2}$$

Due to the discontinuous property of $H(\bar{x}, t)$, we separate the whole space into two half-spaces:

$$\begin{cases} \bar{u}_t + ((3 - b)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \quad \text{for } \bar{x} < 0, \quad t > 0, \\ \bar{v}_t + ((1 - b)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \quad \text{for } \bar{x} < 0, \quad t > 0, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-|\bar{x}|/C}, \end{cases} \tag{3.3}$$

$$\begin{cases} \bar{u}_t + ((3\alpha^2 - b)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \quad \text{for } \bar{x} > 0, \quad t > 0, \\ \bar{v}_t + ((\alpha^2 - b)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \quad \text{for } \bar{x} > 0, \quad t > 0, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-|\bar{x}|/C}. \end{cases} \tag{3.4}$$

The boundary data $\bar{u}(0\pm, t)$, $\bar{v}(0\pm, t)$ will be determined by $C^1$ continuity of $(\bar{u}, \bar{v})$ and two Dirichlet–Neumann maps.

We now apply the procedure given in the preliminary section to obtain the Dirichlet–Neumann maps in the transformed variables. For $\bar{u}$, the Dirichlet–Neumann maps with the zero initial datum in each half space are:

$$\begin{cases} \mathbb{L}_s[\bar{u}]_{\bar{x}}(0-,s) = \frac{3-b+\sqrt{(3-b)^2+4s}}{2}\mathbb{L}_s[\bar{u}](0-,s) \equiv \mathbf{D}_1\mathbb{L}_s[\bar{u}](0-,s), \\ \mathbb{L}_s[\bar{u}]_{\bar{x}}(0+,s) = \frac{3\alpha^2-b-\sqrt{(3\alpha^2-b)^2+4s}}{2}\mathbb{L}_s[\bar{u}](0+,s) \equiv \mathbf{D}_2\mathbb{L}_s[\bar{u}](0+,s). \end{cases} \tag{3.5}$$

Combining (3.5) with $C^1$ continuity of $\bar{u}$, taking the initial datum $\bar{u}_0(\bar{x})$ into account, we have:

$$\begin{cases} \mathbb{L}_s[\bar{u}](0-,s) = \mathbb{L}_s[\bar{u}](0+,s), \\ (3-b)\mathbb{L}_s[\bar{u}](0-,s) - \mathbb{L}_s[\bar{u}]_{\bar{x}}(0-,s) = (3\alpha^2-b)\mathbb{L}_s[\bar{u}](0+,s) - \mathbb{L}_s[\bar{u}]_{\bar{x}}(0+,s), \\ \mathbb{L}_s[\bar{u}]_{\bar{x}}(0-,s) - \mathbb{L}_s[I]_{\bar{x}}(0-,s) = \mathbf{D}_1(\mathbb{L}_s[\bar{u}](0-,s) - \mathbb{L}_s[I](0-,s)), \\ \mathbb{L}_s[\bar{u}]_{\bar{x}}(0+,s) - \mathbb{L}_s[I]_{\bar{x}}(0+,s) = \mathbf{D}_2(\mathbb{L}_s[\bar{u}](0+,s) - \mathbb{L}_s[I](0+,s)), \end{cases} \tag{3.6}$$

where $I(0-,t)$ and $I(0+,t)$ are given by (2.6):

$$I(0-,t) = \int_0^\infty \mathbb{G}^{3-b}(\bar{x}-y,t)\bar{u}_0(y)dy|_{\bar{x}=0},$$

$$I(0+,t) = \int_0^\infty \mathbb{G}^{3\alpha^2-b}(\bar{x}-y,t)\bar{u}_0(y)dy|_{\bar{x}=0}.$$

Solving (3.6), we have

$$\begin{aligned} &(3-3\alpha^2-\mathbf{D}_1+\mathbf{D}_2)\mathbb{L}_s[\bar{u}](0-,s) \\ &= -\mathbf{D}_1\mathbb{L}_s[I](0-,s) + \mathbf{D}_2\mathbb{L}_s[I](0+,s) + \mathbb{L}_s[I]_{\bar{x}}(0-,s) - \mathbb{L}_s[I]_{\bar{x}}(0+,s) \\ &\equiv \mathbb{L}_s[I_0](s) = O(1)\varepsilon. \end{aligned}$$

The operator which converts $\mathbb{L}_s[\bar{u}](0-,s)$ into $\mathbb{L}_s[I_0](s)$ has the following expression:

$$\begin{aligned} &3-3\alpha^2-\mathbf{D}_1+\mathbf{D}_2 \\ &= \frac{3-3\alpha^2-\sqrt{(3-b)^2+4s}-\sqrt{(3\alpha^2-b)^2+4s}}{2} \\ &= 2s\left(\frac{1}{3\alpha^2-b-\sqrt{(3\alpha^2-b)^2+4s}} - \frac{1}{3-b+\sqrt{(3-b)^2+4s}}\right). \end{aligned}$$

Hence,

$$\mathbb{L}_s[\bar{u}](0-,s) = O(1)\frac{C_1+\sqrt{s+C_1}}{s}\mathbb{L}_s[I_0](s),$$

$C_1$ is a positive constant.

Taking the inverse Laplace transformation, we have the following estimates:

$$|\bar{u}(0\pm,t)| = \left| O(1)\int_0^t I_0(\tau)d\tau + O(1)\int_0^t \frac{e^{-C_1\tau}}{2\sqrt{\pi\tau}} * (\partial_\tau + C_1)I_0(\tau)d\tau \right|$$

$$\leq O(1)\varepsilon \int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau. \tag{3.7}$$

Here and in the following, instead of studying the inverse Laplace transform of type $\sqrt{s+C}$, we consider $\frac{\sqrt{s+C}}{s+C} = \frac{1}{\sqrt{s+C}}$ by the usual inversion formula of the Laplace transform:

$$L_t^{-1}(\frac{1}{\sqrt{s+C}}) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{1}{\sqrt{s+C}}ds = \frac{e^{-Ct}}{\sqrt{\pi t}}. \tag{3.8}$$

The inversion of the Laplace transform has the property that division in $s$ corresponds to differentiation in $t$.

Similarly, one could obtain the same estimate for $\bar{v}(0\pm,t)$. For large times, the boundary information obtained in (3.7) implies that there is a non-decaying component caused by initial data.

Moreover, if extracting the boundary information, one can get the pointwise estimate of (3.2) with the help of Green's function for the two half-space problem:

**Lemma 3.1.**

$$\left|(\bar{u}(\bar{x},t), \bar{v}(\bar{x},t)) - diag(\bar{u}(0\pm,t), \bar{v}(0\pm,t))\bar{\Psi}(\bar{x})\right| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C},$$

*$\bar{\Psi}$ is the stationary solution of (3.2) with $\bar{\Psi}(\pm\infty) = 0$. One has the following estimates: $|\bar{\Psi}(\bar{x})| \leq O(1)e^{-\frac{|\bar{x}|}{C}}$, $|diag(\bar{u}(0\pm,t), \bar{v}(0\pm,t))| \leq O(1)\varepsilon\int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau$.*

*For large times, $\int_R diag(\bar{u}(0\pm,t), \bar{v}(0\pm,t))\bar{\Psi}(\bar{x})d\bar{x} = \int_R (\bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}))d\bar{x}$ due to conservation laws.*

*If the initial data $\int_R (\bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}))d\bar{x} = 0$, one has*

$$|(\bar{u}(\bar{x},t), \bar{v}(\bar{x},t))| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C}.$$

**Proof.** The way of obtaining the estimates is similar to that in [1,2], we just take the equation of $\bar{u}(\bar{x},t)$ in system (3.3) as an example and outline the main ingredients of proof here.

One shifts the initial datum to the boundary by

$$w(\bar{x},t) = \bar{u}(\bar{x},t) - \int_0^\infty \mathbb{G}^{3-b}(\bar{x}-y,t)\bar{u}_0(y)dy.$$

Denote the Laplace–Laplace transform of $w(\bar{x},t)$ as $\mathbb{J}[w](\xi,s)$, it is the solution of

$$\begin{cases} (s+(3-b)\xi-\xi^2)\mathbb{J}[w]=(3-b-\xi)\mathbb{L}_s[w](0,s)-\partial_x\mathbb{L}_s[w](0,s), \\ \mathbb{L}_s[w](0,s)=O(1)\varepsilon(\frac{C_1+\sqrt{s+C_1}}{s}-\frac{1}{\sqrt{4s+(3-b)^2}}). \end{cases}$$

Then $\mathbb{J}[w](\xi,s)=O(1)\varepsilon(\frac{C_1+\sqrt{s+C_1}}{s}-\frac{1}{\sqrt{4s+(3-b)^2}})\frac{1}{\xi-\frac{1}{2}((3-b)-\sqrt{(3-b)^2+4s})}$.

The explicit expression of $\mathbb{L}_s[w](\bar{x},s)$ is

$$\begin{aligned} \mathbb{L}_s[w](\bar{x},s) &= O(1)\varepsilon(\frac{C_1+\sqrt{s+C_1}}{s}-\frac{1}{\sqrt{4s+(3-b)^2}})e^{\frac{1}{2}\bar{x}((3-b)-\sqrt{(3-b)^2+4s})} \\ &= O(1)\varepsilon(\frac{C_1+\sqrt{s+C_1}}{s}-\frac{1}{\sqrt{4s+(3-b)^2}})\sqrt{4s+(3-b)^2} \\ &\quad\times\frac{e^{\frac{1}{2}\bar{x}((3-b)-\sqrt{(3-b)^2+4s})}}{\sqrt{4s+(3-b)^2}}. \end{aligned}$$

Since $\bar{x}<0$ and

$$\mathbb{L}_t^{-1}[\sqrt{4s+(3-b)^2}]=(4\partial_t+(3-b)^2\delta(t))*\frac{e^{\frac{-(3-b)^2t}{4}}}{\sqrt{4\pi t}},$$

we have

$$|\mathbb{L}_t^{-1}[s\mathbb{L}_s[w]](\bar{x},t)|=O(1)\varepsilon\frac{e^{-\frac{(\bar{x}-(3-b)t)^2}{4t}}}{\sqrt{4\pi t}}=O(1)\varepsilon e^{-|\bar{x}|/C-t/C},$$

$C$ is a universal constant.

Thus

$$|w(\bar{x},t)|=O(1)\varepsilon e^{-|\bar{x}|/C}.$$

This verifies the non-decaying part of Lemma 3.1. The rest of the proof is quite standard and we won't repeat the analysis here.

With the above observation, we introduce a non-decaying component stacked in the shock region: $\beta(t)\Psi(\bar{x})$, $|\beta(t)|\le O(1)\varepsilon\int_0^t\frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau$, $|\Psi(\bar{x})|\le O(1)e^{-\frac{|\bar{x}|}{C}}$, $\Psi(\bar{x})\equiv(\psi_1,\psi_2)$ is a stationary solution of (3.1) with $\Psi(\pm\infty)=0$. Extracting it from (3.1), setting

$$(z_1,z_2)\equiv(u,v)-\beta(t)\Psi,$$

we have

$$\begin{cases} z_{1t}+(f_1(\phi)z_1)_{\bar{x}}=z_{1\bar{x}\bar{x}}+\beta'(t)\psi_1, \\ z_{2t}+(f_2(\phi)z_2)_{\bar{x}}=z_{2\bar{x}\bar{x}}+\beta'(t)\psi_2, \qquad \square \\ z_i(\bar{x},0)=0, i=1,2. \end{cases} \tag{3.9}$$

*3.2. Pointwise estimate of the approximate problem*

The approximate problem for (3.9) is given as follows:

$$\begin{cases} r_{1t} + (f_1(\phi_L)r_1)_{\bar{x}} = r_{1\bar{x}\bar{x}} + \beta'(t)\psi_1, \\ r_{2t} + (f_2(\phi_L)r_2)_{\bar{x}} = r_{2\bar{x}\bar{x}} + \beta'(t)\psi_2, \\ r_i(\bar{x}, 0) = 0, i = 1, 2, \end{cases} \tag{3.10}$$

$\phi_L$ is defined by (1.9).

Following the framework, we split the whole space domain into three parts. The solvability of problem in the left and right far fields gives two Dirichlet–Neumann maps:

$$\mathbb{L}_s[r_1]_x(-L-, s) = \mathbf{D}_1\mathbb{L}_s[r_1](-L-, s) + I_1(-L-, s), \tag{3.11}$$

$$\mathbb{L}_s[r_1]_x(L+, s) = \mathbf{D}_2\mathbb{L}_s[r_1](L+, s) + I_2(L+, s), \tag{3.12}$$

where

$$I_1(-L-, s) = \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2} + C)},$$

$$I_2(L+, s) = \frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2} + C)},$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are defined by (3.5).

Applying the Laplace transformation to the equation of $r_1$ with respect to time variable $t$, we multiply the transformed equation with $G$ defined by (1.12), integrate in the domain $[-L, L]$ by parts, with the help of $C^1$ continuity of $r_1$ at $\bar{x} = -L$, we get

$$\begin{aligned} &\int_{-L}^{L} G(\bar{x}, s)\mathbb{L}_s[S](\bar{x}, s)d\bar{x} + \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2} + C)} \\ &= (Gf_1(\phi_L)\mathbb{L}_s[r_1] - G\mathbb{L}_s[r_1]_{\bar{x}} + G_{\bar{x}}\mathbb{L}_s[r_1])|_{L-}, \end{aligned} \tag{3.13}$$

which is the second relationship for Dirichlet and Neumann data at $\bar{x} = L$. Here $S(\bar{x}, t) \equiv \beta'(t)\psi_1$.

By (3.12), (3.13) and the $C^1$ continuity of $r_1$ at $\bar{x} = L$, one could solve:

$$\begin{aligned} \mathbb{L}_s[r_1](L-, s) = &\frac{\frac{\int_{-L}^{L} G\mathbb{L}_s[S]d\bar{x}}{G(L-,s)} + \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)G(L-,s)}}{f_1(\phi_L(\infty)) + \mathbf{D}_2 + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)}} \\ &+ \frac{\frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2}+C)}}{f_1(\phi_L(\infty)) + \mathbf{D}_2 + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)}} \end{aligned}$$

$$= \frac{\frac{\int_{-L}^{L} G\mathbb{L}_s[S]d\bar{x}}{G(L-,s)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}} + \frac{\frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)G(L-,s)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}$$
$$+ \frac{\frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2}+C)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}. \tag{3.14}$$

To invert $\mathbb{L}_s[r_1](L-,s)$, we first give some observation on the transition function $G(\bar{x},s)$ which satisfies:

$$\begin{cases} sG - f_1(\phi_L)G_{\bar{x}} - G_{\bar{x}\bar{x}} = 0, \\ G(-L,s) = 1, \\ G_{\bar{x}}(-L,s) = -\mathbf{D}_1 - f_1(\phi_L)(-\infty) = -\frac{3(3-b)+\sqrt{(3-b)^2+4s}}{2} < 0. \end{cases} \tag{3.15}$$

First, $\forall s$, $G(L-,s) \neq 0$. Set $P(\bar{x},s) = -\frac{G_{\bar{x}}(\bar{x},s)}{G(\bar{x},s)}$, then $P$ satisfies the following ODE problem:

$$\begin{cases} P_{\bar{x}} = P^2 - f_1(\phi_L)P - s, \\ P(-L,s) = \frac{3(3-b)+\sqrt{(3-b)^2+4s}}{2}. \end{cases}$$

By the monotone property, we have

$$P(L,s) > \frac{f_1(\phi_L(+\infty)) - \sqrt{f_1{}^2(\phi_L(+\infty)) + 4s}}{2} = \frac{(3\alpha^2-b) - \sqrt{(3\alpha^2-b)^2+4s}}{2}.$$

Therefore, $\forall s$,

$$\frac{3(3\alpha^2-b) - \sqrt{(3\alpha^2-b)^2+4s}}{2} + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)} < 3\alpha^2 - b < 0.$$

We choose a suitable positive constant $C_3$, and conclude that $\mathbb{L}_s[r_1](L-,s)$ is an analytic function of variable $s$ for $Re[s] > -C_3$. Note that $\mathbb{L}_s[r_1](L-,s)$ decays to zero when $s \to \pm i\infty$. Therefore, by the complex analysis we have

$$|r_1(L-,t)| = O(1)\varepsilon e^{-t/C_3} \left| \int_R e^{i\eta t}\mathbb{L}_s[r_1](L-, i\eta - C_3)d\eta \right| \leq O(1)\varepsilon e^{-t/C_4},$$

for some constant $C_4$. One could use the similar approach to estimate the term $r_1(-L+,t)$.

Once all the boundary data are clear, with the help of spectrum gap [8], we solve the problem in finite domain and get the following exponential decaying estimate:

$$\sup_{\bar{x}\in[-L,L]} |r_1(\bar{x},t)| = O(1)\varepsilon e^{-t/C}. \tag{3.16}$$

The problem restricted to the left or right far field is just a constant coefficient problem in half space. Consider the problem in the left far field:

$$\begin{cases} r_{1t} + (f_1(\phi_L(-\infty))r_1)_{\bar{x}} = r_{1\bar{x}\bar{x}} + S(\bar{x},t), \bar{x} < -L, t > 0, \\ r_1(\bar{x},0) = z_1(\bar{x},0), \\ r_1(-L,t) = O(1)\varepsilon e^{-t/C_4}. \end{cases} \tag{3.17}$$

With the Green's identity, we have the solution for (3.17) when $\bar{x} < -L, t > 0$:

$$\begin{aligned} r_1(\bar{x},t) = & \int_{-\infty}^{-L} \mathbb{G}^{f_1(\phi_L(-\infty))}(\bar{x}-y,t)r_1(y,0)dy \\ & + \int_0^t \mathbb{G}^{f_1(\phi_L(-\infty))}(\bar{x},t-\tau)(r_1(-L,\tau) - r_{1\bar{x}}(-L,\tau))d\tau \\ & + \int_0^t \mathbb{G}_{\bar{x}}^{f_1(\phi_L(-\infty))}(\bar{x},t-\tau)r_1(-L,\tau)d\tau \\ & + \int_0^t \int_{-\infty}^{-L} \mathbb{G}^{f_1(\phi_L(-\infty))}(\bar{x}-y,t-\tau)S(y,\tau)dyd\tau. \end{aligned} \tag{3.18}$$

The transformation of Neumann datum $r_{1\bar{x}}(-L,\tau)$ is given by the Dirichlet–Neumann map (3.11). Note that

$$\begin{cases} \mathbb{L}_t^{-1}[\mathbf{D}_1] = \dfrac{(3-b)\delta(t)+(3-b)^2\frac{e^{-(3-b)^2/4t}}{\sqrt{\pi t}}+4\frac{e^{-(3-b)^2/4t}}{\sqrt{\pi t}}*\partial_t}{2}, \\ \mathbb{L}_t^{-1}[I_1(-L-,s)] = O(1)\varepsilon e^{-Ct} * e^{-(3-b)^2 t/4}, \end{cases}$$

one has:

$$\begin{aligned} |r_1(\bar{x},t)| = & \Bigg| O(1)\varepsilon \int_0^t \bar{\mathbb{G}}(\bar{x},t-\tau,f_1(\phi_L(-\infty)))(\mathbb{L}_\tau^{-1}[\mathbf{D}_1] * e^{-\tau/C} + e^{-\tau/C})d\tau \\ & + O(1)\varepsilon \int_0^t \bar{\mathbb{G}}_{\bar{x}}(\bar{x},t-\tau,f_1(\phi_L(-\infty)))e^{-\tau/C}d\tau \\ & + O(1)\varepsilon \int_0^t \int_{-\infty}^{-L} \bar{\mathbb{G}}(\bar{x}-y,t-\tau,f_1(\phi_L(-\infty)))e^{(-|y|-\tau)/C}dyd\tau \Bigg| \\ = & O(1)\varepsilon e^{(-|\bar{x}|-t)/C}, \bar{x} < -L, t > 0. \end{aligned} \tag{3.19}$$

One could do the similar computations in the right far field. Combining (3.16) with (3.19), we get the estimate for the whole space:

$$|r_1(\bar{x},t)| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C}, \text{for } \bar{x} \in R, t > 0, \tag{3.20}$$

$C$ is a universal positive constant. Similar approach can be used to estimate $r_2$.

*3.3. Error term and iterative scheme*

The error function $(e_1, e_2) = (z_1 - r_1, z_2 - r_2)$ of the approximate problem (3.10) to (3.9) satisfies the following initial value problem:

$$\begin{cases} e_{1t} + (f_1(\phi)e_1)_{\bar{x}} = e_{1\bar{x}\bar{x}} + ((f_1(\phi_L) - f_1(\phi))r_1)_{\bar{x}}, \\ e_{2t} + (f_2(\phi)e_2)_{\bar{x}} = e_{2\bar{x}\bar{x}} + ((f_2(\phi_L) - f_2(\phi))r_2)_{\bar{x}}, \\ e_i(\bar{x},0) = 0, i = 1,2, \end{cases} \tag{3.21}$$

here

$$\begin{cases} |r_i| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C}, i = 1,2, \\ |f_i(\phi_L) - f_i(\phi)| = O(1)\varepsilon, i = 1,2. \end{cases}$$

Now we introduce an iterative scheme to construct the solution of (3.9):

$$\begin{cases} r_i^0 = r_i, \\ e_i^0 = z_i, \\ e_i^k \equiv e_i^{k-1} - r_i^{k-1}, k \geq 1, \end{cases} \tag{3.22}$$

$$\begin{cases} e_{1t}^k + (f_1(\phi)e_1^k)_{\bar{x}} = e_{1\bar{x}\bar{x}}^k + ((f_1(\phi_L) - f_1(\phi))r_1^{k-1})_{\bar{x}}, \\ e_{2t}^k + (f_2(\phi)e_2^k)_{\bar{x}} = e_{2\bar{x}\bar{x}}^k + ((f_2(\phi_L) - f_2(\phi))r_2^{k-1})_{\bar{x}}, \\ e_i^k(\bar{x},0) = 0, i = 1,2, \end{cases} \tag{3.23}$$

$$\begin{cases} r_{1t}^k + (f_1(\phi_L)r_1^k)_{\bar{x}} = r_{1\bar{x}\bar{x}}^k + ((f_1(\phi_L) - f_1(\phi))r_1^{k-1})_{\bar{x}}, \\ r_{2t}^k + (f_2(\phi_L)r_2^k)_{\bar{x}} = r_{2\bar{x}\bar{x}}^k + ((f_2(\phi_L) - f_2(\phi))r_2^{k-1})_{\bar{x}}, \\ r_i^k(\bar{x},0) = 0, i = 1,2. \end{cases} \tag{3.24}$$

For each $r_i^k$, we have the following estimate:

$$|r_i^k(\bar{x},t)| = O(1)\varepsilon^{k+1} e^{-(|\bar{x}|+t)/C}, \text{for } \bar{x} \in R, t > 0. \tag{3.25}$$

The solution of (3.9) can be written formally in terms of the iterative scheme:

$$z_i = \sum_{k=0}^{\infty} r_i^k.$$

From (3.25), the series is convergent, and

$$|z_i(\bar{x},t)| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C}, \quad \text{for } \bar{x} \in R, t > 0.$$

To summarize, we have the following theorem:

**Theorem 3.2.** *The solution of (3.1) defines a semi-group $\mathbb{G}^t_\Phi$ as follows:*

$$\mathbb{G}^t_\Phi[u_0, v_0](\bar{x}) \equiv (u(\bar{x},t), v(\bar{x},t))$$

*which satisfies:*

$$|(u(\bar{x},t), v(\bar{x},t)) - \beta(t)\Psi(\bar{x})| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C},$$

*where: $|\Psi(\bar{x})| = O(1)e^{-\frac{|\bar{x}|}{C}}$, $|\beta(t)| \le O(1)\varepsilon \int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau$, $C$ is a universal constant. For large times,*

$$\int_R \beta(t)\Psi(\bar{x})d\bar{x} = \int_R (\bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}))d\bar{x}$$

*due to the conservation laws.*

*If the initial data satisfy: $\int_R (u(\bar{x},0), v(\bar{x},0))d\bar{x} = 0$, one has*

$$|u(\bar{x},t), v(\bar{x},t)| = O(1)\varepsilon e^{-(|\bar{x}|+t)/C}.$$

## 4. Nonlinear stability of over-compressive shock waves (proof of Main Theorem)

The estimate obtained in Theorem 3.2 tells us that there is a stationary structure $O(1)\varepsilon e^{-|\bar{x}|/C}$ determined by the initial data. To get the nonlinear stability, instead of extracting the over-compressive shock profile, the shock profile with a proper shift and a proper shape changed is the real final state (extracting the stationary part). Consider the perturbation of new profile

$$\Phi^* \equiv (\phi^*(\bar{x}-x_0), \psi^*(\bar{x}-x_0)),$$

which satisfies:

$$\begin{cases} \int_R (u_0, v_0)(x)dx = \int_R ((\phi^*, \psi^*) - (\phi, \psi))dx, \\ |\Phi^* - \Phi| = O(1)\varepsilon e^{-|\bar{x}|/C}. \end{cases} \tag{4.1}$$

Note that the phase shift $x_0$ contributes to the integral only for the first component, because the end states $(\tilde{u}_\pm, \tilde{v}_\pm)$ have zero second component. For simplicity, assume that $x_0 = 0$.

The perturbation $\begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \equiv \begin{pmatrix} \tilde{u}(x,t) \\ \tilde{v}(x,t) \end{pmatrix} - \Phi^*$ satisfies:

$$\begin{cases} \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix}_t + \left( (A(\Phi^*(\bar{x})) - bI) \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \right)_{\bar{x}} \\ = \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix}_{\bar{x}\bar{x}} - Q\left( \Phi^*, \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \right)_{\bar{x}}, \\ \left| \begin{pmatrix} u^*(\bar{x},0) \\ v^*(\bar{x},0) \end{pmatrix} \right| = O(1)\varepsilon e^{-|\bar{x}|/C}, \\ \int\limits_R \begin{pmatrix} u^*(\bar{x},0) \\ v^*(\bar{x},0) \end{pmatrix} d\bar{x} = 0, \end{cases} \tag{4.2}$$

where $A(\Phi^*) \equiv |\Phi^*|^2 I + 2\Phi^*\Phi^{*T}$, $Q$ is the quadratic remainder which is easily checked to obey the estimate [3]:

$$|Q(\Phi^*, Z)| \le 3(|\Phi^*| + |Z|)|Z|^2.$$

Due to the second estimate in (4.1), we rewrite the first equation of system (4.2) as follows:

$$\begin{cases} u^*{}_t + (f_1(\phi)u^*)_{\bar{x}} = u^*{}_{\bar{x}\bar{x}} + O(1)\varepsilon(e^{-|\bar{x}|/C}(u^* + v^*))_{\bar{x}} + Q_1(\bar{x},t), \\ v^*{}_t + (f_2(\phi)v^*)_{\bar{x}} = v^*{}_{\bar{x}\bar{x}} + O(1)\varepsilon(e^{-|\bar{x}|/C}(u^* + v^*))_{\bar{x}} + Q_2(\bar{x},t), \end{cases} \tag{4.3}$$

$Q_1$ and $Q_2$ are higher order remainders including terms $O(1)\varepsilon^2(e^{-|\bar{x}|/C}(u^* + v^*))_{\bar{x}}$, $(u^* + v^*)^2$.

Now make an ansatz on the solution:

$$\left| u^*(\bar{x},t), v^*(\bar{x},t) \right| \le O(1)\varepsilon e^{-(|\bar{x}|+t)/C}, \tag{4.4}$$

to deal with the nonlinear term.

To verify the ansatz, we only need to show that under the ansatz, the following holds:

$$\begin{aligned} &\Bigg| \int\limits_R \mathbb{G}_\Phi(\bar{x}-y,t)[u^*(y,0), v^*(y,0)]dy \\ &\quad + O(1)\int\limits_0^t \int\limits_R \mathbb{G}_\Phi(\bar{x}-y,t-s) \\ &\quad [\varepsilon(e^{-|y|/C}(u^*+v^*))_y + Q_1(y,s), \varepsilon(e^{-|y|/C}(u^*+v^*))_y + Q_2(y,s)]dyds \Bigg| \\ &\le O(1)\varepsilon e^{-(|\bar{x}|+t)/C}, \quad \text{for } \bar{x} \in R, \quad t > 0. \end{aligned} \tag{4.5}$$

By Theorem 3.2 $\mathbb{G}^t_\phi[u_0, v_0](\bar{x}) = (u(\bar{x},t), v(\bar{x},t))$,

$$|\mathbb{G}^t_\phi[u_0, v_0](\bar{x})| \le O(1)\varepsilon e^{-(|\bar{x}|+t)/C}.$$

From the ansatz (4.4), we have:

$$|O(1)\varepsilon(e^{-|\bar{x}|/C}(u^* + v^*))_x| \le O(1)\varepsilon^2 e^{-(|\bar{x}|+t)/D},$$

and

$$|Q_i(\bar{x}, t)| \le O(1)\varepsilon^2 e^{-(|\bar{x}|+t)/D}.$$

Therefore (4.5) is true and we prove the Main Theorem 1.1.

## Acknowledgments

The author would like to express her gratitude to Professor Shih-Hsien Yu for his guidance and suggestions. The research is supported by National Natural Science Foundation of China (No. 11526049) and the Fundamental Research Funds for the Central Universities (No. 2232016D-22), and in part by National Natural Science Foundation of China (No. 11571070).

## References

[1] S.-J. Deng, L.-L. Du, S.-H. Yu, Nonlinear stability of Broadwell model with Maxwell diffuse boundary condition, Kinet. Relat. Models 6 (2013) 865–882.
[2] L.-L. Du, Characteristic half space problem for the Broadwell model, Netw. Heterog. Media 9 (2014) 97–110.
[3] H. Freistühler, T.-P. Liu, Nonlinear stability of the overcompressive shock waves in a rotationally invariant system of viscous conservation laws, Comm. Math. Phys. 153 (1993) 147–158.
[4] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Ration. Mech. Anal. 95 (1986) 325–344.
[5] R. Gardner, K. Zumbrun, The Gap Lemma and geometric criteria for instability of viscous shock profiles, Comm. Pure Appl. Math. 51 (7) (1998) 797–855.
[6] G. Kreiss, H.-O. Kreiss, Convergence to steady state of solutions of Burgers' equation, Appl. Numer. Math. 2 (1986) 161–179.
[7] G. Kreiss, H.-O. Kreiss, Stability of systems of viscous conservation laws, Comm. Pure Appl. Math. 51 (1998) 1397–1424.
[8] G. Kreiss, H.-O. Kreiss, J. Lorenz, Stability of viscous shocks on finite intervals, Arch. Ration. Mech. Anal. 187 (2008) 157–183.
[9] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer. Math. Soc. 56 (328) (1985).
[10] T.-P. Liu, Recent mathematical methods in nonlinear wave propagation, in: Nonlinear Hyperbolic-Dissipative Partial Differential Equations, in: Lecture Notes in Mathematics, vol. 1640, 1996, pp. 103–136.
[11] T.-P. Liu, Pointwise convergence to shock waves for viscous conservation laws, Comm. Pure Appl. Math. 50 (11) (1997) 1113–1182.
[12] T.-P. Liu, S.-H. Yu, On boundary relation for some dissipative systems, Bull. Inst. Math. Acad. Sin. (N.S.) 6 (3) (2011) 245–267.
[13] T.-P. Liu, Y. Zeng, Time-asymptotic behavior of wave propagation around a viscous shock profile, Comm. Math. Phys. 290 (2009) 23–82.
[14] C. Mascia, K. Zumbrun, Stability of large-amplitude viscous shock profiles of hyperbolic–parabolic systems, Arch. Ration. Mech. Anal. 172 (1) (2004) 93–131.
[15] D.H. Sattinger, On the stability of waves of nonlinear parabolic systems, Adv. Math. 22 (3) (1976) 312–355.
[16] S.-H. Yu, Nonlinear wave propagation over a Boltzmann shock profile, J. Amer. Math. Soc. 23 (4) (2010) 1040–1118.
[17] K. Zumbrun, P. Howard, Pointwise semigroup methods and stability of viscous shock waves, Indiana Univ. Math. J. 47 (3) (1998) 741–871.