



# Heat equation on the Heisenberg group: Observability and applications <sup>☆</sup>

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## Abstract

We investigate observability and Lipschitz stability for the Heisenberg heat equation on the rectangular domain

$$\Omega = (-1, 1) \times \mathbb{T} \times \mathbb{T}$$

taking as observation regions slices of the form  $\omega = (a, b) \times \mathbb{T} \times \mathbb{T}$  or tubes  $\omega = (a, b) \times \omega_y \times \mathbb{T}$ , with  $-1 < a < b < 1$ . We prove that observability fails for an arbitrary time  $T > 0$  but both observability and Lipschitz stability hold true after a positive minimal time, which depends on the distance between  $\omega$  and the boundary of  $\Omega$ :

$$T_{\min} \geq \frac{1}{8} \min\{(1+a)^2, (1-b)^2\}.$$

Our proof follows a mixed strategy which combines the approach by Lebeau and Robbiano, which relies on Fourier decomposition, with Carleman inequalities for the heat equations that are solved by the Fourier modes. We extend the analysis to the unbounded domain  $(-1, 1) \times \mathbb{T} \times \mathbb{R}$ .

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## 1. Introduction

This article focuses on the heat equation on the Heisenberg group. This equation is usually written in the following way

$$\left(\partial_t - \left(\partial_{x_1} - \frac{x_2}{2}\partial_{x_3}\right)^2 - \left(\partial_{x_2} + \frac{x_1}{2}\partial_{x_3}\right)^2\right)G(t, x_1, x_2, x_3) = 0. \quad (1.1)$$

For technical reasons that will be explained later (see Section 1.4), we will rather work on the following formulation

$$(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2)g = 0 \quad (1.2)$$

which is equivalent to the previous one, by the change of variables and functions

$$G(t, x_1, x_2, x_3) = g\left(t, x = x_1, y = x_2, z = x_3 + \frac{x_1 x_2}{2}\right). \quad (1.3)$$

Thus, this article focuses on the system

$$\begin{cases} (\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2)g = \tilde{h} & \text{in } (0, T) \times \Omega, \\ g(t, \pm 1, y, z) = 0, & (t, y, z) \in (0, T) \times \mathbb{T} \times \mathbb{T}, \\ g(0, x, y, z) = g^0(x, y, z), & (x, y, z) \in \Omega, \end{cases} \quad (1.4)$$

where  $\mathbb{T}$  is the 1D-torus and  $\Omega = (-1, 1) \times \mathbb{T} \times \mathbb{T}$ . In section 2, we will give the precise notion of weak solution to problem (1.4) for

$$g^0 \in L^2(\Omega) \quad \text{and} \quad \tilde{h} \in L^2((0, T) \times \Omega).$$

For the above problem, we will investigate observability and Lipschitz stability. We recall the definition of these two notions below and we state our main results.

### 1.1. Observability and null controllability

**Definition 1** (*Observability*). Let  $T > 0$  and  $\omega$  be an open subset of  $\Omega$ . System (1.4) is *observable in  $\omega$  in time  $T$*  if there exists a constant  $C_T > 0$  such that, for every  $g^0 \in L^2(\Omega)$ , the solution of (1.4) with  $\tilde{h} \equiv 0$  satisfies

$$\int_{\Omega} |g(T, x, y, z)|^2 dx dy dz \leq C_T \int_0^T \int_{\omega} |g(t, x, y, z)|^2 dx dy dz dt. \quad (1.5)$$

**Theorem 1.** *Let*

$$\omega := (a, b) \times \omega_y \times \mathbb{T}, \quad (1.6)$$

where  $-1 < a < b < 1$  and  $\omega_y$  is an open subset of  $\mathbb{T}$ .

Then, there exists  $T_{\min} \geq \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$  such that

- for every  $T > T_{\min}$ , system (1.4) is observable in  $\omega$  in time  $T$ ,
- for every  $T < T_{\min}$ , system (1.4) is not observable in  $\omega$  in time  $T$ .

It is well-known that the Heisenberg laplacian

$$A := -\partial_x^2 - (x\partial_z + \partial_y)^2 \quad (1.7)$$

is an hypoelliptic operator of the form  $X_1^2 + X_2^2$ , where

$$X_1(x, y, z) := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2(x, y, z) := \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix},$$

see [24]. However, no clear connection between hypoellipticity and observability has been established so far.

We observe that, given the width  $\ell = b - a \in (0, 2)$ , there is no location of the slice  $\omega = (a, b) \times \mathbb{T} \times \mathbb{T}$  for which the minimal observability time  $T_{\min}$  vanishes. Such a behavior differs from the one observed for the 2D Grushin case

$$\begin{cases} (\partial_t - \partial_x^2 - x^2 \partial_y^2)g = 0 & \text{in } \Omega_G := (0, T) \times (-1, 1) \times (0, 1), \\ g(t, x, y) = 0, & (t, y, z) \in (0, T) \times \partial\Omega_G \\ g(0, x, y) = g^0(x, y), & (x, y) \in \Omega_G, \end{cases}$$

for which

- $T_{\min} > 0$  when  $\omega = (a, b) \times (0, 1)$  and  $a > 0$  (see [3]),
- $T_{\min} = 0$  when  $\omega = (0, b) \times (0, 1)$  (see [6]).

This difference may be related to the fact that, for the Heisenberg operator, the number of iterated Lie brackets of the vector fields required to generate  $\mathbb{R}^3$  has no jump at  $\{x = 0\}$ :  $X_1$ ,  $X_2$  and  $[X_1, X_2]$  are needed everywhere.

As usual, by the Hilbert uniqueness method (see [34,20]), the observability result of Theorem 1 is equivalent to the following null controllability result.

**Definition 2 (Null controllability).** Let  $T > 0$  and  $\omega$  be an open subset of  $\Omega$ . System (1.4) is said to be *null controllable from  $\omega$  in time  $T$*  if, for every  $g^0 \in L^2(\Omega)$ , there exists  $\tilde{h} \in L^2((0, T) \times \Omega)$ , supported on  $[0, T] \times \omega$ , such that the solution of (1.4) satisfies  $g(T, \cdot) = 0$ .

**Theorem 2.** Let  $\omega$  be as in (1.6). Then, there exists

$$T_{\min} \geq \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$$

such that

- for every  $T > T_{\min}$ , system (1.4) is null controllable from  $\omega$  in time  $T$ ,
- for every  $T < T_{\min}$ , system (1.4) is not null controllable from  $\omega$  in time  $T$ .

## 1.2. Lipschitz stability

In (1.4), we consider a source term of the form

$$\begin{aligned} \tilde{h}(t, x, y, z) &= R(t, x)h(x, y, z) \\ \text{where } R &\in \mathcal{C}([0, T] \times [-1, 1]) \text{ and } h \in L^2(\Omega), \end{aligned} \quad (1.8)$$

and we are interested in the following inverse source problem: is it possible to recover the source term  $h$  from the measurement of  $\partial_t g|_{(T_1, T_2) \times \omega}$  where  $\omega$  is a nonempty open subset of  $\Omega$ ? Precisely, we will obtain Lipschitz stability estimates for (1.4) in the following sense.

**Definition 3** (Lipschitz stability). Let  $T > 0$ , let  $0 \leq T_0 < T_1 \leq T$ , and let  $\omega$  be an open subset of  $\Omega$ . We say that system (1.4), with  $\tilde{h}$  as in (1.8), satisfies a *Lipschitz stability estimate* on  $(T_0, T_1) \times \omega$  if there exists a constant  $\tilde{C}_T > 0$  such that, for every  $g^0 \in L^2(\Omega)$  and  $h \in L^2(\Omega)$ , the solution of (1.4) satisfies

$$\begin{aligned} &\int_{\Omega} |h(x, y, z)|^2 dx dy dz \\ &\leq \tilde{C}_T \left( \int_{T_0}^{T_1} \int_{\omega} |\partial_t g(t, x, y, z)|^2 dx dy dz dt + \int_{\Omega} |Ag(T_1, x, y, z)|^2 dx dy dz \right), \end{aligned}$$

where  $A$  is defined in (1.7).

Notice that the above Lipschitz stability estimate implies the uniqueness of the source term  $h$  via 2 measurements:  $\partial_t g|_{(T_0, T_1) \times \omega}$  and  $Ag(T_1, \cdot)$ .

When  $\omega$  is a slice, parallel to the  $(y, z)$ -plane, we can prove Lipschitz stability in large time under general assumptions on  $R$ .

**Theorem 3.** Let  $a, b \in \mathbb{R}$  be such that  $-1 < a < b < 1$  and  $\omega := (a, b) \times \mathbb{T} \times \mathbb{T}$ . Suppose further that

$$\begin{aligned} &R, \partial_t R \in \mathcal{C}([0, T] \times [-1, 1]) \text{ and} \\ &\exists T_1 \in (0, T] \text{ and } R_0 > 0 \text{ such that } R(T_1, x) \geq R_0, \forall x \in [-1, 1]. \end{aligned} \quad (1.9)$$

Then, there exists  $T^* > 0$  such that system (1.4) satisfies a Lipschitz stability estimate on  $(T_0, T_1) \times \omega$  for every  $T_0, T_1 \in [0, T]$  with  $(T_1 - T_0) > T^*$ .

More generally, when  $\omega$  is a tube along the  $z$ -axis, we can still prove Lipschitz stability in large time under an additional smallness assumption on the source term, which is probably due just to technical reasons.

**Theorem 4.** *Let  $\omega$  be as in (1.6). There exists  $T^* > 0$  and a continuous function  $\eta : (T_*, \infty) \rightarrow (0, \infty)$  such that, if  $R$  satisfies (1.9),  $T_0, T_1 \in [0, T]$ ,  $(T_1 - T_0) > T^*$ , and*

$$\frac{1}{\rho_0} \left( \int_{T_0}^{T_1} \|\partial_t R(t)\|_{L^\infty(-1,1)}^2 dt \right)^{\frac{1}{2}} < \eta(T_1 - T_0), \quad (1.10)$$

*then system (1.4) satisfies a Lipschitz stability estimate on  $(T_0, T_1) \times \omega$ .*

### 1.3. Motivations and bibliographical comments

#### 1.3.1. Motivations

The relevance of the Heisenberg group to quantum mechanics has long been acknowledged. Indeed, it was recognized by Weyl [38] that the Heisenberg algebra generated by the momentum and position operators comes from a Lie algebra representation associated with a corresponding group—namely the Heisenberg group (Weyl group in the traditional language of physicists). In such a group, the role played by the so-called Heisenberg laplacian is absolutely central, being analogous to the standard laplacian in Euclidean spaces, see [22]. On an even larger scale, deep connections have been pointed out between the properties of subriemannian operators, like the Heisenberg laplacian, and other topics of interest to current mathematical research such as isoperimetric problems and systems theory, see, for instance, [19].

#### 1.3.2. Observability

Observability is well known to hold for the linear heat equation in arbitrary positive time  $T$  with any observation domain  $\omega$  (see [21, Theorem 3.3], [31] and [23]). Degenerate parabolic equations exhibit a wider range of behaviors: observability may hold true or not depending on the type of degeneracy. For instance, the case of degenerate parabolic equations on the boundary of the domain in one space dimension is well understood (see [14,15,1,35,12], and [11]). Fewer results are available for multidimensional problems, see [16].

As for parabolic equations with interior degeneracy, a fairly complete analysis is available for Grushin type operators

$$\begin{cases} \partial_t g - \Delta_x g - |x|^{2\gamma} \Delta_y g = 0 & \text{in } (0, \infty) \times \Omega, \\ g(t, x, y) = 0 & (t, x, y) \in (0, \infty) \times \partial\Omega, \\ g(0, x, y) = g^0(x, y), & (x, y) \in \Omega, \end{cases} \quad (1.11)$$

where  $\Omega := \Omega_1 \times \Omega_2$ ,  $\Omega_1$  is a bounded open subset of  $\mathbb{R}^{N_1}$  such that  $0 \in \Omega_1$ ,  $\Omega_2$  is a bounded open subset of  $\mathbb{R}^{N_2}$ ,  $N_1, N_2 \in \mathbb{N}^* := \{1, 2, 3, \dots\}$ , and  $\gamma > 0$ . Indeed, it has been proved [3,4] that the observability inequality:

- holds in any positive time  $T > 0$  with an arbitrary open set  $\omega \subset \Omega$  when  $\gamma \in (0, 1)$ ,

- holds only in large time  $T > T_{\min} > 0$  when  $\gamma = 1$  and  $\omega := \omega_1 \times \Omega_2$  is a strip parallel to the  $y$ -axis not containing the line segment  $x = 0$ , and
- does not hold when  $\gamma > 1$ .

Moreover, the value of  $T_{\min}$  has been explicitly computed for suitable observation regions  $\omega$ , see [6]. The above observability properties may be changed by adding a zero order term with singular coefficient, see [13] and [36]. Similar results have been obtained for Kolmogorov type equations, see [2,7,5].

### 1.3.3. Lipschitz stability

Our formulation of the inverse problem corresponds to a single measurement (see also Bukhgeim and Klibanov [10] who first proposed a methodology based on Carleman estimates). Following [10], many works have been published on this subject. For uniformly parabolic equations we can refer the reader, for example, to Imanuvilov and Yamamoto [26], Isakov [27], Klibanov [28], Yamamoto [39], and the references therein (the present list of references is by no means complete). Inverse problems for boundary-degenerate parabolic equations were studied by Cannarsa, Tort and Yamamoto [17,18]. For Grushin type equations, the inverse source problem was addressed in [4], and an inverse coefficient problem in [8].

### 1.4. Strategy and structure of this article

The strategy developed in this article relies on

- the use of partial Fourier transform on the solution of the 3D-PDE under study,
- a precise analysis of the independent 1D PDEs solved by the Fourier modes, involving appropriate 1D Carleman estimates.

In particular, formulation (1.2) allows to take partial Fourier transform with respect to the 2 space variables  $y$  and  $z$  and to work on a family of independent 1D heat equations on the variable  $x$ , for which this program can be completed. On the contrary, because of the presence of coefficients  $x_2$  and  $x_1$ , equation (1.1) allows to take partial Fourier transform only with respect to the space variable  $x_3$ . Then, one obtains a family of independent 2D heat equations on the variables  $(x_1, x_2)$  with complex valued coefficients. However, we are not able to prove appropriate Carleman estimates on this family to conclude with this strategy. This is why this article focuses on the alternative formulation (1.2).

Sections 2 and 3 are devoted to preliminary results concerning the well posedness of (1.4), the Fourier decomposition of its solutions, and the dissipation speed of the Fourier modes.

In Section 4, we state a Carleman estimate for a 1D-heat equation with parameters  $(n, p)$ , solved by the Fourier modes of the solution of (1.4).

In Section 5, we prove Lipschitz stability with observation on a slice parallel to the  $(y, z)$ -plane (Theorem 3). In this configuration, it is equivalent to prove the uniform Lipschitz stability of the 1D heat equations solved by the Fourier modes i.e. with a constant that does not depend on the Fourier frequency  $(n, p)$ . This is obtained by applying the Carleman estimate of Section 4.1 to the time derivative of the Fourier mode, and by combining it with the dissipation speed of Section 3.2.

In Section 6, we prove Lipschitz stability with observation on a tube parallel to the  $z$ -axis (Theorem 4). The proof relies on a variation of the Lebeau–Robbiano’s method, that was intro-

duced to prove observability inequalities in [31]. Here, we adapt it to prove Lipschitz stability estimates, in the same spirit as in [4].

In Section 7, we prove that observability holds only in large time (Theorem 1). The positive result in large time is a direct consequence of the previous analysis. The negative result in small time is obtained by designing an explicit counterexample, involving eigenfunctions of the Heisenberg operator.

In Section 8, we state and justify analogous results for the Heisenberg heat equation on  $(-1, 1) \times \mathbb{T} \times \mathbb{R}$ . Such a formulation allows to use the above theory to treat the Heisenberg equation written in the alternative form (8.1).

Finally, in Section 9, we discuss conclusions and open problems.

This paper has much to do with estimates. So, keeping track of all constants is definitely an issue. That is why we shall use capital letters, possibly with a subscript, only for those constants  $C$  that are used in different parts of the article. Technical constants  $c$  that are used in a single proof will be labeled by lower case letters, possibly with a subscript.

## 2. Well-posedness and unique continuation

Without further specification, all functions are understood to be real-valued.

### 2.1. Well-posedness

In this section, we recall well-posedness and regularity results for problem (1.4). It is convenient to denote by  $L^2([-1, 1] \times \mathbb{T} \times \mathbb{T})$ , or briefly  $L^2(\Omega)$ , the space of all (equivalence classes of) Lebesgue-measurable functions  $u : [-1, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  such that, for all  $h, k \in \mathbb{Z}$ ,

$$u(x, y + 2h\pi, z + 2k\pi) = u(x, y, z) \quad (x, y, z) \in [-1, 1] \times \mathbb{R} \times \mathbb{R} \text{ a.e.} \quad (2.1)$$

and

$$\|u\|^2 := \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} |u(x, y, z)|^2 dz < \infty.$$

$L^2(\Omega)$  is a Hilbert space over  $\mathbb{R}$  with scalar product

$$\langle u, v \rangle = \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} u(x, y, z)v(x, y, z) dz \quad \forall u, v \in L^2(\Omega).$$

Such a space will be also denoted by  $H$ . Now, consider the dense subspace  $C_{(0)}^\infty(\Omega)$  of  $H$  which consists of all functions  $u \in C^\infty([-1, 1] \times \mathbb{R} \times \mathbb{R})$  satisfying (2.1) such that, for some  $r \in [0, 1)$ ,

$$u(x, y, z) = 0 \quad \forall (x, y, z) \in ([-1, 1] \setminus [-r, r]) \times \mathbb{R} \times \mathbb{R}.$$

The bilinear form  $(\cdot, \cdot) : C_{(0)}^\infty(\Omega) \times C_{(0)}^\infty(\Omega) \rightarrow \mathbb{R}$  defined by

$$(u, v) = \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} \{ \partial_x u \partial_x v + (\partial_y u + x \partial_z u)(\partial_y v + x \partial_z v) \} dz$$

is positive definite because, for all  $u \in C_{(0)}^{\infty}(\Omega)$  we have

$$\|u\|^2 \leq 4 \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} |\partial_x u|^2 dz \leq 4(u, u). \quad (2.2)$$

Denoting by  $|\cdot|$  the norm associated with the scalar product  $(\cdot, \cdot)$ , we introduce the space  $H_{(0)}^1(\Omega)$ , or  $V$ , as the closure of  $C_{(0)}^{\infty}(\Omega)$  with respect to  $|\cdot|$ . Observe that two bounded linear operators  $X_1, X_2 : V \rightarrow H$  are defined by

$$X_1 u = \lim_{k \rightarrow \infty} \partial_x u_k \quad \text{and} \quad X_2 u = \lim_{k \rightarrow \infty} (\partial_y u_k + x \partial_z u_k),$$

where  $\{u_k\}_k$  is any sequence in  $C_{(0)}^{\infty}(\Omega)$  such that  $|u_k - u| \rightarrow 0$  as  $k \rightarrow \infty$ . Moreover,

$$\int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} (X_1 u) v dz = - \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} u \partial_x v dz,$$

and

$$\int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} (X_2 u) v dz = - \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} u (\partial_y v + x \partial_z v) dz,$$

for all  $u \in V$  and  $v \in C_{(0)}^{\infty}(\Omega)$ . Also, the inequality

$$\|u\| \leq 2 \|X_1 u\| \quad \forall u \in V$$

readily follows from (2.2). So,  $V$  is a Hilbert space with the scalar product

$$(u, v) = \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} \{ (X_1 u)(X_1 v) + (X_2 u)(X_2 v) \} dz \quad \forall u, v \in V.$$

Following a well-known procedure [33] we can introduce the *regularly accretive operator*  $A : D(A) \subset H \rightarrow H$  defined by

$$\begin{cases} D(A) = \{u \in V : \exists C > 0 \text{ such that } |(u, v)| \leq C \|v\|, \forall v \in V\} \\ Au = f \quad \forall u \in D(A), \end{cases} \quad (2.3)$$

where  $f$  is the unique element of  $H$  associated (via the Riesz isomorphism) with the extension to  $H$  of the bounded linear functional  $v \mapsto (u, v)$ . Observe that  $D(A)$  is dense in  $H$  because it



contains  $C_{(0)}^\infty(\Omega)$ . Therefore,  $A$  is a positive self-adjoint operator on  $H$  satisfying  $D(A^{1/2}) = V$  ([37, Theorem 2.2.3]), and  $-A$  generates an analytic semigroup of contractions on  $H$  ([37, Theorem 3.6.1]) that will be denoted by  $S(t)$ .

For every  $g^0 \in H$  and  $\tilde{h} \in L^2(0, T; H)$ , problem (1.4) can be recast as follows

$$\begin{cases} g'(t) + Ag(t) = \tilde{h}(t) & t \in (0, T) \\ g(0) = g^0. \end{cases} \quad (2.4)$$

The function  $g \in C^0([0, T]; H) \cap L^2(0, T; V)$  given by

$$g(t) = S(t)g^0 + \int_0^t S(t-s)\tilde{h}(s)ds \quad t \in [0, T]$$

is called the *mild solution* of (2.4). It is well known that the mild solution of (2.4) is also a *weak solution* in the following sense: for every  $v \in D(A)$

- the function  $\langle g(\cdot), v \rangle$  is absolutely continuous on  $[0, T]$ , and
- for a.e.  $t \in [0, T]$

$$\frac{d}{dt} \langle g(t), v \rangle + \langle g(t), Av \rangle = \langle \tilde{h}(t), v \rangle. \quad (2.5)$$

Note that, as showed in [33], condition (2.5) is equivalent to the definition of solution by transposition, that is,

$$\begin{aligned} & \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} \left\{ g(\tau, x, y, z) \varphi(\tau, x, y, z) - g^0(x, y, z) \varphi(0, x, y, z) \right\} dz \\ &= \int_0^{\tau} dt \int_{-1}^1 dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} g \left\{ \partial_x^2 \varphi - (x \partial_z + \partial_y)^2 \varphi \right\} dz \end{aligned}$$

for every  $\tau \in (0, T)$  and every function  $\varphi \in \mathcal{C}^2([0, T] \times [-1, 1] \times \mathbb{T} \times \mathbb{T})$ .

The following proposition describes well-known properties of mild solutions that follow from the analyticity of  $S(t)$ .

**Proposition 1.** *For every  $g^0 \in H$ ,  $T > 0$ , and  $\tilde{h} \in L^2(0, T; H)$ , the mild solution  $g$  of the Cauchy problem (2.4) satisfies*

$$\|g(t)\| \leq \|f_0\| + \sqrt{T} \|\tilde{h}\|_{L^2(0, T; H)} \quad \forall t \in [0, T]. \quad (2.6)$$

Moreover, for every  $\tau \in (0, T]$ ,

$$g \in H^1(\tau, T; H) \cap \mathcal{C}([\tau, T]; V) \cap L^2(\tau, T; D(A)).$$

In particular,  $g(t) \in D(A)$  and  $g'(t) \in H$  for a.e.  $t \in [0, T]$ .

## 2.2. Unique continuation

Observe that, in particular, (1.5) yields a unique continuation property for (1.4). The following more general result, which is a consequence of Holmgren's uniqueness theorem, suggests that no obstruction to observability should be expected for problem (1.4). The proof is given in the Appendix A.

**Proposition 2.** *Let  $T > 0$  and let  $\omega$  be as in (1.6). Any solution*

$$g \in C^0([0, T], L^2(\Omega))$$

*of (1.4) with  $\tilde{h} = 0$ , which vanishes on  $(0, T) \times \omega$  is identically zero.*

## 3. Fourier decomposition and dissipation

### 3.1. Fourier decomposition

We are now going to study the Fourier decomposition of the solution of (1.4). For this purpose, for any  $(n, p) \in \mathbb{Z}^2$  let us consider the operator

$$A_{n,p} : D(A_{n,p}) \subset L^2(-1, 1; \mathbb{C}) \rightarrow L^2(-1, 1; \mathbb{C})$$

defined by

$$\begin{cases} D(A_{n,p}) = H^2 \cap H_0^1(-1, 1; \mathbb{C}) \\ A_{n,p}u(x) = -u''(x) + (px + n)^2u(x) \end{cases} \quad \forall u \in D(A_{n,p}). \quad (3.1)$$

It is well known that  $A_{n,p}$  is a positive self-adjoint operator on  $L^2(-1, 1; \mathbb{C})$  and  $-A_{n,p}$  generates an analytic semigroup of contractions. The notion of mild/weak solutions of the evolution equation associated with  $A_{n,p}$ , that we recalled in section 2, is used in our next proposition.

**Proposition 3.** *Let  $g^0 \in H = L^2(\Omega)$ ,  $T > 0$ , and  $\tilde{h} \in L^2((0, T) \times \Omega)$ . Then the mild solution  $g$  of the Cauchy problem (1.4) satisfies, in  $L^2((0, T) \times \Omega)$ ,*

$$g(t, x, y, z) = \sum_{n,p \in \mathbb{Z}} g_{n,p}(t, x) e^{i(ny + pz)} \quad (3.2)$$

where

$$g_{n,p}(t, x) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} g(t, x, y, z) e^{-i(ny + pz)} dy dz \quad (3.3)$$

*belongs to  $C([0, T]; L^2(-1, 1; \mathbb{C})) \cap L^2(0, T; H_0^1(-1, 1; \mathbb{C}))$ . Moreover, for every  $(n, p) \in \mathbb{Z}^2$ ,  $g_{n,p}$  is the mild solution of the Cauchy problem*

$$\begin{cases} \left( \partial_t - \partial_x^2 + (px + n)^2 \right) g_{n,p}(t, x) = \tilde{h}_{n,p}(t, x), & (t, x) \in (0, T) \times (-1, 1), \\ g_{n,p}(t, \pm 1) = 0, & t \in (0, T), \\ g_{n,p}(0, x) = g_{n,p}^0(x), & x \in (-1, 1), \end{cases} \quad (3.4)$$

where

$$\tilde{h}_{n,p}(t, x) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \tilde{h}(t, x, y, z) e^{-i(ny+pz)} dy dz$$

and

$$g_{n,p}^0(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} g^0(x, y, z) e^{-i(ny+pz)} dy dz.$$

Furthermore, if

$$g_{n,p}^0 \in H^2 \cap H_0^1(-1, 1; \mathbb{C}) \quad \text{and} \quad \tilde{h}_{n,p} \in H^1(0, T; L^2(-1, 1; \mathbb{C})),$$

then the function  $v_{n,p} := \partial_t g_{n,p} \in \mathcal{C}([0, T]; L^2(-1, 1; \mathbb{C})) \cap L^2(0, T; H_0^1(-1, 1; \mathbb{C}))$  is the weak solution of

$$\begin{cases} \left( \partial_t - \partial_x^2 + (px + n)^2 \right) v_{n,p}(t, x) = \partial_t \tilde{h}_{n,p}(t, x), & (t, x) \in (0, T) \times (-1, 1), \\ v_{n,p}(t, \pm 1) = 0, & t \in (0, T), \\ v_{n,p}(0, x) = A_{n,p} g_{n,p}^0(x) + \tilde{h}_{n,p}(0, x), & x \in (-1, 1). \end{cases}$$

**Proof.** The relations (3.2) and (3.3) between  $g$  and the family of the Fourier coefficients of  $g(t, x, \cdot, \cdot)$  is justified by the fact that  $g \in C([0, T]; L^2(\Omega))$ . Thus, we just have to show that, for every  $(n, p) \in \mathbb{Z}^2$ ,  $g_{n,p}$  is the weak solution of problem (3.4). Since

$$g_{n,p} \in \mathcal{C}([0, T]; L^2(-1, 1; \mathbb{C})) \cap L^2(0, T; H_0^1(-1, 1; \mathbb{C}))$$

in view of (3.3), we just need to show that, for every  $\varphi \in H^2 \cap H_0^1(-1, 1; \mathbb{C})$ ,

- (i) the function  $t \mapsto \int_{-1}^1 g_{n,p}(t, x) \varphi(x) dx$  is absolutely continuous on  $[0, T]$ ,
- (ii) for a.e.  $t \in [0, T]$

$$\frac{d}{dt} \int_{-1}^1 g_{n,p}(t, x) \varphi(x) dx + \int_{-1}^1 g_{n,p}(t, x) A_{n,p} \varphi(x) dx = \int_{-1}^1 \tilde{h}(t, x) \varphi(x) dx.$$

Indeed, since

$$\int_{-1}^1 g_{n,p}(t, x) \varphi(x) dx = \frac{1}{(2\pi)^2} \int_{-1}^1 dx \int_{\mathbb{T}^2} g(t, x, y, z) e^{-i(ny+pz)} \varphi(x) dx dy dz,$$

property (i) follows from the fact that  $g$  is the weak solution of (2.4) and the real and imaginary parts,  $u$  and  $v$ , of the complex-valued function

$$w(x, y, z) := e^{-i(ny+pz)}\varphi(x) \quad (x, y, z) \in \Omega$$

belong to  $D(A)$ . As for property (ii), observe that by the same argument

$$\begin{aligned} & \frac{d}{dt} \int_{-1}^1 g_{n,p}(t, x) \varphi(x) dx + \int_{-1}^1 g_{n,p}(t, x) A_{n,p} \varphi(x) dx \\ &= \frac{1}{(2\pi)^2} \frac{d}{dt} \int_{-1}^1 dx \int_{\mathbb{T}^2} g(t, x, y, z) e^{-i(ny+pz)} \varphi(x) dy dz \\ & \quad + \frac{1}{(2\pi)^2} \int_{-1}^1 dx \int_{\mathbb{T}^2} g(t, x, y, z) e^{-i(ny+pz)} (-\varphi''(x) + (px+n)^2 \varphi(x)) dy dz \\ &= \frac{1}{(2\pi)^2} \frac{d}{dt} \int_{-1}^1 dx \int_{\mathbb{T}^2} g(t, x, y, z) (u(x, y, z) + i v(x, y, z)) dy dz \\ & \quad + \frac{1}{(2\pi)^2} \int_{-1}^1 dx \int_{\mathbb{T}^2} g(t, x, y, z) (A u(x, y, z) + i A v(x, y, z)) dy dz \\ &= \frac{1}{(2\pi)^2} \int_{-1}^1 dx \int_{\mathbb{T}^2} \tilde{h}(t, x, y, z) e^{-i(ny+pz)} \varphi(x) dy dz \\ &= \int_{-1}^1 \tilde{h}_{n,p}(t, x) \varphi(x) dx \end{aligned}$$

for a.e.  $t \in [0, T]$ . This completes the proof.  $\square$

### 3.2. Dissipation speed on $(-1, 1)$

For any  $(n, p) \in \mathbb{Z} \times \mathbb{Z}$ , we define

$$\lambda_{n,p} = \inf_{\varphi \in H_0^1(-1,1)} \left\{ \int_{-1}^1 [\varphi'(x)^2 + (px+n)^2 \varphi(x)^2] dx : \int_{-1}^1 \varphi(x)^2 dx = 1 \right\}. \quad (3.5)$$

**Proposition 4.** *The following inequalities hold:*

$$\lambda_{n,p} \geq \frac{1}{4}(|p| + 1), \quad \forall (n, p) \in \mathbb{Z} \times \mathbb{Z} \quad (3.6)$$

$$\lambda_{n,p} \geq \frac{n^2}{4}, \quad \forall (n, p) \in \mathbb{Z} \times \mathbb{Z} \text{ such that } |n| \geq 2|p|. \quad (3.7)$$

**Remark 1.** Observe that, when  $|n| \geq 2|p|$ , the dependence of  $\lambda_{n,p}$  is quadratic with respect to  $n$ . This is the key point to apply the Lebeau–Robbiano strategy with respect to the variable  $y$  ( $n$  has to be negligible with respect to  $\lambda_{n,p}$  when  $n \rightarrow \infty$  and  $p$  is fixed). This is no longer true when  $x$  is free to range in the whole space  $\mathbb{R}$  because of translation invariance.

**Proof of Proposition 4.** If  $p = 0$  then, by Poincaré inequality,  $\lambda_{n,0} \geq (\frac{\pi}{2})^2 + n^2$ . So, (3.6) and (3.7) hold true. Let now  $(n, p) \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$  and observe that, without loss of generality, one may assume  $p > 0$ . By the change of variable

$$\varphi(x) = \sqrt[4]{p} \psi \left( \tilde{x} = \sqrt{p} \left( x + \frac{n}{p} \right) \right),$$

from (3.5) we deduce that

$$\begin{aligned} \lambda_{n,p} &\geq \inf_{\varphi \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}, x^2 dx)} \left\{ \int_{\mathbb{R}} [|\varphi'|^2 + (px + n)^2 |\varphi|^2] dx : \int_{\mathbb{R}} |\varphi|^2 = 1 \right\} \\ &= p \inf_{\psi \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}, \tilde{x}^2 d\tilde{x})} \left\{ \int_{\mathbb{R}} [\psi'(\tilde{x})^2 + \tilde{x}^2 \psi(\tilde{x})^2] d\tilde{x} : \int_{\mathbb{R}} |\psi|^2 = 1 \right\}, \end{aligned}$$

where we have denoted by  $L^2(\mathbb{R}, x^2 dx)$  the space of all Lebesgue measurable functions  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_{\mathbb{R}} |\varphi(x)|^2 x^2 dx < \infty.$$

It is well known that the last infimum above equals 1: it is the first eigenvalue of the harmonic oscillator, associated with the eigenfunction  $e^{-\frac{\tilde{x}^2}{2}}$ . Then  $\lambda_{n,p} \geq p \geq \frac{1}{4}(|p| + 1)$ , i.e. formula (3.6) is proved.

Now, suppose  $|n| \geq 2p$ . Then for every  $x \in [-1, 1]$

$$|px + n| \geq |n| - p \geq \frac{n}{2}.$$

Thus

$$\lambda_{n,p} \geq \inf_{\varphi \in H_0^1(-1,1)} \left\{ \frac{n^2}{4} \int_{-1}^1 \varphi(x)^2 dx : \int_{-1}^1 \varphi(x)^2 dx = 1 \right\} = \frac{n^2}{4},$$

which proves (3.7).  $\square$

#### 4. 1D heat equations with parameters

In this section, we will prove several estimates for 1D heat equations with parameters which will be used in the proof of the main results of the paper.

##### 4.1. Carleman estimates

Let us set  $\mathbb{R}_+ = (0, \infty)$ . For a given  $T > 0$  and any  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+$ , we define the operator

$$\mathcal{P}_{n,p}g = \partial_t g - \partial_x^2 g + (px + n)^2 g$$

acting on functions  $g : [0, T] \times [-1, 1] \rightarrow \mathbb{C}$ .

**Proposition 5.** *Let  $a, b \in \mathbb{R}$  be such that  $-1 \leq a < b \leq 1$ . Then there exist a weight function  $\beta \in C^3([-1, 1]; \mathbb{R}_+)$  and positive constants  $C_1, C_2$  such that for any  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+$ , any  $T > 0$ , and any*

$$g \in C([0, T]; L^2(-1, 1)) \cap L^2(0, T; H_0^1(-1, 1))$$

the following inequality holds

$$\begin{aligned} C_1 \int_{0-1}^T \int_{-1}^1 \left( \frac{M}{t(T-t)} |\partial_x g|^2 + \frac{M^3}{[t(T-t)]^3} |g|^2 \right) e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \\ \leq \int_{0-1}^T \int_{-1}^1 |\mathcal{P}_{n,p}g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt + \int_0^T \int_a^b \frac{M^3}{[t(T-t)]^3} |g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \end{aligned} \quad (4.1)$$

where

$$M := C_2 \max\{T + T^2; (|n| + p)T^2\}. \quad (4.2)$$

In the appendix, we give a complete proof of the above Carleman estimate. In particular the dependency of  $M$  with respect to  $(n, p, T)$  is a key point to apply the Lebeau–Robbiano strategy in Section 6. Thus, tracking the dependency with respect to  $(n, p, T)$  of all the constants, in proof the Carleman estimate, is really needed.

**Remark 2.** The proof of the main results of this article only uses the above result for  $p \in \mathbb{Z}$ . However, we prefer to derive most of our preliminary results for  $p \in \mathbb{R}$  instead of  $p \in \mathbb{Z}$  in order to justify the generalization discussed in Section 8, where the domain is  $(-1, 1) \times \mathbb{T} \times \mathbb{R}$ .

##### 4.2. 1D observability inequality with source term

The goal of this section is the proof of the following result.

**Proposition 6.** Let  $a, b \in \mathbb{R}$  be such that  $-1 \leq a < b \leq 1$ . Then there exist constants  $C_3, C_4 > 0$  such that, for every  $T > 0$ ,  $p \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ ,  $g_{n,p}^0 \in L^2(-1, 1)$ , and  $\tilde{h}_{n,p} \in L^2([0, T] \times [-1, 1])$  the solution of (3.4) satisfies

$$\begin{aligned} \int_{-1}^1 |g_{n,p}(T, x)|^2 dx \leq e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 \min\{|p|, p^2\} T\right)} \int_0^T \int_a^b |g_{n,p}(t, x)|^2 dx dt \\ + \epsilon_{n,p}(T) \int_0^T \int_{-1}^1 |\tilde{h}_{n,p}(t, x)|^2 dx dt, \end{aligned} \quad (4.3)$$

for some constant  $\epsilon_{n,p}(T)$  satisfying

$$|\epsilon_{n,p}(T)| \leq \frac{C_3}{|p| + 1} + e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 \min\{|p|, p^2\} T\right)} =: \epsilon'_p(T) \quad (4.4)$$

for all  $(n, p) \in \mathbb{Z} \times \mathbb{R}$  and

$$|\epsilon_{n,p}(T)| \leq \frac{C_3}{n^2} + e^{C_3 \left(1 + \frac{1}{T} - C_4 n^2 T\right)} =: \epsilon''_n(T) \text{ if } |n| > 2|p|. \quad (4.5)$$

We will use the following preliminary result.

**Lemma 1.** For every  $0 \leq T_1 < T_2 < \infty$ ,  $(n, p) \in \mathbb{Z} \times \mathbb{R}$ ,  $g_{n,p}^0 \in L^2(-1, 1)$ , and  $\tilde{h}_{n,p} \in L^2([0, T] \times [-1, 1])$  the solution of (3.4) satisfies

$$\|g_{n,p}(T_2)\|^2 \leq 2\|g_{n,p}(T_1)\|^2 e^{-2\lambda_{n,p}(T_2-T_1)} + \frac{1}{\lambda_{n,p}} \int_{T_1}^{T_2} \|\tilde{h}_{n,p}(t)\|^2 dt$$

where  $\|\cdot\| = \|\cdot\|_{L^2(-1,1)}$ .

**Proof of Lemma 1.** By Duhamel's formula and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \|g_{n,p}(T_2)\| &\leq e^{-\lambda_{n,p}(T_2-T_1)} \|g_{n,p}(T_1)\| + \int_{T_1}^{T_2} e^{-\lambda_{n,p}(T_2-t)} \|\tilde{h}_{n,p}(t)\| dt \\ &\leq e^{-\lambda_{n,p}(T_2-T_1)} \|g_{n,p}(T_1)\| + \frac{1}{\sqrt{2\lambda_{n,p}}} \left( \int_{T_1}^{T_2} \|\tilde{h}_{n,p}(t)\|^2 dt \right)^{1/2}. \end{aligned}$$

The inequality  $(a + b)^2 \leq 2a^2 + 2b^2$  gives the conclusion.  $\square$

**Proof of Proposition 6.** In this proof, we write  $g$ ,  $g^0$ ,  $\tilde{h}$  and  $\|\cdot\|$  instead of  $g_{n,p}$ ,  $g_{n,p}^0$ ,  $\tilde{h}_{n,p}$  and  $\|\cdot\|_{L^1(-1,1)}$  in order to simplify the notation.

*Step 1: use of dissipation.* Applying Lemma 1 with  $(T_1, T_2) = (t, T)$  and integrating the resulting inequality over  $t \in (T/3, 2T/3)$  yields

$$\int_{-1}^1 |g(T)|^2 dx \leq \frac{6}{T} e^{-2\lambda_{n,p} \frac{T}{3}} \int_{T/3-1}^{2T/3-1} \int |g|^2 dx dt + \frac{1}{\lambda_{n,p}} \int_0^T \int_{-1}^1 |\tilde{h}|^2 dx dt. \quad (4.6)$$

Step 2: existence of a constant  $c_0 > 0$ , independent of  $(T, n, p, g^0, \tilde{h})$ , such that

$$\int_{T/3-1}^{2T/3-1} \int |g|^2 dx dt \leq c_0 T e^{\frac{9M\beta^*}{2T^2}} \left( \int_0^T \int_a^b |g|^2 dx dt + \int_0^T \int_{-1}^1 |\tilde{h}|^2 dx dt \right) \quad (4.7)$$

where  $\beta$ ,  $C_2$ , and  $M = M(n, p, T)$  are as in Proposition 5 and

$$\beta^* := \max\{\beta(x); x \in [-1, 1]\}.$$

By Proposition 5 we have

$$\begin{aligned} C_1 \left( \frac{4M}{T^2} \right)^3 e^{-\frac{9M\beta^*}{2T^2}} \int_{T/3-1}^{2T/3-1} \int |g|^2 dx dt &\leq C_1 \int_{T/3-1}^{2T/3-1} \int \frac{M^3}{[t(T-t)]^3} |g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \\ &\leq C_1 \int_0^T \int_{-1}^1 \frac{M^3}{[t(T-t)]^3} |g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \\ &\leq \int_0^T \int_{-1}^1 |\tilde{h}|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt + \int_0^T \int_a^b \frac{M^3}{[t(T-t)]^3} |g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \\ &\leq \int_0^T \int_{-1}^1 |\tilde{h}|^2 dx dt + c_1 \int_0^T \int_a^b |g|^2 dx dt \end{aligned}$$

where  $c_1 = \sup\{x^3 e^{-\beta_* x}; x \geq 0\}$  and  $\beta_* := \min\{\beta(x); x \in (a, b)\}$ . Thus

$$\int_{T/3-1}^{2T/3-1} \int |g(t)|^2 dx dt \leq \frac{\max\{1, c_1\}}{4^3 C_1} \frac{T^6}{M^3} e^{\frac{9M\beta^*}{2T^2}} \left( \int_0^T \int_{-1}^1 |\tilde{h}|^2 + \int_0^T \int_a^b |g|^2 \right).$$

We remark that  $M \geq C_2 T$  and  $M \geq C_2 T^2$  thus  $T^6/M^3 \leq T/C_2^3$ . Then, the previous inequality gives (4.7) with  $c_0 = \max\{1, c_1\}/(4^3 C_1 C_2^3)$ .



Step 3: combination of (4.6) and (4.7).

$$\begin{aligned} \int_{-1}^1 |g(T)|^2 dx &\leq 6c_0 e^{\frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3}} \int_0^T \int_a^b |g|^2 dx dt \\ &+ \left( \frac{1}{\lambda_{n,p}} + 6c_0 e^{\frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3}} \right) \int_0^T \int_{-1}^1 |\tilde{h}|^2 dx dt. \end{aligned} \quad (4.8)$$

From now on, we introduce the constants

$$C_3 := \ln(6c_0) + \frac{27C_2\beta^*}{2} + 3\alpha^2 + 4 \quad \text{and} \quad C_4 := \frac{1}{12C_3}$$

where  $C_2$  is as in (4.2) and  $\alpha := 27\beta^*C_2/4$ .

Step 4: proof of

$$\ln(6c_0) + \frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3} \leq C_3 \left( 1 + \frac{1}{T} + |p| - C_4 \min\{|p|, p^2\}T \right), \quad (4.9)$$

for every  $(n, p) \in \mathbb{Z} \times \mathbb{R}$ .

Case 1:  $|n| < 2|p|$ . By (3.6) and (4.2) we have that

$$M(T, n, p) \leq C_2 \left( T^2 + T + 3|p|T^2 \right) \quad \text{and} \quad \lambda_{n,p} \geq \frac{1}{4}(|p| + 1).$$

Thus

$$\begin{aligned} \ln(6c_0) + \frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3} &\leq \ln(6c_0) + \frac{9C_2\beta^*}{2} \left( 1 + \frac{1}{T} + 3|p| \right) - \frac{1}{6}|p|T \\ &\leq C_3 \left( 1 + \frac{1}{T} + |p| - C_4|p|T \right) \end{aligned}$$

which gives (4.9).

Case 2:  $|n| \geq 2|p|$ . In view of (3.7) and (4.2),

$$M(T, n, p) \leq C_2 \left( T^2 + T + \frac{3}{2}|n|T^2 \right) \quad \text{and} \quad \lambda_{n,p} \geq \frac{n^2}{4}.$$

Therefore,

$$\begin{aligned} \ln(6c_0) + \frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3} &\leq \ln(6c_0) + \frac{9C_2\beta^*}{2} \left( 1 + \frac{1}{T} + \frac{3}{2}|n| \right) - \frac{1}{6}n^2T \\ &\leq \ln(6c_0) + \frac{9C_2\beta^*}{2} \left( 1 + \frac{1}{T} \right) + \frac{3\alpha^2}{T} - \frac{1}{12}n^2T \end{aligned} \quad (4.10)$$

because the maximal value of the function  $f : s \in (0, \infty) \mapsto \alpha s - \frac{s^2 T}{12}$  is exactly  $\frac{3\alpha^2}{T}$ . Finally, using the assumption  $|n| \geq 2|p|$ , we obtain

$$\begin{aligned} \ln(6c_0) + \frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3} &\leq \ln(6c_0) + \frac{9C_2\beta^*}{2} \left(1 + \frac{1}{T}\right) + \frac{3\alpha^2}{T} - \frac{1}{3}p^2T \\ &\leq C_3 \left(1 + \frac{1}{T} - C_4 p^2 T\right), \end{aligned}$$

which gives (4.9).

Step 5: proof of

$$\ln(6c_0) + \frac{9M\beta^*}{2T^2} - 2\lambda_{n,p} \frac{T}{3} \leq C_3 \left(1 + \frac{1}{T} - C_4 n^2 T\right), \quad (4.11)$$

for every  $(n, p) \in \mathbb{Z} \times \mathbb{R}$  with  $|n| \geq 2|p|$ . It results from (4.10) and the choice of  $C_3$  and  $C_4$ .

Step 6: conclusion. From (4.8), Step 4 and (3.6), we deduce that (4.3) and (4.4) hold. From (4.8), Step 4, Step 5 and (3.7), we obtain (4.3) and (4.5).  $\square$

### 5. 3D-Lipschitz stability estimate when $\omega$ is a slice

The goal of this section is the proof of Theorem 3. We focus on the uniform Lipschitz stability estimate for systems (3.4) in the sense of the following definition. We assume the source term  $\tilde{h}_{n,p}$  in (3.4) takes the form

$$\begin{aligned} \tilde{h}_{n,p}(t, x) &= R(t, x)h_{n,p}(x) \\ \text{where } h_{n,p} &\in L^2(-1, 1) \text{ and } R \in C^0([0, T] \times [-1, 1]). \end{aligned} \quad (5.1)$$

**Definition 4** (Uniform Lipschitz stability). Let  $a, b \in \mathbb{R}$  with  $-1 \leq a < b \leq 1$ ,  $T > 0$  and  $0 < T_0 < T_1 \leq T$ . We say the system (3.4)–(5.1) satisfies a *uniform Lipschitz stability estimate* on  $(T_0, T_1) \times (a, b)$  if, there exists a constant  $C > 0$  such that, for every  $p \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ ,  $g_{n,p}^0 \in L^2(-1, 1)$ ,  $h_{n,p} \in L^2(-1, 1)$ , the solution of (3.4)–(5.1) satisfies

$$\int_{-1}^1 |h_{n,p}|^2 dx \leq C \left( \int_{T_0}^{T_1} \int_a^b |\partial_t g_{n,p}|^2 dx dt + \int_{-1}^1 |A_{n,p} g_{n,p}(T_1, x)|^2 dx \right) \quad (5.2)$$

where  $A_{n,p} := -\partial_x^2 + (px + n)^2$ .

Theorem 3 is a consequence of the next result and Bessel–Parseval identity.

**Proposition 7.** Let  $a, b \in \mathbb{R}$  be such that  $-1 < a < b < 1$  and  $R$  be such that (1.9) holds. The exists  $T^* > 0$  such that, for every  $T_0 \in (0, T_1 - T^*)$  system (3.4)–(5.1) satisfies a uniform Lipschitz stability estimate on  $(T_0, T_1) \times (a, b)$ .

**Remark 3.** Inequality (5.2), with a constant  $C$  that may depend on  $n$  and  $p$  is already known (see [26]). Therefore, in order to prove Proposition 7 it suffices to focus on high frequencies  $(n, p)$ .

**Proof of Proposition 7.** Let  $C_3$  and  $C_4$  be the constants given by Proposition 6. We assume  $(T_1 - T_0) > T^* := 1/C_4$ .

*Step 1: application of Proposition 6.* From (1.9) it follows that

$$R_0|h_{n,p}(x)| \leq |R(T_1, x)h_{n,p}(x)|$$

By Proposition 3 and parabolic smoothing,

$$\partial_t g_{n,p}(T_1, x) - A_{n,p}g_{n,p}(T_1, x) = R(T_1, x)h_{n,p}(x)$$

for almost every  $x \in (-1, 1)$ , thus

$$R_0|h_{n,p}(x)| \leq |\partial_t g_{n,p}(T_1, x)| + |A_{n,p}g_{n,p}(T_1, x)|$$

and

$$\int_{-1}^1 |h_{n,p}|^2 dx \leq \frac{2}{R_0^2} \left( \int_{-1}^1 |\partial_t g_{n,p}(T_1, x)|^2 dx + \int_{-1}^1 |A_{n,p}g_{n,p}(T_1, x)|^2 dx \right). \quad (5.3)$$

Notice that

$$|p| - C_4 \min\{|p|, p^2\}(T_1 - T_0) \leq \begin{cases} 1 - C_4 p^2(T_1 - T_0 - T^*) \leq 1 & \text{if } |p| \leq 1 \\ -C_4 |p|(T_1 - T_0 - T^*) \leq 0 & \text{if } |p| \geq 1. \end{cases}$$

Thus, by Proposition 3 and Proposition 6, applied to  $\partial_t g_{n,p}$ , we get

$$\begin{aligned} \int_{-1}^1 |\partial_t g_{n,p}(T_1, x)|^2 dx &\leq e^{C_3 \left(2 + \frac{1}{T_1 - T_0}\right)} \int_{T_0}^{T_1} \int_a^b |\partial_t g_{n,p}|^2 dx dt \\ &\quad + \epsilon_{n,p} \left( \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt \right) \int_{-1}^1 |h_{n,p}|^2 dx, \end{aligned} \quad (5.4)$$

where  $\|\partial_t R(t)\|_\infty := \|\partial_t R(t, \cdot)\|_{L^\infty(-1,1)}$  and

$$|\epsilon_{n,p}| \leq \begin{cases} \epsilon'_p := \left( \frac{C_3}{(|p|+1)} + e^{C_3 \left(2 + \frac{1}{T_1 - T_0} - C_4 \min\{|p|, p^2\}(T_1 - T_0 - T^*)\right)} \right), & \forall (n, p) \in \mathbb{Z}^2 \\ \epsilon''_n := \left( \frac{C_3}{n^2} + e^{C_3 \left(1 + \frac{1}{T_1 - T_0} - C_4 n^2(T_1 - T_0)\right)} \right), & \text{if } |n| > 2|p|. \end{cases}$$

Step 2: proof of the existence of a constant  $C = C(T_1 - T_0) > 0$  such that, for  $(n, p) \in \mathbb{Z}^2$  large enough, the following inequality holds

$$\frac{2}{R_0^2} \int_{-1}^1 |\partial_t g_{n,p}(T_1, x)|^2 dx \leq C \int_{T_0}^{T_1} \int_a^b |\partial_t g_{n,p}|^2 dx dt + \frac{1}{2} \int_{-1}^1 |h_{n,p}|^2 dx.$$

Note that  $\epsilon'_p \rightarrow 0$  when  $|p| \rightarrow \infty$  and  $\epsilon''_n \rightarrow 0$  when  $|n| \rightarrow \infty$ , thus there exists  $\rho > 0$  such that

$$\frac{2 \max\{\epsilon'_j, \epsilon''_j\}}{R_0^2} \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt < \frac{1}{2}, \quad \forall j \in \mathbb{Z} \text{ with } |j| > \rho. \quad (5.5)$$

Let  $(n, p) \in \mathbb{Z}^2$  be such that  $n^2 + p^2 > 5\rho^2$ .

First case:  $|p| > \rho$ . We have that

$$\frac{2\epsilon_{n,p}}{R_0^2} \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt \leq \frac{2\epsilon'_p}{R_0^2} \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt < \frac{1}{2}.$$

Second case:  $|p| \leq \rho$ . Since  $n^2 > 4\rho^2$ , we have  $|n| > 2|p|$  and  $|n| > \rho$ . Then

$$\frac{2\epsilon_{n,p}}{R_0^2} \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt \leq \frac{2\epsilon''_n}{R_0^2} \int_{T_0}^{T_1} \|\partial_t R(t)\|_\infty^2 dt < \frac{1}{2}.$$

Step 2 follows with  $C := \frac{2}{R_0^2} \exp\left(C_3 \left(2 + \frac{1}{T_1 - T_0}\right)\right)$  thanks to (5.4).

Step 3: conclusion. For  $(n, p) \in \mathbb{Z}^2$  such that  $n^2 + p^2 > 5\rho^2$ , we deduce from (5.3) and Step 2 that

$$\frac{1}{2} \int_{-1}^1 |h_{n,p}|^2 dx \leq C \int_{T_0}^{T_1} \int_a^b |\partial_t g_{n,p}|^2 dx dt + \frac{2}{R_0^2} \int_{-1}^1 |A_{n,p} g_{n,p}(T_1, x)|^2 dx. \quad \square$$

### 6. 3D-Lipschitz stability estimate when $\omega$ is a tube

The goal of this section is the proof of Theorem 4.

For  $n, p \in \mathbb{Z}$ ,  $H_{n,p} := L^2(-1, 1) \otimes e^{i(ny + pz)}$  is a closed subspace of  $L^2(\Omega)$ . For  $j \in \mathbb{N}$ , we define

$$E_{j,p} := \bigoplus_{|n| \leq 2^j} H_{n,p}$$

and denote by  $\Pi_{j,p}$  the orthogonal projection from  $L^2(\Omega)$  onto  $E_{j,p}$ . We also denote by  $\Pi_{\infty,p}$  the orthogonal projection from  $L^2(\Omega)$  onto  $L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz}$ . Moreover,  $Id$  stands for the identity operator on  $L^2(\Omega)$ .

### 6.1. Observability with source for frequency packets

The goal of this section is the proof of the following result.

**Proposition 8.** *Let  $a, b \in \mathbb{R}$  be such that  $-1 \leq a < b \leq 1$  and let  $C_3, C_4 > 0$  be as in Proposition 6. Let  $\omega_y$  be an open subset of  $\mathbb{T}$  and  $\omega := (a, b) \times \omega_y \times \mathbb{T}$ . There exists  $C_5 > C_3$  and  $C_6 \in (0, C_4)$  such that, for every  $T > 0$ ,  $p, j \in \mathbb{Z}$  with*

$$j \geq j_0(p) := \begin{cases} \left\lceil \frac{\ln |p|}{\ln(2)} \right\rceil + 2 & \text{if } p \neq 0 \\ 0 & \text{if } p = 0. \end{cases} \quad (6.1)$$

$g^0 \in L^2(\Omega)$ ,  $\tilde{h} \in L^2((0, T) \times \Omega)$ , the solution of (1.4) satisfies

$$\begin{aligned} \int_{\Omega} |\Pi_{j,p} g(T)|^2 dx dy dz &\leq e^{C_5(2^j + \frac{1}{T} - C_6 |p| T)} \int_0^T \int_{\omega} |\Pi_{j,p} g|^2 dx dy dz \\ &+ \left( C_3 + e^{C_3(1 + \frac{1}{T} + |p| - C_4 |p| T)} \right) \int_0^T \int_{\Omega} |\Pi_{j,p} \tilde{h}|^2 dx dy dz. \end{aligned}$$

The proof of this result relies on the following spectral inequality.

**Proposition 9.** *Let  $\omega_y$  be an open subset of  $\mathbb{T}$ . There exists  $C_{LR} > 0$  such that, for all  $N \in \mathbb{N}^*$  and  $(b_k)_{-N \leq k \leq N} \in \mathbb{C}^{2N+1}$ ,*

$$\left( \sum_{k=-N}^N |b_k|^2 \right)^{\frac{1}{2}} \leq e^{C_{LR} N} \left( \int_{\omega_y} \left| \sum_{k=-N}^N b_k e^{iky} \right|^2 dy \right)^{\frac{1}{2}}.$$

In this statement, the functions  $y \mapsto e^{iky}/\sqrt{2\pi}$  are the orthonormal eigenfunctions of the Laplace operator on the 1D-torus  $\mathbb{T}$ . In arbitrary dimension, for a second-order symmetric elliptic operator, typically the Laplace–Beltrami operator  $\Delta_g$  on a bounded Riemannian manifold  $\mathcal{M}$  of dimension  $d$ , with or without boundary, the spectral inequality takes the form

$$\|u\|_{L^2(\mathcal{M})} \leq C e^{C\sqrt{\mu}} \|u\|_{L^2(\omega)}, \quad u \in \text{Span}\{\phi_j; \mu_j \leq \mu\}, \quad (6.2)$$

where  $\omega \subset \mathcal{M}$  is an open subset of  $\mathcal{M}$  and the functions  $\phi_j$  form a Hilbert basis of  $L^2(\mathcal{M})$  of eigenfunctions of  $-\Delta_g$ , associated with the non-negative eigenvalues  $\mu_j$ ,  $j \in \mathbb{N}$ , counted with their multiplicities. (In the case of a manifold with boundary, one can consider homogeneous Dirichlet or Neuman boundary conditions.) This was proven in [31,30,32].

Inequality (6.2) is a key tool to prove the null controllability of the heat equation by the Lebeau–Robbiano strategy (see [29] for a presentation). This strategy was adapted much later to the case of separated variables, for the null controllability of parabolic equations in stratified media in [9]: in one direction, one has observability by means of a Carleman estimate for a one-dimensional parabolic operator with parameter, and, in the transverse direction, a spectral inequality such as (6.2) is used. This approach was successfully transposed to the study of the null controllability of the Grushin equation in [3] and the Kolmogorov equation in [2]. This approach was also adapted to the study of Lipschitz stability for the Grushin equation in [4]. The strategy we develop in this article is more subtle than the one above. Indeed, the choice of the space variables with respect to which we develop in Fourier series is not arbitrary. For instance, the strategy would not work by developing only with respect to  $z$  because the 2D resulting heat equations would not satisfy appropriate Carleman estimates. This is why we take the Fourier series with respect to both  $y$  and  $z$ . Then, we apply the Lebeau–Robbiano strategy with respect to  $(y, n)$ , paying attention to the behavior of the different constants with respect to  $p$  (the Fourier frequency associated with  $z$ ). Indeed, these constants need to be uniform with respect to  $p$  to get the conclusion.

**Proof of Proposition 8.** Let  $p \in \mathbb{Z}$  and  $j \geq j_0(p)$ , i.e.,  $2^j \geq 2^{j_0} > 2|p|$ . By the Bessel–Parseval equality, (4.3), (4.4) and the previous Lemma, we get

$$\begin{aligned}
 \int_{\Omega} |\Pi_{j,p} g(T)|^2 dx dy dz &= \sum_{|n| \leq 2^j - 1} \int_0^1 |g_{n,p}(T, x)|^2 dx dt \\
 &\leq \sum_{|n| \leq 2^j} \left[ e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 |p| T\right)} \int_0^T \int_a^b |g_{n,p}(t, x)|^2 dx dt \right. \\
 &\quad \left. + \left( \frac{C_3}{|p| + 1} + e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 |p| T\right)} \right) \int_0^T \int_{-1}^1 |\tilde{h}_{n,p}(t, x)|^2 dx dt \right] \\
 &\leq e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 |p| T\right) + C_{LR} 2^j} \int_0^T \int_a^b \int_{\omega_y} \left| \sum_{|n| \leq 2^j} g_{n,p}(t, x) e^{iny} \right|^2 dx dy dt \\
 &\quad + \left( C_3 + e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 |p| T\right)} \right) \sum_{|n| \leq 2^j} \int_0^T \int_{-1}^1 |\tilde{h}_{n,p}(t, x)|^2 dx dt \\
 &\leq e^{C_5 \left(2^j + \frac{1}{T} - C_6 |p| T\right)} \int_0^T \int_{\omega} |\Pi_{j,p} g|^2 dx dy dz dt \\
 &\quad + \left( C_3 + e^{C_3 \left(1 + \frac{1}{T} + |p| - C_4 |p| T\right)} \right) \int_0^T \int_{\Omega} |\Pi_{j,p} \tilde{h}|^2 dx dy dz dt,
 \end{aligned}$$

where  $C_5 := 2C_3 + C_{LR}$  and  $C_6 := \frac{C_3 C_4}{C_5}$ .  $\square$

## 6.2. Lebeau–Robbiano strategy for high frequencies

The goal of this section is the proof of the following result.

**Proposition 10.** *There exists  $C_7 > 0$  such that, for all  $T \geq 1$ ,  $p \in \mathbb{Z}$ ,  $g^0 \in L^2(\Omega)$ , and  $\tilde{h} \in L^2((0, T) \times \Omega)$  the solution of (1.4) satisfies*

$$\|(Id - \Pi_{j_0, p})g(T)\|^2 \leq C_7 T^2 \left( \int_0^T \int_{\omega} |\Pi_{\infty, p} g|^2 dx dy dz dt + \int_0^T \|\Pi_{\infty, p} \tilde{h}(t)\|^2 dt \right)$$

where  $j_0 = j_0(p)$  is as in (6.1).

**Remark 4.** The lower bound  $T \geq 1$  is chosen arbitrarily and may be replaced by any positive lower bound  $T \geq T_* > 0$  with a constant  $C_7 = C_7(T_*) > 0$ . In the proof, assuming  $T \geq 1$  will simplify the expression of the  $T$ -dependence of several constants. Such an assumption is compatible with the fact that the positive result we have in mind only holds in large time.

To prove Proposition 10, we follow the Lebeau–Robbiano strategy, from the observability point of view, with respect to parameter  $n$  keeping parameter  $p$  fixed. We pay attention to the dependence of constants with respect to  $p$ .

In the whole section, we fix  $\rho \in (0, 1)$ ,  $T \geq 1$ ,  $p \in \mathbb{Z}$  and  $j_0 := j_0(p)$  as in (6.1). Note that

$$2^{j_0-1} \leq 2|p| < 2^{j_0} \quad \text{if } p \neq 0. \quad (6.3)$$

Let  $K = K(T, p, \rho) > 0$  be such that

$$T = 2K \sum_{j \geq j_0} 2^{-\rho j} = \frac{2K 2^{-\rho j_0}}{1 - 2^{-\rho}}.$$

From (6.3) it follows that

$$\frac{2^\rho - 1}{2} T |p|^\rho < K \leq 2^\rho \frac{2^\rho - 1}{2} T |p|^\rho \quad \text{if } p \neq 0. \quad (6.4)$$

Then there exists  $K_* = K_*(\rho) > 0$ , independent of  $(T, p)$ , such that

$$K(T, p, \rho) \geq 2K_* T > 0, \quad \forall (T, p) \in (0, \infty) \times \mathbb{Z}. \quad (6.5)$$

We now define times

$$\tau_j = \tau_j(T, p, \rho) = K 2^{-j\rho} \quad \text{and} \quad \alpha_j = \alpha_j(T, p, \rho) = 2 \sum_{k=j_0}^j \tau_k \quad \forall j \geq j_0, \quad (6.6)$$

and time intervals

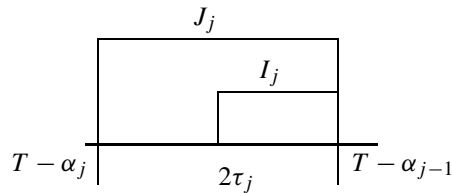
$$I_j := (T - \alpha_{j-1} - \tau_j, T - \alpha_{j-1}) \quad \text{and} \quad J_j := (T - \alpha_j, T - \alpha_{j-1}) \quad \forall j > j_0.$$

We will also use the notation

$$\lambda(2^j) = \frac{2^{2j}}{4} \quad (6.7)$$

so that  $\lambda_{n,p} \geq \lambda(2^j)$  for every  $|n| \geq 2^j$  and  $j \geq j_0(p)$  by (3.7) and (6.3).

We will need the following preliminary result, which is left to the reader: it is a direct consequence of the Bessel–Parseval identity and Lemma 1.



**Lemma 2.** Let  $T_1, T_2 \in \mathbb{R}$ ,  $p \in \mathbb{Z}$ ,  $j_1, j_2 \in \mathbb{N} \cup \{\infty\}$  be such that

$$0 \leq T_1 < T_2 < \infty \quad \text{and} \quad j_0(p) \leq j_1 < j_2 \leq \infty.$$

For every  $g^0 \in L^2(\Omega)$  and  $\tilde{h} \in L^2((0, T) \times \Omega)$ , the solution of (1.4) satisfies

$$\begin{aligned} \|(\Pi_{j_2,p} - \Pi_{j_1,p})g(T_2)\|^2 &\leq 2\|(\Pi_{j_2,p} - \Pi_{j_1,p})g(T_1)\|^2 e^{-2\lambda(2^{j_1})(T_2-T_1)} \\ &\quad + \frac{1}{\lambda(2^{j_1})} \int_{T_1}^{T_2} \|(\Pi_{j_2,p} - \Pi_{j_1,p})\tilde{h}(t)\|^2 dt. \end{aligned}$$

**Proposition 11.** There exist  $C_8, C_9 > 0$  such that for every  $T \geq 1$ ,  $p \in \mathbb{Z}$ ,  $j > j_0(p)$ ,  $g^0 \in L^2(\Omega)$ , and  $\tilde{h} \in L^2((0, T) \times \Omega)$  the solution of (1.4) satisfies

$$\begin{aligned} e^{-C_8 2^j} \|\Pi_{j,p}g(T - \alpha_{j-1})\|^2 &\leq \int_{I_j \times \omega} |\Pi_{\infty,p}g|^2 dx dy dz dt \\ &\quad + \frac{C_9 T}{2^j} \int_{J_j \times \Omega} |\Pi_{\infty,p}\tilde{h}|^2 dx dy dz dt + T e^{-K_* T 2^{(2-\rho)j}} \|\Pi_{\infty,p}g(T - \alpha_j)\|^2, \quad (6.8) \end{aligned}$$

where  $K_* = K_*(\rho) > 0$  is as in (6.5).

**Proof of Proposition 11.** Let  $p \in \mathbb{Z}$ ,  $j > j_0(p)$ ,  $g^0 \in L^2(\Omega)$ ,  $\tilde{h} \in L^2((0, T) \times \Omega)$ . To simplify notations in this proof, we assume that  $g_0 \in L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz}$  and  $\tilde{h} \in L^2(0, T; L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz})$ , so that  $\Pi_{\infty,p}g(t) = g(t)$  and  $\Pi_{\infty,p}\tilde{h}(t) = \tilde{h}(t)$  for every  $t \in [0, T]$ . We also write  $\Pi_j$  instead of  $\Pi_{j,p}$  and omit all integration symbols such as  $dx, dy, dz, dt$ .



By Proposition 8, the solution of (1.4) satisfies

$$\begin{aligned} \|\Pi_j g(T - \alpha_{j-1})\|^2 &\leq e^{C_5\left(2^j + \frac{1}{\tau_j}\right)} \int_{I_j \times \omega} |\Pi_j g|^2 \\ &\quad + \left(C_3 + e^{C_3\left(1 + \frac{1}{\tau_j} + |p| - C_4 |p| \tau_j\right)}\right) \int_{I_j \times \Omega} |\Pi_j \tilde{h}|^2. \end{aligned} \quad (6.9)$$

Moreover, we have

$$\int_{I_j \times \omega} |\Pi_j g|^2 \leq 2 \int_{I_j \times \omega} |g|^2 + 2 \int_{I_j \times \Omega} |(Id - \Pi_j)g|^2, \quad (6.10)$$

and, by Lemma 2 applied with  $T_1 = T - \alpha_j$ ,  $T_2 = t \in I_j$ ,  $j_1 = j$ ,  $j_2 = \infty$ ,

$$\begin{aligned} \int_{I_j \times \Omega} |(Id - \Pi_j)g|^2 &\leq 2\tau_j \|(Id - \Pi_j)g(T - \alpha_j)\|^2 e^{-2\lambda(2^j)\tau_j} \\ &\quad + \frac{\tau_j}{\lambda(2^j)} \int_{J_j} \|(Id - \Pi_j)\tilde{h}\|^2. \end{aligned} \quad (6.11)$$

Therefore,

$$\begin{aligned} \|\Pi_j g(T - \alpha_{j-1})\|^2 &\leq 2e^{C_5\left(2^j + \frac{1}{\tau_j}\right)} \int_{I_j \times \omega} |g|^2 + 2e^{C_5\left(2^j + \frac{1}{\tau_j}\right)} \left(\frac{\tau_j}{\lambda(2^j)} + C_3 e^{-C_5\left(2^j + \frac{1}{\tau_j}\right)}\right) \\ &\quad + e^{C_3\left(1 + \frac{1}{\tau_j} + |p|\right) - C_5\left(2^j + \frac{1}{\tau_j}\right)} \int_{J_j} \|\tilde{h}\|^2 \\ &\quad + 4\tau_j e^{C_5\left(2^j + \frac{1}{\tau_j}\right) - 2\lambda(2^j)\tau_j} \|(Id - \Pi_j)g(T - \alpha_j)\|^2. \end{aligned} \quad (6.12)$$

From (6.6), (6.5), assumptions  $T \geq 1$  and  $\rho < 1$ , we deduce that

$$\frac{1}{\tau_j} = \frac{2^{j\rho}}{K} \leq \frac{2^{j\rho}}{K_* T} \leq \frac{2^j}{K_*}, \quad \forall j > j_0(p).$$

Then there exists  $C_8 > 0$  independent of  $(T, p, g^0, \tilde{h})$  such that

$$2e^{C_5\left(2^j + \frac{1}{\tau_j}\right)} \leq e^{C_8 2^j}, \quad \forall j > j_0(p). \quad (6.13)$$

We also have

$$\begin{aligned}
& C_3 \left( 1 + \frac{1}{\tau_j} + |p| \right) - C_5 \left( 2^j + \frac{1}{\tau_j} \right) \\
&= C_3 - (C_5 - C_3) \frac{1}{\tau_j} - C_3 \left( 2^j - |p| \right) - (C_5 - C_3) 2^j \\
&\leq C_3 - (C_5 - C_3) 2^j \text{ because } C_5 > C_3 \text{ and } 2|p| < 2^{j_0} < 2^j.
\end{aligned}$$

Thus, there exists a constant  $c > 0$  independent of  $(T, p, g^0, \tilde{h})$  such that

$$\begin{aligned}
e^{C_3 \left( 1 + \frac{1}{\tau_j} + |p| \right) - C_5 \left( 2^j + \frac{1}{\tau_j} \right)} &\leq e^{C_3 - (C_5 - C_3) 2^j} \leq \frac{c}{2^j}, \quad \forall j \geq j_0(p), \\
\frac{\tau_j}{\lambda(2^j)} &\leq \frac{2T}{2^{2j}} \leq \frac{cT}{2^j}, \quad \forall j \geq j_0(p),
\end{aligned}$$

and

$$C_3 e^{-C_5 \left( 2^j + \frac{1}{\tau_j} \right)} \leq C_3 e^{-C_5 2^j} \leq \frac{c}{2^j}, \quad \forall j \geq j_0(p).$$

As a consequence, there exists  $C_9 > 0$  independent of  $(T, p, g^0, \tilde{h})$  such that

$$\frac{\tau_j}{\lambda(2^j)} + C_3 e^{-C_5 \left( 2^j + \frac{1}{\tau_j} \right)} + e^{C_3 \left( 1 + \frac{1}{\tau_j} + |p| \right) - C_5 \left( 2^j + \frac{1}{\tau_j} \right)} \leq \frac{C_9 T}{2^j}, \quad \forall j > j_0(p). \quad (6.14)$$

By (6.7), (6.6) and (6.5), we have

$$4\tau_j e^{-2\lambda(2^j)\tau_j} \leq T e^{-K_* T 2^{(2-\rho)j}}, \quad \forall j > j_0(p), \quad (6.15)$$

because  $4\tau_j \leq 2(\tau_j + \tau_{j_0}) \leq T$ . Finally, from (6.12), (6.13), (6.14) and (6.15) we deduce that

$$\begin{aligned}
\|\Pi_j g(T - \alpha_{j-1})\|^2 &\leq e^{C_8 2^j} \int_{I_j \times \omega} |g|^2 + e^{C_8 2^j} \frac{C_9 T}{2^j} \int_{J_j \times \Omega} |\tilde{h}|^2 \\
&\quad + T e^{C_8 2^j - K_* T 2^{(2-\rho)j}} \|(Id - \Pi_j)g(T - \alpha_j)\|^2,
\end{aligned}$$

which ends the proof of Proposition 11.  $\square$

**Proof of Proposition 10.** Let  $p \in \mathbb{Z}$ ,  $j > j_0(p)$ ,  $g^0 \in L^2(\Omega)$  and let  $\tilde{h} \in L^2((0, T) \times \Omega)$ . To simplify notations in this proof, we assume that  $g_0 \in L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz}$  and  $\tilde{h} \in L^2(0, T; L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz})$ , so that  $\Pi_{\infty, p} g(t) = g(t)$  and  $\Pi_{\infty, p} \tilde{h}(t) = \tilde{h}(t)$  for every  $t \in [0, T]$ . We also write  $\Pi_j$  instead of  $\Pi_{j, p}$  and omit all integration symbols such as  $dx, dy, dz, dt$ . Let  $C_8, C_9$  be as in Proposition 11.

Step 1: we prove by induction on  $j \geq j_0 + 1$  that, for every  $j \geq j_0 + 1$ ,

$$\begin{aligned} & \sum_{k=j_0+1}^j e^{-C_8 2^k} \|\Pi_k g(T - \alpha_{k-1})\|^2 \\ & \leq \sum_{k=j_0+1}^j \delta_k \int_{I_k \times \omega} |g|^2 + A_j \int_{T-\alpha_j}^T \|\tilde{h}(t)\|^2 dt + B_j \|g(T - \alpha_j)\|^2 \end{aligned} \quad (\mathcal{P}_j)$$

where

$$\delta_{j_0+1} := 1, \quad A_{j_0+1} := \frac{C_9 T}{2^{j_0+1}}, \quad B_{j_0+1} := T e^{-K_* T 2^{(2-\rho)(j_0+1)}} \quad (6.16)$$

and

$$\delta_{j+1} := 1 + B_j e^{C_8 2^{j+1}}, \quad (6.17)$$

$$A_{j+1} := A_j + \frac{B_j}{2^{2j}} + \frac{\delta_{j+1} C_9 T}{2^{j+1}}, \quad (6.18)$$

$$B_{j+1} := (2B_j + \delta_{j+1} T) e^{-K_* T 2^{(2-\rho)(j+1)}}. \quad (6.19)$$

The inequality  $(\mathcal{P}_{j_0+1})$  is given by [Proposition 11](#) with  $j = j_0 + 1$ . Let us now assume that  $(\mathcal{P}_j)$  holds for some  $j > j_0$  and prove  $(\mathcal{P}_{j+1})$ . We have

$$B_j \|g(T - \alpha_j)\|^2 = B_j \|\Pi_{j+1} g(T - \alpha_j)\|^2 + B_j \|(Id - \Pi_{j+1})g(T - \alpha_j)\|^2.$$

Applying [Lemma 2](#) to the last term (with  $T_1 = T - \alpha_{j+1}$ ,  $T_2 = T - \alpha_j$ ,  $j_1 = j + 1$ ,  $j_2 = \infty$ ) we get

$$\begin{aligned} B_j \|g(T - \alpha_j)\|^2 & \leq B_j \|\Pi_{j+1} g(T - \alpha_j)\|^2 \\ & + 2B_j \|(Id - \Pi_{j+1})g(T - \alpha_{j+1})\|^2 e^{-4\lambda(2^{j+1})\tau_{j+1}} + \frac{B_j}{\lambda(2^{j+1})} \int_{J_{j+1}} \|(Id - \Pi_{j+1})\tilde{h}(t)\|^2 dt. \end{aligned}$$

Moreover, by [\(6.6\)](#), [\(6.7\)](#) and [\(6.5\)](#), we have

$$4\lambda(2^{j+1})\tau_{j+1} = K 2^{(2-\rho)(j+1)} \geq K_* T 2^{(2-\rho)(j+1)} \quad \text{and} \quad \frac{B_j}{\lambda(2^{j+1})} = \frac{B_j}{2^{2j}}.$$

Thus,  $(\mathcal{P}_j)$  implies

$$\begin{aligned} & \sum_{k=j_0+1}^j e^{-C_8 2^k} \|\Pi_k g(T - \alpha_{k-1})\|^2 - B_j \|\Pi_{j+1} g(T - \alpha_j)\|^2 \\ & \leq \sum_{k=1}^j \delta_k \int_{I_k \times \omega} |g|^2 + \left(A_j + \frac{B_j}{2^{2j}}\right) \int_{T - \alpha_{j+1}}^T \|\tilde{h}\|^2 + 2B_j e^{-K_* T 2^{(2-\rho)(j+1)}} \|g(T - \alpha_{j+1})\|^2. \end{aligned} \quad (6.20)$$

Moreover, by Proposition 11, we also have

$$\begin{aligned} e^{-C_8 2^{j+1}} \|\Pi_{j+1} g(T - \alpha_j)\|^2 & \leq \int_{I_{j+1} \times \omega} |g|^2 + \frac{C_9 T}{2^{j+1}} \int_{J_{j+1}} \|\tilde{h}\|^2 \\ & \quad + T e^{-K_* T 2^{(2-\rho)(j+1)}} \|g(T - \alpha_{j+1})\|^2. \end{aligned} \quad (6.21)$$

Note that  $\delta_{j+1}$  is chosen so that

$$\delta_{j+1} e^{-C_8 2^{j+1}} - B_j = e^{-C_9 2^{j+1}}.$$

Thus, summing (6.20) and  $\delta_{j+1} * (6.21)$ , we get  $(\mathcal{P}_{j+1})$ , which ends the first step.

*Step 2: existence of  $B^* > 0$  independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$  such that*

$$\tilde{B}_j := B_j e^{C_8 2^{j+1}} \leq B^* T, \quad \forall j > j_0.$$

From (6.19), (6.17) and assumption “ $T \geq 1$ ”, we deduce that

$$\tilde{B}_{j+1} \leq 3T (\tilde{B}_j + 1) e^{C_8 2^{j+2} - K_* T 2^{(2-\rho)(j+1)}}, \quad \forall j > j_0.$$

Moreover, there exists  $M_1, M_2 > 0$  independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$  such that

$$3T e^{-\frac{K_*}{4} T 2^{(2-\rho)(j+1)}} \leq 3T e^{-\frac{K_*}{4} T 2^{(2-\rho)}} \leq M_1, \quad \forall j > j_0,$$

and

$$e^{C_8 2^{j+2} - \frac{K_*}{4} T 2^{(2-\rho)(j+1)}} \leq e^{C_8 2^{j+2} - \frac{K_*}{4} T 2^{(2-\rho)(j+1)}} \leq M_2, \quad \forall j > j_0.$$

Thus

$$\tilde{B}_{j+1} \leq M (\tilde{B}_j + 1) e^{-\frac{K_*}{2} T 2^{(2-\rho)(j+1)}}, \quad \forall j > j_0, \quad (6.22)$$

where  $M := M_1 M_2$  is independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$  and may be assumed to be  $> 1$ . In particular

$$\tilde{B}_{j+1} \leq M (\tilde{B}_j + 1), \quad \forall j > j_0,$$

or

$$\tilde{B}_{j+1} + \frac{M}{M-1} \leq M \left( \tilde{B}_j + \frac{M}{M-1} \right), \quad \forall j > j_0.$$

Thus,

$$\tilde{B}_j + 1 \leq \tilde{B}_j + \frac{M}{M-1} \leq M^{j-j_0-1} \left( \tilde{B}_{j_0+1} + \frac{M}{M-1} \right)$$

and by (6.22) we deduce that

$$\tilde{B}_{j+1} \leq M^{j-j_0} \left( \tilde{B}_{j_0+1} + \frac{M}{M-1} \right) e^{-\frac{K_*}{2} T 2^{(2-\rho)(j+1)}}, \quad \forall j > j_0.$$

Moreover, there exists  $c_1 > 0$ , independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$ , such that

$$M^{j-j_0} e^{-\frac{K_*}{2} T 2^{(2-\rho)(j+1)}} \leq M^{j-j_0} e^{-\frac{K_*}{2} 2^{(2-\rho)(j+1)}} \leq c_1, \quad \forall j > j_0$$

and, in view of (6.3),

$$\tilde{B}_{j_0+1} = T e^{C_8 2^{j_0+2} - K_* T 2^{(2-\rho)(j_0+1)}} \leq T e^{16 C_8 |p| - K_* |p|^2 - \rho} \leq c_1 T.$$

This ends Step 2, because  $T \geq 1$ .

*Step 3: existence of  $A^* > 0$  independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$  such that*

$$A_j \leq A^* T^2, \quad \forall j > j_0.$$

By definition, we have

$$A_j = A_{j_0+1} + \sum_{k=j_0+1}^{j-1} \left( \frac{B_k}{2^{2k}} + \frac{\delta_{k+1} C_9 T}{2^{k+1}} \right) \leq \frac{C_9 T}{2^{j_0+1}} + \sum_{k=0}^{\infty} \left( \frac{B^* T}{2^{2k}} + \frac{(1 + B^* T) C_9 T}{2^{k+1}} \right)$$

which proves Step 3, because  $T \geq 1$ .

*Step 4: passing to the limit as  $j \rightarrow \infty$  in  $(\mathcal{P}_j)$ .* The last term on the right-hand side of  $(\mathcal{P}_j)$  converges to zero because  $B_j \leq B^* T e^{-C_8 2^{j+1}}$ . Thus, we get

$$\sum_{k=j_0+1}^{\infty} e^{-C_8 2^k} \|\Pi_k g(T - \alpha_{k-1})\|^2 \leq (1 + B^* T) \int_0^T \int_{\omega} |g|^2 + A^* T^2 \int_0^T \|\tilde{h}(t)\|^2 dt. \quad (6.23)$$

*Step 5: conclusion.* Using the Pythagorean theorem and Lemma 2, we get

$$\begin{aligned} \|(Id - \Pi_{j_0})g(T)\|^2 &= \sum_{k=j_0+1}^{\infty} \|(\Pi_k - \Pi_{k-1})g(T)\|^2 \\ &\leq 2 \sum_{k=j_0+1}^{\infty} \|(\Pi_k - \Pi_{k-1})g(T - \alpha_{k-1})\|^2 e^{-2\lambda(2^{k-1})\alpha_{k-1}} \end{aligned}$$

$$+ \sum_{k=j_0+1}^{\infty} \frac{1}{\lambda(2^{k-1})} \int_{T-\alpha_{k-1}}^T \|\tilde{h}\|^2.$$

Moreover, there exists  $c_2 > 0$  independent of  $(T, p) \in [1, \infty) \times \mathbb{Z}$  such that

$$e^{-2\lambda(2^{k-1})\alpha_{k-1}} \leq e^{-2\lambda(2^{k-1})\tau_{k-1}} \leq e^{-K_*2^{(2-\rho)(k-1)}} \leq \frac{c_2}{2} e^{-C_8 2^k}, \forall k > j_0.$$

Thus

$$\|(Id - \Pi_{j_0})g(T)\|^2 \leq c_2 \sum_{k=j_0+1}^{\infty} \|\Pi_k g(T - \alpha_{k-1})\|^2 e^{-C_8 2^k} + \left( \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \right) \int_0^T \|\tilde{h}\|^2.$$

Finally (6.23) gives the conclusion of Proposition 10 because  $T \geq 1$ .  $\square$

### 6.3. 3D observability inequality with source term

The goal of this section is the proof of the following result.

**Proposition 12.** *There exist  $T_* > 0$  and  $c_* : (T_*, \infty) \rightarrow (0, \infty)$  continuous such that, for every  $T > T_*$ ,  $p \in \mathbb{Z}$ ,  $g^0 \in L^2(\Omega)$ , and  $\tilde{h} \in L^2((0, T) \times \Omega)$ , the solution of (1.4) satisfies*

$$\|\Pi_{\infty,p} g(T)\|^2 \leq c_*(T) \left( \int_0^T \int_{\omega} |\Pi_{\infty,p} g|^2 + \int_0^T \int_{\Omega} |\Pi_{\infty,p} \tilde{h}|^2 \right).$$

**Proof of Proposition 12.** Let  $p \in \mathbb{Z}$ ,  $g^0 \in L^2(\Omega)$ , and  $\tilde{h} \in L^2((0, T) \times \Omega)$ . To simplify notations in this proof, we assume that  $g_0 \in L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz}$  and  $\tilde{h} \in L^2(0, T; L^2((-1, 1) \times \mathbb{T}) \otimes e^{ipz})$ , so that  $\Pi_{\infty,p} g(t) = g(t)$  and  $\Pi_{\infty,p} \tilde{h}(t) = \tilde{h}(t)$  for every  $t \in [0, T]$ . We also write  $\Pi_j$  instead of  $\Pi_{j,p}$ . Let  $C_4 > 0$  be as in Proposition 6,  $C_6 \in (0, C_4)$  be as in Proposition 8 and  $T_* := \max\{1, 8/C_6\}$ . We assume that  $T > T_*$ . By orthogonality,

$$\|g(T)\|^2 = \|\Pi_{j_0} g(T)\|^2 + \|(Id - \Pi_{j_0})g(T)\|^2. \quad (6.24)$$

Appealing to Proposition 8, we get

$$\begin{aligned} \int_{\Omega} |\Pi_{j_0} g(T)|^2 &\leq e^{C_5 \left( 2^{j_0} + \frac{2}{T} - C_6 |p| \frac{T}{2} \right)} \int_{T/2}^T \int_{\omega} |\Pi_{j_0} g|^2 \\ &\quad + \left( C_3 + e^{C_3 \left( 1 + \frac{2}{T} + |p| - C_4 |p| \frac{T}{2} \right)} \right) \int_{T/2}^T \int_{\Omega} |\Pi_{j_0} \tilde{h}|^2. \end{aligned}$$

Moreover, invoking (6.3) and the fact that  $T > T_* \geq \frac{8}{C_6} > \frac{8}{C_4}$ , we obtain

$$2^{j_0} - C_6|p|\frac{T}{2} \leq 4|p| - C_6|p|\frac{T}{2} < 0 \quad \text{and} \quad |p| - C_4|p|\frac{T}{2} < 0.$$

Thus, recalling that  $T \geq 1$  once again, we conclude that

$$\int_{\Omega} |\Pi_{j_0} g(T)|^2 \leq e^{2C_5} \int_{T/2}^T \int_{\omega} |\Pi_{j_0} g|^2 + (C_3 + e^{3C_3}) \int_{T/2}^T \int_{\Omega} |\Pi_{j_0} \tilde{h}|^2. \quad (6.25)$$

By Lemma 2, we have

$$\begin{aligned} \int_{T/2}^T \int_{\omega} |\Pi_{j_0} g|^2 &\leq 2 \int_{T/2}^T \int_{\omega} |g|^2 + 2 \int_{T/2}^T \int_{\Omega} |(Id - \Pi_{j_0})g|^2 \\ &\leq 2 \int_{T/2}^T \int_{\omega} |g|^2 + 2T \left\| (Id - \Pi_{j_0})g\left(\frac{T}{2}\right) \right\|^2 + T \int_{T/2}^T \|\tilde{h}\|^2. \end{aligned} \quad (6.26)$$

Therefore, (6.24), (6.25), and (6.26) yield

$$\begin{aligned} \|g(T)\|^2 &\leq 2e^{2C_5} \int_{T/2}^T \int_{\omega} |g|^2 + (Te^{2C_5} + C_3 + e^{3C_3}) \int_{T/2}^T \|\tilde{h}\|^2 \\ &\quad + \|(Id - \Pi_{j_0})g(T)\|^2 + 2Te^{2C_5} \left\| (Id - \Pi_{j_0})g\left(\frac{T}{2}\right) \right\|^2. \end{aligned}$$

We complete the proof by applying Proposition 10 to the last two terms.  $\square$

#### 6.4. Proof of Theorem 4

Let  $T_*$  be as in Proposition 12 and let  $T_0 \in [0, T_1)$  be such that  $T_1 - T_0 > T_*$ . In view of (1.9), we have

$$\int_{\Omega} |h|^2 \leq \frac{1}{R_0^2} \int_{\Omega} |R(T_1)h|^2 \leq \frac{2}{R_0^2} \int_{\Omega} (|\partial_t g(T_1)|^2 + |Ag(T_1)|^2).$$

By the Bessel–Parseval identity (note the particular form of  $\omega = (a, b) \times \omega_y \times \mathbb{T}$ ) and Proposition 12, we obtain

$$\int_{\Omega} |h|^2 \leq \frac{2C_{10}(T)}{R_0^2} \left( \int_{T_0}^{T_1} \int_{\omega} |\partial_t g|^2 + \int_{T_0}^{T_1} \|\partial_t R(t)\|_{\infty}^2 \|h\|^2 dt \right) + \frac{2}{R_0^2} \int_{\Omega} |Ag(T_1)|^2$$

for some constant  $C_{10} > 0$ . The conclusion follows with

$$\eta(T) := \frac{R_0}{2\sqrt{C_{10}(T)}}. \quad \square$$

### 7. 3D-Observability inequality when $\omega$ is a tube

The goal of this section is the proof of [Theorem 1](#).

#### 7.1. Observability in large time

Let  $T_*$  be as in [Proposition 12](#) and  $T > T_*$ . The observability of (1.4) on  $\omega = (a, b) \times \omega_y \times \mathbb{T}$  in time  $T > T_*$  follows from the Bessel–Parseval identity and [Proposition 12](#) (no source term  $\tilde{h}$ ).

#### 7.2. No observability in small time

The goal of this section is the proof of the following result.

**Proposition 13.** *Let  $a, b \in \mathbb{R}$  be such that  $-1 < a < b < 1$  and*

$$\omega := (a, b) \times \mathbb{T} \times \mathbb{T}.$$

*If  $T < \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$ , then (1.4) is not observable in  $\omega$  in time  $T$ .*

**Proof of Proposition 13.** One may assume that  $-1 < a < b = 1$ . Let  $T < \frac{1}{8}(1+a)^2$ . We are going to construct a sequence  $(g_k)_{k \in \mathbb{N}^*}$  of solutions of (1.4) such that

$$\frac{\int_0^T \int_a^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |g_k(t, x, y, z)|^2 dz dy dx dt}{\int_{-1}^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |g_k(T, x, y, z)|^2 dz dy dx dt} \xrightarrow{k \rightarrow \infty} 0. \quad (7.1)$$

Let  $\alpha := \frac{1-a}{2} > 0$  and  $\epsilon > 0$  be such that

$$(-1 + \alpha)^2 - \epsilon > 0, \quad T < \frac{1}{8}(a+1)^2 - \epsilon \quad (7.2)$$

and  $k_1(\epsilon) \in \mathbb{N}^*$  be such that

$$\left( \pm 1 + \frac{[\alpha k]}{k} \right)^2 \geq (\pm 1 + \alpha)^2 - \epsilon, \quad \forall k \geq k_1(\epsilon), \quad (7.3)$$

where  $[\cdot]$  is the floor function.

*Step 1: construction of  $g_k$  from an explicit approximate solution.* The function

$$G(x) := \frac{1}{\sqrt[4]{\pi}} e^{-\frac{x^2}{2}}$$

satisfies

$$\begin{cases} -G''(x) + x^2 G(x) = G(x), & x \in \mathbb{R}, \\ \int_{\mathbb{R}} G(x)^2 dx = 1. \end{cases}$$



Let  $\theta_{\pm} \in C_c^\infty(\mathbb{R})$  be such that

$$\theta_{\pm}(\pm 1) = 1, \quad \theta_{\pm}(\mp 1) = 0 \quad \text{and} \quad \text{Supp}(\theta_-) \cap (a, 1) = \emptyset.$$

For  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+^*$ , the function

$$\mathcal{K}_{n,p}(t, x) := \sqrt[4]{p} \left\{ G\left(\sqrt{p}\left(x + \frac{n}{p}\right)\right) - \sum_{\sigma \in \{-1, 1\}} G\left(\sqrt{p}\left(\sigma + \frac{n}{p}\right)\right) \theta_{\sigma}(x) \right\} e^{-pt}$$

satisfies

$$\begin{cases} (\partial_t - \partial_x^2 + (px + n)^2) \mathcal{K}_{n,p}(t, x) = E_{n,p}(t, x), & (t, x) \in (0, \infty) \times (-1, 1), \\ \mathcal{K}_{n,p}(t, \pm 1) = 0, & t \in (0, \infty), \end{cases}$$

where

$$E_{n,p}(t, x) = \sqrt[4]{p} \sum_{\sigma \in \{-1, 1\}} \left( -p + \partial_x^2 - (px + n)^2 \right) \theta_{\sigma}(x) e^{-pt} G\left(\sqrt{p}\left(\sigma + \frac{n}{p}\right)\right).$$

For  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+^*$ , let  $\mathcal{G}_{n,p}(t, x)$  be the solution of

$$\begin{cases} (\partial_t - \partial_x^2 + (px + n)^2) \mathcal{G}_{n,p}(t, x) = 0, & (t, x) \in (0, \infty) \times (-1, 1), \\ \mathcal{G}_{n,p}(t, \pm 1) = 0, & t \in (0, \infty), \\ \mathcal{G}_{n,p}(0, x) = \mathcal{K}_{n,p}(0, x), & x \in (-1, 1). \end{cases}$$

Then, by Duhamel's formula, there exists  $c_1 > 0$ , independent of  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+^*$ , such that for all  $(t, n, p) \in (0, T) \times \mathbb{Z} \times \mathbb{N}^*$

$$\|(\mathcal{G}_{n,p} - \mathcal{K}_{n,p})(t, \cdot)\|_{L^2(-1,1)}^2 \leq c_1 \int_0^t \|E_{n,p}(s)\|_{L^2(-1,1)}^2 ds.$$

Thus, recalling the definition of  $E_{n,p}$  we conclude that there exists  $c_2 > 0$ , independent of  $(n, p) \in \mathbb{Z} \times \mathbb{R}_+^*$ , such that for every  $(t, n, p) \in (0, T) \times \mathbb{Z} \times \mathbb{R}_+^*$

$$\|(\mathcal{G}_{n,p} - \mathcal{K}_{n,p})(t, \cdot)\|_{L^2(-1,1)} \leq c_2 \frac{p^2 + n^2}{\sqrt[4]{p}} \max_{\sigma \in \{-1, 1\}} e^{-\frac{p}{2}\left(\sigma + \frac{n}{p}\right)^2}. \quad (7.4)$$

We define

$$g_k(t, x, y, z) := \mathcal{G}_{[\alpha k], k}(t, x) e^{i([\alpha k]y + kz)}.$$

*Step 2: we start the proof of (7.1) arguing by contradiction.* Assume that there exists  $c_3 > 0$  such that, for every  $k \in \mathbb{N}^*$ ,

$$\left( \int_{-1}^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |g_k(T, x, y, z)|^2 dz dy dx \right)^{\frac{1}{2}} \leq c_3 \left( \int_0^T \int_a^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |g_k(t, x, y, z)|^2 dz dy dx dt \right)^{\frac{1}{2}}.$$

Thanks to the Bessel–Parseval identity, the above inequality may be written as

$$\left( \int_{-1}^1 |\mathcal{G}_{[\alpha k], k}(T, x)|^2 dx \right)^{\frac{1}{2}} \leq c_3 \left( \int_0^T \int_a^1 |\mathcal{G}_{[\alpha k], k}(t, x)|^2 dx dt \right)^{\frac{1}{2}} \quad \forall k \in \mathbb{N}^*.$$

By the triangular inequality and (7.4), we deduce that, for some constant  $c_4 > 0$ ,

$$\begin{aligned} \left( \int_{-1}^1 |\mathcal{K}_{[\alpha k], k}(T, x)|^2 dx \right)^{\frac{1}{2}} &\leq c_3 \left( \int_0^T \int_a^1 |\mathcal{K}_{[\alpha k], k}(t, x)|^2 dx dt \right)^{\frac{1}{2}} \\ &\quad + c_4 k^{7/8} e^{-\frac{k}{2} [(-1+\alpha)^2 - \epsilon]}, \quad \forall k > k_1(\epsilon). \end{aligned} \quad (7.5)$$

*Step 3: lower bound for the left-hand side of (7.5).* We have

$$\begin{aligned} \left( \int_{-1}^1 |\mathcal{K}_{[\alpha k], k}(T, x)|^2 dx \right)^{\frac{1}{2}} &\geq \left( \int_{-1}^1 \sqrt{k} e^{-k(x + \frac{[\alpha k]}{k})^2} e^{-2kT} dx \right)^{\frac{1}{2}} \\ &\quad - \sum_{\sigma \in \{-1, 1\}} \left( \int_{-1}^1 \sqrt{k} e^{-k(\sigma + \frac{[\alpha k]}{k})^2} \theta_{\sigma}(x)^2 e^{-2kT} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, there exists  $c_5, c_6, c_7 > 0$  and  $k_2(\epsilon) \geq k_1(\epsilon)$  such that, for all  $k > k_2(\epsilon)$ ,

$$\begin{aligned} \left( \int_{-1}^1 |\mathcal{K}_{[\alpha k], k}(T, x)|^2 dx \right)^{\frac{1}{2}} &\geq 2c_5 e^{-kT} - c_6 e^{-\frac{k}{2} [(-1+\alpha)^2 - \epsilon + 2T]} \\ &\geq e^{-kT} \left( 2c_5 - c_6 e^{-\frac{k}{2} [(-1+\alpha)^2 - \epsilon]} \right) \geq c_7 e^{-kT}, \end{aligned}$$

where we have also taken (7.2) into account.

*Step 4: upper bound for the right-hand side of (7.5).* There exist constants  $c_8, c_9, c_{10} > 0$  and  $k_3(\epsilon) > k_2(\epsilon)$  such that, for every  $k > k_3(\epsilon)$ ,

$$\begin{aligned}
 & \left( \int_0^T \int_a^1 |\mathcal{K}_{[\alpha k],k}(t,x)|^2 dx dt \right)^{\frac{1}{2}} \\
 & \leq \left( \int_0^T \int_a^1 \sqrt{k} e^{-k\left(x+\frac{[\alpha k]}{k}\right)^2} e^{-2kt} dx dt \right)^{\frac{1}{2}} + \left( \int_0^T \int_a^1 \sqrt{k} e^{-k\left(1+\frac{[\alpha k]}{k}\right)^2} \theta_+(x)^2 e^{-2kt} dx dt \right)^{\frac{1}{2}} \\
 & \leq \left( \int_{\sqrt{k}\left(a+\frac{[\alpha k]}{k}\right)}^{\infty} \frac{e^{-x^2}}{2k} dx \right)^{\frac{1}{2}} + \frac{c_8}{\sqrt[4]{k}} e^{-\frac{k}{2}[(1+\alpha)^2-\epsilon]} \\
 & \leq c_9 \left( \frac{e^{-k\left(a+\frac{[\alpha k]}{k}\right)^2}}{\sqrt{k}\left(a+\frac{[\alpha k]}{k}\right)} \right)^{\frac{1}{2}} + c_8 e^{-\frac{k}{2}[(1+\alpha)^2-\epsilon]} \leq c_{10} e^{-\frac{k}{2}[(a+\alpha)^2-\epsilon]}
 \end{aligned}$$

where we have used the fact that  $0 < a + \alpha < 1 + \alpha$ .

*Step 5: conclusion.* We deduce from (7.5), Step 3 and Step 4 that

$$c_7 e^{-kT} \leq c_3 c_{10} e^{-\frac{k}{2}[(a+\alpha)^2-\epsilon]} + c_4 k^{7/8} e^{-\frac{k}{2}[-(1+\alpha)^2-\epsilon]}, \quad \forall k > k_3(\epsilon).$$

Moreover, by choice of  $\alpha$ , we have  $(a + \alpha)^2 = (-1 + \alpha)^2$ , thus

$$e^{-kT} \leq [c_3 c_{10} + c_4 k^{7/8}] e^{-\frac{k}{2}[-(1+\alpha)^2-\epsilon]}, \quad \forall k > k_3(\epsilon).$$

This is a contradiction because  $T < (-1 + \alpha)^2 - \epsilon$ .  $\square$

### 7.3. Proof of Theorem 1

Let  $a, b \in \mathbb{R}$  be such that  $-1 < a < b < 1$ ,  $\omega_y$  be an open subset of  $\mathbb{T}$  and  $\omega := (a, b) \times \omega_y \times \mathbb{T}$ . The quantity

$$T_{\min} := \inf \{ T > 0 : \text{system (1.4) is observable in } \omega \text{ in time } T \}$$

is finite by Section 7.1 and  $\geq \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$  by Proposition 13.

## 8. Observability on an unbounded domain

In this section, we consider the Heisenberg equation

$$\begin{cases} \left( \partial_t - \left( \partial_{x_1} - \frac{x_2}{2} \partial_{x_3} \right)^2 - \left( \partial_{x_2} + \frac{x_1}{2} \partial_{x_3} \right)^2 \right) G = 0 & \text{in } (0, T) \times \tilde{\Omega}, \\ G(t, \pm 1, x_2, x_3) = 0, \\ G(t, x_1, -\pi, x_3) = G(t, x_1, \pi, x_3), \\ \partial_y G(t, x_1, -\pi, x_3) = \partial_y G(t, x_1, \pi, x_3), \\ G(0, x) = G_0(x), \end{cases} \quad (8.1)$$

where  $(x_1, x_2, x_3) \in \tilde{\Omega} := (-1, 1) \times (-\pi, \pi) \times \mathbb{R}$ , and we prove the following observability result.

**Theorem 5.** *Let  $-1 < a < b < 1$ ,  $-\pi < c < d < \pi$  and  $\omega := (a, b) \times (c, d) \times \mathbb{R}$ . Then there exists  $T_{\min} \geq \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$  such that*

- for every  $T > T_{\min}$ , system (8.1) is observable in  $\omega$  in time  $T$ ,
- for every  $T < T_{\min}$ , system (8.1) is not observable in  $\omega$  in time  $T$ .

In a similar way, one can extend the Lipschitz stability result of Theorems 3 and 4 to system (8.1).

Observe that the change of variables given in (1.3) transforms system (8.1) into the following auxiliary system

$$\begin{cases} \left( \partial_t - \partial_x^2 - (x \partial_z + \partial_y)^2 \right) g(t, x, y, z) = 0, & (t, x, y, z) \in (0, T) \times \Omega, \\ g(t, \pm 1, y, z) = 0, & (t, y, z) \in (0, T) \times \mathbb{T} \times \mathbb{R}, \\ g(0, x, y, z) = g_0(x, y, z), & (x, y, z) \in \Omega, \end{cases} \quad (8.2)$$

where  $\Omega = (-1, 1) \times \mathbb{T} \times \mathbb{R}$ . This equation is well posed in  $L^2(\Omega)$  as is equation (8.1). Theorem 5 is a direct consequence of the same statement for (8.2). The observability in large time can be proved by following the same arguments than in the previous sections, replacing summations over  $p \in \mathbb{Z}$  by integrals over  $p \in \mathbb{R}$ . In Section 6, the assumption “ $p \in \mathbb{Z}$ ” was used to simplify the writing of several estimates, but the same analysis can be performed for  $p \in \mathbb{R}$  by replacing  $|p|$  by  $\min\{|p|, p^2\}$  at several places, as in Proposition 6. On the other hand, the counter-example we gave to show that observability fails in time  $T < \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$  needs adjustment, which is what we do below.

**Adaptation of the proof of Proposition 13.** Let  $k_1(\epsilon) \in \mathbb{N}^*$  be such that

$$\left( -1 + \frac{k}{p} \right)^2 \geq (-1 + \alpha)^2 - \epsilon, \quad \forall p \in \left( \frac{k}{\alpha}, 1 + \frac{k}{\alpha} \right), \quad \forall k \geq k_1(\epsilon). \quad (8.3)$$

*Step 1: construction of  $g_k$ .* After introducing  $G$ ,  $\mathcal{K}_{n,p}$  and  $\mathcal{G}_{n,p}$  as in the proof of Proposition 13, we define

$$g_k(t, x, y, z) := \frac{e^{iky}}{2\pi} \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \mathcal{G}_{k,p}(t, x) e^{ipz} dp.$$

*Step 2: contradiction argument.* Suppose  $c_1 > 0$  is such that,  $\forall k \in \mathbb{N}^*$ ,

$$\left( \int_{-1}^1 \int_{\mathbb{T}} \int_{\mathbb{R}} |g_k(T, x, y, z)|^2 dz dy dx \right)^{\frac{1}{2}} \leq c_1 \left( \int_0^T \int_a^1 \int_{\mathbb{T}} \int_{\mathbb{R}} |g_k(t, x, y, z)|^2 dz dy dx dt \right)^{\frac{1}{2}}.$$

By Plancherel’s identity, this inequality may be rewritten as

$$\left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{G}_{k,p}(T, x)|^2 dp dx \right)^{\frac{1}{2}} \leq c_1 \left( \int_0^T \int_a^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{G}_{k,p}(t, x)|^2 dp dx dt \right)^{\frac{1}{2}} \quad \forall k \in \mathbb{N}^*.$$

As above, by the triangular inequality and (7.4) we deduce that, for some constant  $c_2 > 0$  and all  $k > k_1(\epsilon)$ ,

$$\begin{aligned} & \left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{K}_{k,p}(T, x)|^2 dp dx \right)^{\frac{1}{2}} \\ & \leq c_2 \left( \int_0^T \int_a^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{K}_{k,p}(t, x)|^2 dp dx dt \right)^{\frac{1}{2}} + c_2 k^{7/8} e^{-\frac{k}{2\alpha} [(-1+\alpha)^2 - \epsilon]}. \end{aligned} \quad (8.4)$$

*Step 3: lower bound for the left-hand side of (8.4).* We have

$$\begin{aligned} \left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{K}_{k,p}(T, x)|^2 dp dx \right)^{\frac{1}{2}} & \geq \left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \sqrt{p} e^{-p\left(x+\frac{k}{p}\right)^2} e^{-2pT} dp dx \right)^{\frac{1}{2}} \\ & \quad - \sum_{\sigma \in \{-1, 1\}} \left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \sqrt{p} e^{-p\left(\sigma+\frac{k}{p}\right)^2} \theta_{\sigma}(x)^2 e^{-2pT} dp dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, there exists  $c_3, c_4 > 0$  and  $k_2(\epsilon) \geq k_1(\epsilon)$  such that for all  $k > k_2(\epsilon)$

$$\begin{aligned} & \left( \int_{-1}^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{K}_{k,p}(T, x)|^2 dp dx \right)^{\frac{1}{2}} \\ & \geq 2c_3 \left( \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} e^{-2pT} dp \right)^{\frac{1}{2}} - c_4 \left( \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} e^{-p[(-1+\alpha)^2 - \epsilon + 2T]} dp \right)^{\frac{1}{2}} \\ & \geq e^{-\frac{kT}{\alpha}} \left\{ 2c_3 - c_4 \left( e^{-\frac{k}{2\alpha} [(-1+\alpha)^2 - \epsilon]} \right) \right\} \geq c_3 e^{-\frac{kT}{\alpha}}, \end{aligned}$$

where we have also used (7.2).

*Step 4: upper bound for the right-hand side of (8.4).* There exist constants  $c_5, c_6, c_7 > 0$  and  $k_3(\epsilon) > k_2(\epsilon)$  such that, for every  $k > k_3(\epsilon)$ ,

$$\begin{aligned}
& \left( \int_0^T \int_a^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} |\mathcal{K}_{k,p}|^2 dp dx dt \right)^{\frac{1}{2}} \\
& \leq \left( \int_0^T \int_a^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \sqrt{p} e^{-p\left(x+\frac{k}{p}\right)^2} e^{-2pt} dp dx dt \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^T \int_a^1 \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \sqrt{p} e^{-p\left(1+\frac{k}{p}\right)^2} \theta_+^2 e^{-2pt} dp dx dt \right)^{\frac{1}{2}} \\
& \leq \left( \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \int_{\sqrt{p}\left(a+\frac{k}{p}\right)}^{\infty} \frac{e^{-x^2}}{2p} dx dp \right)^{\frac{1}{2}} + \frac{c_5}{\sqrt[4]{k}} e^{-\frac{k}{2\alpha}[(1+\alpha)^2-\epsilon]} \\
& \leq c_6 \left( \int_{\frac{k}{\alpha}}^{1+\frac{k}{\alpha}} \frac{e^{-p\left(a+\frac{k}{p}\right)^2}}{\sqrt{p}\left(a+\frac{k}{p}\right)} dp \right)^{\frac{1}{2}} + c_5 e^{-\frac{k}{2\alpha}[(1+\alpha)^2-\epsilon]} \leq c_7 e^{-\frac{k}{2\alpha}[(a+\alpha)^2-\epsilon]},
\end{aligned}$$

where we have used the fact that  $0 < a + \alpha < 1 + \alpha$ .

*Step 4: conclusion.* Combining (8.4), Step 3, and Step 4 we conclude that

$$c_3 e^{-\frac{kT}{\alpha}} \leq c_2 c_7 e^{-\frac{k}{2\alpha}[(a+\alpha)^2-\epsilon]} + c_2 k^{7/8} e^{-\frac{k}{2\alpha}[(-1+\alpha)^2-\epsilon]}$$

for all  $k > k_3(\epsilon)$ . Moreover, the choice of  $\alpha$  yields  $(a + \alpha)^2 = (-1 + \alpha)^2$ . Thus, the above inequality gives a contradiction because  $T < (a + \alpha)^2 - \epsilon$ .  $\square$

## 9. Conclusion and open problems

In this article, we have proved observability inequalities and Lipschitz stability estimates for the Heisenberg heat equation on product-shaped domains in  $\mathbb{R}^3$ . Observations were taken on appropriate slices or tubes. Both results require a minimal time  $T_{\min} > 0$ , a lower bound for which was given in terms of the distance between the observability region and the boundary of the space domain, in the  $x$  direction. The sharp evaluation of  $T_{\min} > 0$  is an open problem for which the techniques developed in [6] for Grushin's operator seem hard to utilize.

The Heisenberg heat equation is also well posed on the unbounded domain  $(x, y, z) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$ . In this situation, the dissipation speed  $\lambda_{n,p}$  does not depend on  $n$ , because of the invariance under translations of variable  $x$  (see Remark 1). Thus the Lebeau–Robbiano method cannot be performed. The validity of the observability inequality in this configuration is a completely open problem.

## Appendix A. Proof of unique continuation

In this appendix, we give a proof of [Proposition 2](#).

Let  $T > 0$ ,  $a, b \in \mathbb{R}$  be such that  $-1 < a < b < 1$ ,  $\omega_y$  be an open subset of  $\mathbb{T}$  and  $g \in C^0([0, T], L^2(\Omega))$  be a solution of (1.4) with  $\tilde{h} = 0$ , which vanishes on  $(0, T) \times (a, b) \times \omega_y \times \mathbb{T}$ .

Let  $\epsilon > 0$  be such that

$$\tilde{T} := \frac{T}{\epsilon} > 2 \operatorname{diam}((-1, 1) \times \mathbb{T} \times \mathbb{T}) \quad (\text{A.1})$$

and  $\tilde{g}(\tau, x, y, z) := g(\epsilon\tau, x, y, z)$  for every  $(\tau, x, y, z) \in (0, \tilde{T}) \times \Omega$ . Then

$$\begin{cases} \left( \partial_\tau - \epsilon \left( \partial_x^2 + (x\partial_z + \partial_y)^2 \right) \right) \tilde{g}(\tau, x, y, z) = 0, & (\tau, x, y, z) \in (0, \tilde{T}) \times \Omega, \\ \tilde{g}(\tau, \pm 1, y, z) = 0, & (\tau, y, z) \in (0, \tilde{T}) \times \mathbb{T} \times \mathbb{T}, \\ \tilde{g}(0, x, y, z) = g^0(x, y, z), & (x, y, z) \in \Omega \end{cases}$$

and  $\tilde{g} = 0$  on  $(0, \tilde{T}) \times (a, b) \times \omega_y \times \mathbb{T}$ .

Let  $\mathcal{O}$  be the maximal open subset of  $(0, \tilde{T}) \times (-1, 1) \times \mathbb{T} \times \mathbb{T}$  such that  $\tilde{g} = 0$  on  $\mathcal{O}$ . Then

$$(0, \tilde{T}) \times (a, b) \times \omega_y \times \mathbb{T} \subset \mathcal{O}. \quad (\text{A.2})$$

Working by contradiction, we assume that  $\mathcal{O} \neq (0, \tilde{T}) \times (-1, 1) \times \mathbb{T} \times \mathbb{T}$ . Let

$$\tau_0 := \frac{\tilde{T}}{2}, \quad x_0 := \frac{a+b}{2}, \quad y_0 \in \omega_y \quad \text{and} \quad z_0 \in \mathbb{T}. \quad (\text{A.3})$$

Then, by (A.2),  $(\tau_0, x_0, y_0, z_0) \in \mathcal{O}$ . Let  $(\tau_*, x_*, y_*, z_*) \in \partial\mathcal{O}$  be such that

$$\|(\tau_0, x_0, y_0, z_0) - (\tau_*, x_*, y_*, z_*)\| = r := \operatorname{dist}((\tau_0, x_0, y_0, z_0), \partial\mathcal{O}).$$

Then, necessarily

$$z_0 = z_*. \quad (\text{A.4})$$

*Step 1: we show that  $\tau_* \in (0, \tilde{T})$ .* Working by contradiction, suppose that  $\tau_* \in \{0, \tilde{T}\}$ . Then, from (A.3) we deduce that  $|\tau_0 - \tau_*| = \tilde{T}/2$ . So,

$$\begin{aligned} r &= \|(\tau_0, x_0, y_0, z_0) - (\tau_*, x_*, y_*, z_*)\| \geq \tilde{T}/2 > \operatorname{diam}((-1, 1) \times \mathbb{T} \times \mathbb{T}) \\ &> \operatorname{diam}(\mathcal{O} \cap [\{t_0\} \times (-1, 1) \times \mathbb{T} \times \mathbb{T}]) > \operatorname{dist}((\tau_0, x_0, y_0), \partial\mathcal{O}) = r, \end{aligned}$$

which is impossible.

*Step 2: we prove that*

$$\begin{pmatrix} x_* - x_0 \\ y_* - y_0 \end{pmatrix} \neq 0. \quad (\text{A.5})$$

From (A.3), (A.2), and Step 1 we deduce that  $(\tau_*, x_0, y_0, z_*)$  belongs to the open subset  $\mathcal{O}$ . So,  $(\tau_*, x_0, y_0, z_*) \neq (\tau_*, x_*, y_*, z_*)$  since the latter point belongs to the boundary of  $\mathcal{O}$ . Thus (A.5) holds.

Step 3: we apply Holmgren's uniqueness theorem. We denote by

$$\sigma((\tau, x, y, z), (s, \xi, \eta, v)) = \epsilon \xi^2 + \epsilon(xv + \eta)^2$$

the principal symbol of the Heisenberg operator  $P := \partial_\tau - \epsilon[\partial_x^2 + (x\partial_z + \partial_y)^2]$ . Let  $\Sigma$  be the sphere with center  $(\tau_0, x_0, y_0, z_0)$  and radius  $r$ . By (A.4), the unit normal to  $\Sigma$  at  $(\tau_*, x_*, y_*, z_*)$  is

$$\vec{n} := (n_\tau, n_x, n_y, n_z) = \frac{1}{r}(\tau_* - \tau_0, x_* - x_0, y_* - y_0, 0).$$

Consequently,  $\sigma((\tau^*, x^*, y^*, z^*), (n_\tau, n_x, n_y, n_z)) = n_x^2 + n_y^2 \neq 0$  by Step 2. Thus  $\Sigma$  is a smooth noncharacteristic surface for  $P$  at  $(\tau_*, x_*, y_*, z_*)$ . Moreover,  $g \equiv 0$  on one side of  $\Sigma$ , in a neighborhood of  $(\tau_*, x_*, y_*, z_*)$ . By Holmgren's theorem [25, Theorem 8.6.5],  $g \equiv 0$  on an open neighborhood of  $(\tau^*, x^*, y^*, z^*)$ . This contradicts the maximality of  $\mathcal{O}$ .  $\square$

## Appendix B. Carleman estimates for the 1D heat equation with parameters

Let us set  $\mathbb{R}_+ = (0, \infty)$  and  $\mathbb{I} = [-1, 1]$ . For any  $T > 0$  let  $\mathbb{I}_T = [0, T] \times [-1, 1]$ . In this appendix section, we use several constants called  $C_1, \dots, C_{12}$ , that are valid only in this section. In particular, they should not be confused with the constants  $C_1, \dots, C_9$  introduced in article sections 4, 5 and 6.

**Proof of Proposition 5.** Fix  $a', b'$  be such that  $a < a' < b' < b$ . Fix a real-valued function  $\beta \in C^3([-1, 1])$  such that

$$\beta \geq 1 \text{ on } [-1, 1], \quad (\text{B.1})$$

$$|\beta'| > 0 \text{ on } [-1, a'] \cup [b', 1], \quad (\text{B.2})$$

$$\beta'(1) > 0, \quad \beta'(-1) < 0, \quad (\text{B.3})$$

$$\beta'' < 0 \text{ on } [-1, a'] \cup [b', 1]. \quad (\text{B.4})$$

For any  $M > 0$  define

$$\alpha(t, x) = \frac{M\beta(x)}{t(T-t)}, \quad (t, x) \in (0, T) \times [-1, 1]. \quad (\text{B.5})$$

Given a complex-valued function  $g \in C^0([0, T]; L^2(-1, 1)) \cap L^2(0, T; H_0^1(-1, 1))$ , let us consider the standard transform

$$z(t, x) := g(t, x)e^{-\alpha(t, x)}, \quad (t, x) \in (0, T) \times [-1, 1]. \quad (\text{B.6})$$

In the following computations we shall assume  $g$  more regular so that we can compute derivatives of all the orders we need in order to obtain estimate (4.1). Such a procedure can be made rigorous assuming  $\mathcal{P}_{n,p}g \in L^2(\mathbb{I}_T)$ . We have



$$e^{-\alpha} \mathcal{P}_{n,p} g = P_1 z + P_2 \bar{z}, \quad (\text{B.7})$$

where we have set

$$\begin{aligned} P_1 z &= -\partial_x^2 z + (\alpha_t - \alpha_x^2 - \alpha_{xx})z + (px + n)^2 z \\ P_2 \bar{z} &= \partial_t z - 2\alpha_x \partial_x z. \end{aligned} \quad (\text{B.8})$$

We follow the classical proof which consists in taking the  $L^2$ -norm of both sides of the identity (B.7). Developing the double product and recalling that  $z$  is complex-valued, we obtain

$$\int_{\mathbb{I}_T} \operatorname{Re}(P_1 z \overline{P_2 z}) dx dt \leq \frac{1}{2} \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_{n,p} g|^2 dx dt, \quad (\text{B.9})$$

where  $\operatorname{Re} z$  denotes the real part of  $z$ . We have

$$\begin{aligned} \operatorname{Re}(P_1 z \overline{P_2 z}) &= -\operatorname{Re}(\partial_x^2 z \partial_t \bar{z} - 2\alpha_x \partial_x^2 z \partial_x \bar{z}) \\ &\quad + (\alpha_t - \alpha_x^2 - \alpha_{xx}) \operatorname{Re}(z \partial_t \bar{z} - 2\alpha_x z \partial_x \bar{z}) + (px + n)^2 \operatorname{Re}(z \partial_t \bar{z} - 2\alpha_x z \partial_x \bar{z}) \\ &=: Q_1 + Q_2 + Q_3. \end{aligned}$$

Now, we compute the integrals of  $Q_1$ ,  $Q_2$ , and  $Q_3$ .

**Evaluation of  $\int_{\mathbb{I}_T} Q_1$ :** integrating by parts, we get

$$\begin{aligned} \int_{\mathbb{I}_T} Q_1 dx dt &= \int_0^T \left[ \alpha_x |\partial_x z|^2 - \operatorname{Re}(\partial_x z \partial_t \bar{z}) \right]_{x=-1}^{x=1} dt \\ &\quad + \int_{\mathbb{I}_T} \left[ \operatorname{Re}(\partial_x z \partial_t \bar{z}) - \alpha_{xx} |\partial_x z|^2 \right] dx dt \\ &= \int_0^T \left[ \alpha_x(t, 1) |\partial_x z(t, 1)|^2 - \alpha_x(t, -1) |\partial_x z(t, -1)|^2 \right] dt - \int_{\mathbb{I}_T} \alpha_{xx} |\partial_x z|^2 dx dt \end{aligned} \quad (\text{B.10})$$

because  $\partial_t z(t, \pm 1) = 0$  and  $z(0, \cdot) \equiv z(T, \cdot) \equiv 0$ .

**Evaluation of  $\int_{\mathbb{I}_T} Q_3$ :** since  $z(0, \cdot) \equiv z(T, \cdot) \equiv 0$  and  $z(\cdot, -1) \equiv z(\cdot, 1) \equiv 0$ , we have

$$\begin{aligned} \int_{\mathbb{I}_T} Q_3 dx dt &= \int_{\mathbb{I}_T} (px + n)^2 \left( \frac{1}{2} \partial_t |z|^2 - \alpha_x \partial_x |z|^2 \right) dx dt \\ &= \int_{\mathbb{I}_T} [(px + n)^2 \alpha_x]_x |z|^2 dx dt. \end{aligned} \quad (\text{B.11})$$

**Evaluation of  $\int_{\mathbb{I}_T} Q_2$ :** again integrating by parts, we have

$$\begin{aligned} \int_{\mathbb{I}_T} Q_2 dx dt &= \frac{1}{2} \int_{\mathbb{I}_T} (\alpha_t - \alpha_x^2 - \alpha_{xx}) \partial_t |z|^2 dx dt - \int_{\mathbb{I}_T} \alpha_x (\alpha_t - \alpha_x^2 - \alpha_{xx}) \partial_x |z|^2 dx dt \quad (\text{B.12}) \\ &= \int_{\mathbb{I}_T} \left\{ \left[ \alpha_x (\alpha_t - \alpha_x^2) \right]_x - \frac{1}{2} (\alpha_t - \alpha_x^2 - \alpha_{xx})_t - (\alpha_x^3)_x \right\} |z|^2 dx dt. \end{aligned}$$

By combining (B.10), (B.11), and (B.12) we obtain

$$\begin{aligned} \int_{\mathbb{I}_T} \operatorname{Re} (P_1 z \overline{P_2 z}) dx dt &= - \int_{\mathbb{I}_T} \alpha_{xx} (|\partial_x z|^2 + 3\alpha_x^2 |z|^2) dx dt \quad (\text{B.13}) \\ &\quad + \int_{\mathbb{I}_T} \left\{ [\alpha_x (\alpha_t - \alpha_{xx})]_x - \frac{1}{2} (\alpha_t - \alpha_x^2 - \alpha_{xx})_t \right\} |z|^2 dx dt \\ &\quad + \int_{\mathbb{I}_T} \left[ (px + n)^2 \alpha_x \right]_x |z|^2 dx dt. \end{aligned}$$

Now, observe that, in view of (B.2) and (B.4),

$$m_1 := \min_{x \in [-1, a'] \cup [b', 1]} |\beta'(x)| > 0 \quad \text{and} \quad m_2 := \min_{x \in [-1, a'] \cup [b', 1]} -\beta''(x) > 0$$

to deduce that

$$-\alpha_{xx} (|\partial_x z|^2 + 3\alpha_x^2 |z|^2) \geq \frac{m_2 M}{t(T-t)} |\partial_x z|^2 + \frac{3m_1^2 m_2 M^3}{[t(T-t)]^3} |z|^2 \quad (\text{B.14})$$

for all  $x \in [-1, a'] \cup [b', 1]$  and  $t \in (0, T)$ . Next, consider the function

$$R_\alpha = [\alpha_x (\alpha_t - \alpha_{xx})]_x - \frac{1}{2} (\alpha_t - \alpha_x^2 - \alpha_{xx})_t \quad (\text{B.15})$$

which is defined on  $(0, T) \times [-1, 1]$ . Recalling (B.5), one can easily check that

$$|R_\alpha(t, x)| \leq \frac{C_0 M^2}{[t(T-t)]^3} \|\beta\|_{\mathcal{C}^3(\mathbb{I})}^2 (T + T^2) \quad \forall (t, x) \in (0, T) \times [-1, 1] \quad (\text{B.16})$$

for some constant  $C_0 > 0$ . Indeed, each of the terms that appear in (B.15) can be bounded by  $M^2/[t(T-t)]^3$  times a polynomial of degree two with no zero order term in  $\beta$  and its derivatives up to the third order, times  $T$  or  $T^2$ . Now, for every

$$M \geq M_1(T, \beta) := \frac{C_0 \|\beta\|_{\mathcal{C}^3(\mathbb{I})}^2}{2m_1^2 m_2} (T + T^2), \quad (\text{B.17})$$

(B.16) implies that

$$\left( \frac{3m_1^2 m_2 M^3}{[t(T-t)]^3} + R_\alpha \right) |z|^2 \geq \frac{m_1^2 m_2 M^3}{[t(T-t)]^3} |z|^2$$

for all  $x \in [-1, a'] \cup [b', 1]$  and  $t \in (0, T)$ . Therefore, owing to (B.13) and (B.14),

$$\begin{aligned} & \int_{0(-1, a') \cup (b', 1)}^T \int \frac{C_1 M}{t(T-t)} |\partial_x z|^2 dx dt \\ & + \int_{0(-1, a') \cup (b', 1)}^T \int \left\{ \frac{C_3 M^3}{[t(T-t)]^3} |z|^2 + [(px+n)^2 \alpha_x]_x |z|^2 \right\} dx dt \\ & \leq \iint_{0 \ a'}^{T \ b'} \left\{ \frac{C_2 M}{t(T-t)} |\partial_x z|^2 + \frac{C_4 M^3}{[t(T-t)]^3} |z|^2 - [(px+n)^2 \alpha_x]_x |z|^2 \right\} dx dt \\ & \quad + \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_{n,p} g|^2 dx dt \quad (\text{B.18}) \end{aligned}$$

for some constant  $C_j = C_j(\beta) > 0$  ( $j = 1, \dots, 4$ ).

Next, observe that, for every  $x \in [-1, 1]$

$$\begin{aligned} |[(px+n)^2 \alpha_x]_x| &= \frac{M}{t(T-t)} |2p(px+n)\beta'(x) + (px+n)^2 \beta''(x)| \\ &\leq \frac{C_5 M(n^2 + p^2)}{t(T-t)} \end{aligned} \quad (\text{B.19})$$

where  $C_5 = C_5(\beta) > 0$ . Let

$$M_2 = M_2(T, \beta, n, p) := \sqrt{\frac{2C_5}{C_3}} \left( \frac{T}{2} \right)^2 (|n| + p)$$

so that, for every  $M \geq M_2$ , we have

$$\frac{C_5 M(n^2 + p^2)}{t(T-t)} \leq \frac{C_3 M^3}{2[t(T-t)]^3}.$$

From now on, we fix

$$M = \max \{M_1(T, \beta), M_2(T, \beta, n, p)\}$$

noting that, in view of (B.17),  $M$  can be represented as in (4.2) for some constant  $C_2(\beta) > 0$ . From (B.18) and (B.19), it follows that

$$\begin{aligned}
& \int_{0(-1,a') \cup (b',1)}^T \int \left\{ \frac{C_1 M}{t(T-t)} |\partial_x z|^2 + \frac{C_3 M^3}{2[t(T-t)]^3} |z|^2 \right\} \\
& \leq \int \int_{0 a'}^{T b'} \left\{ \frac{C_2 M}{t(T-t)} |\partial_x z|^2 + \frac{C_6 M^3}{[t(T-t)]^3} |z|^2 \right\} dx dt + \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_{n,p} g|^2 dx dt
\end{aligned} \tag{B.20}$$

where  $C_6 = C_6(\beta) := C_4 + C_3/2$ .

A this point, we need to recast the above inequality in terms of the original function  $g$ . Since, for every  $\epsilon > 0$ ,

$$\begin{aligned}
& \frac{C_1 M}{t(T-t)} |\partial_x g - \alpha_x g|^2 + \frac{C_3 M^3}{2[t(T-t)]^3} |g|^2 \\
& \geq \left(1 - \frac{1}{1+\epsilon}\right) \frac{C_1 M}{t(T-t)} |\partial_x g|^2 + \frac{M^3}{[t(T-t)]^3} \left(\frac{C_3}{2} - \epsilon C_1 \|\beta'\|_\infty^2\right) |g|^2,
\end{aligned}$$

choosing

$$\epsilon = \epsilon(\beta) := \frac{C_3}{4C_1 \|\beta'\|_\infty^2}$$

from (B.20) we deduce that

$$\begin{aligned}
& \int_{0(-1,a') \cup (b',1)}^T \int \left\{ \frac{C_7 M}{t(T-t)} |\partial_x g|^2 + \frac{C_3 M^3 |g|^2}{4[t(T-t)]^3} \right\} e^{-2\alpha} dx dt \\
& \leq \int \int_{0 a'}^{T b'} \left\{ \frac{C_9 M^3 |g|^2}{[t(T-t)]^3} + \frac{C_8 M}{t(T-t)} |\partial_x g|^2 \right\} e^{-2\alpha} dx dt + \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_n g|^2 dx dt
\end{aligned}$$

where

$$C_7 = C_7(\beta) = [1 - 1/(1+\epsilon)]C_1$$

$$C_8 = C_8(\beta) = 2C_2$$

$$C_9 = C_9(\beta) = C_6 + 2C_2 \sup\{\beta'(x)^2 : x \in [a', b']\}.$$

So, adding the same quantity to both sides, we obtain

$$\begin{aligned}
& \int_{\mathbb{I}_T} \left\{ \frac{C_7 M}{t(T-t)} |\partial_x g|^2 + \frac{C_3 M^3 |g|^2}{4[t(T-t)]^3} \right\} e^{-2\alpha} dx dt \\
& \leq \int \int_{0 a'}^{T b'} \left\{ \frac{C_{11} M^3 |g|^2}{(t(T-t))^3} + \frac{C_{10} M}{t(T-t)} |\partial_x g|^2 \right\} e^{-2\alpha} dx dt + \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_n g|^2 dx dt
\end{aligned} \tag{B.21}$$

where  $C_{10} = C_{10}(\beta) = C_8 + C_7$  and  $C_{11} = C_{11}(\beta) = C_9 + C_3/4$ .

The last step of the proof consists in showing that  $|\partial_x g|^2$  in the right-hand side of the above inequality can be absorbed by the remaining two terms. This fact is a rather standard consequence of a Caccioppoli-type inequality. We give the proof for completeness. Let  $\rho \in C^\infty(\mathbb{R})$  be such that  $0 \leq \rho \leq 1$  and

$$\rho \equiv 1 \text{ on } [a', b'], \quad (\text{B.22})$$

$$\rho \equiv 0 \text{ on } [-1, a] \cup [b, 1]. \quad (\text{B.23})$$

We have

$$\int_{\mathbb{I}_T} (\mathcal{P}_n g) \frac{g \rho e^{-2\alpha}}{t(T-t)} dx dt = \int_{0-1}^T \int \left\{ \partial_t g - \partial_x^2 g + (px+n)^2 g \right\} \frac{g \rho e^{-2\alpha}}{t(T-t)} dx dt.$$

Integrating by parts with respect to time and space, we obtain

$$\int_{\mathbb{I}_T} \frac{1}{2} \partial_t g^2 \frac{\rho e^{-2\alpha}}{t(T-t)} dx dt = \int_{\mathbb{I}_T} \frac{1}{2} |g|^2 \rho \left\{ \frac{2\alpha_t}{t(T-t)} + \frac{T-2t}{[t(T-t)]^2} \right\} e^{-2\alpha} dx dt$$

and

$$\begin{aligned} & - \int_{\mathbb{I}_T} \partial_x^2 g \frac{g \rho e^{-2\alpha}}{t(T-t)} dx dt \\ &= \int_{\mathbb{I}_T} \frac{\rho e^{-2\alpha}}{t(T-t)} |\partial_x g|^2 dx dt - \int_{\mathbb{I}_T} \frac{|g|^2 e^{-2\alpha}}{2t(T-t)} \left\{ \rho'' - 4\rho' \alpha_x + \rho(4\alpha_x^2 - 2\alpha_{xx}) \right\} dx dt. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{\mathbb{I}_T} \mathcal{P}_n g \frac{g \rho e^{-2\alpha}}{t(T-t)} dx dt \geq \int_{\mathbb{I}_T} \frac{\rho e^{-2\alpha}}{t(T-t)} |\partial_x g|^2 dx dt \\ & - \int_{\mathbb{I}_T} \frac{|g|^2 e^{-2\alpha}}{2t(T-t)} \left\{ \rho'' - 4\rho' \alpha_x + \rho \left[ 4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)} \right] \right\} dx dt. \quad (\text{B.24}) \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{0 \ a'}^{T \ b'} \frac{C_{10} M}{t(T-t)} |\partial_x g|^2 e^{-2\alpha} dx dt \leq \int_{\mathbb{I}_T} \frac{C_{10} M \rho}{t(T-t)} |\partial_x g|^2 e^{-2\alpha} dx dt \\ & \leq \int_{\mathbb{I}_T} \mathcal{P}_n g \frac{C_{10} M g \rho e^{-2\alpha}}{t(T-t)} dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{I}_T} \frac{C_{10} M |g|^2 e^{-2\alpha}}{2t(T-t)} \left\{ \rho'' - 4\rho' \alpha_x + \rho \left[ 4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)} \right] \right\} dx dt \\
& \leq \int_{\mathbb{I}_T} |\mathcal{P}_n g|^2 e^{-2\alpha} dx dt + \int_0^T \int_a^b \frac{C_{12} M^3 |g|^2 e^{-2\alpha}}{[t(T-t)]^3} dx dt
\end{aligned}$$

for some constant  $C_{12} = C_{12}(\beta, \rho) > 0$ . Combining (B.21) with the previous inequality, we get

$$\begin{aligned}
& \int_{\mathbb{I}_T} \left\{ \frac{C_7 M}{t(T-t)} |\partial_x g|^2 + \frac{C_3 M^3 |g|^2}{4[t(T-t)]^3} \right\} e^{-2\alpha} dx dt \\
& \leq \frac{3}{2} \int_{\mathbb{I}_T} |e^{-\alpha} \mathcal{P}_n g|^2 dx dt + \int_0^T \int_a^b \frac{C_{13} M^3 |g|^2}{[t(T-t)]^3} e^{-2\alpha} dx dt,
\end{aligned}$$

where  $C_{13} = C_{13}(\beta, \rho) := C_{11} + C_{12}$ . Then, taking

$$C_1 = C_1(\beta) := \frac{\min\{C_7; C_3/4\}}{\max\{3/2; C_{13}\}}$$

we obtain the global Carleman estimate (4.1).  $\square$

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