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Asymptotic stability of superposition of stationary solutions and rarefaction waves for 1D Navier–Stokes/Allen–Cahn system

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Abstract

In this paper, we investigate the large time behavior of the solutions to the inflow problem for the one-dimensional Navier–Stokes/Allen–Cahn system in the half space. First, we assume that the space-asymptotic states (ρ_+, u_+, χ_+) and the boundary data (ρ_b, u_b, χ_b) satisfy some conditions so that the time-asymptotic state of solutions for the inflow problem is a nonlinear wave which is the superposition of a stationary solution and a rarefaction wave. Then, we show the existence of the stationary solution by the center manifold theorem. Finally, we prove that the nonlinear wave is asymptotically stable when the initial data is a small perturbation of the nonlinear wave. The proof is mainly based on the energy method by taking into account the effect of the concentration χ and the complexity of nonlinear wave.

MSC: 35M10; 35M33; 76D33; 35B40; 35B35

Keywords: Navier–Stokes/Allen–Cahn system; Inflow problem; Rarefaction wave; Stationary solution; Stability

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1. Introduction

In the paper, we study the Navier–Stokes/Allen–Cahn system, a combination of the compressible Navier–Stokes system with an Allen–Cahn phase field description. Under some suitable assumptions, mathematically, the model in $\mathbb{R}_+ := (0, \infty)$ takes the following form in the Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = \nu \partial_x^2 u - \dfrac{\delta}{2}\partial_x(\partial_x \chi)^2, \\ \rho\partial_t \chi + \rho u \partial_x \chi = -\mu, \\ \rho\mu = -\delta\partial_x^2 \chi + \dfrac{\rho}{\delta}(\chi^3 - \chi), \end{cases} \tag{1.1}$$

for $(x, t) \in (0, +\infty) \times (0, +\infty)$. Here $\rho(x, t) > 0$ denotes the total density, $u(x, t)$ represents the mean velocity of the fluid mixture, $\chi(x, t)$ is the concentration difference of the two components, $\mu(x, t)$ denotes the chemical potential, the viscous coefficient $\nu > 0$, the constant $\delta > 0$ and $\sqrt{\delta}$ represents the thickness of the interfacial region. The pressure $P(\rho)$ is given by

$$P(\rho) = a\rho^\gamma,$$

where $a > 0$ is a positive constant and $\gamma > 1$ is the adiabatic exponent.

The Navier–Stokes/Allen–Cahn system describes two-phase patterns in a flowing liquid including phase transformations. A phase field variable χ is introduced and a mixing energy is defined in terms of χ and its spatial gradient. As pointed out in [1], the model should be viewed as a first step toward incorporating transport mechanism into the description of phase-formation process. For the detailed derivation of Navier–Stokes/Allen–Cahn system (1.1), please refer to [1,7] and references therein. So far, some important progress has been made for the Navier–Stokes/Allen–Cahn system. Let us recall some known results about the Navier–Stokes/Allen–Cahn system. Feireisl et al. [11] proved the existence of global-in-time weak solutions in $\mathbb{R}^3$ without any restriction on the size of initial data for $\gamma > 6$. Recently, Chen et al. [5] extended Feireisl's result to $\gamma > 2$. Gal and Grasselli [12] showed the existence of the trajectory attractor for both incompressible Navier–Stokes/Allen–Cahn and Navier–Stokes/Cahn–Hilliard systems and also obtained a convergence rate estimate in the phase-space metric. Xu et al. [42] discussed the global regularity of axisymmetric solutions in both large viscosity and small initial data cases in $\mathbb{R}^3$. Kotschote [24] proved the existence and uniqueness of local strong solutions for the Navier–Stokes/Allen–Cahn system with arbitrary initial data. Zhao et al. [44] investigated the vanishing viscosity limit and proved that the solutions of the Navier–Stokes/Allen–Cahn system converged to that of the Euler/Allen–Cahn system in a proper small time interval. Ding et al. [7] proved the existence and uniqueness of global classical solution, the existence of weak solutions and the existence of unique strong solution of the Navier–Stokes/Allen–Cahn system in $\mathbb{R}$ for initial data without vacuum states. Chen and Guo [4] established the global existence and uniqueness of strong and classical solutions of Navier–Stokes/Allen–Cahn system in $\mathbb{R}$ with initial vacuum. Kotschote [25] investigated the stability of traveling wave solutions to the so-called Navier–Stokes/Allen–Cahn system. Luo et al. [28] proved the stability of rarefaction wave to Navier–Stokes/Allen–Cahn system. Moreover, for numerical simulations, such as jet pinching-off and drop formation, we referred the readers to [2,43,27]. We also emphasized for a different

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two-phase model. Evje et al. obtained a series results in [8–10] and references therein. However, to our knowledge, there are few results about the large-time behavior of solutions to an initial boundary value problem for the Navier–Stokes/Allen–Cahn system (1.1). Here, we will partly give a positive answer for this important problem.

Initial data for system (1.1) is given by

$$(\rho, u, \chi)(x, 0) = (\rho_0, u_0, \chi_0)(x), \qquad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0. \tag{1.2}$$

We assume that the initial data at the far field $x = +\infty$ is constant, namely

$$\lim_{x\to+\infty} (\rho_0, u_0, \chi_0)(x) = (\rho_+, u_+, \chi_+), \tag{1.3}$$

and the boundary data for ρ, u, χ at $x = 0$ is given by

$$(\rho, u, \chi)(0, t) = (\rho_b, u_b, \chi_b), \qquad \forall t \geq 0, \tag{1.4}$$

where $\rho_b > 0$, $u_b > 0$ and χ_b are constants and the following compatibility conditions hold

$$\rho_0(0) = \rho_b > 0, \qquad u_0(0) = u_b > 0, \qquad \chi_0(0) = \chi_b. \tag{1.5}$$

The situation $u_b > 0$ means that the gas blows into the region through the boundary $x = 0$ with the velocity u_b, and hence the problem (1.1)–(1.5) is called as inflow problem. We note that for the case $u_b < 0$, the situation is different and we have an outflow problem. For the well-posedness of outflow problem, one cannot impose the boundary condition on ρ at $x = 0$, and the boundary condition turn out to be

$$u(0, t) = u_b, \;\; \chi(0, t) = \chi_b$$

with $u_b < 0$. Such an outflow problem is also interesting and will be studied in the future.

Now we turn back to our inflow problem. The aim of this paper is to present the large time behavior of solutions to the inflow problem (1.1)–(1.5). To construct a classical solution of the Navier–Stokes/Allen–Cahn system $(1.1)_3$ and $(1.1)_4$, it is necessary to require $\chi_+ = 1$. We also assume that the left and right constant states are different, namely, $\rho_+ \neq \rho_b$, $u_+ \neq u_b$ and $\chi_+ \neq \chi_b$. It is expected that as $t \to \infty$, the solution $(\rho, u, \chi)(x, t)$ to the inflow problem (1.1)–(1.5) is asymptotically described by the superposition of a transonic stationary solution and a rarefaction wave, which can be determined by the space-asymptotic states (ρ_+, u_+, χ_+), and the boundary data ρ_b, u_b and χ_b under some suitable assumptions.

Notice that the Navier–Stokes/Allen–Cahn system is a combination of the compressible Navier–Stokes system with an Allen–Cahn phase field description. To our knowledge, there have been a huge numbers of paper in the literature about the large-time behavior of the solutions to Cauchy problem of the compressible Navier–Stokes equations toward the viscous version of the three basic wave patterns, namely, shock wave, rarefaction wave, contact discontinuity and even their compositions. In 1985, Matsumura–Nishihara [31] firstly proved the asymptotic stability of the viscous shock wave to the one-dimensional isentropic compressible Navier–Stokes equations. Since then, many authors had been attracted to study the asymptotic stability of the viscous wave patterns and much progress has been made. For example, we can refer to [13,15,18,26,32,33] and some references therein for details. All these results show that the large-time behavior of the solutions to Cauchy problem for the compressible Navier–Stokes equations are basically governed by the Riemann problem of the corresponding Euler equations. However, in the case of the initial boundary value problem of the compressible Navier–Stokes equations, not only basic wave patterns but also a stationary solution, which is also called the boundary layer solution, may appear due to the boundary effect. Thus the large-time behavior of the solutions to the initial boundary value problem is much more complicated than that of the Cauchy problem. In 1999, Matsumura at Hong Kong [29] gave the complete classification of the large-time behavior of the solutions in terms of the far field state and the boundary data. After then, the impermeable wall problem, the inflow problem and the outflow problem for the compressible Navier–Stokes equations have been extensively studied and improved by many authors in lots of the literatures. We refer to [14,30,34] for the impermeable wall problem, to [17,20,22,23,37,19,38] for the outflow problem, and to [16,35,36,39,40] for the inflow problem. In this paper, motivated by [16,39], we will consider the stability of the nonlinear wave for an inflow problem to the initial boundary value problem of the Navier–Stokes/Allen–Cahn system (1.1)–(1.5).

First, when χ is a nontrivial concentration for the large time behavior, we construct a stationary solution $(\tilde\rho,\tilde u,\tilde\chi)(x)$ to (1.1) independent of a time variable t as follows:

$$\begin{cases} \partial_x(\tilde\rho\tilde u)=0,\\ \partial_x(\tilde\rho\tilde u^2)+\partial_x(a\tilde\rho^\gamma)=\nu\partial_x^2\tilde u-\dfrac{\delta}{2}\partial_x(\partial_x\tilde\chi)^2,\\ \tilde\rho\tilde u\partial_x\tilde\chi=-\tilde\mu,\\ \tilde\rho\tilde\mu=-\delta\partial_x^2\tilde\chi+\dfrac{\tilde\rho}{\delta}(\tilde\chi^3-\tilde\chi), \end{cases} \tag{1.6}$$

with conditions

$$\inf_{x\in\mathbb{R}_+}\tilde\rho(x)>0,\ \lim_{x\to+\infty}(\tilde\rho,\tilde u,\tilde\chi)(x)=(\rho_*,u_*,1),\ \tilde\rho(0)=\rho_b>0,\ \tilde u(0)=u_b>0,\ \tilde\chi(0)=\chi_b. \tag{1.7}$$

Here, constants (ρ_*,u_*) satisfy

$$\rho_b u_b=\rho_* u_*,\quad u_*=c(\rho_*), \tag{1.8}$$

where $c(\rho)=\sqrt{P'(\rho)}=\sqrt{a\gamma}\rho^{\frac{\gamma-1}{2}}$ is sound speed.

Next, when $\chi=\chi_+=1$ is a trivial concentration and the dissipation effects are neglected for the large time behavior, we use the following Euler equation

$$\begin{cases} \partial_t\rho+\partial_x(\rho u)=0,\\ \partial_t(\rho u)+\partial_x(\rho u^2)+\partial_x P(\rho)=0 \end{cases} \tag{1.9}$$

with

$$(\rho,u)(x,0)=\begin{cases}(\rho_*,u_*), & x<0,\\ (\rho_+,u_+), & x>0\end{cases} \tag{1.10}$$

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to construct a 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$. Here (ρ_*, u_*) and (ρ_+, u_+) satisfy

$$u_+ = u_* + \int_{\rho_*}^{\rho_+} \sqrt{\frac{P'(s)}{s^2}} ds, \tag{1.11}$$

and

$$(\rho_+, u_+) \in \{(\rho, u) : u > c(\rho),\ \rho > 0,\ u > 0\}. \tag{1.12}$$

Once we have the rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ and the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\chi})(x)$, we can define the composite wave $(\bar{\rho}, \bar{u})(x, t)$ as follows:

$$(\bar{\rho}, \bar{u})(x, t) = (\tilde{\rho}, \tilde{u})(x) + (\rho^{r_2}, u^{r_2})(x, t) - (\rho_*, u_*), \tag{1.13}$$

where $(\rho^{r_2}, u^{r_2})(x, t)$ is a suitably smoothed function of $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$, and will be stated in Subsection 2.2.

Now we state main results of this paper in the following theorem.

Theorem 1.1. *Assume that the given constants $\rho_b > \rho_* > 0$, $0 < u_b < u_* < u_+$, $\chi_b > 1$, $0 < \rho_* < \rho_+$ satisfy (1.8), (1.11) and (1.12). Suppose further that the initial data satisfy*

$$\rho_0(x) - \bar{\rho}(x, 0), u_0(x) - \bar{u}(x, 0) \in H^1(\mathbb{R}_+),\quad \chi_0(x) - \tilde{\chi}(x) \in H^2(\mathbb{R}_+),\quad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0, \tag{1.14}$$

and that

$$\tilde{\delta} + \epsilon + \|\rho_0(x) - \bar{\rho}(x, 0)\|_{H^1} + \|u_0(x) - \bar{u}(x, 0)\|_{H^1} + \|\chi_0(x) - \tilde{\chi}(x)\|_{H^2} \le \varepsilon_0, \tag{1.15}$$

where ϵ is given in (2.24), $\tilde{\delta} = |\rho_b - \rho_*| + |u_b - u_*| + |\chi_b - 1|$ and the positive constant ε_0 are small enough. Then the inflow problem (1.1)–(1.5) has a unique global classical solution $(\rho, u, \chi)(x, t)$ satisfying

$$\begin{cases} \rho(x, t) - \bar{\rho}(x, t),\quad u(x, t) - \bar{u}(x, t) \in C([0, \infty); H^1(\mathbb{R}_+)), \\ \chi(x, t) - \tilde{\chi}(x, t) \in C([0, \infty); H^2(\mathbb{R}_+)), \\ \partial_x\rho(x, t) - \partial_x\bar{\rho}(x, t) \in L^2([0, \infty); L^2(\mathbb{R}_+)), \\ \partial_x u(x, t) - \partial_x\bar{u}(x, t),\quad \partial_x\chi(x, t) - \partial_x\tilde{\chi}(x, t) \in L^2([0, \infty); H^1(\mathbb{R}_+)), \end{cases}$$

where $\bar{\rho}(x, t), \bar{u}(x, t)$ are defined by (1.13). Moreover, it holds that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}_+} \left|\rho(x, t) - \tilde{\rho}(x) - \rho^{R_2}\left(\frac{x}{t}\right) + \rho_*\right| = 0, \tag{1.16}$$

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}_+} \left|u(x, t) - \tilde{u}(x) - u^{R_2}\left(\frac{x}{t}\right) + u_*\right| = 0, \tag{1.17}$$

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}_+} |\chi(x, t) - \tilde{\chi}(x)| = 0. \tag{1.18}$$

The outline of Theorem 1.1 is as follows. First, we assume that the space-asymptotic states (ρ_+, u_+, χ_+) and the boundary data (ρ_b, u_b, χ_b) satisfy some suitable conditions so that the time-asymptotic state of solution for the inflow problem (1.1)–(1.5) is a nonlinear wave which is the superposition of a stationary solution and a rarefaction wave. On the one hand, notice that when χ is a trivial concentration in the large time, the Navier–Stokes/Allen–Cahn system can reduce to the isentropic Navier–Stokes system and thus we can establish smoothly approximate rarefaction wave by the idea of [32] and [33]. On the other hand, we can show the existence of the stationary solution by the center manifold theory in [3] and one can see the detailed proof in Subsection 2.1. Next, using the arguments and ideas of [35], we can obtain the asymptotic stability of the nonlinear wave by the energy method when the initial data is a small perturbation of the nonlinear wave. Since the complexity of nonlinear wave has been investigated carefully in [35], we only need to focus on the effect of the concentration χ. Compared with the classical Navier–Stokes system, the concentration χ in this model (1.1) brings both benefit and trouble. The benefit lies in the fact that the term $\delta\partial_x^2\chi$ in equation $(1.1)_4$ is a viscous dissipation term which provides extra regularity of $\partial_x\chi$, while the trouble is brought by the term $\frac{\delta}{2}\partial_x(\partial_x\chi)^2$ in equation $(1.1)_2$ which increases the nonlinearity of the system. It is also this term that requires $\|\partial_x\zeta\|_{L^\infty}$ to be small when we deal with the high order nonlinear term J_{32} in Lemma 4.3, which just demands the initial perturbation to be small and also requires the smallness of $\|\zeta\|_{H^2}$ to close the *a priori* assumption (3.11).

Remark 1.1. Theorem 1.1 shows that the superposition of the stationary solution and the rarefaction wave is stable when the strength of the stationary solution is necessarily weak, but the strength of the rarefaction wave is not necessarily weak. Here, we are not concerned with the large initial perturbation.

Remark 1.2. Here, we should point out that we can not directly use the results of the stationary solution (second-order ODE system) for Navier–Stokes system in [35,36] since the third-order ODE system (2.5) should be our focus in the analysis of the existence of the stationary solution.

Remark 1.3. Here we only consider inflow problem to one-dimensional compressible Navier–Stokes/Allen–Cahn system. However, we should mention that the corresponding outflow problem is surely more interesting since we don't know how to deal with the boundary term $\frac{\delta u_b}{2}(\partial_x\zeta)^2(0,t)$ in (4.7), thus more difficult. These are left to the forthcoming paper in the future.

The rest of the paper is organized as follows. In Section 2, we construct the nonlinear wave. We first show the existence of stationary solutions by the center manifold theorem in Subsection 2.1. Then in Subsection 2.2, we shall go over some results obtained in some other references, on smooth approximations of the rarefaction wave. Next, we reformulate the original problem in terms of the perturbed variables in Section 3. Section 4 is the key, in which we establish the *a priori* estimate. Finally, we complete the proof of our main theorem in Section 5.

Notations. Throughout this paper, C denotes some positive constant (generally large) and c denotes some positive constant (generally small), where both C and c may take different values in different places. For a nonnegative integer k, $H^k(\mathbb{R}_+)$ denotes the standard Hilbert spaces of order k. $L^p = L^p(\mathbb{R}_+)$ $(1 \le p \le +\infty)$ denotes the usual Lebesgue space on $\mathbb{R}_+$ with its norm $\|\cdot\|_{L^p}$, and when $p = 2, +\infty$, we write $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$ and $\|\cdot\|_{L^\infty(\mathbb{R}_+)} = \|\cdot\|_\infty$.

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2. The nonlinear wave

In this section, we mainly present a nonlinear wave which is superposition of the stationary solution and rarefaction wave. We first show the existence of the stationary solutions from the center manifold theorem in Subsection 2.1. Then, we present the existence and smooth approximation of the rarefaction wave in Subsection 2.2.

2.1. *The existence of stationary solutions*

In this subsection, let us construct stationary solutions of (1.1). That is, we consider the stationary equation:

$$\begin{cases} \partial_x(\tilde{\rho}\tilde{u}) = 0, \\ \partial_x(\tilde{\rho}\tilde{u}^2) + \partial_x(a\tilde{\rho}^\gamma) = \nu\partial_x^2\tilde{u} - \dfrac{\delta}{2}\partial_x(\partial_x\tilde{\chi})^2, \\ \tilde{\rho}\tilde{u}\partial_x\tilde{\chi} = -\tilde{\mu}, \\ \tilde{\rho}\tilde{\mu} = -\delta\partial_x^2\tilde{\chi} + \dfrac{\tilde{\rho}}{\delta}(\tilde{\chi}^3 - \tilde{\chi}), \end{cases} \tag{2.1}$$

and the corresponding boundary conditions:

$$\begin{cases} (\tilde{\rho}, \tilde{u}, \tilde{\chi}) \to (\rho_*, u_*, 1) \quad \text{as} \quad x \to \infty, \\ (\tilde{\rho}, \tilde{u}, \tilde{\chi})|_{x=0} = (\rho_b, u_b, \chi_b) \end{cases} \tag{2.2}$$

with $\rho_b > 0$ and $u_b > 0$.

From (2.2) and $(2.1)_3$, one can derive

$$\tilde{\mu} \to 0 \quad \text{as} \quad x \to \infty. \tag{2.3}$$

From the fact that $\tilde{\rho}(x) > 0$, $\rho_b > 0$, $u_b > 0$ and $(2.1)_1$, we have

$$\rho_b u_b = \rho_* u_*, \quad \tilde{\rho}(x) = \frac{\rho_* u_*}{\tilde{u}(x)}, \quad \tilde{u}(x) > 0, \quad u_* > 0. \tag{2.4}$$

From (2.1) and (2.2), we have

$$\begin{aligned} \tilde{\rho}\tilde{u} &= \rho_* u_*, \\ \partial_x\tilde{\chi} &= -\frac{1}{\rho_* u_*}\tilde{\mu}, \\ \rho_* u_*(\tilde{u} - u_*) + a(\tilde{\rho}^\gamma - \rho_*^\gamma) &= \nu\partial_x\tilde{u} - \frac{\delta}{2\rho_*^2 u_*^2}\tilde{\mu}^2, \end{aligned}$$

and

$$\tilde{\rho}\tilde{\mu} = \frac{\delta}{\rho_* u_*}\partial_x\tilde{\mu} + \frac{\tilde{\rho}}{\delta}(\tilde{\chi}^3 - \tilde{\chi}).$$

Then the boundary value problem (2.1) and (2.2) can be reduced to

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$$
\begin{cases}
\tilde{\rho}\tilde{u} = \rho_* u_*, \\
\partial_x \tilde{u} = \frac{\rho_* u_*}{\nu}(\tilde{u} - u_*) + \frac{a}{\nu}(\tilde{\rho}^\gamma - \rho_*^\gamma) + \frac{\delta}{2\nu\rho_*^2 u_*^2}\tilde{\mu}^2, \\
\partial_x \tilde{\mu} = \frac{\rho_* u_*}{\delta}\tilde{\rho}\tilde{\mu} - \frac{\rho_* u_*}{\delta^2}\tilde{\rho}(\tilde{\chi}^3 - \tilde{\chi}), \\
\partial_x \tilde{\chi} = -\frac{1}{\rho_* u_*}\tilde{\mu},
\end{cases}
\tag{2.5}
$$

with

$$
\begin{cases}
(\tilde{\rho}, \tilde{u}, \tilde{\chi}, \tilde{\mu}) \to (\rho_*, u_*, 1, 0) \quad \text{as} \quad x \to \infty, \\
(\tilde{\rho}, \tilde{u}, \tilde{\chi})|_{x=0} = (\rho_b, u_b, \chi_b).
\end{cases}
\tag{2.6}
$$

Concerning (2.5) and (2.6), we have

Lemma 2.1. *If the given constants $\rho_b > 0$, $u_b > 0$, χ_b, $\rho_* > 0$, $u_* > 0$ satisfy $\rho_* u_* = \rho_b u_b$ and $u_* = \sqrt{a\gamma}\rho_*^{\frac{\gamma-1}{2}}$ (i.e., it is located at the transonic curve.). If $u_b < u_*$, $\chi_b > 1$ and $\tilde{\delta} = |\rho_b - \rho_*| + |u_b - u_*| + |\chi_b - 1|$ is small enough, then there exists a solution $(\tilde{\rho}, \tilde{u}, \tilde{\chi}, \tilde{\mu})(x)$ to the stationary problem (2.5) and (2.6), such that $\partial_x \tilde{u} > 0$, $\partial_x \tilde{\rho} < 0$, $\partial_x \tilde{\chi} < 0$ and*

$$
|\partial_x^k (\tilde{\rho}(x) - \rho_*, \tilde{u}(x) - u_*, \tilde{\chi}(x) - 1)| \le \frac{C\tilde{\delta}^{k+1}}{(1+\tilde{\delta}x)^{k+1}},
\tag{2.7}
$$

$$
|\partial_x^k \tilde{\mu}(x)| \le \frac{C\tilde{\delta}^{k+2}}{(1+\tilde{\delta}x)^{k+2}},
\tag{2.8}
$$

for $k = 0, 1, 2, \cdots$.

Proof of Lemma 2.1. We first introduce

$$
\bar{u} = \tilde{u} - u_*, \quad \bar{\mu} = \tilde{\mu} - 0, \quad \bar{\chi} = \tilde{\chi} - 1.
\tag{2.9}
$$

Then from (2.5) and (2.6), and using $\tilde{\rho} = \frac{\rho_* u_*}{\tilde{u}}$, we have

$$
\frac{d}{dx}\begin{pmatrix} \bar{u} \\ \bar{\mu} \\ \bar{\chi} \end{pmatrix} = A \begin{pmatrix} \bar{u} \\ \bar{\mu} \\ \bar{\chi} \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix},
\tag{2.10}
$$

and

$$
\begin{cases}
(\bar{u}, \bar{\mu}, \bar{\chi})(x) \to (0, 0, 0) \text{ as } x \to +\infty, \\
(\bar{u}, \bar{\chi})(x)|_{x=0} = (u_b - u_*, \chi_b - 1).
\end{cases}
\tag{2.11}
$$

Here

$$
A = \begin{pmatrix}
\frac{\rho_*(u_*^2 - a\gamma\rho_*^{\gamma-1})}{\nu u_*} & 0 & 0 \\
0 & \frac{\rho_*^2 u_*}{\delta} & -\frac{2\rho_*^2 u_*}{\delta^2} \\
0 & -\frac{1}{\rho_* u_*} & 0
\end{pmatrix},
$$

and

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$$F_1 = \frac{a\gamma\rho_*^\gamma}{\gamma u_*(\bar{u}+u_*)}\bar{u}^2 + \frac{\delta}{2\nu\rho_*^2u_*^2}\bar{\mu}^2 + \frac{a\rho_*^\gamma}{\nu}\left[\left(\frac{u_*}{\bar{u}+u_*}\right)^\gamma - 1 - \gamma\left(\frac{u_*}{\bar{u}+u_*} - 1\right)\right],$$

$$F_2 = -\frac{\rho_*^2u_*}{\delta(\bar{u}+u_*)}\bar{\mu}\bar{u} + \frac{2\rho_*^2u_*}{\delta^2(\bar{u}+u_*)}\bar{\chi}\bar{u} - \frac{\rho_*^2u_*^2}{\delta^2(\bar{u}+u_*)}\bar{\chi}^2(\bar{\chi}+3).$$

To prove the existence of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\chi}, \tilde{\mu})$, it is sufficient to show the existence of the solutions $(\bar{u}, \bar{\mu}, \bar{\chi})$ to the boundary value problem (2.10) and (2.11). For this, we diagonalize the system (2.10). After a simple calculation of $|\lambda I - A| = 0$ and recalling $u_* = \sqrt{a\gamma}\rho_*^{\frac{\gamma-1}{2}}$, we know that the matrix A has the following three eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\rho_*^2u_* - \sqrt{\rho_*^4u_*^2 + 8\rho_*}}{2\delta} < 0, \quad \lambda_3 = \frac{\rho_*^2u_* + \sqrt{\rho_*^4u_*^2 + 8\rho_*}}{2\delta} > 0,$$

and the corresponding eigenvectors:

$$\begin{aligned} r_1 &= (1, 0, 0)^t, \\ r_2 &= \left(0, \rho_*^2u_* - \sqrt{\rho_*^4u_*^2 + 8\rho_*}, -\frac{2\delta}{\rho_*u_*}\right)^t, \\ r_3 &= \left(0, \rho_*^2u_* + \sqrt{\rho_*^4u_*^2 + 8\rho_*}, -\frac{2\delta}{\rho_*u_*}\right)^t. \end{aligned}$$

Define the matrix $P = (r_1, r_2, r_3)$. Notice that $\det P = \frac{4\delta}{\rho_*u_*}\sqrt{\rho_*^4u_*^2 + 8\rho_*} \neq 0$ which shows that P^{-1} exists. Furthermore, we employ new unknown function defined by

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = P^{-1}\begin{pmatrix} \bar{u} \\ \bar{\mu} \\ \bar{\chi} \end{pmatrix}. \tag{2.12}$$

We also define the corresponding boundary data

$$\begin{pmatrix} V_{1b} \\ V_{2b} \\ V_{3b} \end{pmatrix} = P^{-1}\begin{pmatrix} u_b - u_* \\ \bar{\mu}_b \\ \chi_b - 1 \end{pmatrix} \tag{2.13}$$

with $\bar{\mu}_b = \tilde{\mu}(0)$, and nonlinear term by

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = P^{-1}\begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}.$$

Using these, we have

$$\frac{d}{dx}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \tag{2.14}$$

and

$$\lim_{x\to\infty}(V_1, V_2, V_3)(x) \to (0,0,0), \quad (V_1, V_2, V_3)(x)|_{x=0} = (V_{1b}, V_{2b}, V_{3b}). \tag{2.15}$$

Namely,

$$\begin{cases} V_{1x} = F_1, \\ V_{2x} = \frac{\rho_*^2 u_* - \sqrt{\rho_*^4 u_*^2 + 8\rho_*}}{2\delta} V_2 - \frac{F_2}{2\sqrt{\rho_*^4 u_*^2 + 8\rho_*}}, \\ V_{3x} = \frac{\rho_*^2 u_* + \sqrt{\rho_*^4 u_*^2 + 8\rho_*}}{2\delta} V_3 + \frac{F_2}{2\sqrt{\rho_*^4 u_*^2 + 8\rho_*}}, \end{cases} \tag{2.16}$$

from which we assert that there exist local center manifold $V_2 = h_1^c(V_1)$ and $V_3 = h_2^c(V_1)$, and local stable manifold $V_1 = h_1^s(V_2)$ and $V_3 = h_2^s(V_2)$. In order to show the existence of the solution, we have to examine dynamics on the center manifold. To this end, we employ a solution $z = z(x)$ to $(2.16)_1$ restricted on the center manifold satisfying the equation

$$z_x = F_1(z, h_1^c(z), h_2^c(z)). \tag{2.17}$$

By virtue of the center manifold theory in [3], there exists a solution z to (2.17) such that the solution $V = (V_1, V_2, V_3)$ of (2.14)–(2.15) is given by

$$\begin{cases} V_1 = z(x) + O(\tilde{\delta} e^{-cx}), \\ V_2 = h_1^c(z(x)) + O(\tilde{\delta} e^{-cx}), \\ V_3 = h_2^c(z(x)) + O(\tilde{\delta} e^{-cx}). \end{cases} \tag{2.18}$$

Therefore, to obtain the solution (V_1, V_2, V_3) to (2.14)–(2.15), it suffices to show the existence of the solution to (2.17) satisfying $z(x) \to 0$ as $x \to +\infty$, we see that the nonlinear terms F_1 and F_2 satisfy

$$F_1 = \frac{a\gamma(\gamma+1)\rho_*^\gamma}{2\nu u_*^2} V_1^2 + O(|V_1|^3 + |V_2|^2 + |V_3|^2 + |V_2 V_3|),$$

and

$$F_2 = O(|V_2|^3 + |V_3|^3 + |V_1 V_2| + |V_2|^2 + |V_3|^2 + |V_1 V_3| + |V_2 V_3|).$$

Substituting them into (2.17) and using $h_1^c(z) = O(z^2)$, $h_2^c(z) = O(z^2)$, we deduce (2.17) to

$$z_x = \frac{a\gamma(\gamma+1)\rho_*^\gamma}{2\nu u_*^2} z^2 + O(|z|^3), \tag{2.19}$$

which yields that $z(x)$ is monotonically increasing for sufficiently small $z(x)$. Thus, to satisfy $z(x) \to 0$ as $x \to \infty$, the boundary data $z(0)$ should be negative. Namely, for the existence of the solution (V_1, V_2, V_3), the boundary data (V_{1b}, V_{2b}, V_{3b}) should be located in the left region from the local stable manifold, that is, (V_{1b}, V_{2b}, V_{3b}) should satisfy the condition

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$$V_{1b} < h_1^s(V_{2b}), \quad V_{3b} < h_2^s(V_{2b}),$$

which can be arrived by choosing enough small $\tilde{\delta}$ and free $\tilde{\mu}_b$. Moreover, since $\tilde{\mu}_b$ is free, we can choose V_2 satisfying

$$V_{2x} + V_{3x} > 0,$$

which together with (2.12) and (2.9) implies $\partial_x \tilde{\chi} < 0$.

From (2.18) and (2.19), we have

$$V_{1x} = \frac{a\gamma(\gamma+1)\rho_*^\gamma}{2\nu u_*^2} z^2 + O(|z|^3) + O(\tilde{\delta} e^{-cx}), \tag{2.20}$$

which together with (2.12) and (2.9) implies $\partial_x \tilde{u} > 0$ if $z(x)$ and $\tilde{\delta}$ are sufficiently small.

Since $\partial_x \tilde{u} > 0$, we rapidly get $u_b < u_*$. Moreover, from (2.19), (2.18), (2.13) and the fact that $u_b < u_*$, we also see that the solution z satisfies

$$0 < \frac{c\tilde{\delta}}{1+\tilde{\delta}x} \le -z(x) \le \frac{C\tilde{\delta}}{1+\tilde{\delta}x}, \quad |\partial_x^k z(x)| \le \frac{C\tilde{\delta}^{k+1}}{(1+\tilde{\delta}x)^{k+1}} \text{ for } k = 0, 1, 2, \cdots. \tag{2.21}$$

Combining (2.18) and (2.21) with using $h_1^c(z) = O(z^2)$, $h_2^c(z) = O(z^2)$, we have the decay property of (V_1, V_2, V_3):

$$|\partial_x^k (V_1, V_2, V_3)| \le \frac{C\tilde{\delta}^{k+1}}{(1+\tilde{\delta}x)^{k+1}} + C\tilde{\delta} e^{-cx} \quad \text{for} \quad k = 0, 1, 2, \cdots,$$

which together with (2.9) and (2.12) yields (2.7). Then (2.8) was immediately obtained from $(2.5)_4$ and (2.7). □

2.2. *The rarefaction wave*

To study the rarefaction wave, we consider the Riemann problem (1.9)–(1.10). One can see that system (1.9) has two characteristics (cf. [6,41])

$$\begin{cases} \lambda_1(\rho, u) = u - c(\rho), \\ \lambda_2(\rho, u) = u + c(\rho), \end{cases}$$

which are genuinely nonlinear and give rise to the rarefaction wave curves

$$R_1[\rho_*, u_*] \equiv \left\{ [\rho, u] \in \mathbb{R}_+ \times \mathbb{R} \;\middle|\; u = u_* - \int_{\rho_*}^{\rho} \sqrt{\frac{P'(s)}{s^2}} ds, \quad \rho < \rho_*, \ u > u_* \right\},$$

and

$$R_2[\rho_*, u_*] \equiv \left\{ [\rho, u] \in \mathbb{R}_+ \times \mathbb{R} \ \middle| u = u_* + \int_{\rho_*}^{\rho} \sqrt{\frac{P'(s)}{s^2}} ds, \quad \rho > \rho_*, \ u > u_* \right\},$$

respectively.

From (1.11), we see $(\rho_+, u_+) \in R_2[\rho_*, u_*]$. Hence, we consider 2-rarefaction wave here. Since the 2-rarefaction wave $\left[\rho^{R_2}, u^{R_2}\right]\left(\frac{x}{t}\right)$ is a weak solution, we shall construct a smooth approximation for the 2-rarefaction wave in the following. Firstly, consider the Riemann problem for Burger's equation:

$$\begin{cases} \partial_t w + w \partial_x w = 0, \\ w(x, 0) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \end{cases} \tag{2.22}$$

where $w_- < w_+$. Then it is well known that (2.22) has a continuous weak solution $w^R\left(\frac{x}{t}\right)$ whose explicit form is given by

$$w^R\left(\frac{x}{t}\right) = \begin{cases} w_-, & x < w_- t, \\ \dfrac{x}{t}, & w_- t \le x \le w_+ t, \\ w_+, & x > w_+ t. \end{cases} \tag{2.23}$$

Moreover, $w^R\left(\frac{x}{t}\right)$ can be approximated by the smooth function $\overline{w}(x, t)$ which is a solution to

$$\begin{cases} \partial_t \bar{w} + \bar{w} \partial_x \bar{w} = 0, \\ \bar{w}(x, 0) = \bar{w}_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q \bar{\delta} \displaystyle\int_0^{\epsilon x} y^q e^{-y} dy, & x > 0, \end{cases} \end{cases} \tag{2.24}$$

where $\bar{\delta} := w_+ - w_- > 0$, $q \ge 10$ is a constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y} dy = 1$, and $\epsilon \le 1$ is a positive constant to be determined later. Then we have the following lemma.

Lemma 2.2. *Let $\bar{\delta} = w_+ - w_-$ be the wave strength of the 2-rarefaction wave. Then the problem (2.24) has a unique smooth solution $\bar{w}(x, t)$ which satisfies the following properties:*

(i) $w_- < \bar{w}(x, t) < w_+$, $\partial_x \bar{w} \ge 0$ for $x \in \mathbb{R}$ and $t \ge 0$.

(ii) For any p $(1 \le p \le +\infty)$, there exists a constant $C_{p,q}$ such that for $t \ge 0$

$$\begin{cases} \|\partial_x \bar{w}\|_{L^p} \le C_{p,q} \min\{\bar{\delta}\epsilon^{1-\frac{1}{p}}, \bar{\delta}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\}, \\ \|\partial_x^2 \bar{w}\|_{L^p} \le C_{p,q} \min\{\bar{\delta}\epsilon^{2-\frac{1}{p}}, \bar{\delta}^{\frac{1}{q}} \epsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\}. \end{cases}$$

(iii) When $x \le w_- t$, $\bar{w} - w_- = \partial_x \bar{w} = \partial_x^2 \bar{w} = 0$.

(iv) $\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |\bar{w}(x, t) - w^R\left(\frac{x}{t}\right)| = 0$.

Then the smooth approximate rarefaction wave $\left[\rho^{r_2}, u^{r_2}\right](x, t)$ which corresponds to the rarefaction wave $\left[\rho^{R_2}, u^{R_2}\right]\left(\frac{x}{t}\right)$ can be defined as follows:

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$$\begin{cases} u^{r_2} + c(\rho^{r_2}) = \bar{w}(x, 1+t), \ \ w_- = u_* + c(\rho_*) > 0, \ \ w_+ = u_+ + c(\rho_+) > 0, \\ u^{r_2} = u_* + \int_{\rho_*}^{\rho^{r_2}} \sqrt{\frac{P'(s)}{s^2}} ds, \ \ u_+ > u_*, \ \ \rho_+ > \rho_*, \end{cases} \tag{2.25}$$

where $\bar{w}(x,t)$ is given in (2.24). Since $\gamma > 1$, we have $c'(\rho) = \frac{(\gamma-1)\sqrt{a\gamma}}{2} \rho^{\frac{\gamma-3}{2}} > 0$, which shows that $c(\rho)$ is a monotonically increasing function about ρ. Hence, from this and the fact $u_+ > u_*$, $\rho_+ > \rho_*$, we can derive $w_- < w_+$. Moreover, it is easy to obtain $[\rho^{r_2}, u^{r_2}](x,t)$ satisfies

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho \partial_t u + \rho u \partial_x u + \partial_x P(\rho) = 0. \end{cases} \tag{2.26}$$

Here we restrict $[\rho^{r_2}, u^{r_2}](x,t)$ in the half space $\{x \geq 0\}$. Then one has

Lemma 2.3. *Let $\delta_r = |\rho_+ - \rho_*| + |u_+ - u_*|$ be the wave strength of the 2-rarefaction wave. Then the smooth approximate 2-rarefaction wave $[\rho^{r_2}, u^{r_2}](x,t)$ constructed in (2.25) has the following properties:*

(i) $\partial_x u^{r_2} \geq 0$, $\rho_* < \rho^{r_2}(x,t) < \rho_+$, $u_* < u^{r_2}(x,t) < u_+$, $\partial_x u^{r_2} \sim |\partial_x \rho^{r_2}|$ for $x \in \mathbb{R}_+$ and $t \geq 0$.

(ii) For any p $(1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that for $t > 0$,

$$\begin{cases} \|\partial_x[\rho^{r_2}, u^{r_2}]\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta_r \epsilon^{1-\frac{1}{p}}, \delta_r^{\frac{1}{p}} (1+t)^{-1+\frac{1}{p}}\}, \\ \|\partial_x^2[\rho^{r_2}, u^{r_2}]\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta_r \epsilon^{2-\frac{1}{p}}, \delta_r^{\frac{1}{q}} \epsilon^{1-\frac{1}{p}+\frac{1}{q}} (1+t)^{-1+\frac{1}{q}}\}. \end{cases}$$

(iii) $[\rho^{r_2}, u^{r_2}](0,t) = [\rho_*, u_*]$.

(iv) $\lim\limits_{t \to +\infty} \sup\limits_{x \in \mathbb{R}_+} \left|[\rho^{r_2}, u^{r_2}](x,t) - \left[\rho^{R_2}, u^{R_2}\right]\left(\frac{x}{t}\right)\right| = 0.$

3. Reformulation of the problem

In this section, we reformulate the original problem (1.1)–(1.5) in terms of the perturbated variables. To begin with, let us recall the nonlinear wave $(\bar{\rho}, \bar{u})$, which is defined in Section 1. That is,

$$(\bar{\rho}, \bar{u})(x,t) = (\tilde{\rho}, \tilde{u}) + (\rho^{r_2}, u^{r_2}) - (\rho_*, u_*). \tag{3.1}$$

$(\tilde{\rho}, \tilde{u}, \tilde{\chi})$ is the stationary solution which connects the two states (ρ_b, u_b, χ_b) and $(\rho_*, u_*, 1)$ and satisfies for any $x > 0$ that

$$\begin{cases} \partial_x(\tilde{\rho}\tilde{u}) = 0, \\ \partial_x(\tilde{\rho}\tilde{u}^2) + \partial_x p(\tilde{\rho}) = \nu \partial_x^2 \tilde{u} - \frac{\delta}{2} \partial_x (\partial_x \tilde{\chi})^2, \\ \tilde{\rho}\tilde{u}\partial_x \tilde{\chi} = -\tilde{\mu}, \\ \tilde{\rho}\tilde{\mu} = -\delta \partial_x^2 \tilde{\chi} + \frac{\tilde{\rho}}{\delta}(\tilde{\chi}^3 - \tilde{\chi}), \end{cases} \tag{3.2}$$

with the boundary condition $(\tilde{\rho}, \tilde{u}, \tilde{\chi})|_{x=0} = (\rho_b, u_b, \chi_b)$. On the other hand, $(\rho^{r_2}, u^{r_2})(x,t)$ is the smooth rarefaction ware satisfying

$$
\begin{cases}
\partial_t \rho^{r_2} + \partial_x(\rho^{r_2} u^{r_2}) = 0, \\
\rho^{r_2}(\partial_t u^{r_2} + u^{r_2}\partial_x u^{r_2}) + \partial_x p(\rho^{r_2}) = 0.
\end{cases} \tag{3.3}
$$

Thus, from (3.2) and (3.3), we see $(\bar{\rho}, \bar{u}, \tilde{\chi})$ satisfies

$$
\begin{cases}
\partial_t \bar{\rho} + \partial_x(\bar{\rho}\bar{u}) = \bar{f}, \\
\bar{\rho}(\partial_t \bar{u} + \bar{u}\partial_x \bar{u}) + \partial_x p(\bar{\rho}) = \nu \partial_x^2 \bar{u} - \frac{\delta}{2}\partial_x(\partial_x \tilde{\chi})^2 + \bar{g}, \\
\tilde{\rho}\tilde{u}\partial_x \tilde{\chi} = -\tilde{\mu}, \\
\tilde{\rho}\tilde{\mu} = -\delta \partial_x^2 \tilde{\chi} + \frac{\tilde{\rho}}{\delta}(\tilde{\chi}^3 - \tilde{\chi}),
\end{cases} \tag{3.4}
$$

where $\bar{f}$ and $\bar{g}$ are defined by

$$
\bar{f} = \partial_x[(\rho^{r_2} - \rho_*)(\tilde{u} - u_*) + (u^{r_2} - u_*)(\tilde{\rho} - \rho_*)],
$$

and

$$
\begin{aligned}
\bar{g} = {} & (\rho^{r_2} - \rho_*)\tilde{u}\partial_x \tilde{u} + \bar{\rho}[(\tilde{u} - u_*)\partial_x u^{r_2} + (u^{r_2} - u_*)\partial_x \tilde{u}] + [p'(\bar{\rho}) - p'(\rho^{r_2})]\partial_x \rho^{r_2} \\
& + [p'(\bar{\rho}) - p'(\tilde{\rho})]\partial_x \tilde{\rho} - \frac{\tilde{\rho} - \rho_*}{\rho^{r_2}} p'(\rho^{r_2})\partial_x \rho^{r_2} - \nu \partial_x^2 u^{r_2}.
\end{aligned}
$$

From (3.1), (2.25) and $(3.2)_1$, it is easy to know

$$
\begin{cases}
|\bar{f}| + |\bar{g} + \nu \partial_x^2 u^{r_2}| \le C \left\{ \partial_x \tilde{u}(u^{r_2} - u_*) + \partial_x u^{r_2}(u_* - \tilde{u}) \right\}, \\
|\partial_x \bar{f}| \le C \left\{ (|\partial_x^2 \tilde{u}| + (\partial_x \tilde{u})^2)(u^{r_2} - u_*) + \partial_x \tilde{u}\partial_x u^{r_2} + |\partial_x^2 u^{r_2}| + (\partial_x u^{r_2})^2 \right\},
\end{cases} \tag{3.5}
$$

where we have used the fact $\partial_x \tilde{u} > 0$, $\partial_x u^{r_2} \ge 0$ and

$$
u_b \le \tilde{u} \le u_* \le u^{r_2} \le u_+, \tag{3.6}
$$

which can be derived from Lemma 2.1 and Lemma 2.3 (i). Similarly, from the fact $\partial_x \tilde{\rho} < 0$ and $\rho_* < \rho^{r_2}(x,t) < \rho_+$, we have

$$
0 < \rho_* \le \bar{\rho} \le \rho_b + \rho_+ - \rho_*. \tag{3.7}
$$

Now we define the new unknowns

$$
\begin{aligned}
\varphi(x,t) &= \rho(x,t) - \bar{\rho}(x,t), \quad & \psi(x,t) &= u(x,t) - \bar{u}(x,t), \\
\zeta(x,t) &= \chi(x,t) - \tilde{\chi}(x,t), \quad & \omega(x,t) &= \mu(x,t) - \tilde{\mu}(x,t).
\end{aligned}
$$

Then from (1.1) and (3.4), it is easy to check that the perturbated variable $(\varphi, \psi, \zeta, \omega)(x,t)$ satisfies

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$$\begin{cases} \partial_t\varphi + u\partial_x\varphi + \rho\partial_x\psi = f, \\ \rho(\partial_t\psi + u\partial_x\psi) + p'(\rho)\partial_x\varphi = \nu\partial_x^2\psi - \frac{\delta}{2}\partial_x(\partial_x\zeta)^2 - \delta\partial_x(\partial_x\zeta\partial_x\tilde{\chi}) + \frac{\delta}{2\bar{\rho}}\partial_x(\partial_x\tilde{\chi})^2\varphi + g, \\ \rho(\partial_t\zeta + u\partial_x\zeta) + \omega = -\partial_x\tilde{\chi}[\rho\psi + \tilde{u}\varphi + \tilde{u}(\rho^{r_2} - \rho_*) + \rho(u^{r_2} - u_*)], \\ \rho\omega = \frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}\varphi + \frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}(\rho^{r_2} - \rho_*) - \delta\partial_x^2\zeta + \frac{1}{\delta}\left[\rho\zeta^3 + 3\tilde{\chi}\rho\zeta^2 + (3\tilde{\chi}^2 - 1)\rho\zeta\right], \end{cases} \tag{3.8}$$

where the functions f and g are given by

$$f = -\bar{f} - \varphi\partial_x\bar{u} - \psi\partial_x\bar{\rho},$$
$$g = -\rho\partial_x\bar{u}\psi - \nu\partial_x^2\bar{u}\frac{\varphi}{\bar{\rho}} + \left[\frac{\rho}{\bar{\rho}}P'(\bar{\rho}) - P'(\rho)\right]\partial_x\bar{\rho} - \frac{\rho\bar{g}}{\bar{\rho}}.$$

And the boundary and initial conditions turn out to be

$$\begin{cases} \varphi(x,0) = \rho_0(x) - \bar{\rho}(x,0), \quad \psi(x,0) = u_0(x) - \bar{u}(x,0), \quad \zeta(x,0) = \chi_0(x) - \tilde{\chi}(x), \\ \varphi(0,t) = \rho_b - \bar{\rho}(0,t) = \rho_* - \rho^{r_2}(0,t) = 0, \\ \psi(0,t) = u_b - \bar{u}(0,t) = u_* - u^{r_2}(0,t) = 0, \\ \zeta(0,t) = \chi_b - \tilde{\chi}(0) = 0. \end{cases} \tag{3.9}$$

Therefore, we can now restate our main results in terms of the perturbated variable $(\varphi,\psi,\zeta)(x,t)$ as follows.

Theorem 3.1. *Under the assumptions of Theorem 1.1, there exists a unique global solution $(\varphi,\psi,\zeta)(x,t)$ to the problem (3.8) and (3.9), satisfying*

$$\varphi(x,t), \psi(x,t) \in C([0,\infty); H^1(\mathbb{R}_+)), \quad \zeta(x,t) \in C([0,\infty); H^2(\mathbb{R}_+)),$$
$$\partial_x\varphi(x,t) \in L^2((0,\infty); L^2(\mathbb{R}_+)), \quad \partial_x\psi(x,t), \partial_x\zeta(x,t) \in L^2((0,\infty); H^1(\mathbb{R}^+)),$$

and

$$\sup_{x\in\mathbb{R}^+} |(\varphi,\psi,\zeta)(x,t)| \to 0 \quad as \quad t \to +\infty. \tag{3.10}$$

To prove this theorem, we employ the standard continuation argument based on a local exist result and *a priori* estimates stated in the following

Proposition 3.1. *Assume that all the conditions in Theorem 1.1 hold true. Let $[\varphi,\psi,\zeta]$ be a smooth solution to the initial boundary value problem (3.8) and (3.9) on $0 \le t \le T$ for $T > 0$. There are constants $\varepsilon_0 > 0$, $C > 0$ such that if*

$$\sup_{0\le t\le T} \left(\|[\varphi,\psi](t)\|_{H^1} + \|\zeta(t)\|_{H^2}\right) + \tilde{\delta} + \epsilon \le \varepsilon_0, \tag{3.11}$$

then one has the solutions $[\varphi,\psi,\zeta]$ satisfy

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$$\sup_{0\le t\le T}\Big(\|[\varphi,\psi](t)\|_{H^1}^2+\|\zeta(t)\|_{H^2}^2\Big)+\int_0^T(\|\partial_x\varphi\|^2+\|\partial_x[\psi,\zeta]\|_{H^1}^2)dt$$
$$\le C\|[\varphi_0,\psi_0]\|_{H^1}^2+C\|\zeta_0\|_{H^2}^2+C\tilde\delta^{\frac19}+C\epsilon^{\frac{1}{10}}. \tag{3.12}$$

It is easy to get

$$\|[\varphi,\psi,\zeta]\|_\infty\le\sqrt2\varepsilon_0,\quad \|\partial_x\zeta\|_\infty\le\sqrt2\varepsilon_0, \tag{3.13}$$

where the following Sobolev inequality

$$|h(x)|\le\sqrt2\|h\|^{1/2}\|\partial_xh\|^{1/2}\quad\text{for}\quad h(x)\in H^1(\mathbb{R}_+) \tag{3.14}$$

is used.

The complexity of nonlinear wave leads to many complicated terms in the course of establishing the *a priori* estimates, however those terms are of two basic types and can be evaluated suitably by using the decay (in both time and space variables) estimates of each component of nonlinear wave. Hence for later use and clear reference, in Lemma 3.1, we will give the following important inequalities which deal with the low order dissipation terms and the complex terms mentioned above before the energy estimates.

Lemma 3.1. *(i) For any function h and $(k+1)j>2$, there is a positive constant C such that,*

$$\int_{\mathbb{R}_+}|\partial_x^k(\tilde u-u_*)|^j|h|^2dx\le C\tilde\delta^{(k+1)j-2}\Big[\tilde\delta h^2(0,t)+\|\partial_xh(t)\|^2\Big]. \tag{3.15}$$

(ii) For any functions f, h and $2(k+1)j>3$, there is a positive constant C such that,

$$\int_{\mathbb{R}_+}|\partial_x^k(\tilde u-u_*)|^j|h\partial_xf|dx\le\tilde\delta\|\partial_xf(t)\|^2+C\tilde\delta^{2(k+1)j-3}\Big[\tilde\delta h^2(0,t)+\|\partial_xh(t)\|^2\Big]. \tag{3.16}$$

(iii) For any $\theta\in[0,1]$, we have

$$\|\partial_x[\rho^{r_2}-\rho_*,u^{r_2}-u_*]\|_\infty\le C\epsilon^\theta(1+t)^{-(1-\theta)}. \tag{3.17}$$

(iv) Since $\rho^{r_2}(0,t)=\rho_*$ and $u^{r_2}(0,t)=u_*$, we have

$$\rho^{r_2}(x,t)-\rho_*\le x\|\partial_x\rho^{r_2}\|_\infty, \tag{3.18}$$

and

$$u^{r_2}(x,t)-u_*\le x\|\partial_xu^{r_2}\|_\infty. \tag{3.19}$$

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(v) For any $\theta \in [0,1]$, $q \geq 10$, *we have*

$$\int_{\mathbb{R}_+} \left(|\bar{f}| + |\bar{g} + \nu \partial_x^2 u^{r_2}| \right) dx \leq C \frac{\tilde{\delta}}{1+\tilde{\delta}t} + C\epsilon^{\theta}(1+t)^{-(1-\theta)} \ln(1+\tilde{\delta}t) \tag{3.20}$$

and

$$\int_{\mathbb{R}_+} |\partial_x \bar{f}| dx \leq C\tilde{\delta}(1+t)^{-1} + C\epsilon^{\theta}(1+t)^{-(1-\theta)} + C\epsilon^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}. \tag{3.21}$$

(vi) For $q \geq 10$, *we have*

$$\int_{\mathbb{R}_+} |\bar{g}|^2 dx \leq C\epsilon^{1+\frac{2}{q}}(1+t)^{-2+\frac{2}{q}} + C\tilde{\delta}(1+t)^{-2}, \tag{3.22}$$

and

$$\int_{\mathbb{R}_+} (|\bar{f}|^2 + |\partial_x \bar{f}|^2) dx \leq C\tilde{\delta}(1+t)^{-2} + C\tilde{\delta}\epsilon^{2+\frac{2}{q}}(1+t)^{-2+\frac{2}{q}}. \tag{3.23}$$

Proof. (i) Using (2.7) and the following Poincaré type inequalities

$$|h(x,t)| \leq |h(0,t)| + x^{\frac{1}{2}} \|\partial_x h(t)\|, \tag{3.24}$$

for $(k+1)j > 2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h|^2 dx \\
&\leq 2 \int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j \left(h^2(0,t) + x\|\partial_x h(t)\|^2 \right) dx \\
&\leq Ch^2(0,t) \int_{\mathbb{R}_+} \frac{\tilde{\delta}^{(k+1)j}}{(1+\tilde{\delta}x)^{(k+1)j}} dx + C\|\partial_x h(t)\|^2 \int_{\mathbb{R}_+} \frac{x\tilde{\delta}^{(k+1)j}}{(1+\tilde{\delta}x)^{(k+1)j}} dx \\
&\leq C\tilde{\delta}^{(k+1)j-2} \left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2 \right].
\end{aligned}$$

(ii) By the Young inequality and Lemma 3.1 (i), for $2(k+1)j > 3$, we have

$$\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h \partial_x f| dx$$

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$$
\begin{aligned}
&\le \tilde{\delta}\|\partial_x f(t)\|^2 + C\int\limits_{\mathbb{R}_+} \frac{\tilde{\delta}^{2(k+1)j-1}}{(1+\tilde{\delta}x)^{2(k+1)j}} h^2 dx \\
&\le \tilde{\delta}\|\partial_x f(t)\|^2 + C\tilde{\delta}^{2(k+1)j-3}\left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2\right].
\end{aligned}
$$

(iii) From Lemma 2.3 (ii), we have

$$
\|\partial_x[\rho^{r_2} - \rho_*, u^{r_2} - u_*]\|_\infty \le C\min\{\epsilon, (1+t)^{-1}\}.
$$

Thus we have

$$
\|\partial_x[\rho^{r_2} - \rho_*, u^{r_2} - u_*]\|_\infty \le C\epsilon^\theta (1+t)^{-(1-\theta)}.
$$

Here we have used the fact that if $0 < C \le A$ and $0 < C \le B$, then $C \le A^\theta B^{1-\theta}$ for any $0 \le \theta \le 1$.

(iv) Notice that

$$
u^{r_2}(x,t) - u^{r_2}(0,t) = \int\limits_0^x u^{r_2}_y dy \le x\|\partial_x u^{r_2}\|_\infty.
$$

Since $u^{r_2}(0,t) = u_*$, we complete the proof of (3.19). Similarly, we have (3.18).

(v) Using (2.7), (3.5), Lemma 2.3 (ii), Lemma 3.1 (iii) and (iv), we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}_+} \left(|\bar{f}| + |\bar{g} + \nu\partial_x^2 u^{r_2}|\right) dx \\
&\le C\int\limits_{\mathbb{R}_+} \{\partial_x \tilde{u}(u^{r_2} - u_*) + \partial_x u^{r_2}(u_* - \tilde{u})\} dx \\
&= C\int\limits_{\mathbb{R}_+} \partial_x[(u^{r_2} - u_*)(\tilde{u} - u_*)] dx + 2C\int\limits_{\mathbb{R}_+} \partial_x u^{r_2}(u_* - \tilde{u}) dx \\
&= 2C\int\limits_0^t \partial_x u^{r_2}(u_* - \tilde{u}) dx + 2C\int\limits_t^{+\infty} \partial_x u^{r_2}(u_* - \tilde{u}) dx \\
&\le C\|\partial_x u^{r_2}\|_\infty \int\limits_0^t \frac{\tilde{\delta}}{1+\tilde{\delta}x} dx + C\frac{\tilde{\delta}}{1+\tilde{\delta}t}\int\limits_t^{+\infty} \partial_x u^{r_2} dx \\
&\le C\|\partial_x u^{r_2}\|_\infty \ln(1+\tilde{\delta}t) + C\frac{\tilde{\delta}}{1+\tilde{\delta}t}\|\partial_x u^{r_2}\|_{L^1} \\
&\le C\epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t) + C\frac{\tilde{\delta}}{1+\tilde{\delta}t},
\end{aligned}
$$

where we have used $u^{r_2}(0,t) = u_*$ and $\tilde{u} \to u_*$ as $x \to +\infty$.

Similarly, we can obtain that

$$\begin{aligned}
&\int_{\mathbb{R}_+} |\partial_x \bar{f}| dx \\
&\le C \int_{\mathbb{R}_+} \left\{ (|\partial_x^2 \tilde{u}| + (\partial_x \tilde{u})^2)(u^{r_2} - u_*) + \partial_x \tilde{u} \partial_x u^{r_2} + |\partial_x^2 u^{r_2}| + (\partial_x u^{r_2})^2 \right\} dx \\
&\le C\|\partial_x u^{r_2}\|_\infty \int_{\mathbb{R}_+} x(|\partial_x^2 \tilde{u}| + (\partial_x \tilde{u})^2) dx + C\|\partial_x u^{r_2}\|_\infty \|\partial_x \tilde{u}\|_{L^1} + C\|\partial_x^2 u^{r_2}\|_{L^1} + C\|\partial_x u^{r_2}\|^2 \\
&\le C\tilde{\delta}(1+t)^{-1} + C\epsilon^\theta (1+t)^{-(1-\theta)} + C\epsilon^{\frac{1}{q}} (1+t)^{-1+\frac{1}{q}}.
\end{aligned}$$

(vi) Noticing (3.5) and Lemma 3.1 (iv), and applying Lemma 2.3, (2.25), $(3.2)_1$ and (2.7), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}_+} |\bar{g}|^2 dx &\le C \int_{\mathbb{R}_+} \{ |\partial_x^2 u^{r_2}|^2 + |\partial_x \tilde{u}|^2 |(u^{r_2} - u_*)|^2 + |\partial_x u^{r_2}|^2 |(u_* - \tilde{u})|^2 \} dx \\
&\le C\|\partial_x^2 u^{r_2}\|^2 + C\|\partial_x u^{r_2}\|_\infty^2 \int_{\mathbb{R}_+} |\partial_x \tilde{u}|^2 x^2 dx + C\|\partial_x u^{r_2}\|_\infty^2 \int_{\mathbb{R}_+} |(u_* - \tilde{u})|^2 dx \\
&\le C\epsilon^{1+\frac{2}{q}} (1+t)^{-2+\frac{2}{q}} + C\tilde{\delta}(1+t)^{-2}, \qquad (3.25)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+} (|\bar{f}|^2 + |\partial_x \bar{f}|^2) dx &\le C \int_{\mathbb{R}_+} [(\partial_x \tilde{u})^2 + (\partial_x^2 \tilde{u})^2](u^{r_2} - u_*)^2 dx + C \int_{\mathbb{R}_+} (\partial_x \tilde{u})^2 (\partial_x u^{r_2})^2 dx \\
&\quad + C \int_{\mathbb{R}_+} [(\partial_x u^{r_2})^2 + (\partial_x^2 u^{r_2})^2](\tilde{u} - u_*)^2 dx \\
&\le C\|\partial_x u^{r_2}\|_\infty^2 \int_{\mathbb{R}_+} x^2 [(\partial_x \tilde{u})^2 + (\partial_x^2 \tilde{u})^2] dx + C\|\partial_x u^{r_2}\|_\infty^2 \int_{\mathbb{R}_+} (\partial_x \tilde{u})^2 dx \\
&\quad + C\|\partial_x [u^{r_2}, \partial_x u^{r_2}]\|_\infty^2 \int_{\mathbb{R}_+} (\tilde{u} - u_*)^2 dx \\
&\le C\tilde{\delta}(1+t)^{-2} + C\tilde{\delta}\epsilon^{2+\frac{2}{q}} (1+t)^{-2+\frac{2}{q}}. \quad \square
\end{aligned}$$

4. The energy estimates

Lemma 4.1. *Assume the conditions in Proposition 3.1 hold, then we have the following energy estimate for $t \in [0, T]$,*

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$$\|[\varphi,\psi,\zeta,\partial_x\zeta]\|^2+\int_0^t\left(\|\partial_x\psi\|^2+\|\omega\|^2\right)d\tau+\int_0^t\|\sqrt{\partial_x\bar{u}}[\varphi,\psi,\partial_x\zeta]\|^2d\tau+\int_0^t\|\sqrt{|\partial_x\tilde{\chi}|}\zeta\|^2d\tau$$

$$\leq C\,\|[\varphi_0,\psi_0,\zeta_0,\partial_x\zeta_0]\|^2+C(\tilde{\delta}^{\frac{2}{3}}+\epsilon^{\frac{1}{10}})\int_0^t\|\partial_x[\varphi,\zeta]\|^2d\tau+C\tilde{\delta}^{\frac{1}{9}}+C\epsilon^{\frac{1}{10}}. \tag{4.1}$$

Proof. Multiply $(3.8)_1$ and $(3.8)_2$ by $\frac{p(\rho)-p(\bar{\rho})}{\rho}$ and ψ, respectively. Direct calculations and then summing up two resulting equations give rise to

$$\begin{aligned}
&\partial_t\left[\rho\Phi(\bar{\rho},\rho)+\frac{\rho}{2}\psi^2\right]\\
&\quad+\partial_x\left\{\rho u\Phi+\frac{\rho u}{2}\psi^2+[p(\rho)-p(\bar{\rho})]\psi-\nu\psi\partial_x\psi+\frac{\delta}{2}\psi(\partial_x\zeta)^2+\delta\partial_x\zeta\partial_x\tilde{\chi}\psi\right\}\\
&\quad+\nu(\partial_x\psi)^2+\partial_x\bar{u}\left[p(\rho)-p(\bar{\rho})-p'(\bar{\rho})\varphi+\rho\psi^2\right]\\
&=-\nu\frac{\partial_x^2\bar{u}}{\bar{\rho}}\varphi\psi-\frac{p'(\bar{\rho})}{\bar{\rho}}\varphi\bar{f}-\frac{\rho}{\bar{\rho}}\bar{g}\psi+\frac{\delta}{2}\partial_x\psi(\partial_x\zeta)^2+\delta\partial_x\zeta\partial_x\tilde{\chi}\partial_x\psi+\frac{\delta}{2\bar{\rho}}\varphi\psi\partial_x(\partial_x\tilde{\chi})^2,
\end{aligned} \tag{4.2}$$

where

$$\Phi(\bar{\rho},\rho)=\int_{\bar{\rho}}^{\rho}\frac{p(s)-p(\bar{\rho})}{s^2}ds. \tag{4.3}$$

It is easy to see that $\Phi(\bar{\rho},\rho)$ is equivalent to $|\varphi|^2$, i.e.,

$$c|\varphi|^2\leq\Phi(\bar{\rho},\rho)\leq C|\varphi|^2, \tag{4.4}$$

since there exist positive constants c and C such that ρ and $\bar{\rho}$ satisfying

$$0<c\leq\rho,\bar{\rho}\leq C,$$

which can be derived from (3.7) and (3.11).

Next, multiplying $(3.8)_3$ by ω and together with the help of $(1.1)_1$ and $(3.8)_4$, we obtain

$$\begin{aligned}
&\partial_t\left[\frac{\rho}{\delta}\left(\frac{\zeta^4}{4}+\tilde{\chi}\zeta^3+\frac{3\tilde{\chi}^2-1}{2}\zeta^2\right)+\frac{\delta}{2}(\partial_x\zeta)^2+\frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}\varphi\zeta\right]\\
&\quad+\partial_x\left[\frac{\rho u}{\delta}\left(\frac{\zeta^4}{4}+\tilde{\chi}\zeta^3+\frac{3\tilde{\chi}^2-1}{2}\zeta^2\right)-\frac{\delta u}{2}(\partial_x\zeta)^2-\delta\partial_x\zeta\partial_t\zeta\right]+\frac{\delta}{2}\partial_x\bar{u}(\partial_x\zeta)^2+\omega^2\\
&=-\frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}(\rho^{r_2}-\rho_*)(\partial_t\zeta+u\partial_x\zeta)-\frac{\delta}{2}\partial_x\psi(\partial_x\zeta)^2+\frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}\zeta\partial_t\varphi-\frac{\delta}{\bar{\rho}}\partial_x^2\tilde{\chi}u\varphi\partial_x\zeta\\
&\quad+\frac{\rho u}{\delta}\partial_x\tilde{\chi}(\zeta^3+3\tilde{\chi}\zeta^2)-\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\omega.
\end{aligned} \tag{4.5}$$

Using $(3.8)_1$ and $(3.8)_3$ together with (4.5), one has

$$\begin{aligned}
&\partial_t\left[\frac{\rho}{\delta}\left(\frac{\zeta^4}{4}+\tilde{\chi}\zeta^3+\frac{3(\tilde{\chi}^2-1)}{2}\zeta^2+\zeta^2\right)+\frac{\delta}{2}(\partial_x\zeta)^2+\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\varphi\zeta\right]\\
&\quad+\partial_x\left[\frac{\rho u}{\delta}\left(\frac{\zeta^4}{4}+\tilde{\chi}\zeta^3+\frac{3\tilde{\chi}^2-1}{2}\zeta^2\right)-\frac{\delta u}{2}(\partial_x\zeta)^2-\delta\partial_x\zeta\partial_t\zeta\right]+\frac{\delta}{2}\partial_x\bar{u}(\partial_x\zeta)^2+\omega^2\\
&=\frac{\delta}{\tilde{\rho}\rho}\partial_x^2\tilde{\chi}(\rho^{r_2}-\rho_*)\left\{\omega+[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\partial_x\tilde{\chi}\right\}-\frac{\delta}{2}\partial_x\psi(\partial_x\zeta)^2\\
&\quad-\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\left(u\partial_x\varphi+\rho\partial_x\psi+\partial_x\bar{u}\varphi+\partial_x\bar{\rho}\psi+\bar{f}\right)\zeta-\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}u\varphi\partial_x\zeta+\frac{\rho u}{\delta}\partial_x\tilde{\chi}(\zeta^3+3\tilde{\chi}\zeta^2)\\
&\quad-\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\omega. \qquad (4.6)
\end{aligned}$$

Taking the summation of (4.2) and (4.6), integrating the resulting equation with respect to x over $\mathbb{R}_+$, and using boundary condition (3.9) and $u_b>0$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+}\left[\rho\Phi(\bar{\rho},\rho)+\frac{\rho}{2}\psi^2+\frac{\rho}{\delta}\left(\frac{\zeta^4}{4}+\tilde{\chi}\zeta^3+\frac{3(\tilde{\chi}^2-1)}{2}\zeta^2+\zeta^2\right)+\frac{\delta}{2}(\partial_x\zeta)^2+\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\varphi\zeta\right]dx\\
&\quad+\frac{\delta u_b}{2}(\partial_x\zeta)^2(0,t)+\int_{\mathbb{R}_+}\left[\nu(\partial_x\psi)^2+\omega^2\right]dx+\int_{\mathbb{R}_+}\frac{3\rho u}{\delta}(-\partial_x\tilde{\chi})\zeta^2dx\\
&\quad+\int_{\mathbb{R}_+}\partial_x\bar{u}\left[p(\rho)-p(\bar{\rho})-p'(\bar{\rho})\varphi+\rho\psi^2+\frac{\delta}{2}(\partial_x\zeta)^2\right]dx\\
&=-\int_{\mathbb{R}_+}\frac{p'(\bar{\rho})}{\bar{\rho}}\varphi\bar{f}dx-\int_{\mathbb{R}_+}\frac{\rho}{\bar{\rho}}\bar{g}\psi dx-\nu\int_{\mathbb{R}_+}\frac{\partial_x^2\bar{u}}{\bar{\rho}}\varphi\psi dx+\int_{\mathbb{R}_+}\frac{\delta}{2\bar{\rho}}\varphi\psi\partial_x(\partial_x\tilde{\chi})^2dx\\
&\quad+\int_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}\rho}\partial_x^2\tilde{\chi}(\rho^{r_2}-\rho_*)\left\{\omega+\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\right\}dx\\
&\quad+\delta\int_{\mathbb{R}_+}\partial_x\zeta\partial_x\tilde{\chi}\partial_x\psi dx-\int_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\left(u\partial_x\varphi+\rho\partial_x\psi+\partial_x\bar{u}\varphi+\partial_x\bar{\rho}\psi+\bar{f}\right)\zeta dx\\
&\quad-\int_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}u\varphi\partial_x\zeta dx-\int_{\mathbb{R}_+}\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\omega dx\\
&\quad+\int_{\mathbb{R}_+}\frac{\rho u}{\delta}\partial_x\tilde{\chi}\left[\zeta^3+3(\tilde{\chi}-1)\zeta^2\right]dx\\
&=\sum_{l=1}^{10}J_l, \qquad (4.7)
\end{aligned}$$

where J_l $(1\le l\le 10)$ denote the corresponding terms on the left hand side of (4.7).

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We use $\partial_x\bar{u} = \partial_x\tilde{u} + \partial_x u^{r_2} > 0$ to derive $0 < u_b < \bar{u} < u_+$, which together with (3.11), and imply $0 < \frac{u_b}{2} < u < 2u_+$. Combining this with $\partial_x\tilde{\chi} < 0$, we have the following estimates:

$$\int_{\mathbb{R}_+} \frac{3\rho u}{\delta}(-\partial_x\tilde{\chi})\zeta^2 dx \geq c\|\sqrt{|\partial_x\tilde{\chi}|}\zeta\|^2,$$

$$\int_{\mathbb{R}_+} \partial_x\bar{u}\left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\varphi + \rho\psi^2 + \frac{\delta}{2}(\partial_x\zeta)^2\right]dx \geq c\|\sqrt{\partial_x\bar{u}}[\varphi,\psi,\partial_x\zeta]\|^2.$$

Before our estimates, we take $q = 10$ and $\theta = \frac{1}{8}$ in Lemma 3.1, Lemma 2.3 in the following for brevity. By applying Sobolev inequality (3.14), Cauchy inequality, and using Lemma 3.1, Lemma 2.3, boundary condition (3.9), (2.7) as well as (3.11), it is direct to obtain the following estimates:

$$\begin{aligned}
J_1 &\leq C\|\varphi\|_\infty\|\bar{f}\|_{L^1} \\
&\leq C\|\varphi\|^{\frac{1}{2}}\|\partial_x\varphi\|^{\frac{1}{2}}\left[\frac{\tilde{\delta}}{1+\tilde{\delta}t} + \epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t)\right] \\
&\leq C(\tilde{\delta}^{\frac{2}{3}} + \epsilon^{\frac{1}{10}})\|\partial_x\varphi\|^2 + C\frac{\tilde{\delta}^{\frac{10}{9}}}{(1+\tilde{\delta}t)^{\frac{4}{3}}} + C\epsilon^{\frac{2}{15}}(1+t)^{-\frac{13}{12}},
\end{aligned}$$

$$\begin{aligned}
J_2 &\leq C\|\psi\|_\infty\|\bar{g}\|_{L^1} \\
&\leq C\|\psi\|_\infty\|\bar{g} + \nu\partial_x^2 u^{r_2}\|_{L^1} + C\|\psi\|_\infty\|\nu\partial_x^2 u^{r_2}\|_{L^1} \\
&\leq C\|\psi\|^{\frac{1}{2}}\|\partial_x\psi\|^{\frac{1}{2}}\left[\frac{\tilde{\delta}}{1+\tilde{\delta}t} + \epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t)\right] + C\epsilon^{\frac{1}{q}}\|\psi\|^{\frac{1}{2}}\|\partial_x\psi\|^{\frac{1}{2}}(1+t)^{-1+\frac{1}{q}} \\
&\leq C(\tilde{\delta}^{\frac{2}{3}} + \epsilon^{\frac{1}{10}})\|\partial_x\psi\|^2 + C\frac{\tilde{\delta}^{\frac{10}{9}}}{(1+\tilde{\delta}t)^{\frac{4}{3}}} + C\epsilon^{\frac{2}{15}}(1+t)^{-\frac{13}{12}} + C\epsilon^{\frac{1}{10}}(1+t)^{-\frac{6}{5}} \\
&\leq C(\tilde{\delta}^{\frac{2}{3}} + \epsilon^{\frac{1}{10}})\|\partial_x\psi\|^2 + C\frac{\tilde{\delta}^{\frac{10}{9}}}{(1+\tilde{\delta}t)^{\frac{4}{3}}} + C\epsilon^{\frac{1}{10}}(1+t)^{-\frac{13}{12}},
\end{aligned}$$

$$\begin{aligned}
J_3 &\leq C\int_{\mathbb{R}_+}(|\partial_x^2\tilde{u}| + |\partial_x^2 u^{r_2}|)(\varphi^2+\psi^2)dx \\
&\leq C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 + C\|[\varphi,\psi]\|_\infty^2\|\partial_x^2 u^{r_2}\|_{L^1} \\
&\leq C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 + C\epsilon^{\frac{1}{q}}\|[\varphi,\psi]\|\|\partial_x[\varphi,\psi]\|(1+t)^{-1+\frac{1}{q}} \\
&\leq C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 + C\epsilon^{\frac{1}{10}}\|\partial_x[\varphi,\psi]\|^2 + C\epsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}},
\end{aligned}$$

$$J_4 \leq C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2,$$

$$
\begin{aligned}
J_5 &\le \frac{1}{4}\|\omega\|^2 + C\int_{\mathbb{R}_+}(\partial_x\tilde{\chi})^2(\varphi^2+\psi^2)dx + C\int_{\mathbb{R}_+}[(\partial_x\tilde{\chi})^2+(\partial_x^2\tilde{\chi})^2](u^{r_2}-u_*)^2dx \\
&\le \frac{1}{4}\|\omega\|^2 + C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2 + C\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}x^2(|\partial_x\tilde{\chi}|^2+|\partial_x^2\tilde{\chi}|^2)dx \\
&\le \frac{1}{4}\|\omega\|^2 + C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2 + C\tilde{\delta}(1+t)^{-2},
\end{aligned}
$$

$$
J_6 \le C\tilde{\delta}\|\partial_x[\zeta,\psi]\|^2,
$$

$$
\begin{aligned}
J_7 \le{}& C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\zeta||\partial_x[\varphi,\psi]|dx + C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\varphi\zeta||\partial_x\tilde{u}|dx + C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\psi\zeta||\partial_x\tilde{\rho}|dx \\
&+ C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\varphi\zeta||\partial_x u^{r_2}|dx + C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\psi\zeta||\partial_x\rho^{r_2}|dx + C\int_{\mathbb{R}_+}|\partial_x^2\tilde{\chi}||\zeta\bar{f}|dx \\
\le{}& C(\tilde{\delta}+\tilde{\delta}^2)\|\partial_x[\varphi,\psi,\zeta]\|^2 + C(\tilde{\delta}^3+\tilde{\delta}^4)\|\partial_x\zeta\|^2 + C\|\zeta\|_\infty\|\bar{f}\|_{L^1} \\
\le{}& C\tilde{\delta}\|\partial_x[\varphi,\psi,\zeta]\|^2 + C(\tilde{\delta}^{\frac{2}{3}}+\epsilon^{\frac{1}{10}})\|\partial_x\zeta\|^2 + C\frac{\tilde{\delta}^{\frac{10}{9}}}{(1+\tilde{\delta}t)^{\frac{4}{3}}} + C\epsilon^{\frac{2}{15}}(1+t)^{-\frac{13}{12}},
\end{aligned}
$$

$$
J_8 \le C\tilde{\delta}\|\partial_x\zeta\|^2 + C\tilde{\delta}^3\|\partial_x\varphi\|^2,
$$

$$
\begin{aligned}
J_9 &\le \frac{1}{4}\|\omega\|^2 + C\int_{\mathbb{R}_+}(\partial_x\tilde{\chi})^2(\varphi^2+\psi^2)dx + C\int_{\mathbb{R}_+}(\partial_x\tilde{\chi})^2(u^{r_2}-u_*)^2dx \\
&\le \frac{1}{4}\|\omega\|^2 + C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2 + C\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}x^2|\partial_x\tilde{\chi}|^2dx \\
&\le \frac{1}{4}\|\omega\|^2 + C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2 + C\tilde{\delta}(1+t)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
J_{10} &\le C\|\zeta\|_\infty\|\sqrt{|\partial_x\tilde{\chi}|}\zeta\|^2 + C\|\tilde{\chi}-1\|_\infty\|\sqrt{|\partial_x\tilde{\chi}|}\zeta\|^2 \\
&\le C(\varepsilon_0+\tilde{\delta})\|\sqrt{|\partial_x\tilde{\chi}|}\zeta\|^2.
\end{aligned}
$$

Therefore, (4.1) follows by plugging the above estimations into (4.7), integrating the resulting inequality with respect to t and applying (4.4), (2.8) and Cauchy inequality, where we recall that ε_0, ϵ and $\tilde{\delta}$ can be small enough. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Assume the conditions in Proposition 3.1 hold, then we have the following energy estimate for $t \in [0, T]$,*

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$$
\begin{aligned}
&\|\partial_x\varphi\|^2+\int_0^t\|\partial_x\varphi\|^2d\tau\\
&\le C(\|\psi_0\|^2+\|[\varphi_0,\zeta_0]\|_{H^1}^2)+C(\tilde\delta+\varepsilon_0)\int_0^t\|\partial_x[\zeta,\partial_x\zeta]\|^2d\tau\\
&+(C\varepsilon_0+\eta)\int_0^t\|\partial_x^2\psi\|^2d\tau+C\tilde\delta^{\frac19}+C\epsilon^{\frac{1}{10}}.
\end{aligned}
\tag{4.8}
$$

Proof. We first differentiate $(3.8)_1$ with respect to x to obtain

$$
\begin{aligned}
&\partial_t\partial_x\varphi+\partial_xu\partial_x\varphi+u\partial_x^2\varphi+\partial_x\rho\partial_x\psi+\rho\partial_x^2\psi+\partial_x^2\bar u\varphi+\partial_x\bar u\partial_x\varphi+\partial_x\bar\rho\partial_x\psi\\
&+\partial_x^2\bar\rho\psi+\partial_x\bar f=0.
\end{aligned}
\tag{4.9}
$$

Then multiplying $(3.8)_2$ and (4.9) by $\frac{\partial_x\varphi}{\rho}$ and $\nu\frac{\partial_x\varphi}{\rho^2}$, respectively, and integrating the resulting equalities over $\mathbb{R}_+$, one has

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\partial_t\psi\partial_x\varphi dx+\int_{\mathbb{R}_+}u\partial_x\psi\partial_x\varphi dx+\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho}(\partial_x\varphi)^2dx\\
&=\int_{\mathbb{R}_+}\nu\partial_x^2\psi\frac{\partial_x\varphi}{\rho}dx-\int_{\mathbb{R}_+}\nu\frac{\partial_x^2\bar u}{\rho\bar\rho}\varphi\partial_x\varphi dx-\int_{\mathbb{R}_+}\partial_x\bar u\psi\partial_x\varphi dx+\int_{\mathbb{R}_+}\left[\frac{p'(\bar\rho)}{\bar\rho}-\frac{p'(\rho)}{\rho}\right]\partial_x\bar\rho\partial_x\varphi dx\\
&\quad-\int_{\mathbb{R}_+}\frac{\delta}{2\rho}\partial_x(\partial_x\zeta)^2\partial_x\varphi dx-\int_{\mathbb{R}_+}\delta\frac{\partial_x\varphi}{\rho}\partial_x(\partial_x\tilde\chi\partial_x\zeta)dx+\int_{\mathbb{R}_+}\frac{\delta}{2\rho\bar\rho}\varphi\partial_x\varphi\partial_x(\partial_x\tilde\chi)^2dx\\
&\quad-\int_{\mathbb{R}_+}\frac{\bar g}{\bar\rho}\partial_x\varphi dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu\int_{\mathbb{R}_+}\frac{\partial_x\varphi}{\rho^2}\partial_t\partial_x\varphi dx+\nu\int_{\mathbb{R}_+}\partial_xu\frac{(\partial_x\varphi)^2}{\rho^2}dx+\nu\int_{\mathbb{R}_+}u\frac{\partial_x\varphi\partial_x^2\varphi}{\rho^2}dx\\
&\quad+\nu\int_{\mathbb{R}_+}\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi dx+\nu\int_{\mathbb{R}_+}\partial_x\bar u\frac{(\partial_x\varphi)^2}{\rho^2}dx\\
&=-\nu\int_{\mathbb{R}_+}\partial_x^2\psi\frac{\partial_x\varphi}{\rho}dx-\nu\int_{\mathbb{R}_+}\partial_x^2\bar u\varphi\frac{\partial_x\varphi}{\rho^2}dx-\nu\int_{\mathbb{R}_+}\partial_x\bar\rho\partial_x\psi\frac{\partial_x\varphi}{\rho^2}dx\\
&\quad-\nu\int_{\mathbb{R}_+}\partial_x^2\bar\rho\psi\frac{\partial_x\varphi}{\rho^2}dx-\nu\int_{\mathbb{R}_+}\partial_x\bar f\frac{\partial_x\varphi}{\rho^2}dx.
\end{aligned}
$$

Using integration by parts, boundary condition (3.9), (1.1)$_1$ and the summation of the equalities above further implies

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+}\left(\psi\partial_x\varphi+\frac{\nu}{2\rho^2}(\partial_x\varphi)^2\right)dx+\int_{\mathbb{R}_+}\frac{p'(\rho)}{\rho}(\partial_x\varphi)^2dx\\
&=\nu\frac{u_b}{2\rho_b^2}(\partial_x\varphi)^2(0,t)-\int_{\mathbb{R}_+}\partial_x\psi\partial_t\varphi dx+\nu\int_{\mathbb{R}_+}\frac{\partial_x u}{2\rho^2}(\partial_x\varphi)^2dx-\int_{\mathbb{R}_+}u\partial_x\psi\partial_x\varphi dx\\
&\quad-\nu\int_{\mathbb{R}_+}\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi dx-\nu\int_{\mathbb{R}_+}\frac{\partial_x^2\bar{u}}{\rho}(\rho^{-1}+\bar{\rho}^{-1})\varphi\partial_x\varphi dx-\nu\int_{\mathbb{R}_+}\partial_x\bar{\rho}\partial_x\psi\frac{\partial_x\varphi}{\rho^2}dx\\
&\quad-\nu\int_{\mathbb{R}_+}\partial_x^2\bar{\rho}\psi\frac{\partial_x\varphi}{\rho^2}dx-\int_{\mathbb{R}_+}\partial_x\bar{u}\psi\partial_x\varphi dx+\int_{\mathbb{R}_+}\left[\frac{p'(\bar{\rho})}{\bar{\rho}}-\frac{p'(\rho)}{\rho}\right]\partial_x\bar{\rho}\partial_x\varphi dx\\
&\quad-\int_{\mathbb{R}_+}\frac{\delta}{2\rho}\partial_x(\partial_x\zeta)^2\partial_x\varphi dx-\int_{\mathbb{R}_+}\delta\frac{\partial_x\varphi}{\rho}\partial_x(\partial_x\tilde{\chi}\partial_x\zeta)dx-\nu\int_{\mathbb{R}_+}\frac{\partial_x\bar{u}}{\rho^2}(\partial_x\varphi)^2dx\\
&\quad+\int_{\mathbb{R}_+}\frac{\delta}{2\rho\bar{\rho}}\varphi\partial_x\varphi\partial_x(\partial_x\tilde{\chi})^2dx-\int_{\mathbb{R}_+}\frac{\bar{g}}{\bar{\rho}}\partial_x\varphi dx-\nu\int_{\mathbb{R}_+}\partial_x\bar{f}\frac{\partial_x\varphi}{\rho^2}dx\\
&=\sum_{l=11}^{26}J_l,
\end{aligned}\tag{4.10}
$$

where J_l $(11\le l\le 26)$ denote the corresponding terms on the left hand side of (4.10).

Before our estimates for J_l $(12\le l\le 26)$, we first deal with boundary term J_{11}. From (3.8)$_1$ together with boundary condition (1.4) and (3.9), we have

$$u_b\partial_x\varphi(0,t)+\rho_b\partial_x\psi(0,t)=-\bar{f}(0,t).$$

Then using the Sobolev inequality and Cauchy–Schwarz's inequality with $0<\eta<1$, and using (3.23) with $q=10$, we have

$$
\begin{aligned}
(\partial_x\varphi)^2(0,t)&\le C(\partial_x\psi)^2(0,t)+C\bar{f}^2(0,t)\\
&\le C\|\partial_x\psi\|_\infty^2+C\|\bar{f}\|_\infty^2\\
&\le C\|\partial_x\psi\|\|\partial_x^2\psi\|+C\|\bar{f}\|^2+C\|\partial_x\bar{f}\|^2\\
&\le\eta\|\partial_x^2\psi\|^2+C_\eta\|\partial_x\psi\|^2+C\tilde{\delta}(1+t)^{-\frac{9}{5}}.
\end{aligned}\tag{4.11}
$$

We now turn to estimate J_l $(12\le l\le 26)$ term by term. By applying Hölder inequality, Cauchy–Schwarz's inequality with $0<\eta<1$, (3.8)$_1$, (3.11), (3.13), (3.23), (2.7), Lemma 3.1, Lemma 2.3, Sobolev inequality (3.14) and integration by parts, it is direct to derive the following estimates:

$$
\begin{aligned}
J_{12} &= -\int_{\mathbb{R}_+} \partial_x\psi\partial_t\varphi dx \\
&= \int_{\mathbb{R}_+} \partial_x\psi(u\partial_x\varphi + \rho\partial_x\psi + \partial_x\bar{u}\varphi + \partial_x\bar{\rho}\psi + \bar{f})dx \\
&\le \eta\|\partial_x\varphi\|^2 + (C+C_\eta)\|\partial_x\psi\|^2 + \tilde{\delta}\|\partial_x\psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 \\
&\quad + C\|\partial_x u^{r_2}\|_\infty\|[\varphi,\psi]\|\|\partial_x\psi\| + C\|\bar{f}\|^2 \\
&\le \eta\|\partial_x\varphi\|^2 + (C+C_\eta)\|\partial_x\psi\|^2 + \tilde{\delta}\|\partial_x\psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 \\
&\quad + C\|\partial_x u^{r_2}\|_\infty^2\|[\varphi,\psi]\|^2 + C\tilde{\delta}(1+t)^{-\frac{9}{5}} \\
&\le \eta\|\partial_x\varphi\|^2 + (C+C_\eta)\|\partial_x\psi\|^2 + \tilde{\delta}\|\partial_x\psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 \\
&\quad + C\epsilon^{2\theta}(1+t)^{-2(1-\theta)} + C\tilde{\delta}(1+t)^{-\frac{9}{5}} \\
&\le \eta\|\partial_x\varphi\|^2 + (C+C_\eta)\|\partial_x\psi\|^2 + \tilde{\delta}\|\partial_x\psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 \\
&\quad + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}} + C\tilde{\delta}(1+t)^{-\frac{9}{5}},
\end{aligned}
$$

$$
\begin{aligned}
J_{13} + J_{23} &\le C(\|\partial_x\psi\|_\infty + \|\partial_x\bar{u}\|_\infty)\|\partial_x\varphi\|^2 \\
&\le C(\tilde{\delta}+\epsilon)\|\partial_x\varphi\|^2 + C(\|\partial_x\psi\| + \|\partial_x^2\psi\|)\|\partial_x\varphi\|^2 \\
&\le C(\tilde{\delta}+\epsilon+\varepsilon_0)\|\partial_x\varphi\|^2 + C\varepsilon_0\|\partial_x^2\psi\|^2,
\end{aligned}
$$

$$
J_{14} \le \eta\|\partial_x\varphi\|^2 + C_\eta\|\partial_x\psi\|^2,
$$

$$
\begin{aligned}
J_{15} + J_{17} &\le \nu\int_{\mathbb{R}} \left|\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi\right| dx + \nu\int_{\mathbb{R}} |\partial_x\bar{\rho}||\partial_x\psi|\frac{|\partial_x\varphi|}{\rho^2}dx \\
&\le C\|\partial_x\bar{\rho}\|_\infty\|\partial_x\varphi\|\|\partial_x\psi\| + C\|\partial_x\psi\|_\infty\|\partial_x\varphi\|^2 \\
&\le C(\tilde{\delta}+\epsilon)(\|\partial_x\varphi\|^2 + \|\partial_x\psi\|^2) + C(\|\partial_x\psi\| + \|\partial_x^2\psi\|)\|\partial_x\varphi\|^2 \\
&\le C(\tilde{\delta}+\epsilon+\varepsilon_0)(\|\partial_x\varphi\|^2 + \|\partial_x\psi\|^2) + C\varepsilon_0\|\partial_x^2\psi\|^2,
\end{aligned}
$$

$$
\begin{aligned}
J_{16} + J_{18} &\le \tilde{\delta}\|\partial_x\varphi\|^2 + C\tilde{\delta}^3\|\partial_x[\varphi,\psi]\|^2 + C\|\partial_x^2 u^{r_2}\|_\infty\|[\varphi,\psi]\|\|\partial_x\varphi\| \\
&\le C(\tilde{\delta}+\varepsilon_0)\|\partial_x\varphi\|^2 + C\tilde{\delta}^3\|\partial_x[\varphi,\psi]\|^2 + C\epsilon^{2+\frac{2}{q}}(1+t)^{-2+\frac{2}{q}} \\
&\le C(\tilde{\delta}+\varepsilon_0)\|\partial_x\varphi\|^2 + C\tilde{\delta}^3\|\partial_x[\varphi,\psi]\|^2 + C\epsilon^{\frac{11}{5}}(1+t)^{-\frac{9}{5}},
\end{aligned}
$$

$$
\begin{aligned}
J_{19} + J_{20} &\le \tilde{\delta}\|\partial_x\varphi\|^2 + C\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 + C\|\partial_x u^{r_2}\|_\infty\|[\varphi,\psi]\|\|\partial_x\varphi\| \\
&\le C(\tilde{\delta}+\varepsilon_0)\|\partial_x\varphi\|^2 + C\tilde{\delta}\|\partial_x\psi\|^2 + C\epsilon^{2\theta}(1+t)^{-2(1-\theta)} \\
&\le C(\tilde{\delta}+\varepsilon_0)\|\partial_x\varphi\|^2 + C\tilde{\delta}\|\partial_x\psi\|^2 + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}},
\end{aligned}
$$

$$
\begin{aligned}
J_{21} &\le C\|\partial_x\zeta\|_\infty(\|\partial_x\varphi\|^2 + \|\partial_x^2\zeta\|^2) \\
&\le C(\|\partial_x\zeta\| + \|\partial_x^2\zeta\|)(\|\partial_x\varphi\|^2 + \|\partial_x^2\zeta\|^2) \\
&\le C\varepsilon_0\|\partial_x[\varphi,\partial_x\zeta]\|^2,
\end{aligned}
$$

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$$J_{22} \leq C\tilde{\delta}\|\partial_x[\varphi,\zeta,\partial_x\zeta]\|^2,$$

$$J_{24} \leq \tilde{\delta}\|\partial_x\varphi\|^2 + C\tilde{\delta}\|\partial_x\varphi\|^2,$$

$$\begin{aligned} J_{25} &\leq \eta\|\partial_x\varphi\|^2 + C_\eta\|\bar{g}\|^2 \\ &\leq \eta\|\partial_x\varphi\|^2 + C_\eta\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})} + C_\eta\tilde{\delta}(1+t)^{-2} \\ &\leq \eta\|\partial_x\varphi\|^2 + C_\eta\epsilon^{\frac{6}{5}}(1+t)^{-\frac{9}{5}} + C_\eta\tilde{\delta}(1+t)^{-2}, \\ J_{26} &\leq \eta\|\partial_x\varphi\|^2 + C_\eta\|\partial_x\bar{f}\|^2 \leq \eta\|\partial_x\varphi\|^2 + C_\eta\tilde{\delta}(1+t)^{-\frac{9}{5}}. \end{aligned}$$

Inserting the above estimations for J_l $(11 \leq l \leq 26)$ into (4.10) and then choosing ε_0, ϵ, $\tilde{\delta}$ and η so small, and integrating (4.10) over $[0, T]$ and using (4.1), Cauchy–Schwarz's inequality with $0 < \eta < 1$, we get (4.8). This completes the proof of Lemma 4.2. □

Lemma 4.3. *Assume the conditions in Proposition 3.1 hold, then we have the following energy estimate for $t \in [0, T]$,*

$$\begin{aligned} \|\partial_x\psi\|^2 + \int_0^t \|\partial_x^2\psi\|^2 d\tau \leq C\|[\varphi_0,\psi_0,\zeta_0]\|_{H^1}^2 + C(\tilde{\delta}+\varepsilon_0)\int_0^t \|\partial_x[\zeta,\partial_x\zeta]\|^2 d\tau \\ + C\tilde{\delta}^{\frac{1}{9}} + C\epsilon^{\frac{1}{10}}. \end{aligned} \tag{4.12}$$

Proof. Multiplying $(3.8)_2$ by $-\frac{\partial_x^2\psi}{\rho}$ and then integrating the resulting equation over $\mathbb{R}_+$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+}(\partial_x\psi)^2 dx + \int_{\mathbb{R}_+}\frac{\nu}{\rho}(\partial_x^2\psi)^2 dx \\ &= \int_{\mathbb{R}_+} u\partial_x\psi\partial_x^2\psi dx + \int_{\mathbb{R}_+}\frac{\partial_x^2\psi}{\rho}\partial_x\varphi p'(\rho)dx + \int_{\mathbb{R}_+}\nu\frac{\partial_x^2\bar{u}}{\rho\bar{\rho}}\varphi\partial_x^2\psi dx + \int_{\mathbb{R}_+}\psi\partial_x^2\psi\partial_x\bar{u}dx \\ &\quad + \int_{\mathbb{R}_+}\left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}}\right)\partial_x\bar{\rho}\partial_x^2\psi dx + \int_{\mathbb{R}_+}\frac{\delta}{2\rho}\partial_x^2\psi\partial_x(\partial_x\zeta)^2 dx + \int_{\mathbb{R}_+}\frac{\delta}{\rho}\partial_x^2\psi\partial_x(\partial_x\tilde{\chi}\partial_x\zeta)dx \\ &\quad - \int_{\mathbb{R}_+}\frac{\delta}{2\rho\bar{\rho}}\varphi\partial_x^2\psi\partial_x(\partial_x\tilde{\chi})^2 dx + \int_{\mathbb{R}_+}\frac{\bar{g}}{\rho}\partial_x^2\psi dx \\ &= \sum_{l=27}^{35} J_l, \end{aligned} \tag{4.13}$$

where we have used boundary condition (3.9) and J_l $(27 \leq l \leq 35)$ denote the corresponding terms on the left hand side of (4.13).

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We now turn to estimate J_l $(27 \le l \le 35)$ term by term. By applying Cauchy–Schwarz's inequality with $0 < \eta < 1$, Sobolev inequality (3.14), (3.22), Lemma 2.3, (2.7), (3.11) and (3.13), it is direct to derive the following estimates:

$$
\begin{aligned}
J_{27} + J_{28} &\le C\|\partial_x \psi\|\|\partial_x^2 \psi\| + C\|\partial_x \varphi\|\|\partial_x^2 \psi\| \\
&\le \eta\|\partial_x^2 \psi\|^2 + C_\eta \|[\partial_x[\varphi, \psi]\|^2, \\
J_{29} &\le \tilde{\delta}\|\partial_x^2 \psi\|^2 + C\tilde{\delta}^3\|\partial_x \varphi\|^2 + C\|\partial_x^2 u^{r_2}\|_\infty \|\varphi\|\|\partial_x^2 \psi\| \\
&\le C(\tilde{\delta} + \varepsilon_0)\|\partial_x^2 \psi\|^2 + C\tilde{\delta}^3\|\partial_x \varphi\|^2 + C\epsilon^{2+\frac{2}{q}}(1+t)^{-2+\frac{2}{q}} \\
&\le C(\tilde{\delta} + \varepsilon_0)\|\partial_x^2 \psi\|^2 + C\tilde{\delta}^3\|\partial_x \varphi\|^2 + C\epsilon^{\frac{11}{5}}(1+t)^{-\frac{9}{5}}, \\
J_{30} + J_{31} &\le \tilde{\delta}\|\partial_x^2 \psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi, \psi]\|^2 + C\|\partial_x u^{r_2}\|_\infty \|[\varphi, \psi]\|\|\partial_x^2 \psi\| \\
&\le C(\tilde{\delta} + \varepsilon_0)\|\partial_x^2 \psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi, \psi]\|^2 + C\epsilon^{2\theta}(1+t)^{-2(1-\theta)} \\
&\le C(\tilde{\delta} + \varepsilon_0)\|\partial_x^2 \psi\|^2 + C\tilde{\delta}\|\partial_x[\varphi, \psi]\|^2 + C\epsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}}, \\
J_{32} &\le C\|\partial_x \zeta\|_\infty \|\partial_x^2 \zeta\|\|\partial_x^2 \psi\| \le C\varepsilon_0(\|\partial_x^2 \zeta\|^2 + \|\partial_x^2 \psi\|^2), \\
J_{33} &\le C\tilde{\delta}\|\partial_x[\zeta, \partial_x \psi, \partial_x \zeta]\|^2, \\
J_{34} &\le C\tilde{\delta}\|\partial_x[\varphi, \partial_x \psi]\|^2, \\
J_{35} &\le \eta\|\partial_x^2 \psi\|^2 + C_\eta \|\bar{g}\|^2 \\
&\le \eta\|\partial_x^2 \psi\|^2 + C_\eta \epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})} + C_\eta \tilde{\delta}(1+t)^{-2} \\
&\le \eta\|\partial_x^2 \psi\|^2 + C_\eta \epsilon^{\frac{6}{5}}(1+t)^{-\frac{9}{5}} + C_\eta \tilde{\delta}(1+t)^{-2}.
\end{aligned}
$$

Inserting the above estimations for J_l $(27 \le l \le 35)$ into (4.13) and then choosing ε_0, ϵ, $\tilde{\delta}$ and η so small, and integrating (4.13) over $[0, T]$, using (4.1) and (4.8), we get (4.12). □

Lemma 4.4. *Assume the conditions in Proposition 3.1 hold, then we have the following energy estimate for $t \in [0, T]$,*

$$
\|\partial_x \zeta\|^2 + \int_0^t (\|\partial_x \zeta\|^2 + \|\partial_x^2 \zeta\|^2) d\tau \le C\|[\varphi_0, \psi_0, \zeta_0]\|_{H^1}^2 + C\tilde{\delta}^{\frac{1}{9}} + C\epsilon^{\frac{1}{10}}. \tag{4.14}
$$

Proof. Let us rewrite $(3.8)_3$ and $(3.8)_4$ as follows:

$$
\begin{aligned}
\rho^2 \partial_t \zeta + \rho^2 u \partial_x \zeta =& -\rho \partial_x \tilde{\chi}[\rho\psi + \tilde{u}\varphi + \tilde{u}(\rho^{r_2} - \rho_*) + \rho(u^{r_2} - u_*)] - \frac{\delta}{\tilde{\rho}}\partial_x^2 \tilde{\chi}\varphi \\
&+ \delta\partial_x^2 \zeta - \frac{1}{\delta}\left[\rho\zeta^3 + 3\rho\tilde{\chi}\zeta^2 + (3\tilde{\chi}^2 - 1)\rho\zeta\right] - \frac{\delta}{\tilde{\rho}}\partial_x^2 \tilde{\chi}(\rho^{r_2} - \rho_*).
\end{aligned} \tag{4.15}
$$

Then, multiplying (4.15) by $-\partial_x^2 \zeta$, noticing $u_b > 0$, using boundary condition (3.9), and integrating the resulting equation over $\mathbb{R}_+$, we arrive at

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$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}_+}\rho^2(\partial_x\zeta)^2dx+\frac{\rho_b^2u_b}{2}(\partial_x\zeta)^2(0,t)+\int\limits_{\mathbb{R}_+}\frac{2\rho}{\delta}(\partial_x\zeta)^2dx+\int\limits_{\mathbb{R}_+}\delta(\partial_x^2\zeta)^2dx$$
$$=-2\int\limits_{\mathbb{R}_+}\rho\partial_x\rho\partial_x\zeta\partial_t\zeta dx-2\int\limits_{\mathbb{R}_+}\rho u\partial_x\rho(\partial_x\zeta)^2dx-\frac{3}{2}\int\limits_{\mathbb{R}_+}\rho^2\partial_x u(\partial_x\zeta)^2dx$$
$$+\int\limits_{\mathbb{R}_+}\rho[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\partial_x\tilde{\chi}\partial_x^2\zeta dx+\int\limits_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\varphi\partial_x^2\zeta dx$$
$$-\frac{1}{\delta}\int\limits_{\mathbb{R}_+}\partial_x\rho\zeta^3\partial_x\zeta dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\rho\zeta^2(\partial_x\zeta)^2dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\partial_x\tilde{\chi}\rho\zeta^2\partial_x\zeta dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\tilde{\chi}\partial_x\rho\zeta^2\partial_x\zeta dx$$
$$-\frac{6}{\delta}\int\limits_{\mathbb{R}_+}\tilde{\chi}\rho\zeta(\partial_x\zeta)^2dx-\frac{6}{\delta}\int\limits_{\mathbb{R}_+}\tilde{\chi}\partial_x\tilde{\chi}\rho\zeta\partial_x\zeta dx-\frac{1}{\delta}\int\limits_{\mathbb{R}_+}(3\tilde{\chi}^2-1)\partial_x\rho\zeta\partial_x\zeta dx$$
$$-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\rho(\tilde{\chi}^2-1)(\partial_x\zeta)^2dx+\int\limits_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}(\rho^{r_2}-\rho_*)\partial_x^2\zeta dx$$
$$=\sum_{l=36}^{49}J_l. \tag{4.16}$$

We now turn to estimate J_l $(36\le l\le 49)$ term by term. By applying Cauchy–Schwarz's inequality with $0<\eta<1$, Sobolev inequality (3.14), Lemma 2.1, Lemma 3.1, (3.11), (3.13), $(2.5)_4$ and $(3.8)_3$, it is direct to derive the following estimates:

$$J_{36}=2\int\limits_{\mathbb{R}_+}\partial_x\rho\partial_x\zeta\{\omega+\rho u\partial_x\zeta+\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\}dx$$
$$=2\int\limits_{\mathbb{R}_+}\partial_x\bar{\rho}\partial_x\zeta\{\omega+\rho u\partial_x\zeta+\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\}dx$$
$$+2\int\limits_{\mathbb{R}_+}\partial_x\varphi\partial_x\zeta\{\omega+\rho u\partial_x\zeta+\partial_x\tilde{\chi}[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\}dx$$
$$\le C(\tilde{\delta}+\epsilon)(\|\omega\|^2+\|\partial_x[\zeta,\varphi]\|^2)+C\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2+C\|\partial_x\zeta\|_\infty\|\partial_x\varphi\|(\|\omega\|+\|\partial_x\zeta\|)$$
$$+C\int\limits_{\mathbb{R}_+}(\partial_x\tilde{\chi})^2(u^{r_2}-u_*)^2dx$$
$$\le C(\tilde{\delta}+\epsilon)(\|\omega\|^2+\|\partial_x[\zeta,\varphi,\psi]\|^2)+C\varepsilon_0(\|\partial_x\varphi\|^2+\|\omega\|^2+\|\partial_x\zeta\|^2)$$
$$+C\|\partial_x u^{r_2}\|_\infty^2\int\limits_{\mathbb{R}_+}x^2|\partial_x\tilde{\chi}|^2dx$$
$$\le C(\tilde{\delta}+\epsilon+\varepsilon_0)(\|\partial_x[\zeta,\varphi,\psi]\|^2+\|\omega\|^2)+C\tilde{\delta}(1+t)^{-2},$$

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$$\begin{aligned}
J_{37} &\le C\|\partial_x \bar\rho\|_\infty \|\partial_x\zeta\|^2 + C\|\partial_x\zeta\|_\infty\|\partial_x\varphi\|\|\partial_x\zeta\| \\
&\le C(\tilde\delta+\epsilon)\|\partial_x\zeta\|^2 + C\varepsilon_0(\|\partial_x\varphi\|^2+\|\partial_x\zeta\|^2), \\
J_{38} &\le C\|\partial_x \bar u\|_\infty \|\partial_x\zeta\|^2 + C\|\partial_x\zeta\|_\infty\|\partial_x\psi\|\|\partial_x\zeta\| \\
&\le C(\tilde\delta+\epsilon)\|\partial_x\zeta\|^2 + C\varepsilon_0(\|\partial_x\psi\|^2+\|\partial_x\zeta\|^2),
\end{aligned}$$

$$\begin{aligned}
J_{39}+J_{49} &\le (\tilde\delta+\eta)\|\partial_x^2\zeta\|^2 + C\tilde\delta\|\partial_x[\varphi,\psi]\|^2 + C_\eta\int_{\mathbb{R}_+}[(\partial_x\tilde\chi)^2+(\partial_x^2\tilde\chi)^2](u^{r_2}-u_*)^2dx \\
&\le (\tilde\delta+\eta)\|\partial_x^2\zeta\|^2 + C\tilde\delta\|\partial_x[\varphi,\psi]\|^2 + C_\eta\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}x^2[(\partial_x\tilde\chi)^2+(\partial_x^2\tilde\chi)^2]dx \\
&\le (\tilde\delta+\eta)\|\partial_x^2\zeta\|^2 + C\tilde\delta\|\partial_x[\varphi,\psi]\|^2 + C_\eta\tilde\delta(1+t)^{-2},
\end{aligned}$$

$$J_{40} \le \tilde\delta\|\partial_x^2\zeta\|^2 + C\tilde\delta\|\partial_x\varphi\|^2,$$

$$\begin{aligned}
&J_{41}+J_{43}+J_{44}+J_{46}+J_{47} \\
&\le C\varepsilon_0\|\partial_x[\varphi,\zeta]\|^2 + \tilde\delta\|\partial_x\zeta\|^2 + C\tilde\delta\|\partial_x[\zeta,\varphi]\|^2 + C\|\partial_x\rho^{r_2}\|_\infty\|\zeta\|\|\partial_x\zeta\| \\
&\le C(\varepsilon_0+\tilde\delta)\|\partial_x[\varphi,\zeta]\|^2 + \tilde\delta\|\partial_x\zeta\|^2 + C\epsilon^{2\theta}(1+t)^{-2(1-\theta)} \\
&\le C(\varepsilon_0+\tilde\delta)\|\partial_x[\varphi,\zeta]\|^2 + \tilde\delta\|\partial_x\zeta\|^2 + C\epsilon^{\frac14}(1+t)^{-\frac74},
\end{aligned}$$

$$J_{42}+J_{45}+J_{48} \le C(\varepsilon_0+\tilde\delta)\|\partial_x\zeta\|^2.$$

Inserting the above estimations for J_l $(36 \le l \le 49)$ into (4.16) and then choosing ε_0, ϵ, $\tilde\delta$ and η so small, and integrating (4.16) over $[0,T]$, using (4.1), (4.8) and (4.12), we get (4.14). □

Lemma 4.5. *Assume the conditions in Proposition 3.1 hold, then we have the following energy estimate for $t \in [0,T]$,*

$$\|\partial_t\zeta\|^2 + \int_0^t(\|\partial_t\zeta\|^2+\|\partial_{xt}\zeta\|^2)d\tau \le C\|[\varphi_0,\psi_0]\|_{H^1}^2 + C\|\zeta_0\|_{H^2}^2 + C\tilde\delta^{\frac19} + C\epsilon^{\frac{1}{10}}. \tag{4.17}$$

Proof. Differentiating (4.15) with respect to t, we deduce that

$$\begin{aligned}
&\rho^2\partial_t^2\zeta + \partial_t(\rho^2)\partial_t\zeta + \partial_t(\rho^2 u)\partial_x\zeta + \rho^2 u\partial_{xt}\zeta \\
&\quad = -\partial_t\rho\partial_x\tilde\chi[\rho\psi+\tilde u\varphi+\tilde u(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)] + \delta\partial_{xxt}\zeta \\
&\qquad - \rho\partial_x\tilde\chi\partial_t[\rho\psi+\tilde u\varphi+\tilde u(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)] - \frac{\delta}{\tilde\rho}\partial_x^2\tilde\chi\partial_t\varphi \\
&\qquad - \frac{1}{\delta}\partial_t\left[\rho\zeta^3+3\rho\tilde\chi\zeta^2+(3\tilde\chi^2-1)\rho\zeta\right] - \frac{\delta}{\tilde\rho}\partial_x^2\tilde\chi\partial_t\rho^{r_2}.
\end{aligned} \tag{4.18}$$

Then, multiplying (4.18) by $\partial_t\zeta$ and using boundary condition (3.9), and integrating the resulting equation over $\mathbb{R}_+$ and using integrating by parts, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}_+}\rho^2(\partial_t\zeta)^2dx+\int\limits_{\mathbb{R}_+}\frac{2\rho}{\delta}(\partial_t\zeta)^2dx+\int\limits_{\mathbb{R}_+}\delta(\partial_{xt}\zeta)^2dx\\
&\quad=-\int\limits_{\mathbb{R}_+}\rho\partial_t\rho(\partial_t\zeta)^2dx+\frac{1}{2}\int\limits_{\mathbb{R}_+}\partial_x(\rho^2u)(\partial_t\zeta)^2dx-\int\limits_{\mathbb{R}_+}\partial_t(\rho^2u)\partial_x\zeta\partial_t\zeta dx\\
&\qquad-\int\limits_{\mathbb{R}_+}\partial_t\rho[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\partial_x\tilde{\chi}\partial_t\zeta dx\\
&\qquad-\int\limits_{\mathbb{R}_+}\rho\partial_t[\rho\psi+\tilde{u}\varphi+\tilde{u}(\rho^{r_2}-\rho_*)+\rho(u^{r_2}-u_*)]\partial_x\tilde{\chi}\partial_t\zeta dx-\int\limits_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\partial_t\varphi\partial_t\zeta dx\\
&\qquad-\frac{1}{\delta}\int\limits_{\mathbb{R}_+}\partial_t\rho\zeta^3\partial_t\zeta dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\rho\zeta^2(\partial_t\zeta)^2dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}\tilde{\chi}\partial_t\rho\zeta^2\partial_t\zeta dx-\frac{6}{\delta}\int\limits_{\mathbb{R}_+}\tilde{\chi}\rho\zeta(\partial_t\zeta)^2dx\\
&\qquad-\frac{1}{\delta}\int\limits_{\mathbb{R}_+}(3\tilde{\chi}^2-1)\partial_t\rho\zeta\partial_t\zeta dx-\frac{3}{\delta}\int\limits_{\mathbb{R}_+}(\tilde{\chi}^2-1)\rho(\partial_t\zeta)^2dx-\int\limits_{\mathbb{R}_+}\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\partial_t\rho^{r_2}\partial_t\zeta dx\\
&\quad=\sum_{l=50}^{62}J_l.
\end{aligned}
\tag{4.19}
$$

We now turn to estimate J_l $(50\le l\le 62)$ term by term. By applying Hölder inequality, Cauchy–Schwarz's inequality with $0<\eta<1$, Sobolev inequality (3.14), Lemma 2.1, Lemma 3.1, (3.11), (3.13), (3.25), (3.23), (4.15), $(1.1)_1$, $(3.8)_1$, $(3.8)_2$ and (3.3), it is direct to derive the following estimates:

$$
\begin{aligned}
J_{50}+J_{51}&=\frac{1}{2}\int\limits_{\mathbb{R}_+}\partial_x\rho\rho u(\partial_t\zeta)^2dx+\frac{3}{2}\int\limits_{\mathbb{R}_+}\rho\partial_x(\rho u)(\partial_t\zeta)^2dx\\
&=\frac{1}{2}\int\limits_{\mathbb{R}_+}\partial_x\bar{\rho}\rho u(\partial_t\zeta)^2dx+\frac{1}{2}\int\limits_{\mathbb{R}_+}\partial_x\varphi\rho u(\partial_t\zeta)^2dx\\
&\quad+\frac{3}{2}\int\limits_{\mathbb{R}_+}\rho(\partial_x\bar{\rho}u+u\partial_x\varphi+\rho\partial_x\psi+\rho\partial_x\bar{u})(\partial_t\zeta)^2dx\\
&\le C\|\partial_x[\bar{\rho},\bar{u}]\|_\infty\|\partial_t\zeta\|^2+C\|\partial_t\zeta\|_\infty\|\partial_x[\varphi,\psi]\|\|\partial_t\zeta\|\\
&\le C(\tilde{\delta}+\epsilon)\|\partial_t\zeta\|^2+C(\|\partial_t\zeta\|+\|\partial_{xt}\zeta\|)\|\partial_x[\varphi,\psi]\|\|\partial_t\zeta\|\\
&\le C(\tilde{\delta}+\epsilon)\|\partial_t\zeta\|^2+C\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_{xt}\zeta\|^2),
\end{aligned}
$$

$$
\begin{aligned}
J_{52}&\le C\|\partial_x\zeta\|_\infty\|\partial_t\zeta\|\|\partial_x[\varphi,\psi]\|+C\|\partial_x[\bar{\rho},\bar{u}]\|_\infty\|\partial_t\zeta\|\|\partial_x\zeta\|+C\|\partial_x\zeta\|_\infty\|\partial_t\zeta\|\|\partial_x^2\psi\|\\
&\quad+C\|[\varphi,\psi]\|_\infty\|\partial_x[\bar{\rho},\bar{u}]\|_\infty\|\partial_t\zeta\|\|\partial_x\zeta\|+C\|\partial_x\zeta\|_\infty^2\|\partial_t\zeta\|\|\partial_x^2\zeta\|\\
&\quad+C\|\partial_x\tilde{\chi}\|_\infty\|\partial_x\zeta\|_\infty\|\partial_t\zeta\|\|\partial_x^2\zeta\|+C\|\partial_x^2\tilde{\chi}\|_\infty\|\partial_x\zeta\|_\infty\|\partial_t\zeta\|\|\partial_x\zeta\|\\
&\quad+C\|\partial_x^2\bar{u}\|_\infty\|\partial_t\zeta\|\|\partial_x\zeta\|+C\|\partial_x\tilde{\chi}\|_\infty\|\partial_x^2\tilde{\chi}\|_\infty\|\varphi\|_\infty\|\partial_t\zeta\|\|\partial_x\zeta\|
\end{aligned}
$$

$$
\begin{aligned}
&\le C(\|\partial_x\zeta\|+\|\partial_x^2\zeta\|)\|\partial_t\zeta\|\|\partial_x[\varphi,\psi]\|+C(\tilde\delta+\epsilon)(\|\partial_t\zeta\|^2+\|\partial_x\zeta\|^2)\\
&\quad+C(\tilde\delta+\epsilon)\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_x\zeta\|^2)+C(\varepsilon_0^2+\varepsilon_0)(\|\partial_t\zeta\|^2+\|\partial_x^2\zeta\|^2+\|\partial_x^2\psi\|^2)\\
&\quad+C\tilde\delta\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_x^2\zeta\|^2)\\
&\le C(\tilde\delta+\epsilon+\varepsilon_0)(\|\partial_t\zeta\|^2+\|\partial_x\zeta\|^2+\|\partial_x^2\zeta\|^2+\|\partial_x^2\psi\|^2),
\end{aligned}
$$

$$
\begin{aligned}
J_{53}&\le C\|[\varphi,\psi]\|_\infty\|\partial_x\tilde\chi\|_\infty\|\partial_t\zeta\|\|\partial_x[\varphi,\psi]\|+C\|\partial_x\tilde\chi\|_\infty\|\partial_t\zeta\|\|\partial_x[\varphi,\psi]\|\\
&\quad+C\|\partial_x\tilde\chi\|_\infty^2\|\partial_t\zeta\|^2+C\tilde\delta^2\|\partial_x\varphi\|^2+C\|\partial_x u^{r_2}\|_\infty^2\|[\varphi,\psi]\|^2+C\|\bar f\|^2\\
&\quad+\eta\|\partial_t\zeta\|^2+C_\eta\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}(\partial_x\tilde\chi)^2(u^{r_2}-u_*)^2dx\\
&\le C(\varepsilon_0\tilde\delta+\tilde\delta)(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+(C\tilde\delta^2+\eta)\|\partial_t\zeta\|^2+C\tilde\delta^2\|\partial_x\varphi\|^2\\
&\quad+C_\eta\|\partial_x u^{r_2}\|_\infty^4\int_{\mathbb{R}_+}x^2(\partial_x\tilde\chi)^2dx+C\epsilon^{2\theta}(1+t)^{-2(1-\theta)}+C\tilde\delta(1+t)^{-\frac95}\\
&\le C(\varepsilon_0\tilde\delta+\tilde\delta)(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+(C\tilde\delta^2+\eta)\|\partial_t\zeta\|^2\\
&\quad+C\tilde\delta^2\|\partial_x\varphi\|^2+C_\eta\tilde\delta(1+t)^{-4}+C\epsilon^{\frac14}(1+t)^{-\frac74}+C\tilde\delta(1+t)^{-\frac95},
\end{aligned}
$$

$$
\begin{aligned}
J_{54}&\le C\|\partial_x\tilde\chi\|_\infty\|\partial_t\zeta\|\|\partial_x[\varphi,\psi]\|+C\|\partial_x\tilde\chi\|_\infty\|\partial_x\zeta\|_\infty\|\partial_t\zeta\|\|\partial_x^2\zeta\|+C\|\bar g\|^2+C\|\bar f\|^2\\
&\quad+(C\tilde\delta+\eta+C\epsilon)\|\partial_t\zeta\|^2+C\tilde\delta\|\partial_x[\zeta,\partial_x\zeta,\partial_x\psi]\|^2+C\tilde\delta^2\|\partial_x[\varphi,\psi]\|^2\\
&\le C\tilde\delta(\|\partial_t\zeta\|^2+\|\partial_x^2\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+(C\tilde\delta+\eta+C\epsilon)\|\partial_t\zeta\|^2+C\tilde\delta\|\partial_x[\zeta,\partial_x\zeta,\partial_x\psi]\|^2\\
&\quad+C\epsilon^{1+\frac2q}(1+t)^{-2(1-\frac1q)}+C\tilde\delta(1+t)^{-2}+C\tilde\delta\epsilon^{2+\frac2q}(1+t)^{-2+\frac2q}\\
&\le C\tilde\delta(\|\partial_t\zeta\|^2+\|\partial_x^2\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+(C\tilde\delta+\eta+C\epsilon)\|\partial_t\zeta\|^2+C\tilde\delta\|\partial_x[\zeta,\partial_x\zeta,\partial_x\psi]\|^2\\
&\quad+C\epsilon^{1+\frac2q}(1+t)^{-\frac95}+C\tilde\delta(1+t)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
J_{55}&\le C\int_{\mathbb{R}_+}|\partial_x^2\tilde\chi|(|\partial_x\bar\rho\psi|+|\partial_x\varphi|+|\partial_x\psi|+|\varphi||\partial_x\bar u|+|\bar f|)|\partial_t\zeta|dx\\
&\le C(\tilde\delta+\epsilon\tilde\delta)(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+C\|\bar f\|^2\\
&\le C(\tilde\delta+\epsilon\tilde\delta)(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+C\tilde\delta(1+t)^{-\frac95},
\end{aligned}
$$

$$
\begin{aligned}
&J_{56}+J_{58}+J_{60}\\
&\le C\int_{\mathbb{R}_+}(|\zeta|+|\zeta|^2+|\zeta|^3)(|\partial_x\bar\rho\psi|+|\partial_x\varphi|+|\partial_x\psi|+|\varphi||\partial_x\bar u|+|\bar f|+|\partial_x u^{r_2}|)|\partial_t\zeta|dx\\
&\le C\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+C\varepsilon_0\tilde\delta^2\|\partial_x[\varphi,\psi]\|^2+C\|\bar f\|^2+\eta\|\partial_t\zeta\|^2\\
&\quad+C_\eta\|\partial_x u^{r_2}\|_\infty^2\|\zeta\|^2\\
&\le C\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+C\varepsilon_0\tilde\delta^2\|\partial_x[\varphi,\psi]\|^2+C\tilde\delta(1+t)^{-\frac95}+\eta\|\partial_t\zeta\|^2\\
&\quad+C_\eta\epsilon^{2\theta}(1+t)^{-2(1-\theta)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\varepsilon_0(\|\partial_t\zeta\|^2+\|\partial_x[\varphi,\psi]\|^2)+C\varepsilon_0\tilde{\delta}^2\|\partial_x[\varphi,\psi]\|^2+C\tilde{\delta}(1+t)^{-\frac{9}{5}}\\
&\quad+\eta\|\partial_t\zeta\|^2+C_\eta\epsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}},
\end{aligned}
$$

$$
J_{57}+J_{59}+J_{61}\le C(\varepsilon_0+\tilde{\delta})\|\partial_t\zeta\|^2,
$$

$$
J_{62}\le\eta\|\partial_t\zeta\|^2+C_\eta\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}(\partial_x^2\tilde{\chi})^2dx\le\eta\|\partial_t\zeta\|^2+C_\eta\tilde{\delta}(1+t)^{-2}.
$$

From (4.15), we notice that $\partial_t\zeta(x,0)$ satisfies

$$
\begin{aligned}
&\rho_0^2\partial_t\zeta(x,0)+\rho_0^2u_0\partial_x\zeta(x,0)\\
&\quad=-\rho_0\partial_x\tilde{\chi}[\rho_0\psi_0+\tilde{u}\varphi_0+\tilde{u}(\rho^{r_2}(x,0)-\rho_*)+\rho_0(u^{r_2}(x,0)-u_*)]-\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}\varphi_0\\
&\qquad+\delta\partial_x^2\zeta(x,0)-\frac{1}{\delta}\left[\rho_0\zeta_0^3+3\rho_0\tilde{\chi}\zeta_0^2+(3\tilde{\chi}^2-1)\rho_0\zeta_0\right]-\frac{\delta}{\tilde{\rho}}\partial_x^2\tilde{\chi}(\rho^{r_2}(x,0)-\rho_*).
\end{aligned}
\tag{4.20}
$$

Inserting the above estimations for J_l $(50\le l\le 62)$ into (4.19) and then choosing ε_0, $\tilde{\delta}$, ϵ and η so small, and integrating (4.19) over $[0,T]$, using (4.1), (4.8), (4.12), (4.14), (4.20) and (2.7), we get (4.17). □

Proof of Proposition 3.1. Combinations of the estimates (4.1), (4.8), (4.12), (4.14), (4.17) and taking ε_0, $\tilde{\delta}$, ϵ and η sufficiently small, we can obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|[\varphi,\psi,\zeta](t)\|_{H^1}^2+\|\partial_t\zeta\|^2\right)+\int_0^T(\|\partial_x\varphi\|^2+\|\partial_x[\psi,\zeta]\|_{H^1}^2+\|\partial_t\zeta\|_{H^1}^2)dt\\
&\quad\le C\|[\varphi_0,\psi_0,]\|_{H^1}^2+C\|\zeta_0\|_{H^2}^2+C\tilde{\delta}^{\frac{1}{9}}+C\epsilon^{\frac{1}{10}}.
\end{aligned}
\tag{4.21}
$$

From (4.15) and using Lemma 3.1, Lemma 2.3, we have

$$
\begin{aligned}
\|\partial_x^2\zeta\|^2&\le C(\|\partial_t\zeta\|^2+\|\zeta\|_{H^1}^2+\|[\varphi,\psi]\|^2)+C\int_{\mathbb{R}_+}[(\partial_x\tilde{\chi})^2+(\partial_x^2\tilde{\chi})^2](u^{r_2}-u_*)^2dx\\
&\le C(\|\partial_t\zeta\|^2+\|\zeta\|_{H^1}^2+\|[\varphi,\psi]\|^2)+C\|\partial_x u^{r_2}\|_\infty^2\int_{\mathbb{R}_+}x^2[(\partial_x\tilde{\chi})^2+(\partial_x^2\tilde{\chi})^2]dx\\
&\le C(\|\partial_t\zeta\|^2+\|\zeta\|_{H^1}^2+\|[\varphi,\psi]\|^2)+C\tilde{\delta}(1+t)^{-2}.
\end{aligned}
\tag{4.22}
$$

Then from (4.22) and (4.21), we can get the desired estimate (3.12). Thus the proof of Proposition 3.1 is completed. □

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5. Global existence and large time behavior

We are now in a position to complete the following.

Proof of Theorem 1.1. This section is concerned with the proof of our main theorem. In order to prove Theorem 1.1, we employ the standard continuation argument based on a local existence theorem and the *a priori* estimates. Similar to [21], we can prove the local existence theorem, so we omit the details. On the other hand, the *a priori* estimates have been given in Proposition 3.1. Therefore, to complete the proof of Theorem 1.1, we need only to investigate the large-time behavior of the solution $(\rho, u, \chi)(x,t)$ to the problem (1.1)–(1.5) as time tends to infinity. Using the energy estimates, we first prove that

$$\sup_{x\in\mathbb{R}_+} |(\rho-\bar{\rho}, u-\bar{u}, \chi-\tilde{\chi})(x,t)| \to 0, \tag{5.1}$$

namely,

$$\sup_{x\in\mathbb{R}_+} |(\varphi, \psi, \zeta)(x,t)| \to 0, \tag{5.2}$$

as $t\to\infty$. For the large time behavior in (5.2), one can verify that

$$\lim_{t\to+\infty} \|\partial_x[\varphi,\psi,\zeta](t)\|_{L^2}^2 = 0. \tag{5.3}$$

To prove (5.3), we get from (4.9), (4.13), (4.11), (4.17) and (3.23) that

$$\begin{aligned}
&\int_0^{+\infty} \left| \frac{d}{dt}\|\partial_x[\varphi,\psi,\zeta]\|^2 \right| dt \\
&= 2\int_0^{+\infty} \left| \int_R \partial_t\partial_x\varphi\partial_x\varphi dx \right| dt + 2\int_0^{+\infty} \left| \int_R \partial_t\partial_x\zeta\partial_x\zeta dx \right| dt + \int_0^{+\infty} \left| \frac{d}{dt}\|\partial_x\psi\|^2 \right| dt \\
&\le C + C\int_0^{+\infty} \|\partial_x[\varphi,\psi,\zeta,\partial_x[\psi,\zeta]]\|^2 dt < +\infty.
\end{aligned} \tag{5.4}$$

Consequently, (5.4) together with (3.12) gives (5.3). Then (5.2) follows from (5.3) and sobolev's inequality.

Finally, by the construction of the smooth approximation function of the rarefaction wave, and in terms of (iv) in Lemma 2.3, we have

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}_+} \left| \rho(x,t) - \tilde{\rho}(x) - \rho^{R_2}\left(\frac{x}{t}\right) + \rho_* \right| = 0,$$

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}_+} \left| u(x,t) - \tilde{u}(x) - u^{R_2}\left(\frac{x}{t}\right) + u_* \right| = 0,$$

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$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}_+}|\chi(x,t)-\tilde{\chi}(x)|=0.$$

This ends the proof of Theorem 1.1. □

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