Oscillatory traveling wave solutions to an attractive chemotaxis system

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Abstract

This paper investigates oscillatory traveling wave solutions to an attractive chemotaxis system. The convective part of this system changes its type when crossing a parabola in the phase space. The oscillatory nature of the traveling wave comes from the fact that one far-field state is in the elliptic region and another in the hyperbolic region. Such traveling wave solutions are shown to be linearly unstable. Detailed construction of some traveling wave solutions is presented.
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1. Introduction

Traveling waves are solutions to partial differential equations that move with constant speed while maintaining their shape. Such waves play an important role in many applied problems such as signal pulses in mathematical biology, flame fronts in chemical reactions, light waves in nonlinear optics, solitons in water waves, and viscous shock profiles in conservation laws that model, for instance, problems in fluid and gas dynamics.

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This work is concerned with traveling wave solutions to the following system

$$\partial_t u + \partial_x(uv) = \partial_x^2 u, \tag{1.1a}$$

$$\partial_t v - \partial_x(u + \epsilon v^2) = \epsilon \partial_x^2 v \tag{1.1b}$$

where $\epsilon > 0$ is a parameter, $(u(x,t), v(x,t))$ are the unknown functions of position $x \in \mathbb{R}$ and time $t > 0$. The physical motivation for studying this system is the propagation of traveling pulses in the chemotactic movement of particles. The main feature of this system is that its convection part is of mixed type, i.e., it changes its type over elliptic, parabolic and hyperbolic types. We are concerned with traveling wave solutions connecting a state from the elliptic region and another state in the hyperbolic region. In particular, we study the interesting spiral–saddle connections. The main difficulty is that the traveling wave is oscillatory, both existence and stability analysis are challenging. The fundamental question we address is to understand how the attractive chemotaxis influences the characteristics of front propagation through construction of traveling wave solutions connecting two states in hyperbolic and elliptic region, respectively.

1.1. The model description

The coupled system (1.1) is derived from the following chemotaxis model

$$\partial_t p = D\partial_x \left(p\partial_x ln \left(\frac{p}{\Phi(w)} \right) \right), \tag{1.2a}$$

$$\partial_t w = w(\lambda p - \mu) + \epsilon \partial_x^2 w \tag{1.2b}$$

where $p(x,t)$ denotes the particle density and $w(x,t)$ is the concentration of chemicals. $D > 0$ is the diffusion rate of particles. The function Φ is commonly referred to as the chemotactic potential given by

$$\Phi(w) = w^{-\alpha} \tag{1.3}$$

where $\alpha > 0$ or $\alpha < 0$. The first term on the right hand side of (1.2) is the chemical kinetics with $\lambda > 0$ and $\mu \geq 0$.

The model (1.2) when setting $\epsilon = 0$ is a canonical example of models proposed by Othmer and Stevens [12] to describe the chemotactic movement of particles where the chemicals are non-diffusible and can modify the local environment for succeeding passages. For example, myxobacteria produce slime over which their cohorts can move more readily and ants can follow trails left by predecessors [12]. Another direct application of (1.2) is to model haptotaxis where cells move towards an increasing concentration of immobilized signals such as surface or matrix-bound adhesive molecules. The model (1.2) with $\epsilon = 0$ and (1.3) was presented in [6] and a comprehensive qualitative and numerical analysis was provided there. (1.2a) is parabolic in p, hence $p \geq 0$ and there is an infinite propagation speed in p. Since there is no diffusion present in (1.2), thus there is a zero propagation speed in w. As analyzed in [6], this is accountable for some interaction of the characteristics, leading to solutions for which p either blows up in finite time, collapses to a spatially uniform constant, or collapses to a piecewise constant stationary solution.

We are interested in traveling wave solutions of system (1.1), hence we consider the model in which diffusion of both the population density and the chemotactic agent occur. The derivation of (1.1) from (1.2) and (1.3) is straightforward. Substitution of (1.3) into (1.2) yields the following system

$$\begin{cases} \partial_t p = D\partial_x \left(\partial_x p + \alpha p \frac{\partial_x w}{w} \right), \\ \partial_t w = \lambda p w - \mu w + \epsilon \partial_x^2 w \end{cases} \tag{1.4}$$

which is a special case of the Keller–Segel model [2,4,5] that describes the chemotactic motion in the macroscopic level. We call the chemotaxis attractive if $\alpha < 0$, and repulsive for $\alpha > 0$. In the case of attractive chemotaxis $\alpha < 0$, the global existence and blow up of solutions for (1.2) (1.3) subject to zero flux boundary condition were investigated in [14,15]. The existence and stability of spike solutions of (1.2) (1.3) were established for bounded domain in [13]. For the case of repulsive chemotaxis $\alpha > 0$, global existence and the nonlinear stability of traveling wave solutions for (1.1) were given in [3,7–11]. Existence and stability of the oscillatory traveling wave solutions for $\alpha < 0$ in the mixed-type regime are important open problems. The study of oscillatory traveling wave solutions in the mixed-type system poses new difficulties which are different from the previous works [7,10], where the convective part of the system is hyperbolic and the traveling waves are monotone. We innovatively construct an invariant region to enclose the unstable spiral to ultimately show the existence of the connected orbit.

1.2. Present investigation and main results

Since $p(x,t)$ represents cell density, we assume that $p \geq 0$. By the same reason we suppose $w(x,t) \geq 0$. One may transform system (1.4) by

$$u = p, \quad v = \partial_x(lnw), \tag{1.5}$$

so that $u \geq 0$ and

$$\partial_t u - \alpha D\partial_x(uv) = D\partial_x^2 u,$$

$$\partial_t v - \partial_x(\lambda u + \epsilon v^2) = \epsilon \partial_x^2 v.$$

For $\alpha < 0$, we substitute the scalings

$$\tilde{t} = \lambda t, \ \tilde{x} = \sqrt{\frac{\lambda}{-\alpha D}}x, \ \tilde{v} = \sqrt{\frac{-\alpha D}{\lambda}}v, \ \tilde{\epsilon} = \frac{\epsilon}{-\alpha D}, \ \tilde{D} = -\frac{1}{\alpha} \tag{1.6}$$

in the above system, dropping the tildes for convenience, to obtain the transformed system

$$\partial_t u + \partial_x(uv) = D\partial_x^2 u, \tag{1.7a}$$

$$\partial_t v - \partial_x(u + \epsilon v^2) = \epsilon \partial_x^2 v. \tag{1.7b}$$

By a direct calculation of eigenvalues of the coefficient matrix of the convection part in (1.7),

$$A = \begin{pmatrix} v & u \\ -1 & -2\epsilon v \end{pmatrix},$$

we see that the characteristic speeds of the convection part are

$$\lambda_1 = \frac{(1 - 2\epsilon)v - \sqrt{(2\epsilon + 1)^2 v^2 - 4u}}{2},$$

$$\lambda_2 = \frac{(1 - 2\epsilon)v + \sqrt{(2\epsilon + 1)^2 v^2 - 4u}}{2}.$$

Hence for the attractive chemotaxis model, the rescaled system (1.7) when dropping the diffusion terms changes type when crossing the parabola

$$u = \left(\epsilon + \frac{1}{2}\right)^2 v^2$$

in phase space.

We are interested in traveling wave solutions to the Keller–Segel model (1.4), which describes the directed movement of cells toward the chemical which is consumed by cells when they encounter, and the traveling wave is an "invasion" pattern. That is, the wave profile of p decreases from its tail to front and that of w increases from its tail to the front, which requires $w_x > 0$. Due to the transformation (1.5), we have $v(x, t) \geq 0$. Therefore typical end states in our consideration are such that $0 < u_+ < u_-$ and $v_+ > v_- \geq 0$ as those given in Theorem 1.1. We point out that when changing (u, v, x, t) to $(u, -v, -x, t)$ the transformed system (1.7) remains unchanged. Our results remain valid for corresponding cases with $v \leq 0$.

The aim of this paper is to investigate oscillatory traveling wave solutions and their stability properties for the transformed system (1.7) instead of chemotaxis model (1.4). We shall focus on (1.7) with $D > 0$, to be more specific, we set $D = 1$ in (1.7) to obtain (1.1).

Thus the convection part in system (1.1) is elliptic in the open set

$$E = \{(u, v) | \quad u > (\epsilon + 0.5)^2 v^2\}, \tag{1.8}$$

but hyperbolic in

$$H = \{(u, v) | \quad 0 \leq u < (\epsilon + 0.5)^2 v^2\}. \tag{1.9}$$

We say that the state (u, v) is on the parabola of degeneracy if $u = (\epsilon + 0.5)^2 v^2$. Therefore system (1.1) when dropping the diffusion terms changes type when crossing the parabola $u = (\epsilon + 0.5)^2 v^2$ in phase space.

Traveling wave solutions of (1.1) are of the form

$$(u, v) = (U, V)(z), \quad z = x - st, \tag{1.10}$$

where s is the traveling wave speed, satisfying far field conditions

$$(U, V)(\pm\infty) = (u_\pm, v_\pm), \quad (U', V')(\pm\infty) = 0 \tag{1.11}$$

where $(u_\pm, v_\pm)$ are the end states at $\pm\infty$ and where s is the traveling wave speed determined by the end states.

We focus on the interesting mixed-type region, i.e., one end state is in the elliptic region while the other is in the hyperbolic region,

$$(u_-, v_-) \in E, \quad (u_+, v_+) \in H. \tag{1.12}$$

The first issue is whether a traveling wave solution connecting such two states exists? The answer is summarized in the following.

Theorem 1.1. *Assume $\epsilon > 0$ in system (1.1). Let $(U, V)(x - st)$ be a traveling wave solution of system (1.1) connecting $(u_-, v_-) \in E$ and $(u_+, v_+) \in H$, then these two states necessarily satisfy*

$$(u_+ - u_-)^2 + \epsilon(u_+ - u_-)(v_+^2 - v_-^2) + (v_+ - v_-)(u_+ v_+ - u_- v_-) = 0.$$

A unique traveling wave solution connecting an unstable spiral (u_-, v_-) and a saddle (u_+, v_+) is explicitly constructed for
(i) $\epsilon = 1/2$ and $(u_-, v_-) = (5, 0)$, $(u_+, v_+) = (1, 4)$; and
(ii) $\epsilon = 1$ and $(u_-, v_-) = (3, 0)$, $(u_+, v_+) = (1, 2)$.
Moreover, the common property in these two cases is that the wave profile is oscillatory when entering in the elliptic region E, yet monotone in the hyperbolic region H.

Remark 1.1. Far field states of traveling wave solutions are classified in Section 2.1, and the structural stability of the saddle–spiral connection is discussed in Section 2.2. We therefore only construct traveling wave solutions for the interesting mixed-type connections, which are further identified in Lemma 2.1

Remark 1.2. One may translate the results for the transformed system (1.7) back to the original Keller–Segel model (1.4) by

$$p(x, t) = u\left(\sqrt{\frac{\lambda}{-\alpha D}} x, \lambda t\right), \quad w(x, t) = w_0 e^{\int_0^x v\left(\sqrt{\frac{\lambda}{-\alpha D}} z, \lambda t\right) dz}.$$

The traveling wave V as obtained here implies that in (1.4) the chemical satisfies $\partial_x w/w \to v_+ > 0$ and increases exponentially, indicating a fast condensation at far field which is a typical behavior of invasion and attractive chemicals such as that of cancer cells [2,6,12].

Next issue of our interest is the stability property of such traveling waves. In other words, we consider (1.1) subject to initial data

$$(u, v)(x, 0) = (u_0, v_0) \to (u_\pm, v_\pm) \quad \text{as } x \to \pm\infty.$$

If such an initial data is close to the traveling wave, stability would imply that the solution converges toward the traveling wave solution as time evolves, unless the traveling wave solution is unstable. Regarding this issue, we have the following result.

Theorem 1.2. *Assume $\epsilon > 0$ in system (1.1), $(u_-, v_-) \in E$ and $(u_+, v_+) \in H$. If $(U(x - st), V(x - st))$ is a traveling wave solution of system (1.1) connecting the two states (u_-, v_-) and (u_+, v_+), then it must be linearly unstable against perturbations in certain Sobolev spaces, say, in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.*

The remainder of the paper is organized as follows. In Section 2, we present some qualitative analysis of traveling wave solutions and the types of two end states. Section 3 is devoted to the detailed construction of oscillatory traveling wave solutions by means of the method of phase plane analysis. Section 4 is devoted to the proof of a general result on the linear instability of traveling wave solutions connecting states from the elliptic region to the hyperbolic region.

2. Traveling wave solutions

In this section, we investigate necessary conditions for the existence of traveling wave solutions for (1.1), and the types of two end states.

Insertion of the traveling wave solution (1.10) into (1.1) gives

$$-sU' + (UV)' = U'',$$
$$-sV' - (U + \epsilon V^2)' = \epsilon V''.$$

Integration using one end condition leads to the first-order nonlinear system of differential equations

$$U' = UV - sU + su_- - u_-v_- =: P, \tag{2.1a}$$

$$V' = -\frac{s}{\epsilon}V - V^2 - \frac{1}{\epsilon}U + \frac{s}{\epsilon}v_- + v_-^2 + \frac{1}{\epsilon}u_- =: Q, \tag{2.1b}$$

while the right far field (u_+, v_+) is necessarily on the curve parameterized by

$$s(u - u_-) = uv - u_-v_-, \tag{2.2a}$$

$$s(v - v_-) = -(u + \epsilon v^2 - u_- - \epsilon v_-^2). \tag{2.2b}$$

The traveling wave speed s is determined by

$$s = (u_+v_+ - u_-v_-)/(u_+ - u_-). \tag{2.3}$$

2.1. Classification of critical points

System (2.1) has two isolated critical points (u_-, v_-) and (u_+, v_+) in the phase plane, denoted by (u^*, v^*). The coefficient matrix of system (2.1), when linearized about (u^*, v^*), is

$$\begin{pmatrix} v^* - s & u^* \\ -\frac{1}{\epsilon} & -\frac{s}{\epsilon} - 2v^* \end{pmatrix}, \tag{2.4}$$

which admits two eigenvalues

$$\lambda_1^* = \frac{-s - s\epsilon^{-1} - v^*}{2} - \frac{\sqrt{(3v^* - s + s\epsilon^{-1})^2 - 4u^*\epsilon^{-1}}}{2}, \tag{2.5a}$$

$$\lambda_2^* = \frac{-s - s\epsilon^{-1} - v^*}{2} + \frac{\sqrt{(3v^* - s + s\epsilon^{-1})^2 - 4u^*\epsilon^{-1}}}{2}. \tag{2.5b}$$

This will be used to classify the types of the two critical points.

First, in order to find admissible pairs $(u_+, v_+) \in H$ to connect to $(u_-, v_-) \in E$, we solve for $(u_-, v_-) \in E$ from the shock curve relation (2.2) to have

$$(u_+ - u_-)^2 + \epsilon(u_+ - u_-)(v_+^2 - v_-^2) + (v_+ - v_-)(u_+v_+ - u_-v_-) = 0. \tag{2.6}$$

This clearly yields two distinctive real roots

$$2\frac{u_+ - u_-}{v_+ - v_-} = -(\epsilon v_+ + (\epsilon + 1)v_-) \pm \sqrt{(\epsilon v_+ + (\epsilon + 1)v_-)^2 - 4u_+},$$

if

$$u_+ < \frac{1}{4}(\epsilon v_+ + (\epsilon + 1)v_-)^2.$$

This implies that given that $(u_-, v_-) \in E$, for $(u_+, v_+) \in H$ on the shock curve (2.6), it necessarily satisfies

$$u_+ < \frac{1}{4}\min\left\{(\epsilon v_+ + (\epsilon + 1)v_-)^2, (2\epsilon + 1)^2 v_+^2\right\}. \tag{2.7}$$

The eigenvalues of system (2.1), when linearized about (u_-, v_-) are

$$\lambda_1^- = \frac{-s - s\epsilon^{-1} - v_-}{2} - \frac{\sqrt{(3v_- - s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1}}}{2}, \tag{2.8a}$$

$$\lambda_2^- = \frac{-s - s\epsilon^{-1} - v_-}{2} + \frac{\sqrt{(3v_- - s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1}}}{2}. \tag{2.8b}$$

The product of the two eigenvalues is

$$\epsilon\lambda_1^-\lambda_2^- = \epsilon(v_- - s)(-s\epsilon^{-1} - 2v_-) + u_-$$

$$= s^2 + (2\epsilon - 1)v_- s + u_- - 2\epsilon v_-^2$$

$$= (s + (\epsilon - 1/2)v_-)^2 + (u_- - (\epsilon + \frac{1}{2})^2 v_-^2) > 0.$$

This suggests that $(u_-, v_-) \in E$ cannot be a saddle, but can be a spiral if

$$(3v_- - s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1} < 0, \tag{2.9}$$

or a node if

$$(3v_- - s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1} > 0.$$

On the other hand the eigenvalues of the critical point (u_+, v_+), are

$$\lambda_1^+ = \frac{-s - s\epsilon^{-1} - v_+}{2} - \frac{\sqrt{(3v_+ - s + s\epsilon^{-1})^2 - 4u_+\epsilon^{-1}}}{2}, \tag{2.10a}$$

$$\lambda_2^+ = \frac{-s - s\epsilon^{-1} - v_+}{2} + \frac{\sqrt{(3v_+ - s + s\epsilon^{-1})^2 - 4u_+\epsilon^{-1}}}{2}. \tag{2.10b}$$

The product of the two eigenvalues is

$$\begin{aligned}
\epsilon\lambda_1^+\lambda_2^+ &= \epsilon(v_+ - s)(-s\epsilon^{-1} - 2v_+) + u_+ \\
&= s^2 + (2\epsilon - 1)v_+s + u_+ - 2\epsilon v_+^2 \\
&= (s + (\epsilon - 1/2)v_+)^2 - ((\epsilon + 1/2)^2 v_+^2 - u_+).
\end{aligned}$$

This indicates that (u_+, v_+) can be a saddle if

$$|s + (\epsilon - 1/2)v_+| < \sqrt{(\epsilon + 1/2)^2 v_+^2 - u_+}. \tag{2.11}$$

Here $(\epsilon + 1/2)^2 v_+^2 - u_+ > 0$ is ensured by the fact that $(u_+, v_+) \in H$. Otherwise it can be a node for

$$(3v_+ - s + s\epsilon^{-1})^2 - 4u_+\epsilon^{-1} > 0,$$

or a spiral for

$$(3v_+ - s + s\epsilon^{-1})^2 - 4u_+\epsilon^{-1} < 0.$$

2.2. Saddle–spiral connection and structural stability

We are interested in the oscillatory traveling waves connecting a saddle in H and a spiral in E for each parameter $\epsilon > 0$. To be more specific, we set $q = (u, v)$, and restrict to the following set

$$W = \{(\epsilon, q_-, q_+)\} \subset \mathbb{R}^+ \times E \times H,$$

where (ϵ, q_-, q_+) satisfies the following:

$$(u_+ - u_-)^2 + \epsilon(u_+ - u_-)(v_+^2 - v_-^2) + (v_+ - v_-)(u_+v_+ - u_-v_-) = 0, \tag{2.12a}$$

$$(3v_- - s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1} < 0, \tag{2.12b}$$

$$s < -\frac{\epsilon}{1 + \epsilon}v_-, \tag{2.12c}$$

$$|s + (\epsilon - 1/2)v_+| < \sqrt{(\epsilon + 1/2)^2 v_+^2 - u_+}. \tag{2.12d}$$

Here s is determined by (2.3). From (2.9) and (2.11), we see that this set corresponds to cases with unstable spiral $q_- \in E$ and saddle $q_+ \in H$.

For each $(\epsilon, q_-, q_+) \in W$, (P, Q) forms a vector field so that system (2.1) satisfies the following properties:

- System (2.1) has only two critical points q_- and q_+, both are hyperbolic (the real parts of the eigenvalues are nonzero).
- System (2.1) does not have any orbit connecting a saddle point to a saddle point.
- System (2.1) does not have any closed orbits.

The last statement may be verified by investigating the critical points at infinity. As such system (2.1) is structurally stable [16, Chapter III] in the sense that any perturbed system of (2.1) obtained by slightly changing (ϵ, q_-) to $(\tilde{\epsilon}, \tilde{q}_-)$, also q_+ to $\tilde{q}_+$ satisfying (2.12a), with $(\tilde{\epsilon}, \tilde{q}_-, \tilde{q}_+)$ still within set W, there exists a homeomorphism which maps oriented orbits in (2.1) to oriented orbits in the perturbed system.

By continuity, the set of orbits connecting spirals to saddles with (ϵ, q_-, q_+) in W forms an open set, and existence of such connections follows if we can construct a saddle–spiral connection for a particular pair in W. In Section 3, we will then carry out construction of such oscillatory traveling wave solutions for two specific cases with $v_- = 0$. One case is $(1/2, 5, 0, 1, 4) \in W$ with $s = -1$, and another case is $(1, 3, 0, 1, 2) \in W$ also with $s = -1$. We now show that these two cases can be continuously connected within W through a one-dimensional curve

$$\left(1 - \theta/2, 2\theta + 3, 0, 1, \frac{2(1+\theta)}{\sqrt{1 + \theta - \theta^2}}\right) \in W$$

with θ running from 1 to 0. Indeed, if we set $\epsilon(\theta) = 1 - \theta/2$, $q_-(\theta) = (2\theta + 3, 0)$, we can determine $q_+(\theta)$ by using (2.12a) and $q_+(1) = (1, 2)$, $q_+(0) = (1, 4)$ to obtain

$$q_+(\theta) = \left(1, \frac{2(1+\theta)}{\sqrt{1 + \theta - \theta^2}}\right).$$

By (2.3) we see that

$$s = -\frac{1}{\sqrt{1 + \theta - \theta^2}} < 0,$$

which satisfies (2.12c). The left hand side of (2.12b) then reduces to

$$\frac{4\theta^2(1+\theta)^2}{(1 + \theta - \theta^2)(2 - \theta)^2} - \frac{8(2\theta + 3)}{2 - \theta} = \frac{4(-3\theta^4 + 8\theta^3 + 15\theta^2 - 14\theta - 12)}{(2 - \theta)^2(1 + \theta - \theta^2)}$$

which is clearly negative for $\theta \in [0, 1]$. A detailed calculation suggests that (2.12d) holds true if any only if

$$4\theta^3 + \theta^2 - 11\theta - 8 < 0,$$

which is indeed the case for $\theta \in [0, 1]$.

2.3. Special cases with $v_- = 0$

When $v_- = 0$, we give a more precise account on the range of W. In such cases, a traveling wave solution that connects $(u_-, 0)$ and (u_+, v_+) satisfies

$$U' = UV - sU + su_- = P, \tag{2.13a}$$

$$V' = -s\epsilon^{-1}V - V^2 - \epsilon^{-1}(U - u_-) = Q. \tag{2.13b}$$

System (2.13) has only two isolated critical points $(u_-, 0)$ and (u_+, v_+) which can be classified in detail as follows.

Lemma 2.1. *Let $0 < \epsilon \le 4$ in system (1.1). Given $(u_-, v_-) = (u_-, 0) \in E$. For $(u_+, v_+) \in H$ to be connected to (u_-, v_-) by a traveling wave, it necessarily satisfies*

$$0 \le u_+ < \min\left\{ \frac{\epsilon^2}{4} v_+^2, \frac{\epsilon}{1+\epsilon} u_- \right\}. \tag{2.14}$$

The critical point $(u_-, 0)$ is a spiral provided that

$$0 \le u_+ < \frac{2\epsilon u_-}{1 + \epsilon + \sqrt{2 + 2\epsilon^2}}.$$

Such spiral is unstable for $v_+ > 0$ and stable for $v_+ < 0$. The critical point (u_+, v_+) is a saddle.

Proof. From (2.12a) with $v_- = 0$ it follows that

$$(u_+ - u_-)^2 + \epsilon v_+^2 (u_+ - u_-) + u_+ v_+^2 = 0. \tag{2.15}$$

Hence u_+ necessarily satisfies

$$\epsilon(u_+ - u_-) + u_+ < 0 \tag{2.16}$$

and (2.7), which together yields (2.14).

From (1.8) and (1.9) we know that $u_- > 0$ and $u_+ \ge 0$ from (1.9). We introduce $\beta = \frac{u_+}{u_-} \ge 0$, and (2.16) requires that $\beta < \frac{\epsilon}{1+\epsilon}$. It follows from (2.3) and (2.15) that

$$s = -\frac{\beta}{1 - \beta} v_+, \quad s^2 = \frac{\beta^2}{\epsilon - (1 + \epsilon)\beta} u_-.$$

Note that the left hand side of (2.12b) is reduced to

$$(-s + s\epsilon^{-1})^2 - 4u_-\epsilon^{-1} = (1 - \epsilon^{-1})^2 \frac{\beta^2}{\epsilon - (1 + \epsilon)\beta} u_- - 4\epsilon^{-1}u_-$$

$$= \frac{u_-}{\epsilon - (1 + \epsilon)\beta} \left[(1 - \epsilon^{-1})^2 \beta^2 + 4(1 + \epsilon^{-1})\beta - 4 \right]$$

which has a positive zero $\beta_1 > 0$, given by

$$\beta_1 = \frac{2\epsilon}{1 + \epsilon + \sqrt{2 + 2\epsilon^2}}.$$

It can be checked directly that indeed $0 < \beta_1 < \frac{\epsilon}{1+\epsilon}$. Hence (2.12b) is satisfied and thus $(u_-, 0)$ is a spiral if $0 < \beta < \beta_1$.

Due to symmetry in v_+ in relation (2.15), we may take $v_+ > 0$, then $s = \frac{u_+ v_+}{u_+ - u_-} < 0$. The spiral $(u_-, 0)$ is unstable in this case since (2.12c) is met. For the case $v_+ < 0$, we have $s > 0$ and the spiral is stable.

In order to show that (u_+, v_+) is a saddle, it suffices to show that (2.12d) holds. From (2.14), we see that $0 \le \beta < \frac{\epsilon}{1+\epsilon}$, i.e., $0 < \frac{\beta}{1-\beta} < \epsilon$, which with (2.14) yields

$$|s + (\epsilon - 1/2)v_+|^2 - \left((\epsilon + 1/2)^2 v_+^2 - u_+\right) = v_+^2 \left(-\frac{\beta}{1-\beta} - 1\right)\left(-\frac{\beta}{1-\beta} + 2\epsilon\right) + u_+$$

$$\le v_+^2 \left[\frac{1}{1-\beta}\left(\frac{\beta}{1-\beta} - 2\epsilon\right) + \frac{\epsilon^2}{4}\right]$$

$$< \epsilon v_+^2 \left[\frac{\epsilon}{4} - \frac{1}{1-\beta}\right] < 0$$

for $0 < \beta < \beta_1$ and for $0 < \epsilon \le 4$. Thus (u_+, v_+) is a saddle point under the conditions of the lemma. The proof is thus complete. $\square$

3. Construction of oscillatory traveling wave solutions

In this section we present a detailed construction of some traveling wave solutions connecting between an unstable spiral $(u_-, 0)$ and a saddle (u_+, v_+) with $s < 0$. To be specific, consider two cases as claimed in Theorem 1.1: (i) $\epsilon = 1/2$ such that system (1.1) has two distinctive viscous coefficients, and (ii) $\epsilon = 1$ such that system (1.1) has the same viscous coefficient in the two equations.

We prove case (i) in Lemma 3.1. For a typical pair of far field states consisting of an unstable spiral and a saddle, we have the following.

Lemma 3.1. *Let $\epsilon = 1/2$ in system (1.1). Given that $(5, 0) \in E$ and $(1, 4) \in H$, then there exists a unique traveling wave $(U, V)(x + t)$ connecting $(u_-, v_-) = (5, 0)$ to $(u_+, v_+) = (1, 4)$.*

Proof. We prove the existence in several steps.

1. *Types of the far field states.* For $(u_-, v_-) = (5, 0) \in E$ and $(u_+, v_+) = (1, 4) \in H$, we have $s = [uv]/[u] = -1$. From (2.13), we have that traveling wave solutions of system (1.1) with $\epsilon = 1/2$ satisfy

$$U' = UV + U - 5 =: P, \tag{3.1a}$$

$$V' = 2V - V^2 - 2U + 10 =: Q. \tag{3.1b}$$

This system has two isolated critical points $(u_-, v_-) = (5, 0)$ and $(u_+, v_+) = (1, 4)$. The coefficient matrix and eigenvalues of system (3.1), when linearized about (u^*, v^*), are respectively

$$\begin{pmatrix} v^* + 1 & u^* \\ -2 & 2 - 2v^* \end{pmatrix}, \qquad \lambda_{\pm} = \frac{3 - v^*}{2} \pm \frac{\sqrt{(3v^* - 1)^2 - 8u^*}}{2}. \tag{3.2}$$

Thus $(u_-, v_-) = (5, 0)$ is an unstable spiral since $\lambda_{\pm} = \frac{3}{2} \pm \frac{\sqrt{39}}{2}i$, and $(u_+, v_+) = (1, 4)$ is a saddle since $\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{113}}{2}$.

2. *Construction of a closed domain around the unstable spiral such that all trajectories originating from the spiral flow out of the domain.*

According to the phase plane theory for ODE systems, any non-trivial orbit, if it exists, necessarily connects the two critical points, unless it is a limit cycle. Also the connecting orbit must tend to (u_+, v_+) at $+\infty$ and (u_-, v_-) at $-\infty$.

In order to show the existence of the connecting orbit, we first select an ellipse centered at $(5, 0)$ and determined by $H_1(u, v) = 0$ with

$$H_1(u, v) = \frac{2}{5}(u - 5)^2 + v^2 - 1. \tag{3.3}$$

We now look at the vector field of system (3.1) on $H_1(U, V) = 0$:

$$\frac{d}{dz}H_1(U, V) = \frac{4}{5}(U - 5)U' + 2VV' = \frac{4}{5}(V + 1)(U - 5)^2 + V^2(2 - V) > 0 \tag{3.4}$$

for $-1 \leq V \leq 1$. Hence all orbits of system (1.1) within the ellipse, which must be originated from the critical point $(5, 0)$, will leave the region enclosed by the ellipse, including those entering into the domain Ω to be identified below.

3. *Construction of a domain Ω such that on the domain boundary $\partial\Omega$, except the part on the ellipse (3.3), all trajectories point outside of the domain Ω.*

The idea of finding the desired connection is to construct a domain Ω such that the other critical point $(1, 4) \in \partial\Omega$ and $\Omega \cap \{(u, v),\ H_1(u, v) = 0\} \neq \emptyset$. Moreover, on all sides of the domain boundary $\partial\Omega$, vector fields of system (3.1) point outside of the domain except the part on the ellipse (3.3). Then by continuity of the vector field, there must be one and only one orbit originated from $(5, 0)$ entering into the saddle point $(1, 4)$ from interior of Ω.

We will show that the domain enclosed by the following curves is a desired domain Ω, see Fig. 1.

$$l_1 = \{(u, v)\mid\ u = \frac{5}{v + 1}, \qquad V_0 \leq v \leq 4\}, \tag{3.5}$$

$$l_2 = \{(u, v)\mid\ \text{left branch of}\ H_1(u, v) = 0, \qquad V_1 \leq v \leq V_0\}, \tag{3.6}$$

$$l_3 = \{(u, v)\mid\ v - V_1 = u - U_1,\ U_2 \leq u \leq U_1\}, \tag{3.7}$$

$$l_4 = \{(u, v)\mid\ v - V_2 = 0.6(u - U_2),\ U_3 \leq u \leq U_2\}, \tag{3.8}$$

$$l_5 = \{(u, v)\mid\ v = -2,\ 1 \leq u \leq U_3\}, \tag{3.9}$$

$$l_6 = \{(u, v)\mid\ u = 1,\ -2 \leq v \leq 4\} \tag{3.10}$$

where (U_i, V_i), $i = 0, 1, 2, 3$ will be defined below.

First of all, let (U_1, V_1) and (U_0, V_0) be the lower and upper intersection points of the nullcline
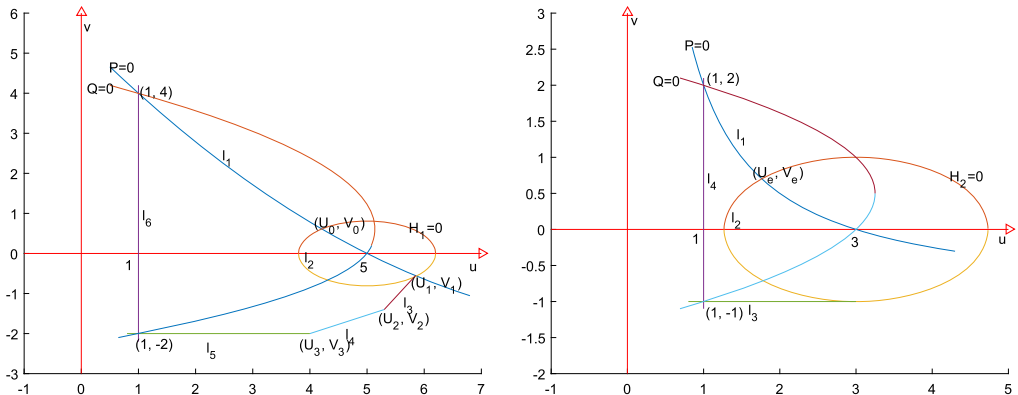
Fig. 1. Phase diagrams for cases in Lemma 3.1 (left) and Lemma 3.2 (right).

$$uv + u - 5 = 0$$

and the ellipse

$$H_1(u, v) = 0$$

defined in (3.3). It is easy to see that there exist two intersection points of the nullcline and the ellipse since the center of the ellipse is on the nullcline which is a hyperbola. Moreover

$$-1 < V_1 < 0 < V_0 < 1. \tag{3.11}$$

Secondly, it can be verified that on sides, l_i, $i = 1, 5, 6$ of $\partial\Omega$, vector fields of system (3.1) are pointing outside of Ω.

Noticing that $U' = P = 0$ on l_1 and (3.11), we have

$$V' = 2V - V^2 - 2U + 10 = \frac{V(4-V)(3+V)}{V+1} > 0, \ V_0 < v < 4$$

which means that vector fields on this curve are pointing outside of Ω.

On l_5 and l_6, the vector field $(U', V') = (P, Q)$ as defined in (3.1) points outwards of Ω since

$$(P, Q)|_{l_5} = (-U - 5, -2U + 2), \quad \text{sign of } (P, Q) = (-, -) \text{ for } U > 1, \tag{3.12}$$

$$(P, Q)|_{l_6} = (V - 4, (4 - V)(2 + V)), \quad \text{sign of } (P, Q) = (-, +) \text{ for } -2 < V < 4. \tag{3.13}$$

On l_2, (3.4) holds true which means that vector fields on this curve are pointing inside of Ω.

To finish the construction, we proceed to use a couple of line segments to form a desired curve.

From (U_i, V_i), which when $i = 1$ is approximately $(6.5368, -0.2351)$, we draw a line l_{i+2}: $v = a_i u + b_i$ with $b_i = V_i - a_i U_i$ until (U_{i+1}, V_{i+1}) such that the difference between the vector fields on line l_{i+2} and the slope of l_{i+2} is positive

$$\left. \frac{dV}{dU} \right|_{l_{i+2}} - a_i = \frac{Q - a_i P}{P} > 0$$

so the vector fields point outside of the domain Ω. This procedure continues for $i = 1, 2, ...m - 1$ until $V_m = -2$ and $u^* = U_m$. In fact, we get there when $m = 3$.

Since Ω is below the nullcline $uv - v - 5 = 0$, we have that $U' = P < 0$, see (3.1). We only need to find a_i and V_{i+1} such that $D_i := Q - a_i P < 0$ for $V_{i+1} \leq v \leq V_i$ for $i = 1, 2, ...m - 1$. Substituting $u = \frac{v}{a_i} - \frac{b_i}{a_i}$ into D_i, a simple calculation shows that

$$D_i = (2v - v^2 - 2u + 10) - a_i(uv + u - 5)$$
$$= -2v^2 + \left(b_i + 1 - \frac{2}{a_i} \right) v + 10 + 5a_i + b_i + \frac{2b_i}{a_i} \qquad (3.14)$$

for $i = 1, 2, ..., m - 1$.

Taking $i = 1$, $a_1 = 1$ and $b_1 = V_1 - U_1$ in (3.14), we then obtain D_1. We show that $D_1 < 0$ for $v \geq -0.8$. Indeed, D_1 is approximately

$$-2v^2 - 7.7719v - 5.3157$$

which is negative for $v \geq -0.8$. Counting the cut-off errors, we still have $D_1 < 0$ for $v \geq -0.8$.

We now take l_3 to be the segment of $v = a_1 u + b_1$ where $V_2 = -0.8 \leq v \leq V_1$ and

$$U_2 = \frac{1}{a_1}(V_2 - b_1)$$

which is approximately 5.9719.

We then continue with the next line segment $v = a_2 u + b_2$ from (U_2, V_2). Taking $i = 2$, $a_2 = 0.6$ and $b_2 = V_2 - a_2 U_2$, in (3.14), D_2 is constructed. We show that $D_2 < 0$ for $-2 \leq v \leq -0.8$. Indeed, D_2 is approximately

$$-2v^2 - 6.7165v - 5.9938$$

which is negative for $v \in [-2, -0.8]$. Counting the cut-off errors, we still have $D_2 < 0$ for $v \in [-2, -0.8]$.

We take l_4 to be the segment of $v = a_2 u + b_2$ where $V_3 = -2 \leq v \leq V_2 = -0.8$ and

$$U_3 = \frac{1}{a_2}(V_3 - b_2)$$

which is approximately 3.9719.

We complete verifying that on all sides of the domain boundary $\partial\Omega$, vector fields of system (3.1) point outside of the domain Ω except l_2 which is on the ellipse (3.3), see Fig. 1. Therefore, all the orbits enter Ω through l_2. These orbits are necessarily originated from $(5, 0)$, the only critical point inside the ellipse.

Furthermore, it can be calculated that the slope of the stable manifold at the saddle point $(1, 4)$ is $-5 - \frac{1}{2} - \frac{\sqrt{113}}{2}$ which is less than -5, the slope of l_1 there, see (3.2). Hence this orbit is the desired spiral–saddle connection between $(1, 4)$ and $(5, 0)$. $\quad\square$

We next turn to prove case (ii) in Theorem 1.1. The two equations in system (1.1) have equal viscous coefficients, i.e., $\epsilon = 1$, in such a case the connection is possible only when

$$u_+ < \frac{1}{2}u_-.$$

For a typical pair of far field states consisting of an unstable spiral and a saddle, we have the following.

Lemma 3.2. *Let $\epsilon = 1$ in system (1.1). Given that $(3, 0) \in E$ and $(1, 2) \in H$, then there exists a unique traveling wave $(U, V)(x + t)$ connecting $(u_-, v_-) = (3, 0)$ to $(u_+, v_+) = (1, 2)$.*

Proof. A similar analysis applies well to the case $\epsilon = 1$, the construction of the domain Ω is even simpler. For

$$(u_-, v_-) = (3, 0) \in E, \quad (u_+, v_+) = (1, 2) \in H,$$

the corresponding traveling wave speed is $s = -1$. In such a case, the traveling wave system becomes

$$U' = UV + U - 3, \tag{3.15a}$$

$$V' = V - V^2 - U + 3. \tag{3.15b}$$

This system has only two isolated critical points $(u_-, v_-) = (3, 0)$ and $(u_+, v_+) = (1, 2)$. The coefficient matrix and eigenvalues of system (3.15), when linearized about (u^*, v^*), are respectively

$$\begin{pmatrix} v^* + 1 & u^* \\ -1 & 1 - 2v^* \end{pmatrix}, \quad \lambda_\pm = \frac{2 - v^*}{2} \pm \frac{\sqrt{9(v^*)^2 - 4u^*}}{2}.$$

Thus $(u_-, v_-) = (3, 0)$ is an unstable spiral since $\lambda_\pm = 1 \pm \sqrt{3}i$, and $(u_+, v_+) = (1, 2)$ is a saddle since $\lambda_\pm = \pm 2\sqrt{2}$. According to the phase plane theory, any non-trivial orbit, if it exists, necessarily connects the two critical points, unless it is a limit cycle. Also the connecting orbit must tend to (u_+, v_+) at $+\infty$ and (u_-, v_-) at $-\infty$.

We follow the same argument as in the proof of Lemma 3.1 to show the existence of the connecting orbit.

We draw an ellipse centered at $(3, 0)$, determined by $H_2(u, v) = 0$ with

$$H_2(u, v) = \frac{1}{3}(u - 3)^2 + v^2 - 1.$$

We now look at the vector field of system (3.15) on $H_2(U, V) = 0$:

$$\begin{aligned}
\frac{d}{dz}H_2(U, V) &= \frac{2}{3}(U - 3)U' + 2VV' \\
&= \frac{2}{3}(V + 1)(U - 3)^2 + 2V^2(1 - V) \\
&> 0
\end{aligned}$$

for $-1 \le V \le 1$ and $0 \le U \le 3$. Hence all orbits of system (1.1) within the ellipse will leave the region.

Consider the closed domain Ω enclosed by the following:

$$l_1 = \{(u, v)| \quad u = \frac{3}{v+1}, \quad 0 < v_e \leq v \leq 2\}, \tag{3.16}$$

$$l_2 = \{(u, v)| \quad \text{left branch of } H_2(u, v) = 0, \quad -1 \leq v \leq v_e\}, \tag{3.17}$$

$$l_3 = \{(u, v)| \quad v = -1, \ 1 \leq u \leq 3\}, \tag{3.18}$$

$$l_4 = \{(u, v)| \quad u = 1, \ -1 \leq v \leq 2\} \tag{3.19}$$

where (u_e, v_e) is the upper intersection point of the nullcline $uv + u - 3 = 0$ and the ellipse $H_2(u, v) = 0$. We also see that l_3 is tangent to $H_2(u, v) = 0$ at $(3, -1)$ and that l_4 intersects with l_1 at $(1, 2)$.

It can be verified that on l_1, l_3 and l_4 all trajectories of system (1.1) leave Ω while on l_2 trajectories enter the domain Ω. The existence of the connecting orbit then follows from the same argument in Lemma 3.1 in case (i). $\quad\square$

The proof of Theorem 1.1 is thus completed.

4. Instability of the traveling wave solutions

In this section, we study the stability of traveling wave solutions $(U(x - st), V(x - st))$ of system (1.1) including those oscillatory ones obtained in Section 2.

Consider the Cauchy problem of system (1.1) in the moving coordinate $z = x - st$

$$\partial_t u - s\partial_z u + \partial_z(uv) = \partial_z^2 u, \tag{4.1a}$$

$$\partial_t v - s\partial_z v - \partial_z(u + \epsilon v^2) = \epsilon \partial_z^2 v \tag{4.1b}$$

with initial data

$$(u, v)(z, 0) = (u_0, v_0)(z)$$

where $(u_0, v_0)(z)$ is a perturbation of the traveling wave solution $(U(z), V(z))$ of the form

$$(u, v)(z, 0) = (U, V)(z) + (\phi_z(z, 0), \psi_z(z, 0)) \tag{4.2}$$

where (ϕ, ψ) is the antiderivatives of the perturbation $(u - U, v - V)$.

Then the solution to the Cauchy problem of conservation laws (1.1) can be written as

$$(u, v)(z, t) = (U, V)(z) + (\phi_z, \psi_z). \tag{4.3}$$

Upon substitution and integration in z once, the linear part of the resulting system yields

$$\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = L \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

where

$$L = \begin{pmatrix} \partial_z^2 + (s - V)\partial_z & -U \\ \partial_z & \epsilon\partial_z^2 + (s + 2\epsilon V)\partial_z \end{pmatrix}$$

which is a closed operator in certain Sobolev spaces, say

$$X = \{(\phi, \psi)| \quad (\phi, \psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\}.$$

By standard spectral theory [1, Appendix of Chapter 5], the boundary of the essential spectrum of L can be described by that of its limiting operators $L^\pm$ as $z \to \pm\infty$

$$L^\pm = \begin{pmatrix} \partial_z^2 + (s - v_\pm)\partial_z & -u_\pm \\ \partial_z & \epsilon\partial_z^2 + (s + 2\epsilon v_\pm)\partial_z \end{pmatrix}.$$

The boundary of the essential spectrum of $L^\pm$ is described by the following curves

$$S^\pm = \{\lambda \in \mathbb{C}| \quad \det(\lambda I - A^\pm(\xi)) = 0, \text{ for some } \xi \in \mathbb{R}\}$$

where

$$A^\pm(\xi) = \begin{pmatrix} -\xi^2 + i(s - v_\pm)\xi & -u_\pm \\ i\xi & -\epsilon\xi^2 + i(s + 2\epsilon v_\pm)\xi \end{pmatrix}.$$

Let $\lambda \in S^\pm$, then

$$\lambda^2 + a(\xi)\lambda + b(\xi) = 0$$

where

$$a(\xi) = (\epsilon + 1)\xi^2 - i(2s + (2\epsilon - 1)v_\pm)\xi,$$
$$b(\xi) = iu_\pm\xi + \epsilon\xi^4 - i\xi^3((\epsilon + 1)s + \epsilon v_\pm) - (s - v_\pm)(s + 2\epsilon v_\pm)\xi^2.$$

A detailed calculation shows that

$$Re(\lambda) < 0 \Leftrightarrow \Lambda(\xi) < 0$$

where

$$\Lambda(\xi) = (Im(b))^2 - 4Re(b)(Re(a))^2 + 2(Re(a)Im(a))^2 - 4Re(a)Im(a)Im(b).$$

Note that

$$Re(a) = (\epsilon + 1)\xi^2,$$
$$Im(a) = -(2s + (2\epsilon - 1)v_\pm)\xi,$$
$$Re(b) = \epsilon\xi^4 - (s - v_\pm)(s + 2\epsilon v_\pm)\xi^2,$$
$$Im(b) = u_\pm\xi - \xi^3((\epsilon + 1)s + \epsilon v_\pm).$$

Substitution and rearrangement of terms lead to

$$\Lambda(\xi) = \xi^2 \left[-4\epsilon(\epsilon+1)^2 \xi^6 + F\xi^4 + 2u_{\pm}\xi^2(3(\epsilon+1)s + (4\epsilon^2+\epsilon-2)v_{\pm}) + u_{\pm} \right]$$

where

$$F = 2(\epsilon+1)^2(2s + (2\epsilon-1)v_{\pm})^2 - 4(\epsilon+1)(2s + (2\epsilon-1)v_{\pm})((\epsilon+1)s + \epsilon v_{\pm})$$
$$+ ((\epsilon+1)s + \epsilon v_{\pm})^2 + 4(\epsilon+1)^2(s - v_{\pm})(s + 2\epsilon v_{\pm}).$$

This implies that for $u_{\pm} > 0$ and $\epsilon > 0$, then $\Lambda(\xi) < 0$ for large $|\xi|$, and $\Lambda(\xi) > 0$ for small $|\xi|$. Therefore $Re(\lambda)$ changes its sign for $u_{\pm} > 0$ and $\epsilon > 0$.

We thus prove the following theorem which asserts that the traveling wave solutions $(U(z), V(z))$ of system (1.1), when in existence, are linearly unstable against perturbations (4.2) as claimed in Theorem 1.2.

Theorem 4.1. *For $u_{\pm} > 0$ and $\epsilon > 0$, then*

$$\sigma(L) \cap \{\lambda \in \mathbb{C} \mid Re(\lambda) > 0\} \neq \emptyset$$

where $\sigma(L)$ is the spectrum of L in X.

Acknowledgments

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