



# Neumann problem for $p$ -Laplace equation in metric spaces using a variational approach: Existence, boundedness, and boundary regularity <sup>☆</sup>

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## Abstract

We employ a variational approach to study the Neumann boundary value problem for the  $p$ -Laplacian on bounded smooth-enough domains in the metric setting, and show that solutions exist and are bounded. The boundary data considered are Borel measurable bounded functions. We also study boundary continuity properties of the solutions. One of the key tools utilized is the trace theorem for Newton–Sobolev functions, and another is an analog of the De Giorgi inequality adapted to the Neumann problem.

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## 1. Overview

Amongst the two types of boundary value problems in PDEs, Dirichlet and Neumann problems, the Dirichlet problem is currently the most well-studied. In the Euclidean setting, much

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of the research on Neumann boundary value problem focused on the zero boundary value problem, the so-called natural boundary value. The general Neumann boundary value problem for the  $p$ -Laplacian is the following: find  $u$  in the appropriate Sobolev class such that

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ -|\nabla u|^{p-2} \partial_\eta u = f & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

which, in its weak formulation, would mean finding  $u$  in the Sobolev class  $W^{1,p}(\Omega)$  such that whenever  $\varphi \in W^{1,p}(\Omega)$ ,

$$\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) dx + \int_{\partial\Omega} |\nabla u(\zeta)|^{p-2} \varphi(\zeta) \partial_\eta u(\zeta) d\mathcal{H}^{n-1}(\zeta) = 0,$$

that is,

$$\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) dx - \int_{\partial\Omega} \varphi(\zeta) f(\zeta) d\mathcal{H}^{n-1}(\zeta) = 0,$$

where  $\partial_\eta u(\zeta)$  is the directional derivative of  $u$  in the direction of the outer normal to  $\partial\Omega$  at  $\zeta$ . Its variational formulation is to find  $u$  that minimizes the functional

$$I(u) = \int_{\Omega} |\nabla u|^p dx + \int_{\partial\Omega} u f dP_\Omega,$$

where  $|\nabla u|$  will be replaced by the (minimal  $p$ -weak) upper gradient of  $u$  in the metric setting.

In [30] issues of existence and stability of solutions to the general Neumann boundary value problem for a class of  $p$ -Laplace-type operators were considered. The paper [47] gave a computational scheme for constructing solutions to the general Neumann boundary value problem for the Laplacian in three-dimensional Euclidean domains with piecewise smooth boundary. The paper [15] of Cianchi and Maz'ya studied regularity of solutions to the Neumann boundary value problem, for Lipschitz domains, related to the  $p$ -Laplace and more general operators, but the Neumann data they consider is the natural boundary condition, i.e., constant zero data. The work of Agmon, Douglis and Nirenberg [1,2], M. Taylor [50], Cranny [16], and the recent work of Kenig, Lin, and Shen [32], Milakis and Silvestre [44], studied regularity of solutions to the general Neumann problem for homogeneous and rapidly oscillating elliptic PDEs for  $C^{1,\alpha}$ -domains in Euclidean setting. The work of Dancer, Daners and Hauer [17] explored the behavior of solutions to the zero (natural) Neumann boundary value problem for the  $p$ -Laplacian on Euclidean domains whose complement is a compact set, showing that solutions that have a certain decay property at  $\infty$  (decay to zero) have to vanish identically on the domain.

The study of the Neumann problem in non-smooth settings is currently sparse. In the more general setting of Carnot groups, Nhieu [46] studied the existence and uniqueness of solutions to the Neumann boundary value problem for the sub-Laplacian operator (corresponding to  $p = 2$ ) on bounded Lipschitz domains. More explicit computations, in terms of Green's functions, were given by Dubey, Kumar, and Misra in [18]. Mixed boundary problems and homogenization for domains in Heisenberg groups were considered by Tchou [51], Biroli, Tchou, and Zhikov [9]. In the non-smooth metric setting, an analog of the Dirichlet problem for the  $p$ -Laplacian was

initiated in [36] and is currently an active area of research (see for example [11]). In this paper, we propose an analog of the Neumann boundary value problem adapted to the non-smooth setting by using the tools of calculus of variation and ideas due to De Giorgi, Giaquinta, and Giusti.

The challenge in our situation is three-fold. First, unlike [1,2,50,32,46,18,51,9], our problem is non-linear even in the Euclidean setting ( $p$ -Laplacian with  $1 < p < \infty$ ); second, lack of smoothness structure, especially at the boundary, and so we have no notion of  $C^1$ -domain in the metric setting; third, unlike in [46,18,51,9], lack of the Euler–Lagrange (PDE) equation corresponding to the energy minimization problem, since the upper gradient structure on the metric space might not come from an inner product structure. We therefore do not have access to tools such as the Euler–Lagrange equation nor layer potentials as used for example in the work of Maz’ya and Poborchi [43]. Thus the results obtained in this paper are, not surprisingly, weaker than those of [1,2,50,32], but on the other hand, they are applicable to a wider class of operators than linear elliptic operators and are applicable to a wider range of domains even in the Euclidean setting (such as Lipschitz domains that might perhaps not be  $C^1$ -domains). We show that solutions exist (Theorem 4.3) and are bounded at the boundary of the domain (Theorem 5.2). As mentioned above, the key step is to identify an analog of the De Giorgi inequality adapted to the problem, see Theorem 5.3. Furthermore, we apply this version of De Giorgi inequality to prove continuity of solutions for certain values of  $p$  at a.e. boundary point as well as at every boundary point in whose neighborhood the Neumann data does not change its sign (Theorem 7.2 and Theorem 7.12).

The paper [20] by García-Azorero, Manfredi, Peral and Rossi studied the Neumann boundary value problem for the  $p$ -Laplace operator in the Euclidean setting and showed for smooth domains with continuous boundary data that the solutions for a given data are unique up to additive constants. Their proof used the Euler–Lagrange formulation of the problem, an approach that is not available to us in the non-smooth setting. We obtain a weaker uniqueness property, namely that the minimal  $p$ -weak upper gradients of the solutions are all equal, see Lemma 4.5. However, if the metric measure space  $X$  has a Cheeger-type differential structure (that is, a first order Taylor theorem is satisfied for Lipschitz functions with respect to a vector bundle on  $X$ ) such that the minimal  $p$ -weak upper gradient of  $u \in N^{1,p}(X)$  is equal to the norm of its Cheeger derivative, then we can conclude from Lemma 4.5 that solution is, in fact, unique (up to an additive constant), taking into consideration also that the set of solutions form a convex set. Metric spaces endowed with such a differential structure are said to be infinitesimally Hilbertian, see [6] and [21]. Even in some weighted Euclidean setting, we do not have this Hilbertian property, see [37]. In infinitesimally Hilbertian spaces, one has also access to the corresponding Euler–Lagrange equation, which enables obtaining the uniqueness of solutions (up to an additive constant) via PDE methods, see [22]. With the PDE structure developed in [22], the tools from [38] would give better control in terms of obtaining Hölder continuity up to the boundary *provided that* the Sobolev class  $N^{1,2}(X)$  is a Hilbert space. In our more general setting, we do not have access to these tools, and hence we obtain only the weaker conclusion that for small  $p$  the solution is continuous up to the parts of the boundary where the boundary datum does not switch sign. Surprisingly, we obtain the full strength of the local “boundedness of the solution at the boundary” result without needing this additional structure. Hence, much of what we do in this paper is applicable even in metric measure spaces that are neither almost Hilbertian nor infinitesimally strictly convex (note that at least one of these two properties is needed to apply the theory developed in [22]). For instance, the results we obtain hold also for the non-Hilbertian example given in [37].

There is also much interest in studying potential theory in weighted metric space setting, see [8]. It was shown in [8] that under some natural conditions, the two possible natural extensions of Sobolev spaces  $N^{1,p}(X)$  to the setting where the measure  $\mu$  is modified via a weight  $\rho \in L^1_{\text{loc}}(X)$  agree. Namely, the space  $N^{1,p}(X, \rho d\mu)$  obtained by looking at the weighted space as a metric measure space in its own right, and the space  $N^{1,p}_\rho(X)$  consisting of functions in  $N^{1,1}_{\text{loc}}(X)$  that belong to  $L^p(X, \rho d\mu)$  with an upper gradient in  $L^p(X, \rho d\mu)$  agree. The traditional approach in the weighted Euclidean setting uses the space  $N^{1,p}_\rho(X)$ , see [28]. The natural conditions assumed in [8] are that the weighted metric measure space is doubling, supports a 1-Poincaré inequality for Lipschitz functions (and hence for all functions, thanks to the results of Keith [31]), and  $\rho$  satisfies the asymptotic condition

$$\liminf_{n \rightarrow \infty} \frac{1}{n^p} \left( \int_X \rho^n d\mu \right)^{1/n} \left( \int_X \rho^{-n} d\mu \right)^{1/n} < \infty.$$

Therefore our results hold true in the weighted setting studied in [8] as well, even without the asymptotic condition above, as we do not need the perspective of weighted Sobolev spaces taken by [28].

The rest of this paper is organized as follows. We give the needed definitions in Section 2. In Section 3 we describe the Neumann problem in the metric setting using the language of calculus of variation, and discuss the needed tool of boundary trace of Sobolev functions. Existence of solutions for bounded boundary data is studied in Section 4, with Theorem 4.3 declaring the existence of solutions. A weak analog of uniqueness of solutions is given in Lemma 4.5 in this section as well. The focus of Section 5 is to prove that the solutions are necessarily bounded, see Theorem 5.2. The key inequality that is an analog of the De Giorgi inequality is also given in this section, in Theorem 5.3. In Section 6 we discuss regularity of solutions at the boundary, and show that at boundary points where the boundary data is non-negative the solution must necessarily be a subminimizer and hence is upper semicontinuous there. For metric spaces with measure  $\mu$  that have a strong regularity, known as Ahlfors regularity, we show in the final section of this paper that the solutions are continuous at boundary points where the boundary data does not change sign (Theorem 7.2) and that the solutions are continuous at Hausdorff co-dimension 1-almost every boundary point (Theorem 7.12).

## 2. Preliminaries

The triplet  $(X, d, \mu)$  denotes a metric measure space. We say that  $\mu$  is *doubling* if there is a constant  $C_D$  such that for each  $x \in X$  and  $r > 0$ ,

$$0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) < \infty.$$

**Lemma 2.1** (See e.g. [11, Lemma 3.3]). *There is  $s > 0$  such that*

$$\frac{\mu(B(x, r))}{\mu(B(y, R))} \geq C \left( \frac{r}{R} \right)^s \quad (2.2)$$

for all  $0 < r \leq R$ ,  $y \in X$ , and  $x \in B(y, R)$ .

Note that we can always take  $s$  to be as large as we wish. Therefore from now onwards we assume that  $s > 1$ . We also say that  $\mu$  is *Ahlfors  $s$ -regular at scale  $r_0 > 0$*  if there is a constant  $C > 0$  such that whenever  $x \in X$  and  $0 < r < r_0$ , we have

$$\frac{1}{C} r^s \leq \mu(B(x, r)) \leq C r^s.$$

In what follows, the space  $L^1_{\text{loc}}(X)$  consists of functions on  $X$  that are integrable on *bounded* subsets of  $X$ .

A Borel function  $g : X \rightarrow [0, \infty]$  is an upper gradient of  $u : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  if the following inequality holds for all (rectifiable) curves  $\gamma : [a, b] \rightarrow X$ , (denoting  $x = \gamma(a)$  and  $y = \gamma(b)$ ),

$$|u(y) - u(x)| \leq \int_{\gamma} g \, ds$$

whenever  $u(x)$  and  $u(y)$  are both finite, and  $\int_{\gamma} g \, ds = \infty$  otherwise. The notion of upper gradients, first formulated in [27] (with the terminology “very weak gradients”), plays the role of  $|\nabla u|$  in the metric setting where no natural distributional derivative structure exists.

**Definition 2.3** (cf. [48]). The *Newtonian space*  $N^{1,p}(X)$  is defined by

$$N^{1,p}(X) = \left\{ u \in L^p(X) : \|u\|_{N^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)} < \infty \right\},$$

where the infimum is taken over all upper gradients  $g$  of  $u$ .

Let us point out that we assume that functions are defined everywhere, and not just up to equivalence classes  $\mu$ -almost everywhere. This is essential for the notion of upper gradients since they are defined by a pointwise inequality.

**Definition 2.4.** Given a ball  $B = B(x, r) \subset X$  and a set  $E \subset B$ , the *relative  $p$ -capacity* of  $E$  with respect to  $2B = B(x, 2r)$  is given by

$$\text{cap}_p(E, 2B) := \inf_u \int_{2B} g_u^p \, d\mu,$$

where the infimum is over all functions  $u \in N^{1,p}(X)$  for which  $u \geq 1$  on  $E$  and  $u = 0$  on  $X \setminus 2B$ .

It follows from [11, Proposition 6.16] that

$$\frac{\mu(E)}{C r^p} \leq \text{cap}_p(E, 2B) \leq C \frac{\mu(B)}{r^p}. \quad (2.5)$$

**Definition 2.6** (cf. [3]). A metric space  $X$  supports a  *$p$ -Poincaré inequality* with  $p \in [1, \infty)$  if there exist positive constants  $\lambda$  and  $C$  such that for all balls  $B \subset X$  and all  $u \in L^1_{\text{loc}}(X)$ ,

$$\int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \left( \int_{\lambda B} g^p d\mu \right)^{1/p}. \quad (2.7)$$

Here and in the rest of the paper,  $f_A$  denotes the *integral mean* of a function  $f \in L^0(X)$  over a measurable set  $A \subset X$  of finite positive measure, defined as

$$f_A = \frac{1}{\mu(A)} \int_A f d\mu$$

whenever the integral on the right-hand side exists, not necessarily finite though. Furthermore, given a ball  $B = B(x, r) \subset X$  and  $\lambda > 0$ , the symbol  $\lambda B$  denotes the inflated ball  $B(x, \lambda r)$ . For a detailed treatise on analysis in metric spaces and the Newtonian approach in particular, we refer the interested reader to [26, 48, 11, 29].

We next give an analog of the notion of sets of finite perimeter, as formulated in [45], see [19, 5, 52] for the Euclidean setting.

**Definition 2.8.** A Borel set  $E \subset X$  is said to be of finite perimeter if there is a sequence  $(u_k)_{k \in \mathbb{N}}$  from  $N^{1,1}(X)$  such that  $u_k \rightarrow \chi_E$  in  $L^1(X)$  and

$$\liminf_{k \rightarrow \infty} \int_X g_{u_k} d\mu < \infty.$$

The *perimeter*  $P_E(X)$  of  $E$  is the infimum of the above limit infima over all such sequences  $(u_k)_k$  as above. Given an open set  $U \subset X$ , the perimeter of  $E$  in  $U$  is

$$P_E(U) = \inf \left\{ \liminf_{k \rightarrow \infty} \int_U g_{u_k} d\mu : (u_k)_{k \in \mathbb{N}} \subset N^{1,1}(U), u_k \rightarrow \chi_{E \cap U} \text{ in } L^1(U) \right\}.$$

An analogous notion (using Lipschitz functions rather than functions in  $N^{1,1}(X)$ ) was proposed in [45], but the notion given there agrees with ours when the measure on  $X$  is doubling and  $X$  supports a 1-Poincaré inequality. A direct translation of the proof given in [45] shows that the Carathéodory extension of  $P_E$  to subsets of  $X$  is a finite Radon measure on  $X$ . In [3], Ambrosio demonstrated that if the measure on  $X$  is doubling and supports a 1-Poincaré inequality, then the Radon measure  $P_E$  is equivalent to the co-dimension 1 Hausdorff measure restricted to the measure-theoretic boundary  $\partial_m E$  of  $E$ . Here,  $x \in \partial_m E$  if and only if  $x \in X$  and

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

Given  $A \subset X$ , we define its *co-dimension 1 Hausdorff measure*  $\mathcal{H}(A)$  by

$$\mathcal{H}(A) = \lim_{\delta \rightarrow 0^+} \inf \left\{ \sum_i \frac{\mu(B_i)}{\operatorname{rad}(B_i)} : B_i \text{ balls in } X, \operatorname{rad}(B_i) < \delta, A \subset \bigcup_i B_i \right\}. \quad (2.9)$$

Thus, the results of [3] show that there is a constant  $C \geq 1$  such that whenever  $E \subset X$  is of finite perimeter and  $K \subset X$  is a Borel set, we must have

$$\frac{1}{C} \mathcal{H}(K \cap \partial_m E) \leq P_E(K) \leq C \mathcal{H}(K \cap \partial_m E).$$

See [45, 3, 7, 5] for more on sets of finite perimeter and associated functions of bounded variation in the metric setting. The paper [4] studies connections between the relaxation of the co-dimension 1 Minkowski content of the boundary and the perimeter measure.

### 3. Statement of the problem and standing assumptions

In this paper,  $1 < p < \infty$  and  $X$  is a complete metric space equipped with a doubling measure  $\mu$  supporting a  $p$ -Poincaré inequality.

**Definition 3.1.** Let  $\Omega$  be a bounded domain (non-empty, connected open set) in  $X$  with  $X \setminus \Omega$  of positive measure such that  $\Omega$  is also of finite perimeter with perimeter measure  $P_\Omega$ . Let  $f : \partial\Omega \rightarrow \mathbb{R}$  be a bounded  $P_\Omega$ -measurable function with  $\int_{\partial\Omega} f dP_\Omega = 0$ . We say that a function  $u : \Omega \rightarrow \mathbb{R}$  is a  $p$ -harmonic solution to the Neumann boundary value problem with boundary data  $f$  if  $u \in N^{1,p}(\Omega)$  and

$$I(u) := \int_{\Omega} g_u^p d\mu + \int_{\partial\Omega} Tu f dP_\Omega \leq \int_{\Omega} g_v^p d\mu + \int_{\partial\Omega} Tv f dP_\Omega = I(v) \quad (3.2)$$

for every  $v \in N^{1,p}(\Omega)$ . Here  $g_u$  and  $g_v$  are the minimal  $p$ -weak upper gradients of  $u$  and  $v$  in  $\Omega$ , respectively, and  $Tu$  and  $Tv$  denote the traces of  $u$  and  $v$  on  $\partial\Omega$ , respectively.

When considering the original Neumann boundary value problem (1.1), we see that adding a constant to a solution gives us another solution. Thus, the Neumann boundary data  $f$  has to satisfy the compatibility condition

$$\int_{\partial\Omega} f dP_\Omega = 0$$

so that the value of the functional  $I$  as defined in (3.2) is invariant with respect to adding a constant to a solution.

**Definition 3.3** (Assumptions on  $\Omega$ ). We will assume in this paper that there is a constant  $C \geq 1$  such that for all  $x \in \partial\Omega$ ,  $z \in \Omega$ , and  $0 < r \leq \text{diam}(\Omega)$ , we have

$$\mu(B(z, r) \cap \Omega) \geq C^{-1} \mu(B(z, r)), \quad (3.4)$$

and

$$C^{-1} \frac{\mu(B(x, r))}{r} \leq P_\Omega(B(x, r)) \leq C \frac{\mu(B(x, r))}{r}. \quad (3.5)$$

We also assume that  $(\Omega, d|_{\Omega}, \mu|_{\Omega})$  admits a  $p$ -Poincaré inequality with dilation factor  $\lambda = 1$ , where  $p \in (1, \infty)$  is equal to the exponent in (3.2).

Under the above assumptions, we also have a Sobolev-type inequality for  $\Omega$ ,

$$\|u - u_{\Omega}\|_{L^p(\Omega)} \leq C \|g_u\|_{L^p(\Omega)}, \quad (3.6)$$

where  $C = C(\Omega, C_D, p, \dots)$ . This Sobolev-type embedding follows from classical embedding results of [25].

The property of satisfying (3.5) will be called *Ahlfors codimension 1 regularity* of  $P_{\Omega}$ .

The condition (3.4) together with condition (3.5) implies that  $\mu(\partial\Omega) = 0$ , and that  $\Omega$  is of finite perimeter. It follows by the results of Ambrosio [3] that if  $X$  supports a 1-Poincaré inequality, then  $P_{\Omega} \approx \mathcal{H}|_{\partial\Omega}$ ; thus the above condition (3.5) remains valid (with a different constant  $C$  perhaps) if  $P_{\Omega}$  is replaced with  $\mathcal{H}$ . Examples of domains satisfying the above conditions include domains with quasiminimal boundary surfaces as studied in [33].

Domains that are sets of finite perimeter are the natural class of domains for which the Neumann boundary value problem makes sense, as this is the largest class of domains for which, at least in the Euclidean setting, a form of Gauss–Green theorem holds true, see the work [14] of Chen, Torres and Ziemer (for metric space analogs see [42]).

The assumption that  $\lambda = 1$  in the  $p$ -Poincaré inequality supported by  $\Omega$  is satisfied for example if  $\Omega$  is a geodesic domain, that is, for each  $x, y \in \Omega$  there is a curve  $\gamma \subset \Omega$  with end points  $x, y$  such that the length of  $\gamma$  is equal to  $d(x, y)$ . It then follows from the results of [25] that the factor  $\lambda$  in the  $p$ -Poincaré inequality can be chosen to equal 1 (perhaps at the expense of a larger constant  $C$ ). The assumption that  $\lambda = 1$  is a mere technicality here, assumed for the sake of simplifying the computations; they get more complicated when  $\lambda > 1$ , but the results still remain true as an interested reader can verify.

**Definition 3.7** (*Traces of Sobolev functions on  $\partial\Omega$* ). Under the standing assumptions on  $\Omega$  given above in Definition 3.3, there is a bounded linear *trace operator*

$$T : N^{1,p}(\Omega) \rightarrow L^{\tilde{p}}(\partial\Omega)$$

for every  $\tilde{p} < p^*$ , where  $p^* = p(s-1)/(s-p)$  if  $p < s$ , and  $p^* = \infty$  if  $p \geq s$ . This trace operator is given as follows. For  $u \in N^{1,p}(\Omega)$ ,  $\mathcal{H}$ -almost every  $x \in \partial\Omega$ , there exists  $Tu(x) \in \mathbb{R}$  such that

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - Tu(x)| d\mu = 0.$$

Here,  $s$  is the lower mass bound exponent from (2.2). If  $p > s$ , then we can allow for  $\tilde{p} = \infty$  as well, though this is not of importance to us in this paper.

Existence of such a trace operator follows from [40, Theorem 3.4], see also [24, 39]. The following trace theorem is a specific case of the trace theorem found in [41].

**Proposition 3.8** (cf. [41, Proposition 4.21]). Assume that  $\Omega$  is a length space and that the dilation factor  $\lambda = 1$  in the Poincaré inequality (2.7). Suppose that  $p < s$ . Let  $\tilde{p} \in (p, p^*)$ . Then, the trace



operator  $T : N^{1,p}(\Omega) \rightarrow L^{\tilde{p}}(\partial\Omega)$  is linear and bounded. Moreover, there is  $C > 0$  such that  $T$  satisfies

$$\begin{aligned} \|Tu\|_{L^{\tilde{p}}(\partial\Omega \cap B)} &\leq C \left( r^{(s-1)(\frac{1}{p} - \frac{1}{p^*})} \|g_u\|_{L^p(\Omega \cap B)} + \frac{P_\Omega(\partial\Omega \cap B)^{1/\tilde{p}}}{\mu(\Omega \cap B)} \|u\|_{L^1(\Omega \cap B)} \right) \\ &= C \left( r^{1-\frac{1}{p}-\aleph} \|g_u\|_{L^p(\Omega \cap B)} + \frac{P_\Omega(\partial\Omega \cap B)^{1/\tilde{p}}}{\mu(\Omega \cap B)} \|u\|_{L^1(\Omega \cap B)} \right) \end{aligned}$$

for every ball  $B = B(z, r)$  with  $z \in \partial\Omega$ , where  $\aleph = s(\frac{1}{p} - \frac{1}{p^*})$ .

**Remark 3.9.** The requirement that  $\lambda = 1$  is not restrictive, since length spaces supporting a  $p$ -Poincaré inequality will support such an inequality with  $\lambda = 1$  (perhaps at the expense of a larger constant  $C$ ), see for example [25, 11].

From now on, for ease of notation, the trace  $Tu$  of  $u$  will also be denoted by  $u$ .

Throughout the paper  $C$  represents various constants that depend solely on the doubling constant, constants related to the Poincaré inequality, and the constants related to (3.4) and (3.5). The precise value of  $C$  is not of interest to us at this time, and its value may differ in each occurrence. Given expressions  $a$  and  $b$ , we say that  $a \approx b$  if there is a constant  $C \geq 1$  such that  $C^{-1}a \leq b \leq Ca$ .

#### 4. Existence of a minimizer

The natural space to look for a minimizer of  $I$  would be  $W^{1,p}(\Omega)$  if we worked in the Euclidean setting. In the metric setting, we will make use of the Newtonian space  $N^{1,p}(\Omega)$  as a suitable counterpart of the Sobolev space.

Since we aim to obtain a unique representative of a solution and adding a constant to a solution yields another solution, we will make use of the following normalization

$$N_*^{1,p}(\Omega) = \left\{ u \in N^{1,p}(\Omega) : \int_{\Omega} u \, dx = 0 \right\}.$$

Observe that  $u \equiv 0$  is a candidate for the infimum in the definition of  $I(u)$ , (3.2). Therefore,

$$\inf_{u \in N_*^{1,p}(\Omega)} I(u) \leq 0.$$

To show existence of a minimizer, we need to prove that  $I(u)$  is bounded below for  $u \in N_*^{1,p}(\Omega)$  and that the functional is sequentially lower semi-continuous. Based on the relation between  $p \in (1, \infty)$  and the “upper measure dimension”  $s$  given by (2.2), we will a priori distinguish two possible integrability conditions of the Neumann boundary data.

**Proposition 4.1.** Let  $u \in N_*^{1,p}(\Omega)$  and  $f \in L^q(\partial\Omega)$ , where  $q = 1$  if  $p > s$ , and  $\frac{p(s-1)}{s(p-1)} < q \leq \infty$  if  $p \leq s$ . Then,

$$I(u) \geq \|g_u\|_{L^p(\Omega)}^p - C \|g_u\|_{L^p(\Omega)} \|f\|_{L^q(\partial\Omega)}.$$

**Proof.** The Hölder inequality yields that

$$I(u) \geq \int_{\Omega} |g_u|^p d\mu - \int_{\partial\Omega} |uf| dP_{\Omega} \geq \|g_u\|_{L^p(\Omega)}^p - \|u\|_{L^{q'}(\partial\Omega)} \|f\|_{L^q(\partial\Omega)}.$$

It follows from the (proof of the) trace theorem for  $N^{1,p}$  functions in  $p$ -Poincaré spaces [41, Corollary 4.14] that  $\|u\|_{L^{q'}(\partial\Omega)} \leq C \|g_u\|_{L^p(\Omega)}$  provided that  $\int_{\Omega} u d\mu = 0$ . Thus,

$$I(u) \geq \|g_u\|_{L^p(\Omega)} (\|g_u\|_{L^p(\Omega)}^{p-1} - C \|f\|_{L^q(\partial\Omega)}). \quad \square$$

Note that functions that are bounded and  $P_{\Omega}$ -measurable on  $\partial\Omega$  are automatically in  $L^q(\partial\Omega)$ .

**Corollary 4.2.** *There is a constant  $C > 0$ , depending on  $p, q$ , and on the norm of the trace operator  $T : N^{1,p}(\Omega) \rightarrow L^{q'}(\partial\Omega)$  such that*

$$I(u) \geq -C \|f\|_{L^q(\partial\Omega)}^{p'}$$

for every  $u \in N_*^{1,p}(\Omega)$ .

**Proof.** The estimate can be shown by finding the absolute minimum of the function  $t \mapsto t^p - Ct \|f\|_{L^q(\partial\Omega)}$ , where  $t \geq 0$ .  $\square$

**Theorem 4.3.** *There is  $u \in N_*^{1,p}(\Omega)$  such that  $I = I(u)$ .*

**Proof.** Let  $I = \inf_{u \in N_*^{1,p}(\Omega)} I(u)$  and let  $\{u_k\}_{k=1}^{\infty} \subset N_*^{1,p}(\Omega)$  be a minimizing sequence, i.e.,  $I = \lim_{k \rightarrow \infty} I(u_k)$ . Let  $g_k$  denote the  $p$ -weak minimal upper gradients of  $u_k$ ,  $k = 1, 2, \dots$ . Using Proposition 4.1, we see that  $I(v) \leq 0$  requires that  $\|g_v\|_{L^p(\Omega)} \leq C^{1/(p-1)} \|f\|_{L^q(\partial\Omega)}^{1/(p-1)}$ . Hence, the sequence  $\{g_k\}_{k=1}^{\infty}$  is bounded in  $L^p(\Omega)$ . Using (3.6), we obtain that  $\{u_k\}_{k=1}^{\infty}$  is also bounded in  $L^p(\Omega)$  since  $(u_k)_{\Omega} = 0$  by definition of  $N_*^{1,p}(\Omega)$ . The reflexivity of  $L^p(\Omega)$  yields that there are subsequences (which will also be denoted by  $\{u_k\}_{k=1}^{\infty}$  and  $\{g_k\}_{k=1}^{\infty}$ ) and  $u, g \in L^p(\Omega)$  such that  $u_k \rightharpoonup u$  and  $g_k \rightharpoonup g$  as  $k \rightarrow \infty$ .

By Mazur's lemma, there are convex combinations

$$\tilde{u}_k = \sum_{i=k}^{N(k)} \alpha_{k,i} u_i \quad \text{and} \quad \tilde{g}_k = \sum_{i=k}^{N(k)} \alpha_{k,i} g_i, \quad k = 1, 2, \dots,$$

such that  $\tilde{u}_k \rightarrow u$  and  $\tilde{g}_k \rightarrow g$  in  $L^p(\Omega)$ . Observe that  $\tilde{g}_k$  are  $p$ -weak upper gradients of  $\tilde{u}_k$  (not necessarily minimal, though). By [11, Proposition 2.3], we can modify  $u$  on a set of measure zero to obtain a good representative such that  $g$  is its  $p$ -weak upper gradient. In what follows, we will consider  $u$  to be such a good representative and hence  $u \in N^{1,p}(\Omega)$ . Applying [11, Proposition 2.3 and Corollary 6.3] and passing to a subsequence if necessary, we obtain that

$$\int_{\Omega} g_u^p d\mu \leq \liminf_{k \rightarrow \infty} \int_{\Omega} g_{\tilde{u}_k}^p d\mu,$$

where  $g_u$  and  $g_{\tilde{u}_k}$  are the minimal  $p$ -weak upper gradients of  $u$  and  $\tilde{u}_k$ , respectively.

Since  $\int_{\Omega} u_k = 0$  for every  $k = 1, 2, \dots$  and  $u_k \rightarrow u$ , we have that  $\int_{\Omega} u = 0$ . Hence,  $u \in N_*^{1,p}(\Omega)$ .

Considering that the trace operator  $T : N_*^{1,p}(\Omega) \rightarrow L^{q'}(\partial\Omega)$  is linear and the energy functional  $v \mapsto \int_{\Omega} g_v^p d\mu$  is convex, we see that

$$I \leq I(\tilde{u}_k) = I\left(\sum_{i=k}^{N(k)} \alpha_{k,i} u_i\right) \leq \sum_{i=k}^{N(k)} \alpha_{k,i} I(u_i) \rightarrow I \quad \text{as } k \rightarrow \infty.$$

The continuity of the trace operator yields that

$$\begin{aligned} I &\leq I(u) = \int_{\Omega} g_u^p d\mu + \int_{\partial\Omega} u f dP_{\Omega} \\ &\leq \liminf_{k \rightarrow \infty} \left( \int_{\Omega} g_{\tilde{u}_k}^p d\mu + \int_{\partial\Omega} \tilde{u}_k f dP_{\Omega} \right) = \liminf_{k \rightarrow \infty} I(\tilde{u}_k) = I. \quad \square \end{aligned}$$

**Lemma 4.4.** *The set  $M_I = \{u \in N_*^{1,p}(\Omega) : I(u) = I\}$  of minimizers of  $I(\cdot)$  is norm-closed and convex.*

**Proof.** Let  $\lambda \in (0, 1)$  and let  $u, v \in M_I$ , then  $w = \lambda u + (1 - \lambda)v$  satisfies

$$I(w) = I(\lambda u + (1 - \lambda)v) \leq \lambda I(u) + (1 - \lambda)I(v) = I$$

due to convexity of the functional  $I(\cdot)$ . Therefore,  $w \in M_I$ .

The set  $M_I$  is closed due to sequential lower semi-continuity of  $I(\cdot)$ .  $\square$

**Lemma 4.5.** *Suppose that  $u, v \in M_I$ . Then  $\int_{\partial\Omega} u f dP_{\Omega} = \int_{\partial\Omega} v f dP_{\Omega}$  and  $g_u = g_v$  a.e. in  $\Omega$ . Furthermore, if  $u, v \in M_I$  then the functions  $w_+, w_-$  given by*

$$w_+ := \max\{u, v\} - \int_{\Omega} \max\{u, v\} d\mu$$

and

$$w_- := \min\{u, v\} - \int_{\Omega} \min\{u, v\} d\mu$$

also belong to  $M_I$ .

**Proof.** For any  $u$  and  $v$  as in the hypothesis, set  $w = \frac{u+v}{2}$ . Then  $g_w \leq \frac{1}{2}[g_u + g_v]$ .

By the uniform convexity of  $t \mapsto t^p$  on  $[0, \infty)$ , we know that for each  $\delta > 0$  there exists a positive constant  $\varepsilon = \delta^p(2^{-1} - 2^{-p})$  such that

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2} - \varepsilon$$

whenever  $a, b \in [0, \infty)$  with  $|a - b| \geq \delta$ .

Suppose that  $\{x \in \Omega : g_v(x) \neq g_u(x)\}$  has positive measure. Then there is some  $\delta > 0$  such that the measure of the set

$$A_\delta := \{x \in \Omega : |g_v(x) - g_u(x)| > \delta\}$$

is positive. Then

$$\begin{aligned} I(u) = I(v) &\leq I(w) \leq \int_{\Omega} \left(\frac{g_u + g_v}{2}\right)^p d\mu + \int_{\partial\Omega} wf dP_{\Omega} \\ &\leq \int_{A_\delta} \left[\frac{g_u^p + g_v^p}{2} - \varepsilon\right] d\mu + \int_{\Omega \setminus A_\delta} \frac{g_u^p + g_v^p}{2} d\mu + \int_{\partial\Omega} wf dP_{\Omega} \\ &= \int_{\Omega} \frac{g_u^p + g_v^p}{2} d\mu + \int_{\partial\Omega} \frac{u+v}{2} f dP_{\Omega} - \varepsilon \mu(A_\delta) \\ &\leq I(u) - \varepsilon \mu(A_\delta), \end{aligned}$$

which is not possible. Therefore  $g_u = g_v$   $\mu$ -a.e. in  $\Omega$ , and hence it also follows from  $I(u) = I(v)$  that  $\int_{\partial\Omega} uf dP_{\Omega} = \int_{\partial\Omega} vf dP_{\Omega}$ .

To prove the last part of the lemma, it suffices to show that  $w_+^0 = \max\{u, v\}$  and  $w_-^0 = \min\{u, v\}$  are minimizers of the functional  $I$  corresponding to  $f$ . Note that  $g_{w_-^0} \leq g_u \chi_{\{u < v\}} + g_v \chi_{\{u \geq v\}} = g_u$  and similarly  $g_{w_+^0} \leq g_u$ . Therefore

$$I(w_{\pm}^0) \leq \int_{\Omega} g_{\pm}^p d\mu + \int_{\partial\Omega} w_{\pm}^0 f dP_{\Omega}.$$

Note that

$$\int_{\partial\Omega} [w_+^0 + w_-^0] f dP_{\Omega} = \int_{\partial\Omega} [u + v] f dP_{\Omega} = 2 \int_{\partial\Omega} uf dP_{\Omega}.$$

It follows that if  $\int_{\partial\Omega} w_+^0 f dP_{\Omega} > \int_{\partial\Omega} uf dP_{\Omega}$ , then  $\int_{\partial\Omega} w_-^0 f dP_{\Omega} < \int_{\partial\Omega} uf dP_{\Omega}$ , which would violate the minimality of  $I(u)$ . Therefore we must have  $\int_{\partial\Omega} w_+^0 f dP_{\Omega} \leq \int_{\partial\Omega} uf dP_{\Omega}$  and similarly,  $\int_{\partial\Omega} w_-^0 f dP_{\Omega} \leq \int_{\partial\Omega} uf dP_{\Omega}$ , which in turn implies that  $I(w_{\pm}^0) \leq I(u)$ , as desired.  $\square$

Observe that in infinitesimally Hilbertian spaces, the above uniqueness of the minimal  $p$ -weak upper gradient together with convexity of the set  $M_I$  imply that the solution of the Neumann problem is in fact unique (up to an additive constant).

## 5. Boundedness of solutions, at the boundary

We will use the De Giorgi method to prove that the minimizers are bounded near the boundary of  $\Omega$ . Local boundedness inside  $\Omega$  follows from previously known results on  $p$ -energy minimizers in the metric setting [36].

Let  $u \in N^{1,p}(\Omega)$  be a minimizer of

$$I(u) = \int_{\Omega} g_u^p d\mu + \int_{\partial\Omega} f u dP_{\Omega}, \quad (5.1)$$

where  $f \in L^{\infty}(\partial\Omega)$  is a Borel function. The main goal of this section is to prove that solutions are bounded whenever the boundary data  $f$  is bounded.

**Theorem 5.2.** *Let  $\Omega$  be a bounded domain in  $X$  satisfying the assumptions given in Definition 3.3, and let  $f$  and  $u$  be as above. Fix  $R_0 \in (0, \text{diam } \Omega)$ . Then for each  $x \in \partial\Omega$  and  $0 < R < R_0/4$  we have that  $|u| \leq C_R$  on  $\overline{\Omega} \cap B(x, R)$ , where  $C_R$  depends on the doubling and Poincaré inequality constants,  $p$ ,  $R$ ,  $\|u\|_{L^1(B(x,R) \cap \partial\Omega)}$ ,  $\|u\|_{L^p(B(x,R) \cap \Omega)}$ , and on  $\|f\|_{L^{\infty}(\partial\Omega \cap B(x, 2R))}$  alone.*

To prove the above theorem we make use of the technique developed by De Giorgi [23]. To do so we first derive a De Giorgi type inequality associated with the Neumann type problem considered here.

**Theorem 5.3.** *There is a constant  $C \geq 1$  such that given a minimizer  $u$  as above on the bounded domain  $\Omega \subset X$ ,  $x \in \partial\Omega$ ,  $0 < r < R \leq R_0 < \text{diam}(\Omega)/10$ , and  $k \in \mathbb{R}$ , we have*

$$\begin{aligned} \int_{\Omega \cap B(x,r)} g_{(u-k)_+}^p d\mu &\leq \frac{C}{(R-r)^p} \int_{\Omega \cap B(x,R)} (u-k)_+^p d\mu \\ &+ C \int_{\partial\Omega \cap B(x,R)} |f| \cdot (u-k)_+ dP_{\Omega}. \end{aligned} \quad (5.4)$$

The constant  $C$  depends solely on the doubling constant of  $\mu$ , the Poincaré inequality constants, and  $p$ .

**Proof.** Let  $x, r, R$  be as in the statement of the theorem, and let

$$\eta_{r,R}(y) = \eta(y) = (1 - \text{dist}(y, B(x, r))/(R - r))_+ \quad (5.5)$$

be a Lipschitz cut-off function. For  $k \in \mathbb{R}$  and  $\rho > 0$ , define

$$A(k, \rho) = \{y \in B(x, \rho) \cap \Omega : u(y) > k\} \cup \{y \in B(x, \rho) \cap \partial\Omega : T(u)(y) > k\}.$$

Note that by our standing assumptions on  $\partial\Omega$ , we automatically have  $\mu(\partial\Omega) = 0$ , and so integrating over  $A(k, \rho) \cap \Omega$  with respect to  $\mu$  is the same as integrating over  $A(k, \rho)$  with respect to  $\mu$ . For the function

$$v = u - \eta \cdot (u - k)_+ = \begin{cases} (1 - \eta)(u - k) + k & \text{in } A(k, R), \\ u & \text{otherwise,} \end{cases}$$

by the properties of upper gradient (see [11]) such as the Leibniz rule, we have

$$g_v \leq \begin{cases} (1 - \eta)g_u + \frac{u-k}{R-r}\chi_{B(x,R)\setminus B(x,r)} & \text{in } A(k, R), \\ g_u & \text{otherwise.} \end{cases} \quad (5.6)$$

Since  $v$  is a candidate for the minimizer of  $I$ , we have  $I(u) \leq I(v)$ . Thus,

$$\int_{\Omega \cap B(x,R)} g_u^p d\mu + \int_{\partial\Omega \cap B(x,R)} fu dP_\Omega \leq \int_{\Omega \cap B(x,R)} g_v^p d\mu + \int_{\partial\Omega \cap B(x,R)} fv dP_\Omega.$$

Subtracting  $\int_{\Omega \cap B(x,R) \setminus A(k,R)} g_u^p d\mu + \int_{\partial\Omega \cap B(x,R)} fu dP_\Omega$  from both sides of the inequality yields that

$$\int_{A(k,R)} g_u^p d\mu \leq \int_{A(k,R)} g_v^p d\mu - \int_{\partial\Omega \cap A(k,R)} f\eta \cdot (u - k) dP_\Omega. \quad (5.7)$$

From (5.6), we obtain the almost everywhere pointwise estimate

$$g_v^p \leq 2^p \left( g_u^p (1 - \chi_{A(k,r)}) + \frac{(u - k)^p}{(R - r)^p} \right) \quad \text{on } A(k, R).$$

Plugging in this estimate into (5.7) and making the integration domain on the left-hand side smaller, we have

$$\begin{aligned} \int_{A(k,r)} g_u^p d\mu &\leq 2^p \int_{A(k,R) \setminus A(k,r)} g_u^p d\mu \\ &\quad + \frac{2^p}{(R - r)^p} \int_{A(k,R)} (u - k)^p d\mu - \int_{\partial\Omega \cap A(k,R)} f\eta \cdot (u - k) dP_\Omega. \end{aligned}$$

Adding  $2^p \int_{A(k,r)} g_u^p d\mu$ , and then dividing by  $(1 + 2^p)$  leads to

$$\begin{aligned} \int_{A(k,r)} g_u^p d\mu &\leq \theta \int_{A(k,R)} g_u^p d\mu \\ &\quad + \frac{\theta}{(R - r)^p} \int_{A(k,R)} (u - k)^p d\mu - \frac{1}{C} \int_{\partial\Omega \cap A(k,R)} f\eta \cdot (u - k) dP_\Omega, \end{aligned} \quad (5.8)$$

where  $\theta = 2^p / (1 + 2^p) \in (0, 1)$  and  $C = 1 + 2^p \geq 1$ .

Now, we can apply [23, Lemma 6.1] with (5.8) as the starting inequality to obtain

$$\int_{A(k,r)} g_u^p d\mu \leq \frac{C}{(R-r)^p} \int_{A(k,R)} (u-k)^p d\mu + C \int_{\partial\Omega \cap A(k,R)} |f| \cdot (u-k) dP_\Omega.$$

This verifies (5.4) and completes the proof of the theorem.  $\square$

**Remark 5.9.** If  $f > 0$  on  $B(x, R_0)$ , then the inequality (5.8) can be made simpler by omitting the last term, viz.,

$$\int_{A(k,r)} g_u^p d\mu \leq \theta \int_{A(k,R)} g_u^p d\mu + \frac{1}{(R-r)^p} \int_{A(k,R)} (u-k)^p d\mu.$$

In such a case [23, Lemma 6.1] provides us with an estimate

$$\int_{A(k,r)} g_u^p d\mu \leq \frac{C}{(R-r)^p} \int_{A(k,R)} (u-k)^p d\mu,$$

which holds for every  $0 < r < R < R_0$ .

**Lemma 5.10.** Let  $x \in \partial\Omega$  and  $0 < r < R < R_0$  as above, and let  $C_f = \|f\|_{L^\infty(\partial\Omega \cap B(x, R_0))}^{1/p}$ ,

$$u(k, r) = \left( \int_{\Omega \cap B(x, r)} (u-k)_+^p d\mu \right)^{1/p},$$

and

$$\psi(k, R) = \int_{\partial\Omega \cap B(x, R)} (u-k)_+ dP_\Omega.$$

If  $N_{\text{loc}}^{1,p}(\Omega) \subset L_{\text{loc}}^{\kappa p}(\Omega)$  and the trace operator  $T : N^{1,p}(\Omega) \rightarrow L^{\tilde{\kappa}p}(\partial\Omega)$  is bounded for some  $\kappa, \tilde{\kappa} > 1$  and  $0 < \aleph < 1$ , then for all real numbers  $h, k$  with  $h < k$ , all positive  $R, r$  with  $R/2 \leq r < R < R_0$ , setting  $\alpha := 1 - \frac{1}{\kappa}$ , and  $\beta := 1 - \frac{1}{\tilde{\kappa}p}$  yields that

$$\begin{aligned} u(k, r) &\leq C \left( \frac{u(h, R)}{k-h} \right)^\alpha \left( \frac{R}{R-r} u(h, R) + C_f R^{1-1/p} \psi(h, R)^{1/p} \right), \quad \text{and} \\ \psi(k, r) &\leq C \left( \frac{\psi(h, R)}{k-h} \right)^\beta \left( \frac{R^{1-\aleph}}{R-r} u(h, R) + C_f R^{1-1/p-\aleph} \psi(h, R)^{1/p} \right). \end{aligned} \quad (5.11)$$

If in addition  $\mu$  is Ahlfors  $s$ -regular at scale  $r_0 > 0$ , then we also have

$$\psi(k, r) \leq C \left( \frac{\psi(h, R)}{k-h} \right)^\beta \left[ \frac{R}{R-r} u(h, R) + C_f R^{1-1/p} \psi(h, R)^{1/p} \right]. \quad (5.12)$$

We can always choose such  $\kappa, \tilde{\kappa}$ , for instance, by choosing  $1 < \kappa < s/(s-p)$  and  $1 < \tilde{\kappa} < (s-1)/(s-p)$  as in Proposition 3.8. If  $p$  is close to  $s$  then  $\kappa$  and  $\tilde{\kappa}$  can be chosen to be as large as we like.

**Proof.** Due to self-improvement of  $(1, p)$ -Poincaré inequality, there is  $\kappa > 1$  such that  $\Omega$  supports a  $(\kappa p, p)$ -Poincaré inequality, see for example [25, 11]. Here any choice of  $1 < \kappa \leq s/(s-p)$  works, where  $s$  is the upper mass bound exponent of the doubling measure  $\mu$  as in (2.2).

Let  $\tilde{\eta}$  be the cut-off function  $\eta_{r, (r+R)/2}$  as in (5.5). Then, the Hölder inequality and the  $(\kappa p, p)$ -Poincaré inequality for functions in  $N^{1,p}(X)$  vanishing on  $X \setminus B(x, (r+R)/2)$  yield

$$\begin{aligned} \int_{\Omega \cap B(x, r)} (u-k)_+^p d\mu &\leq \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{1-1/\kappa} \left( \int_{\Omega \cap B(x, r)} (u-k)_+^{\kappa p} d\mu \right)^{1/\kappa} \\ &\leq C \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{1-1/\kappa} \left( \int_{\Omega \cap B(x, (r+R)/2)} (\tilde{\eta}(u-k)_+)^{\kappa p} d\mu \right)^{1/\kappa} \\ &\leq C \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{1-1/\kappa} R^p \int_{\Omega \cap B(x, (r+R)/2)} g_{\tilde{\eta}(u-k)_+}^p d\mu \\ &\leq C \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{1-1/\kappa} R^p \int_{\Omega \cap B(x, (r+R)/2)} g_{(u-k)_+}^p + \frac{(u-k)_+^p}{(R-r)^p} d\mu, \end{aligned}$$

where the product rule (Leibniz rule) for  $(p$ -weak) upper gradients was used in the last step. Estimating the integral of  $g_{(u-k)_+}^p$  via (5.4) gives

$$\begin{aligned} \int_{\Omega \cap B(x, r)} (u-k)_+^p d\mu &\leq C \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{1-1/\kappa} \left[ \frac{R^p}{(R-r)^p} \int_{\Omega \cap B(x, R)} (u-k)_+^p d\mu \right. \\ &\quad \left. + R^{p-1} \int_{\partial \Omega \cap B(x, R)} |f|(u-k)_+ dP_\Omega \right]. \end{aligned}$$

It follows that

$$u(k, r) \leq C \left( \frac{\mu(A(k, r))}{\mu(B(x, r))} \right)^{\frac{\kappa-1}{\kappa p}} \left( \frac{R}{R-r} u(k, R) + C_f R^{1-1/p} \psi(k, R)^{1/p} \right) \quad (5.13)$$

We will now show that  $(\mu(A(k, r))/\mu(B(x, r)))^{1/p} < Cu(h, R)/(k-h)$  whenever  $h < k$ . Since  $u \geq k$  on  $A(k, R)$ , we have

$$(k-h)^p \mu(A(k, r)) \leq \int_{A(k, r)} (u-h)^p d\mu \leq \int_{A(h, r)} (u-h)^p d\mu$$



$$= \mu(B(x, r))u(h, r)^p \leq C\mu(B(x, r))u(h, R)^p$$

as desired.

Using this estimate as well as the inequalities  $u(k, R) \leq u(h, R)$  and  $\psi(k, R) \leq \psi(h, R)$  in (5.13) yields that

$$u(k, r) \leq C \left( \frac{u(h, R)}{k - h} \right)^{\frac{\kappa-1}{\kappa}} \left( \frac{R}{R-r} u(h, R) + C_f R^{1-1/p} \psi(h, R)^{1/p} \right). \quad (5.14)$$

Thus we have verified the first of the two inequalities claimed in the lemma.

Let us now establish an analogous inequality for  $\psi(k, r)$ . Let  $\tilde{\kappa} > 1$  be such that  $\tilde{\kappa}p = \tilde{p}$ , where  $\tilde{p}$  is an admissible target exponent for the trace operator, see Proposition 3.8. It follows from the Hölder inequality that

$$\begin{aligned} \psi(k, r) &= \int_{\partial\Omega \cap B(x, r)} (u - k)_+ dP_\Omega \\ &\leq \left( \int_{\partial\Omega \cap B(x, r)} (u - k)_+^{\tilde{\kappa}p} dP_\Omega \right)^{1/\tilde{\kappa}p} \cdot \left( \frac{P_\Omega(A(k, r) \cap \partial\Omega)}{P_\Omega(B(x, r) \cap \partial\Omega)} \right)^{1-1/\tilde{\kappa}p}. \end{aligned}$$

Then, Proposition 3.8 yields that

$$\begin{aligned} \left( \int_{\partial\Omega \cap B(x, r)} (u - k)_+^{\tilde{\kappa}p} dP_\Omega \right)^{1/\tilde{\kappa}p} &\leq Cr^{1-1/\tilde{\kappa}p-\aleph} \left( \int_{\Omega \cap B(x, r)} g_{(u-k)_+}^p d\mu \right)^{1/p} \\ &\quad + CP_\Omega(\partial\Omega \cap B(x, r))^{1/\tilde{\kappa}p} \int_{\Omega \cap B(x, r)} (u - k)_+ d\mu. \end{aligned}$$

Combining these two inequalities together with the assumption of co-dimension 1 Ahlfors regularity of  $P_\Omega$  results in

$$\begin{aligned} \psi(k, r) &\leq C \left( \frac{P_\Omega(A(k, r) \cap \partial\Omega)}{P_\Omega(B(x, r) \cap \partial\Omega)} \right)^{1-1/\tilde{\kappa}p} \\ &\quad \cdot \left( r^{1-\aleph} \mu(B(x, r))^{\frac{\tilde{\kappa}-1}{\tilde{\kappa}p}} \left( \int_{\Omega \cap B(x, r)} g_{(u-k)_+}^p d\mu \right)^{1/p} + u(k, r) \right). \end{aligned} \quad (5.15)$$

For an arbitrary  $h < k$ , we have

$$\begin{aligned} (k - h)P_\Omega(A(k, r) \cap \partial\Omega) &\leq \int_{A(k, r) \cap \partial\Omega} (u - h) dP_\Omega \\ &\leq \int_{A(h, r) \cap \partial\Omega} (u - h) dP_\Omega \leq P_\Omega(B(x, r) \cap \partial\Omega) \psi(h, r). \end{aligned}$$

Applying this inequality together with (5.4) to (5.15) yields that

$$\begin{aligned} \psi(k, r) &\leq C \left( \frac{\psi(h, r)}{k-h} \right)^{\frac{\tilde{\kappa}p-1}{\tilde{\kappa}p}} \left( r^{1-\aleph} \mu(B(x, r))^{\frac{\tilde{\kappa}-1}{\tilde{\kappa}p}} \left( \int_{\Omega \cap B(x, r)} g_{(u-k)_+}^p d\mu \right)^{1/p} + u(k, r) \right) \\ &\leq C \left( \frac{\psi(h, R)}{k-h} \right)^{\frac{\tilde{\kappa}p-1}{\tilde{\kappa}p}} \left[ r^{1-\aleph} \mu(B(x, r))^{\frac{\tilde{\kappa}-1}{\tilde{\kappa}p}} \left( \frac{u(k, R)}{R-r} + \frac{(C_f \psi(k, R))^{1/p}}{R^{1/p}} \right) + u(k, R) \right] \\ &\leq C \left( \frac{\psi(h, R)}{k-h} \right)^{\frac{\tilde{\kappa}p-1}{\tilde{\kappa}p}} \left[ \left( 1 + \frac{R^{1-\aleph}}{R-r} \right) u(h, R) + C_f R^{1-1/p-\aleph} \psi(h, R)^{1/p} \right], \end{aligned}$$

where the crude estimate  $\mu(B(x, r)) \leq \mu(\Omega)$  was used in the last line. Since  $R-r \leq R/2 \leq R_0/2$ , and since  $0 < 1-\aleph < 1$ , the desired inequality for  $\psi$  follows.

If  $\mu$  happens to be Ahlfors  $s$ -regular at scale  $r_0 > 0$ , then a finer estimate  $\mu(B(x, r)) \leq Cr^s$  is to be used above. Since  $\aleph = s(\frac{1}{p} - \frac{1}{\tilde{p}})$  and  $\tilde{p} = \tilde{\kappa}p$ , we have

$$r^{-\aleph} \mu(B(x, r))^{\frac{\tilde{\kappa}-1}{\tilde{\kappa}p}} \leq Cr^{s(\frac{1}{\tilde{\kappa}p} - \frac{1}{p})} r^{s\frac{\tilde{\kappa}-1}{\tilde{\kappa}p}} = C.$$

Then, it follows from the penultimate line of the estimate of  $\psi(k, r)$  above that

$$\begin{aligned} \psi(k, r) &\leq C \left( \frac{\psi(h, R)}{k-h} \right)^{\frac{\tilde{\kappa}p-1}{\tilde{\kappa}p}} \left[ r \left( \frac{u(k, R)}{R-r} + \frac{(C_f \psi(k, R))^{1/p}}{R^{1/p}} \right) + u(k, R) \right] \\ &\leq C \left( \frac{\psi(h, R)}{k-h} \right)^{\frac{\tilde{\kappa}p-1}{\tilde{\kappa}p}} \left[ \left( 1 + \frac{R}{R-r} \right) u(h, R) + C_f R^{1-1/p} \psi(h, R)^{1/p} \right]. \end{aligned}$$

Again noting that  $R-r \leq R/2$ , we obtain the inequality (5.12).  $\square$

We are now ready to prove the main theorem of this section. Recall that the minimizer  $u$  necessarily belongs to  $L^1(\Omega)$  and its trace belongs to  $L^1(\partial\Omega, P_\Omega)$ . The boundedness estimates we obtain in the proof indicate that the bound on  $u$  is determined by its trace's average value on the boundary of  $\Omega$  with respect to the measure  $P_\Omega$  as well as on the average of  $u$  on the ball, and on the bound on  $f$  on the boundary of  $\Omega$ . This is in contrast to the local boundedness estimates of [36] for  $p$ -energy minimizers in the interior of  $\Omega$ , where the bound is determined by the average value of  $u$  alone.

**Proof of Theorem 5.2.** In order to prove that  $u$  is bounded from above near the boundary, it suffices to show that for a fixed  $R > 0$  with  $R < R_0/4$  and  $k_0 \in \mathbb{R}$  we can find  $d \geq 0$  such that  $u(k_0 + d, R/2) = 0$ , where  $u(k, r)$  is as in Lemma 5.10.

If  $u(k_0, R) = 0$ , then we immediately obtain the upper bound that  $u \leq k_0$  in  $B(x, R)$ . In what follows, suppose that  $u(k_0, R) > 0$ .

Let  $r_n = (1 + 2^{-n}) \cdot R/2$  and  $k_n = k_0 + d(1 - 2^{-n})$ , where the precise value of  $d > 0$  will be determined later. Setting  $h = k_n$ ,  $k = k_{n+1}$ ,  $\rho = r_n$ , and  $r = r_{n+1}$  in (5.11) yields that

$$\begin{aligned}
 u(k_{n+1}, r_{n+1}) &\leq C \left( \frac{u(k_n, r_n)}{2^{-n-1}d} \right)^\alpha \left( \frac{1+2^{-n}}{2^{-n-1}} u(k_n, r_n) + C_f r_n^{1-1/p} \psi(k_n, r_n)^{1/p} \right) \\
 &\leq C_{f,R} \frac{2^{n(\alpha+1)}}{d^\alpha} (u(k_n, r_n)^{1+\alpha} + u(k_n, r_n)^\alpha \psi(k_n, r_n)^{1/p})
 \end{aligned} \quad (5.16)$$

and analogously

$$\psi(k_{n+1}, r_{n+1}) \leq C_{f,R} \frac{2^{n(\beta+1)}}{d^\beta} (u(k_n, r_n) \psi(k_n, r_n)^\beta + \psi(k_n, r_n)^{\beta+1/p}), \quad (5.17)$$

where  $C_{f,R} = C \cdot (1 + C_f R^{1-1/p} + R^{-\aleph} + C_f R^{1-1/p-\aleph})$ . By induction, we will show that

$$u(k_n, r_n) \leq 2^{-\sigma n} u(k_0, R) \quad \text{and} \quad \psi(k_n, r_n) \leq 2^{-\tau n} \psi(k_0, R) \quad (5.18)$$

for a suitable choice of positive constants  $\sigma$ ,  $\tau$ , and  $d$ . In such a case, we will have  $u(k_0 + d, R/2) = \lim_{n \rightarrow \infty} u(k_n, r_n) = 0$ . Observe that both inequalities in (5.18) are satisfied for  $n = 0$ .

If  $\psi(k_0, R) = 0$ , then the second inequality in (5.18) is vacuously satisfied. If  $\psi(k_0, R) \neq 0$ , then (5.17) together with (5.18) lead to

$$\begin{aligned}
 \psi(k_{n+1}, r_{n+1}) &\leq \frac{\psi(k_0, R)}{2^{\tau(n+1)}} \cdot \frac{2^{\tau(n+1)}}{\psi(k_0, R)} \\
 &\quad \cdot C_{f,R} \frac{2^{n(\beta+1)}}{d^\beta} \left[ \frac{u(k_0, R)}{2^{\sigma n}} \left( \frac{\psi(k_0, R)}{2^{\tau n}} \right)^\beta + \left( \frac{\psi(k_0, R)}{2^{\tau n}} \right)^{\beta+1/p} \right] \\
 &\leq \frac{\psi(k_0, R)}{2^{\tau(n+1)}} \cdot \frac{C_{f,R}}{d^\beta \psi(k_0, R)^{1-\beta}} 2^{\tau+n(\tau+\beta+1-\tau\beta)} \left( \frac{u(k_0, R)}{2^{\sigma n}} + \frac{\psi(k_0, R)^{1/p}}{2^{\tau n/p}} \right).
 \end{aligned}$$

Thus, if (5.18) is to be satisfied when  $\psi(k_0, R) \neq 0$ , we need

$$\tau + \beta + 1 - \tau\beta - \sigma \leq 0 \quad \text{and} \quad \tau + \beta + 1 - \tau\beta - \frac{\tau}{p} \leq 0 \quad (5.19)$$

as well as

$$d \geq \left( \frac{C_{f,R} 2^\tau (u(k_0, R) + \psi(k_0, R)^{1/p})}{\psi(k_0, R)^{1-\beta}} \right)^{1/\beta}. \quad (5.20)$$

Analogously, inequalities (5.16) and (5.18) provide us with the estimate

$$\begin{aligned}
 u(k_{n+1}, r_{n+1}) &\leq \frac{u(k_0, R)}{2^{\sigma(n+1)}} \\
 &\quad \cdot \frac{C_{f,R}}{d^\alpha u(k_0, R)^{1-\alpha}} 2^{\sigma+n(\sigma+\alpha+1-\sigma\alpha)} \left( \frac{u(k_0, R)}{2^{\sigma n}} + \frac{\psi(k_0, R)^{1/p}}{2^{\tau n/p}} \right).
 \end{aligned}$$

Therefore, we need

$$\alpha + 1 - \sigma\alpha \leq 0 \quad \text{and} \quad \sigma + \alpha + 1 - \sigma\alpha - \frac{\tau}{p} \leq 0 \quad (5.21)$$

as well as

$$d \geq \left( \frac{C_{f,R} 2^\sigma (u(k_0, R) + \psi(k_0, R)^{1/p})}{u(k_0, R)^{1-\alpha}} \right)^{1/\alpha}. \quad (5.22)$$

Simplifying (5.19) and (5.21) yields

$$\max \left\{ 1 + \frac{1}{\alpha}, \tau(1 - \beta) + 1 + \beta \right\} \leq \sigma \leq \frac{\frac{\tau}{p} - (1 + \alpha)}{1 - \alpha} \quad \text{and} \quad \tau \geq \frac{\beta + 1}{\beta + \frac{1}{p} - 1}.$$

Recall that  $\alpha = 1 - \frac{1}{\kappa}$  and  $\beta = 1 - \frac{1}{\kappa p}$ , where  $\kappa > 1$  is chosen such that  $N^{1,p}(\Omega) \subset L^{\kappa p}(\Omega)$  while  $\tilde{\kappa} > 1$  is chosen such that the trace operator maps  $N^{1,p}(\Omega)$  into  $L^{\tilde{\kappa} p}(\partial\Omega)$ . Choosing

$$\tau \geq \max \left\{ \frac{2\tilde{\kappa}p - 1}{\tilde{\kappa} - 1}, p(\kappa - 1), \frac{2p + 2\kappa - 1 - 1/\tilde{\kappa}}{\kappa - 1/\tilde{\kappa}} \right\}$$

will allow us to find  $\sigma$  so that both (5.19) and (5.21) are fulfilled, which will then enable us to use (5.20) and (5.22) to find a sufficiently big value of  $d$ .

For such a constant  $d$ , we have

$$0 = u\left(k_0 + d, \frac{R}{2}\right) = \left( \int_{\Omega \cap B(x, R/2)} (u - k_0 - d)_+^p d\mu \right)^{1/p},$$

which shows that  $u \leq k_0 + d$   $\mu$ -a.e. in  $B(x, R/2)$ . Analogously, we have the trace  $Tu \leq k_0 + d$   $P_\Omega$ -a.e. in  $\partial\Omega \cap B(x, R/2)$ . Running the argument once more with  $u$  and  $f$  replaced by  $-u$  and  $-f$ , respectively, we obtain that  $u \in L^\infty(\Omega_R)$  and  $Tu \in L^\infty(\partial\Omega)$ , where  $\Omega_R = \{z \in \Omega : \text{dist}(z, \partial\Omega) < R/2\}$ .

Letting  $k_0 = 0$  yields the desired conclusion.  $\square$

## 6. Further boundary regularity

In PDE literature, the part of the boundary where the Neumann data  $f$  vanishes is called the natural boundary. If  $x \in \partial\Omega$  and  $r > 0$  such that  $f = 0$  on  $\partial\Omega \cap B(x, r)$ , then

$$\int_{\Omega \cap B(x, r)} g_u^p d\mu \leq \int_{\Omega \cap B(x, r)} g_{u+\varphi}^p d\mu$$

for every  $\varphi \in N^{1,p}(X)$  with compact support in  $B(x, r)$ , i.e.,  $u$  is  $p$ -harmonic in  $\Omega \cup (\partial\Omega \cap \overline{B(x, r)})$ . Thus, given our standing assumptions on  $\Omega$ , the results of [36] apply to  $u$  on  $B(x, r) \cap \overline{\Omega}$ , to yield that  $u$  is locally Hölder continuous in  $B(x, r) \cap \overline{\Omega}$ . We have so far no boundary Hölder continuity of  $u$  at other parts of  $\partial\Omega$ . In the Euclidean setting, we know from the work of [1, 2, 32, 50] that if  $\Omega$  is a bounded Euclidean domain of class  $C^1$ , and the boundary data  $f$  is Hölder continuous, then  $u$  is Hölder continuous at  $\partial\Omega$ . On the other hand, we obtain partial regularity results for  $u$  near sets of positivity of  $f$  (and correspondingly, sets of negativity of  $f$ ) in this section using the results from [34, 35, 11] on nonlinear potential theory on metric measure spaces.

These will allow us to prove continuity of  $u$  up to the boundary on open subsets of positivity (or negativity) of  $f$  for values of  $p$  close to 1 or close to  $s$  in Section 7.

**Definition 6.1.** Let  $(Y, d_Y, \mu_Y)$  be a metric measure space. A function  $v$  on an open set  $A \subset Y$  is a  $p$ -subminimizer if

$$\int_A g_v^p d\mu_Y \leq \int_A g_{v+\varphi}^p d\mu_Y$$

for every non-positive  $\varphi \in N^{1,p}(Y)$  that is compactly supported in  $A$ .

The notion of subminimizers in the metric setting is extensively studied; a non-exhaustive listing of papers about subminimizers in the metric setting is [49,34,35,10,13,12]. The book [11] contains a nice discussion of nonlinear potential theory in metric setting.

It is known that if  $\mu_Y$  is doubling,  $Y$  is complete, and supports a  $p$ -Poincaré inequality, then subminimizers are  $p$ -finely continuous in  $A$  (see [13] or [11, Theorem 11.38]) and are upper semicontinuous in  $A$  (see [34] or [11, Theorem 8.22]). Recall that a function is  $p$ -finely continuous at  $z \in A$  if it is continuous with respect to the  $p$ -fine topology on  $Y$ . Here, a set  $U \subset Y$  is  $p$ -finely open if  $Y \setminus U$  is  $p$ -finely thin at each  $x \in U$ , that is,

$$\int_0^1 \left( \frac{\text{cap}_p(B(x, \rho) \setminus U, B(x, 2\rho))}{\text{cap}_p(B(x, \rho), B(x, 2\rho))} \right)^{1/(p-1)} \frac{d\rho}{\rho} < \infty. \quad (6.2)$$

Here, for  $E \subset B(x, \rho)$ , the quantity  $\text{cap}_p(E, B(x, 2\rho))$  is the relative variational  $p$ -capacity of  $E$  with respect to  $B(x, 2\rho)$  as given in Definition 2.4; see [11, Section 11.6].

**Proposition 6.3.** Let  $x \in \partial\Omega$  and  $r > 0$  such that  $f \geq 0$  on  $B(x, r) \cap \partial\Omega$ . Then  $u$  is a  $p$ -subminimizer on  $B(x, r) \cap \overline{\Omega}$ , and hence is upper semicontinuous at  $x$ , that is,

$$u(x) \geq \limsup_{B(x,r) \cap \overline{\Omega} \ni y \rightarrow x} u(y),$$

and  $u$  is  $p$ -finely continuous in  $B(x, r) \cap \overline{\Omega}$ .

**Proof.** From our standing hypothesis that  $\Omega$  supports a  $p$ -Poincaré inequality and that the restriction of  $\mu$  to  $\Omega$  satisfies (3.4), we know that  $\overline{\Omega}$ , equipped with the inherited metric and the restriction of  $\mu$  to  $\overline{\Omega}$  is doubling and supports a  $p$ -Poincaré inequality. Hence the results regarding  $p$ -subharmonic functions mentioned above would yield the desired conclusions regarding  $u$  provided we demonstrate that  $u$  is a  $p$ -subminimizer on  $B(x, r) \cap \overline{\Omega}$ .

To this end, let  $\varphi \in N^{1,p}(\overline{\Omega})$  be a non-positive function such that  $\varphi = 0$  on  $\overline{\Omega} \setminus B(x, r)$ . With  $u + \varphi$  as a competitor, we know that  $I(u) \leq I(u + \varphi)$ , that is,

$$\int_{\Omega} g_u^p d\mu + \int_{\partial\Omega} u f dP_{\Omega} \leq \int_{\Omega} g_{u+\varphi}^p d\mu + \int_{\partial\Omega} (u + \varphi) f dP_{\Omega}.$$

It follows from  $g_u = g_{u+\varphi}$   $\mu$ -a.e. in  $\overline{\Omega} \setminus B(x, r)$  and from  $\mu(\partial\Omega) = 0$  that

$$\int_{B(x,r) \cap \overline{\Omega}} g_u^p d\mu \leq \int_{B(x,r) \cap \overline{\Omega}} g_{u+\varphi}^p d\mu + \int_{\partial\Omega \cap B(x,r)} \varphi f dP_\Omega.$$

Because  $f \geq 0$  on  $\partial\Omega \cap B(x, r)$  and  $\varphi \leq 0$  there, it follows that

$$\int_{B(x,r) \cap \overline{\Omega}} g_u^p d\mu \leq \int_{B(x,r) \cap \overline{\Omega}} g_{u+\varphi}^p d\mu$$

as desired.  $\square$

We next show that if  $u$  is constant in a neighborhood of a point in the boundary, then that point belongs to the natural boundary (that is,  $f$  vanishes in a relative neighborhood of that point).

**Proposition 6.4.** *Let  $u$  be a  $p$ -harmonic solution to the Neumann boundary value problem on  $\Omega$  with continuous boundary data  $f$ , and if  $x \in \partial\Omega$  and  $r > 0$  such that  $u$  is constant on  $B(x, r) \cap \Omega$ , then  $f = 0$  on  $B(x, r/2)$ .*

**Proof.** It suffices to show that for each such  $x$  and  $r > 0$  we have  $f(x) = 0$ . Suppose that  $f(x) > 0$  (by replacing  $f$  with  $-f$  and  $u$  with  $-u$  if necessary). Then for sufficiently small  $r > 0$  we have in addition to  $u$  being constant on  $B(x, r) \cap \Omega$  that  $f > 0$  on  $B(x, r) \cap \partial\Omega$ .

Let the constant value of  $u$  on  $B(x, r) \cap \Omega$  be  $M$ . For  $k \in \mathbb{R}$  with  $k < M$ , with the choice of  $v = u - \eta_{r/2,r}(u - k)_+$  as in (5.6) that

$$\begin{aligned} M \int_{\partial\Omega \cap B(x,r)} f dP_\Omega &= \int_{\Omega \cap B(x,r)} g_u^p d\mu + \int_{\partial\Omega \cap B(x,r)} u f d\mu \\ &\leq \int_{\Omega \cap B(x,r)} g_v^p d\mu + \int_{\partial\Omega \cap B(x,r)} v f dP_\Omega. \end{aligned}$$

Since  $g_v \leq (1 - \eta)g_u - k + \frac{2}{r}(u - k)_+ = \frac{2}{r}(u - k)_+$  on  $B(x, r) \setminus B(x, r/2)$   $\mu$ -a.e., it follows that

$$\int_{\partial\Omega \cap B(x,r)} M f dP_\Omega \leq \frac{2^p}{r^p} (M - k)^p \mu(B(x, r) \setminus B(x, r/2)) + \int_{\partial\Omega \cap B(x,r)} v f dP_\Omega.$$

Thus

$$\begin{aligned} \int_{\partial\Omega \cap B(x,r)} (M - v) f dP_\Omega &\leq \frac{2^p (M - k)^p}{r^p} \mu([B(x, r) \setminus B(x, r/2)] \cap \Omega) \\ &\leq \frac{2^p}{r^p} \mu(B(x, r)) (M - k)^p. \end{aligned}$$

Since  $v \leq M$ , and as  $\lim_{k \rightarrow M^-} \frac{M-v}{M-k} = 1$  on  $B(x, r/2) \cap \partial\Omega$ , we have

$$\begin{aligned} \int_{\partial\Omega \cap B(x, r/2)} \frac{M-v}{M-k} f \, dP_\Omega &\leq \frac{2^p (M-k)^{p-1}}{r^p} \mu([B(x, r) \setminus B(x, r/2)] \cap \Omega) \\ &\leq \frac{2^p}{r^p} \mu(B(x, r)) (M-k)^{p-1}, \end{aligned}$$

and letting  $k \rightarrow M^-$  we obtain

$$0 \leq \int_{\partial\Omega \cap B(x, r/2)} f \, dP_\Omega \leq 0.$$

Here, we used the fact that  $p > 1$ .  $\square$

As a consequence of the above proposition, we know that if the boundary data  $f$  is not constant (equivalently, not the zero function), then  $u$  is not constant on  $\Omega$ . This agrees with our intuitive understanding of the boundary data  $f$  controlling the “outer normal derivative” of  $u$  at  $\partial\Omega$  — if the derivative cannot vanish on the boundary, then the function cannot be constant. This is in spite of the fact that we do not have analogous differential equation in the metric setting.

## 7. Boundary continuity for $p$ close to 1 or the natural dimension $s$ when $\mu$ is Ahlfors $s$ -regular at small scales

In this section we need the strong version (5.12). We therefore assume from now on that  $\mu$  is Ahlfors  $s$ -regular at scale  $r_0 > 0$ .

Recall that the exponents  $\alpha$  and  $\beta$  used in Section 5 to prove boundedness of the solution  $u$  depend on  $p$  and the exponent  $s$  from (2.2). When  $p$  is close to either 1 or  $s$ , then it is possible to find values of  $\alpha$  and  $\beta$  such that

$$\alpha + \frac{1}{p} - 1 > 0 \quad \text{and} \quad \beta + \frac{1}{p} - 1 > 0. \quad (7.1)$$

The above conditions are satisfied whenever  $p^2 - sp + s > 0$ . In particular, they allow for all  $p > 1$  if the dimension  $s < 4$ . In this section we will show that when  $p$  satisfies (7.1), the function  $u$  is continuous up to the boundary of  $\Omega$ .

**Theorem 7.2.** *Suppose that  $\mu$  is Ahlfors  $s$ -regular at scale  $r_0 > 0$ . Under the standard assumptions on  $\Omega$  and  $\mu$ , if  $f : \partial\Omega \rightarrow \mathbb{R}$  is a bounded Borel measurable function on  $\partial\Omega$ ,  $x \in \partial\Omega$ , and  $r_0 > 0$  such that  $f \geq 0$  on  $B(x, r_0) \cap \partial\Omega$  or  $f \leq 0$  on  $B(x, r_0) \cap \partial\Omega$ , then  $u$  is continuous at  $x$  relative to  $\overline{\Omega}$ .*

**Proof.** Without loss of generality we may assume that  $f \leq 0$  on  $\partial\Omega \cap B(x, r_0)$ , for if  $f \geq 0$  at each point in  $\partial\Omega \cap B(x, r_0)$  then we apply the following analysis to  $-u$ , which is a solution for the boundary data  $-f$ .

Suppose that  $u$  is not continuous at  $x$ . For  $R > 0$  we set

$$M(R) := \sup_{y \in B(x, R) \cap \Omega} u(y) \quad \text{and} \quad m(R) := \inf_{y \in B(x, R) \cap \Omega} u(y).$$

Then by assumption we have that  $\lim_{R \rightarrow 0^+} M(R) =: M > \lim_{R \rightarrow 0^+} m(R) =: m$ .

For  $0 < R < \min\{1, r_0\}$ ,  $k_0 \in \mathbb{R}$  with  $k_0 < M(R)$ , and for  $n \in \mathbb{N}$  we set  $r_n = (1 + 2^{-n})R/2$  and  $k_n = k_0 + d(1 - 2^{-n})$ , where we want to choose  $d > 0$  such that we have  $u \leq k_0 + d$  on  $B(x, R/2) \cap \Omega$ . In other words, we repeat the proof of boundedness of  $u$ , but now we modify the choice of  $d$  by modifying (5.18). As in Lemma 5.10, we set

$$u(k, r) = \left( \int_{\Omega \cap B(x, r)} (u - k)_+^p d\mu \right)^{1/p} \quad \text{and} \quad \psi(k, R) = \int_{\partial \Omega \cap B(x, R)} (u - k)_+ dP_\Omega.$$

Suppose that  $k_0 \in \mathbb{R}$  such that

$$\frac{\mu(A(k_0, R))}{\mu(B(x, R) \cap \Omega)} \leq \frac{1}{(4D)^p},$$

then we wish to show that there exist  $\sigma, \tau > 0$  such that for each  $n \in \mathbb{N}$ ,

$$u(k_n, r_n) \leq \frac{2^{-\sigma n}(M(R) - k_0)}{4D} \quad \text{and} \quad \psi(k_n, r_n) \leq 2^{-\tau n}(M(R) - k_0). \quad (7.3)$$

Here in the above, we just replaced  $4C[1 + C_f]$  with  $C$ , and we remind the reader that we are not particularly concerned with the precise value of the constants  $C$  as long as they are independent of  $R$ . This holds when  $n = 0$ . Suppose we know that the above holds for some non-negative integer  $n$ . Observe that by Theorem 5.2 we have  $|M(R)| < \infty$  and  $|m(R)| < \infty$ . By (5.11) of Lemma 5.10 we have

$$\begin{aligned} u(k_{n+1}, r_{n+1}) &\leq C \left[ \frac{u(k_n, r_n)}{k_{n+1} - k_n} \right]^\alpha \left[ \frac{r_n}{r_n - r_{n+1}} u(k_n, r_n) + r_n^{1-1/p} \psi(k_n, r_n)^{1/p} \right] \\ &\leq C \left[ \frac{2^{-n(\sigma-1)}(M(R) - k_0)}{4Dd} \right]^\alpha \left[ \frac{(M(R) - k_0)}{4D 2^{n(\sigma-1)}} + \frac{R^{1-1/p}(M(R) - k_0)^{1/p}}{2^{\tau n/p}} \right] \end{aligned}$$

and by (5.12),

$$\begin{aligned} \psi(k_{n+1}, r_{n+1}) &\leq C \left[ \frac{\psi(k_n, r_n)}{k_{n+1} - k_n} \right]^\beta \left[ \frac{r_n}{r_n - r_{n+1}} u(k_n, r_n) + r_n^{1-1/p} \psi(k_n, r_n)^{1/p} \right] \\ &\leq C \left[ \frac{2^{-n(\tau-1)}(M(R) - k_0)}{d} \right]^\beta \left[ \frac{(M(R) - k_0)}{4D 2^{n(\sigma-1)}} + \frac{R^{1-1/p}(M(R) - k_0)^{1/p}}{2^{\tau n/p}} \right]. \end{aligned}$$

Therefore (7.3) would hold for  $n + 1$  if we can ensure that



$$C \left[ \frac{2^{-n(\sigma-1)}(M(R) - k_0)}{4Dd} \right]^\alpha \left[ \frac{(M(R) - k_0)}{4D 2^{n(\sigma-1)}} + \frac{R^{1-1/p}(M(R) - k_0)^{1/p}}{2^{\tau n/p}} \right] \\ \leq \frac{2^{-\sigma(n+1)}(M(R) - k_0)}{4D},$$

and

$$C \left[ \frac{2^{-n(\tau-1)}(M(R) - k_0)}{d} \right]^\beta \left[ \frac{(M(R) - k_0)}{4D 2^{n(\sigma-1)}} + \frac{R^{1-1/p}(M(R) - k_0)^{1/p}}{2^{\tau n/p}} \right] \\ \leq 2^{-\tau(n+1)}(M(R) - k_0).$$

The above two inequalities are satisfied if we can guarantee that

$$\begin{aligned} \sigma &\geq \frac{\alpha + 1}{\alpha}, \\ \tau &\geq p[\sigma(1 - \alpha) + \alpha], \\ \tau &\geq \frac{\beta}{\beta + \frac{1}{p} - 1}, \\ \tau &\leq \frac{\sigma - (1 + \beta)}{1 - \beta}, \\ d &\geq \max\{C^{1/\alpha}, C^{1/\beta}\} \frac{(M(R) - k_0)}{4D}, \\ d &\geq C^{1/\alpha} \left[ (M(R) - k_0)^{\alpha + \frac{1}{p} - 1} R^{1-1/p} (4D)^{1-\alpha} \right]^{1/\alpha}, \\ d &\geq C^{1/\beta} \left[ (M(R) - k_0)^{\beta + \frac{1}{p} - 1} R^{1-1/p} \right]^{1/\beta}. \end{aligned} \tag{7.4}$$

In the above, we choose  $D > 1$  such that

$$D \geq \max\{C^{1/\alpha}, C^{1/\beta}\}.$$

Given the assumptions (7.1) on  $p$ , the above are guaranteed by the choices of  $\sigma$ ,  $\tau$ , and  $d$  such that

$$\max \left\{ \frac{\alpha + 1}{\alpha}, 1 + \beta + \frac{\beta(1 - \beta)}{\beta + \frac{1}{p} - 1}, \frac{1 + \beta}{1 - p(1 - \alpha)(1 - \beta)} \right\} = \sigma, \\ \max \left\{ \frac{\beta}{\beta + \frac{1}{p} - 1}, p[\sigma - (\sigma - 1)\alpha] \right\} \leq \tau \leq \frac{\sigma - (1 + \beta)}{1 - \beta}, \tag{7.5}$$

and it suffices to choose  $d$  as follows:

$$\max \left\{ \frac{(M(R) - k_0)}{4}, C \left[ R^{1-1/p} (M(R) - k_0)^{\alpha + \frac{1}{p} - 1} \right]^{1/\alpha}, \right. \\ \left. C \left[ R^{1-1/p} (M(R) - k_0)^{\beta + \frac{1}{p} - 1} \right]^{1/\beta} \right\} = d. \quad (7.6)$$

The above choice of  $\tau$  is possible because of the assumptions (7.1) on  $p$ . Thus given  $k_0 < M(R)$  we have the above choice of  $d$ ,  $\sigma$ , and  $\tau$  such that, by letting  $n \rightarrow \infty$  in (7.3), we can conclude that  $u \leq k_0 + d$  on  $B(x, R/2) \cap \Omega$ .

We only consider  $0 < R < \max\{1, r_0\}$  for which

$$0 < M - m \leq M(R) - m(R) \leq 2(M - m).$$

Finally, for  $v \in \mathbb{N}$  set  $\kappa_v = M(R) - 2^{-v-1}(M(R) - m(R))$ . By Proposition 6.3,  $u$  is lower semicontinuous at  $x$ , and so  $m = Tu(x)$ . Furthermore, by this proposition we have that  $u$  is finely continuous at  $x$ , and so by (6.2) together with [11, Proposition 6.16] (see (2.5)),  $\lim_{R \rightarrow 0^+} \frac{\mu(A(\kappa_v, R))}{\mu(B(x, R) \cap \Omega)} = 0$  for sufficiently large  $v$ . Fix such  $v \geq 3$  and we further restrict  $R$  for which

$$\frac{\mu(A(\kappa_v, r))}{\mu(B(x, r) \cap \Omega)} \leq \frac{1}{(4D)^p}$$

whenever  $0 < r \leq R$ . Then by the above, with  $\kappa_v$  playing the role of  $k_0$ , we have  $M(R) - \kappa_v = 2^{-(v+1)}(M(R) - m(R))$ , and so

$$M(R/2) - m(R/2) \leq \kappa_v - m(R/2) + d \leq \kappa_v - m(R) + d \\ = [1 - 2^{-(v+1)}](M(R) - m(R)) + d.$$

We further restrict  $R$  so that

$$\lambda_1 := 1 - 2^{-(v+1)} + \frac{CR^{(1-1/p)/\alpha} 2^{-(v+1)\widehat{\alpha}}}{(M - m)^{1-\widehat{\alpha}}} < 1 \quad (7.7)$$

and

$$\lambda_2 := 1 - 2^{-(v+1)} + \frac{CR^{(1-1/p)/\beta} 2^{-(v+1)\widehat{\beta}}}{(M - m)^{1-\widehat{\beta}}} < 1. \quad (7.8)$$

Here,

$$\widehat{\alpha} = \left[ \alpha + \frac{1}{p} - 1 \right] / \alpha < 1, \quad \widehat{\beta} = \left[ \beta + \frac{1}{p} - 1 \right] / \beta < 1.$$

If  $d = \frac{1}{4}(M(R) - \kappa_v)$ , then we see by the choice of  $v \geq 3$  as outlined above that

$$M(R/2) - m(R/2) \leq [1 - 2^{-(v+1)}](M(R) - m(R)) + \frac{2^{-(v+1)}}{4}(M(R) - m(R)) \\ \leq [1 - 2^{-(v+2)}](M(R) - m(R)). \quad (7.9)$$

If  $d = C \left[ R^{1-1/p} (M(R) - k_0)^{\alpha + \frac{1}{p} - 1} \right]^{1/\alpha}$ , then by the restriction (7.7) we have from  $M(R) - m(R) \approx M - m$  that

$$\begin{aligned} M(R/2) - m(R/2) &\leq [1 - 2^{-(v+1)}](M(R) - m(R)) \\ &\quad + \frac{C R^{(1-1/p)/\alpha} 2^{-(v+1)\widehat{\alpha}}}{(M - m)^{1-\widehat{\alpha}}} (M(R) - m(R)) \\ &\leq \lambda_1 (M(R) - m(R)). \end{aligned} \quad (7.10)$$

If  $d = C \left[ R^{1-1/p} (M(R) - k_0)^{\beta + \frac{1}{p} - 1} \right]^{1/\beta}$ , then similarly we obtain

$$M(R/2) - m(R/2) \leq \lambda_2 (M(R) - m(R)). \quad (7.11)$$

Combining (7.9), (7.10), and (7.11), setting

$$\lambda = \max\{1 - 2^{-(v+2)}, \lambda_1, \lambda_2\},$$

and noting that  $0 < \lambda < 1$ , we obtain in all three cases that for all small  $R > 0$ ,

$$M(R/2) - m(R/2) \leq \lambda (M(R) - m(R)).$$

An iterated application of the above tells us that

$$M(r) - m(r) \leq 2^{1+\theta_0} \left( \frac{r}{R} \right)^{\theta_0} [M - m]$$

for all  $0 < r < R$ , where  $\theta_0 = \log_2(1/\lambda)$ . It follows that  $u$  must have  $\theta_0$ -Hölder continuous decay to  $Tu(x)$  at  $x$ , which contradicts our assumption that  $u$  is not continuous at  $x$ .

Thus we conclude that  $u$  must be continuous at  $x$  from  $\Omega$ , that is,

$$\lim_{\Omega \ni y \rightarrow x} u(y) = Tu(x).$$

This holds for each  $x \in \partial\Omega \cap B(y, r)$  on which  $f$  does not change sign. Since  $Tu$  is the trace of  $u$  on  $\partial\Omega$ , it follows that  $u$  is continuous at  $x$  relative to  $\overline{\Omega}$ .  $\square$

Note that the above proof does not permit us to conclude that  $u$  must be Hölder continuous at the boundary point  $x$ . From the work of [16,32] we know that in the Euclidean setting, with  $\Omega$  a bounded smooth domain,  $u$  is Hölder continuous at the boundary. As far as we know, this remains open in the metric setting.

The above proof does not permit us to draw any conclusions at boundary points where  $f$  changes sign. On the other hand, an analysis of the proof above shows that if there is some  $\xi \in [m(R), M(R)]$  for which

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - \xi| d\mu = 0,$$

then when  $\lim_{r \rightarrow 0^+} M(r) = M > m = \lim_{r \rightarrow 0^+} m(r)$ , we must have either

$$\lim_{r \rightarrow 0^+} \frac{\mu(\{u > (m + M)/2\} \cap B(x, r) \cap \Omega)}{\mu(B(x, r) \cap \Omega)} = 0$$

or

$$\lim_{r \rightarrow 0^+} \frac{\mu(\{u < (m + M)/2\} \cap B(x, r) \cap \Omega)}{\mu(B(x, r) \cap \Omega)} = 0.$$

By considering  $u$  in the first case and  $-u$  in the second case, for sufficiently large  $\nu$ , with  $\kappa_\nu = M(R) - 2^{-(\nu+1)}[M(R) - m(R)]$  we have

$$\lim_{r \rightarrow 0^+} \frac{\mu(A(\kappa_\nu, r))}{\mu(B(x, r) \cap \Omega)} = 0,$$

and so the proof of Theorem 7.2 will show that  $u$  has to be continuous at  $x$ . Note that here we will obtain that  $\xi = Tu(x)$ . By the definition of the trace function  $Tu$ , we have

$$\lim_{r \rightarrow 0^+} \int_{B(x, r) \cap \Omega} |u - Tu(x)| d\mu = 0 \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega.$$

Thus, we have the following theorem.

**Theorem 7.12.** *Under the standard assumptions on  $\Omega$  and  $\mu$ , if  $f : \partial\Omega \rightarrow \mathbb{R}$  is a bounded Borel measurable function on  $\partial\Omega$ , then for  $\mathcal{H}$ -almost every  $x \in \partial\Omega$ ,  $u$  is continuous at  $x$  relative to  $\overline{\Omega}$ .*

## References

- [1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.* 12 (1959) 623–727.
- [2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, *Comm. Pure Appl. Math.* 17 (1964) 35–92.
- [3] L. Ambrosio, Fine properties of sets of finite perimeter in doubling metric measure spaces, *calculus of variations, nonsmooth analysis and related topics, Set-Valued Anal.* 10 (2–3) (2002) 111–128.
- [4] L. Ambrosio, S. Di Marino, N. Gigli, Perimeter as relaxed Minkowski content in metric measure spaces, <http://cvgmt.sns.it/paper/2956/>, 2016, pp. 1–12.
- [5] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
- [6] L. Ambrosio, N. Gigli, G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, *Duke Math. J.* 163 (2014) 1405–1490.
- [7] L. Ambrosio, M. Miranda Jr., D. Pallara, Special functions of bounded variation in doubling metric measure spaces, in: *Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi*, in: *Quad. Mat.*, vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45.
- [8] L. Ambrosio, A. Pinamonti, G. Speight, Weighted Sobolev spaces on metric measure spaces, *J. Reine Angew. Math.*, Ahead of Print (2016), <https://doi.org/10.1515/crelle-2016-0009>.
- [9] M. Biroli, N.A. Tchou, V.V. Zhikov, Homogenization for Heisenberg operator with Neumann boundary conditions. *Papers in memory of Ennio De Giorgi (Italian)*, *Ric. Mat.* 48 (suppl) (1999) 45–59.
- [10] A. Björn, Characterizations of  $p$ -superharmonic functions on metric spaces, *Studia Math.* 169 (1) (2005) 45–62.
- [11] A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics, vol. 17, European Mathematical Society (EMS), Zürich, 2011, xii+403 pp.

- [12] A. Björn, J. Björn, M. Parviainen, Lebesgue points and the fundamental convergence theorem for superharmonic functions on metric spaces, *Rev. Mat. Iberoam.* 26 (1) (2010) 147–174.
- [13] J. Björn, Fine continuity on metric spaces, *Manuscripta Math.* 125 (3) (2008) 369–381.
- [14] G.-Q. Chen, M. Torres, W.P. Ziemer, Gauss–Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws, *Comm. Pure Appl. Math.* 62 (2) (2009) 242–304.
- [15] A. Cianchi, V.G. Maz'ya, Global boundedness of the gradient for a class of nonlinear elliptic systems, *Arch. Ration. Mech. Anal.* 212 (1) (2014) 129–177.
- [16] T.R. Cranny, Regularity of solutions for the generalized inhomogeneous Neumann boundary value problem, *J. Differential Equations* 126 (2) (1996) 292–302.
- [17] E.N. Dancer, D. Daners, D.A. Hauer, Liouville theorem for  $p$ -harmonic functions on exterior domains, *Positivity* 19 (3) (2015) 577–586.
- [18] S. Dubey, A. Kumar, M.M. Mishra, The Neumann problem for the Kohn–Laplacian on the Heisenberg group, *Potential Anal.* 45 (1) (2016) 119–133.
- [19] L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics Series, CRC Press, Boca Raton, 1992, viii+268 pp.
- [20] J. García-Azorero, J.J. Manfredi, I. Peral, J.D. Rossi, The Neumann problem for the  $\infty$ -Laplacian and the Monge–Kantorovich mass transfer problem, *Nonlinear Anal.* 66 (2) (2007) 349–366.
- [21] N. Gigli, On the differential structure of metric measure spaces and applications, *Mem. Amer. Math. Soc.* 236 (1113) (2015).
- [22] N. Gigli, A. Mondino, A PDE approach to nonlinear potential theory in metric measure spaces, *J. Math. Pures Appl.* (9) 100 (4) (2013) 505–534.
- [23] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing, River Edge, NJ, 2003.
- [24] A. Gogatishvili, P. Koskela, N. Shanmugalingam, Interpolation properties of Besov spaces defined on metric spaces, *Math. Nachr.* 283 (2) (2010) 215–231.
- [25] P. Hajlasz, P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* 145 (2000) 688.
- [26] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001, x+141 pp.
- [27] J. Heinonen, P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* 181 (1) (1998) 1–61.
- [28] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd edn., Dover, Mineola, NY, 2006.
- [29] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients*, New Mathematical Monographs, vol. 27, Cambridge University Press, 2015, xi+448 pp.
- [30] V.O. Kapustyan, O.P. Kogut, On the existence of optimal controls in the coefficients for the nonlinear Neumann boundary value problem, *Differ. Uravn.* 46 (7) (2010) 915–930; translation in *Differ. Equ.* 46 (7) (2010) 923–938.
- [31] S. Keith, Modulus and the Poincaré inequality on metric measure spaces, *Math. Z.* 245 (2) (2003) 255–292.
- [32] C. Kenig, F. Lin, Z. Shen, Homogenization of elliptic systems with Neumann boundary conditions, *J. Amer. Math. Soc.* 26 (4) (2013) 901–937.
- [33] J. Kinnunen, R. Korte, A. Lorent, N. Shanmugalingam, Regularity of sets with quasiminimal boundary surfaces in metric spaces, *J. Geom. Anal.* 23 (2013) 1607–1640.
- [34] J. Kinnunen, O. Martio, Nonlinear potential theory on metric spaces, *Illinois J. Math.* 46 (3) (2002) 857–883.
- [35] J. Kinnunen, O. Martio, Sobolev space properties of superharmonic functions on metric spaces, *Results Math.* 44 (1–2) (2003) 114–129.
- [36] J. Kinnunen, N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, *Manuscripta Math.* 105 (2001) 401–423.
- [37] P. Koskela, N. Shanmugalingam, Y. Zhou, Geometry and analysis of Dirichlet forms (II), *J. Funct. Anal.* 267 (2014) 2437–2477.
- [38] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York–London, 1968, xviii+495 pp.
- [39] P. Lahti, Extensions and traces of functions of bounded variation on metric spaces, *J. Math. Anal. Appl.* 423 (1) (2015) 521–537.
- [40] P. Lahti, N. Shanmugalingam, Trace theorems for functions of bounded variation in metric setting, *J. Funct. Anal.* 274 (10) (2018) 2754–2791.
- [41] L. Malý, Trace and extension theorems for Sobolev-type functions in metric spaces, preprint available at: arXiv: 1704.06344.
- [42] N. Marola, M. Miranda Jr., N. Shanmugalingam, Boundary measures, generalized Gauss–Green formulas, and mean value property in metric measure spaces, *Rev. Mat. Iberoam.* 31 (2) (2015) 497–530.

- [43] V. Maz'ya, S. Poborchi, On solvability of boundary integral equations of potential theory for a multidimensional cusp domain, Problems in mathematical analysis No. 43, J. Math. Sci. (N. Y.) 164 (3) (2010) 403–414.
- [44] E. Milakis, L.E. Silvestre, Regularity for fully nonlinear elliptic equations with Neumann boundary data, Comm. Partial Differential Equations 31 (7–9) (2006) 1227–1252.
- [45] M. Miranda Jr., Functions of bounded variation on “good” metric spaces, J. Math. Pures Appl. (9) 82 (8) (2003) 975–1004.
- [46] D-M. Nhieu, The Neumann problem for sub-Laplacians on Carnot groups and the extension theorem for Sobolev spaces, Ann. Mat. Pura Appl. (4) 180 (1) (2001) 1–25.
- [47] G.V. Ryzhakov, A.V. Setukha, On the convergence of the vortex loop method with regularization for the Neumann boundary value problem on a plane screen, Differ. Uravn. 47 (9) (2011) 1352–1358; translation in Differ. Equ. 47 (9) (2011) 1365–1371.
- [48] N. Shanmugalingam, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoam. 16 (2) (2000) 243–279.
- [49] N. Shanmugalingam, Some convergence results for  $p$ -harmonic functions on metric measure spaces, Proc. Lond. Math. Soc. (3) 87 (1) (2003) 226–246.
- [50] M.E. Taylor, Tools for PDE, in: Pseudodifferential Operators, Paradiifferential Operators, and Layer Potentials, in: Mathematical Surveys and Monographs, vol. 81, AMS, Providence, RI, 2000.
- [51] N.A. Tchou, Homogenization for the Heisenberg operator, in: Homogenization and Applications to Material Sciences, Nice, 1995, in: GAKUTO Internat. Ser. Math. Sci. Appl., vol. 9, Gakkotosho, Tokyo, 1995, pp. 413–420.
- [52] W.P. Ziemer, Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.