



Asymptotic expansion of the L^2 -norm of a solution of the strongly damped wave equation

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Abstract

The Fourier transform, \mathcal{F} , on \mathbb{R}^N ($N \geq 3$) transforms the Cauchy problem for the strongly damped wave equation $u_{tt} - \Delta u_t - \Delta u = 0$ to an ordinary differential equation in time. We let $u(t, x)$ be the solution of the problem given by the Fourier transform, and $v(t, \xi)$ be the asymptotic profile of $\mathcal{F}(u)(t, \xi) = \hat{u}(t, \xi)$ found by Ikehata in the paper *Asymptotic profiles for wave equations with strong damping* (2014).

In this paper we study the asymptotic expansions of the squared L^2 -norms of $u(t, x)$, $\hat{u}(t, \xi) - v(t, \xi)$, and $v(t, \xi)$ as $t \rightarrow \infty$. With suitable initial data $u(0, x)$ and $u_t(0, x)$, we establish the rate of decay of the squared L^2 -norms of $u(t, x)$ and $v(t, \xi)$ as $t \rightarrow \infty$. By noting the cancellation of leading terms of their respective expansions, we conclude that the rate of convergence between $\hat{u}(t, \xi)$ and $v(t, \xi)$ in the L^2 -norm occurs quickly relative to their individual behaviors. This observation is similar to the diffusion phenomenon, which has been well studied.

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1. Introduction

For $N \in \{3, 4, 5, \dots\}$ we begin by considering the strongly damped wave equation in \mathbb{R}^N

$$\begin{aligned} u_{tt}(t, x) - \Delta u_t(t, x) - \Delta u(t, x) &= 0, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 \in H^1(\mathbb{R}^N), \quad u_t(0, \cdot) &= u_1 \in L^2(\mathbb{R}^N). \end{aligned} \quad (1.1)$$

It was determined in [1] by Ikehata, Todorova, and Yordanov that (1.1) admits a unique weak solution $u \in C([0, \infty); H^1(\mathbb{R}^N)) \cap C^1([0, \infty); L^2(\mathbb{R}^N))$.

It is the goal of this paper to investigate two main problems.

1. Given suitable additional assumptions on the initial data u_0 and u_1 , determine the asymptotic expansion of the squared L^2 -norm of the weak solution $u(t, x)$ as $t \rightarrow \infty$.

To state the second problem, we need further context about the weak solution $u(t, x)$. We will determine $u(t, x)$ with the help of the Fourier transform \mathcal{F} . In the Fourier space, Ikehata [2] found an asymptotic profile $v(t, \xi)$ of $\mathcal{F}(u)(t, \xi) := \hat{u}(t, \xi)$ such that for space dimension $N \in \{1, 2, 3, \dots\}$,

$$\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 = O(t^{-N/2}) \quad (1.2)$$

as $t \rightarrow \infty$ given additional assumptions on the initial data. In the same paper, Ikehata found a sharp decay rate for the weak solution $u(t, x)$ in space dimension $N \geq 3$,

$$\|u(t, \cdot)\|_2^2 = O(t^{-N/2+1}) \quad (t \rightarrow \infty). \quad (1.3)$$

(1.2) and (1.3) reveal that $\hat{u}(t, \xi)$ and $v(t, \xi)$ tend to each other in norm faster than the decay of $u(t, x)$ in norm. This is a phenomenon similar to the diffusion phenomenon studied by Volkmer in [3]. In his paper Volkmer studied the dissipative wave equation and the heat equation. Given additional assumptions on the initial conditions of each problem, he was able to exhibit the diffusion phenomenon and provide asymptotic expansions of the L^2 -norms of the solutions and the L^2 -norm of the difference of the solutions. We may now state the second main problem.

2. Given suitable additional assumptions on the initial data u_0 and u_1 , determine the asymptotic expansion of the squared L^2 -norm of the difference of the solution $\hat{u}(t, \xi)$ and its profile $v(t, \xi)$ in the Fourier space.

To determine the necessary additional assumptions on the initial data, we follow the methods of Ikehata [2], [4], [5] and Volkmer [3] in that we use weighted L^1 -data. We may then utilize Taylor expansions of the solution $\hat{u}(t, \xi)$ about $\xi = 0$ to obtain asymptotic expansions of the desired norms. The number of terms in the expansion will be dependent upon the L^1 -conditions, and the leftover error term will be of lower order than all other terms. In [4] and [5], Ikehata used a prototype of this idea to study the dissipative wave equation. In both papers Ikehata applied the mean value theorem to the transformed initial data $(\hat{u}_0(\xi))$ and $\hat{u}_1(\xi)$ to obtain sharp decay estimates of the norm of the solution.

We first introduce some notation.

Notation. First we let $\mathbb{N} = \{1, 2, \dots\}$ denote the positive integers and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the non-negative integers. Let $N \in \mathbb{N}$. Throughout this paper the $L^q(\mathbb{R}^N)$ -norm is denoted by $\|\cdot\|_q$. We also define for all $\epsilon > 0$ $\bar{B}_\epsilon := \{x \in \mathbb{R}^N : |x| \leq \epsilon\}$ to be the closed ball of radius ϵ in \mathbb{R}^N . Then we denote the $L^q(\bar{B}_\epsilon)$ -norm by $\|\cdot\|_{q,\epsilon}$.

For all $\theta \geq 0$ we define the weighted L^1 -space

$$L^{1,\theta}(\mathbb{R}^N) := \left\{ \phi \in L^1(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} (1 + |x|)^\theta |\phi(x)| dx < \infty \right. \right\}.$$

The norm $\|\cdot\|_{L^{1,\theta}(\mathbb{R}^N)}$ on the space $L^{1,\theta}(\mathbb{R}^N)$ is defined

$$\|\phi\|_{L^{1,\theta}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (1 + |x|)^\theta |\phi(x)| dx.$$

Lastly for all $\phi \in L^1(\mathbb{R}^N)$ we define the Fourier transform

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \phi(x) e^{-ix \cdot \xi} dx,$$

and the Fourier inverse

$$\mathcal{F}^{-1}(\phi)(x) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \phi(\xi) e^{i\xi \cdot x} d\xi.$$

We remark that for arbitrary $\phi \in L^2(\mathbb{R}^N)$, the Fourier transform is not given by the above integral definition. Rather, for functions $\phi_j \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that $\|\phi - \phi_j\|_2 \rightarrow 0$ as $j \rightarrow \infty$, $\hat{\phi}$ is defined to be the unique $L^2(\mathbb{R}^N)$ -limit of $\hat{\phi}_j$ as $j \rightarrow \infty$. More details can be found on p. 189 in Chapter 4 of Evans' Partial Differential Equations [6].

1.1. Assumptions

We assume that the space dimension is $N \in \mathbb{N}$ and that there exists $K \in \mathbb{N}$ such that

$$u_0, u_1 \in L^{1,2K}(\mathbb{R}^N)$$

where the functions $u_0 \in H^1(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N)$ are the initial data of (1.1). Under these assumptions both \hat{u}_0 and \hat{u}_1 are $2K$ -times continuously differentiable on \mathbb{R}^N , and in particular at $\xi = 0$. Thus we may choose $0 < \delta < 1$ such that the Taylor approximations

$$\begin{aligned} \hat{u}_1(\xi) &= \sum_{|\sigma| \leq 2K-1} a_\sigma \xi^\sigma + O(|\xi|^{2K}), \\ \hat{u}_0(\xi) &= \sum_{|\sigma| \leq 2K-1} b_\sigma \xi^\sigma + O(|\xi|^{2K}), \end{aligned} \tag{1.4}$$

hold in the closed δ -neighborhood of $\xi = 0$, where $\sigma \in \mathbb{N}_0^N$ is a multi-index of order $|\sigma| = \sigma_1 + \dots + \sigma_N$ (see Appendix A, p. 701 of [6] for further details), and for all $0 \leq |\sigma| \leq 2K - 1$

$$a_\sigma = \frac{(-i)^{|\sigma|}}{\sigma!(2\pi)^{N/2}} \int_{\mathbb{R}^N} x^\sigma u_1(x) dx,$$

$$b_\sigma = \frac{(-i)^{|\sigma|}}{\sigma!(2\pi)^{N/2}} \int_{\mathbb{R}^N} x^\sigma u_0(x) dx.$$

Remark. We fix this chosen value of $0 < \delta < 1$, and whenever δ is hereafter referred to, we mean this fixed value.

In this fixed closed δ -neighborhood of $\xi = 0$, we also have the Taylor approximations

$$\frac{|\xi|^2}{2} \hat{u}_0(\xi) = \sum_{|\sigma| \leq 2K-1} b'_\sigma \xi^\sigma + O(|\xi|^{2K}),$$

$$\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) = \sum_{|\sigma| \leq 2K-1} c_\sigma \xi^\sigma + O(|\xi|^{2K}), \quad (1.5)$$

$$\left| \hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right|^2 = \sum_{|\sigma| \leq 2K-1} d_\sigma \xi^\sigma + O(|\xi|^{2K}), \quad (1.6)$$

$$|\hat{u}_0(\xi)|^2 = \sum_{|\sigma| \leq 2K-1} f_\sigma \xi^\sigma + O(|\xi|^{2K}), \quad (1.7)$$

$$\left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) \bar{\hat{u}}_0(\xi) = \sum_{|\sigma| \leq 2K-1} l_\sigma \xi^\sigma + O(|\xi|^{2K}), \quad (1.8)$$

where, with $e_j = (\delta_{ij})_{i=1}^N$ denoting the unit multi-index in the j -direction,

$$b'_\sigma = \begin{cases} 0 & \text{all entries of } \sigma \text{ are } \leq 1 \\ \sum_{\{j|\sigma_j \geq 2\}} \frac{1}{2} b_{\sigma-2e_j} & \text{otherwise,} \end{cases}$$

$$c_\sigma = a_\sigma + b'_\sigma,$$

$$d_\sigma = \sum_{\psi+\omega=\sigma} c_\psi \bar{c}_\omega,$$

$$f_\sigma = \sum_{\psi+\omega=\sigma} b_\psi \bar{b}_\omega,$$

$$l_\sigma = \sum_{\psi+\omega=\sigma} c_\psi \bar{b}_\omega.$$

With the definitions $P_0 := (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_0(x) dx$ and $P_1 := (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_1(x) dx$, it is easy to see that $a_0 = P_1$ and $b_0 = P_0$. Hence $c_0 = P_1$, $d_0 = |P_1|^2$, $f_0 = |P_0|^2$, and $l_0 = P_1 \bar{P}_0$.

1.2. Main results

Let us consider the strongly damped wave equation in \mathbb{R}^N as given in (1.1). Applying the Fourier transform to (1.1), we obtain an ordinary differential equation in t

$$\begin{aligned} \hat{u}_{tt}(t, \xi) + |\xi|^2 \hat{u}_t(t, \xi) + |\xi|^2 \hat{u}(t, \xi) &= 0, & (t, \xi) \in (0, \infty) \times \mathbb{R}^N, \\ \hat{u}(0, \cdot) &= \hat{u}_0, & \hat{u}_t(0, \cdot) = \hat{u}_1. \end{aligned} \quad (1.9)$$

The solution to (1.9) is given by

$$\hat{u}(t, \xi) = \left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) h_1(t, \xi) + \hat{u}_0(\xi) h_2(t, \xi),$$

where

$$h_1(t, \xi) = e^{-t|\xi|^2/2} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{|\xi|\sqrt{1-|\xi|^2/4}} \quad (1.10)$$

and

$$h_2(t, \xi) = e^{-t|\xi|^2/2} \cos(t|\xi|\sqrt{1-|\xi|^2/4}) \quad (1.11)$$

Therefore the weak solution of (1.1) under the Fourier transform is the inverse Fourier transform of $\hat{u}(t, \xi)$, i.e., $u(t, x) = \mathcal{F}^{-1}(\hat{u})(t, x)$. By Plancherel's theorem the Fourier transform as defined is an $L^2(\mathbb{R}^N)$ -isometry (see Theorem 1 on p. 187 in Chapter 4 of [6]). Thus we will rarely need to appeal to the weak solution $u(t, x)$, instead using $\hat{u}(t, \xi)$ whose form is given explicitly.

We define

$$\begin{aligned} \mu_1(t, \xi) &:= \left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) h_1(t, \xi), \\ \mu_2(t, \xi) &:= \hat{u}_0(\xi) h_2(t, \xi), \\ v_1(t, \xi) &:= P_1 e^{-t|\xi|^2/2} \frac{\sin(t|\xi|)}{|\xi|}, \\ v_2(t, \xi) &:= P_0 e^{-t|\xi|^2/2} \cos(t|\xi|). \end{aligned}$$

Then $\hat{u}(t, \xi) = \mu_1(t, \xi) + \mu_2(t, \xi)$. With the definition $v(t, \xi) := v_1(t, \xi) + v_2(t, \xi)$, we wish to find the asymptotic expansions of

$$\|u(t, \cdot)\|_2^2 \quad \text{and} \quad \|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 \quad \text{as } t \rightarrow \infty.$$

Indeed $v(t, \xi)$ is the profile determined by Ikehata in his paper [2]. His main result from that paper concerns the asymptotic behavior of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$.

Theorem (Ikehata, 2014 [2]). Let $N \in \mathbb{N}$. Assume the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$. Then there exist constants $\alpha > 0$ and $C > 0$ such that as $t \rightarrow \infty$

$$\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 \leq C \left((\|u_1\|_1^2 + \|u_0\|_1^2 + \|u_1\|_{L^{1,1}(\mathbb{R}^N)}^2) t^{-N/2} + \|u_0\|_{L^{1,1}(\mathbb{R}^N)}^2 t^{-N/2-1} + e^{-\alpha t} (\|u_1\|_2^2 + \|u_0\|_2^2) \right).$$

Additionally, in [2] Ikehata cites the paper [7], which he wrote with Natsume, to provide the asymptotic behavior of $\|u(t, \cdot)\|_2^2$. We give this result as a theorem as well.

Theorem (Ikehata and Natsume, 2012 [7]). Let $N \geq 3$ be an integer. Assume the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$. Then there exist constants $\eta > 0$ and $C > 0$ such that as $t \rightarrow \infty$

$$\|u(t, \cdot)\|_2^2 \leq C \left(|P_1|^2 (1+t)^{-N/2+1} + (|P_0|^2 + \|u_1\|_{L^{1,1}(\mathbb{R}^N)}^2) (1+t)^{-N/2} + \|u_0\|_{L^{1,1}(\mathbb{R}^N)}^2 (1+t)^{-N/2-1} + e^{-\eta t} (\|u_0\|_2^2 + \|u_1\|_2^2) \right).$$

We wish to extend the results of Ikehata [2] and Ikehata–Natsume [7] given above to full asymptotic expansions. To do so, we use the fact that

$$\begin{aligned} \|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 &= \int_{\mathbb{R}^N} |\hat{u}(t, \xi) - v(t, \xi)|^2 d\xi, \\ &= \|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2,\epsilon}^2 + \int_{|\xi| \geq \epsilon} |\hat{u}(t, \xi) - v(t, \xi)|^2 d\xi, \end{aligned}$$

where $\epsilon > 0$ is an arbitrary constant. In the proof of the main theorem from [2], Ikehata proved the following fact, which is helpful in determining the desired expansions. We state the fact as a lemma.

Lemma (Ikehata, 2014 [2]). Let $\epsilon > 0$ be given. Then there exists some $\eta_1 > 0$ depending on $\epsilon > 0$ such that as $t \rightarrow \infty$

$$\int_{|\xi| \geq \epsilon} |\hat{u}(t, \xi)|^2 d\xi = O(e^{-\eta_1 t}).$$

Furthermore, it is a routine exercise to verify that for any $\epsilon > 0$ and $t \geq 1$,

$$\int_{|\xi| \geq \epsilon} |v(t, \xi)|^2 d\xi = O(e^{-t\epsilon^2/2}).$$

Therefore for all $\epsilon > 0$, let $\eta_1 > 0$ be as given in the preceding lemma. Then as $t \rightarrow \infty$

$$\int_{|\xi| \geq \epsilon} |\hat{u}(t, \xi) - v(t, \xi)|^2 d\xi \leq 2 \left(\int_{|\xi| \geq \epsilon} |\hat{u}(t, \xi)|^2 d\xi + \int_{|\xi| \geq \epsilon} |v(t, \xi)|^2 d\xi \right) = O(e^{-\eta t}),$$

where $\eta = \min\{\eta_1, \epsilon^2/2\}$. We have shown that for all $\epsilon > 0$ there is some $\eta > 0$ such that

$$\begin{aligned} \|u(t, \cdot)\|_2^2 &= \|\hat{u}(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_{2, \epsilon}^2 + O(e^{-\eta t}) \quad (t \rightarrow \infty), \\ \|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 &= \|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2, \epsilon}^2 + O(e^{-\eta t}) \quad (t \rightarrow \infty). \end{aligned}$$

Thus all the interesting asymptotic behaviors of $\|u(t, \cdot)\|_2^2$ and $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$ are captured in any ϵ -neighborhood of the origin. We let ϵ equal the fixed value of $0 < \delta < 1$, and hence we are interested in the expansions of

$$\|\hat{u}(t, \cdot)\|_{2, \delta}^2 \quad \text{and} \quad \|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2, \delta}^2 \quad \text{as } t \rightarrow \infty.$$

We will compute these expansions by first noting $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2, \delta}^2 = X(t) + Y(t) + Z(t)$, where

$$X(t) = \|\mu_1(t, \cdot) - v_1(t, \cdot)\|_{2, \delta}^2, \quad (1.12)$$

$$Y(t) = \|\mu_2(t, \cdot) - v_2(t, \cdot)\|_{2, \delta}^2, \quad (1.13)$$

$$Z(t) = 2\Re\langle \mu_1(t, \cdot) - v_1(t, \cdot), \mu_2(t, \cdot) - v_2(t, \cdot) \rangle_{2, \delta}, \quad (1.14)$$

and with $\langle \cdot, \cdot \rangle_{2, \delta}$ denoting the L^2 -inner product over the closed ball of radius δ about the origin. Then $X(t) = X_1(t) + X_2(t) + X_3(t)$, $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$, and $Z(t) = Z_1(t) + Z_2(t) + Z_3(t) + Z_4(t)$, where

$$X_1(t) = \|\mu_1(t, \cdot)\|_{2, \delta}^2, \quad X_2(t) = \|v_1(t, \cdot)\|_{2, \delta}^2, \quad X_3(t) = -2\Re\langle \mu_1(t, \cdot), v_1(t, \cdot) \rangle_{2, \delta};$$

$$Y_1(t) = \|\mu_2(t, \cdot)\|_{2, \delta}^2, \quad Y_2(t) = \|v_2(t, \cdot)\|_{2, \delta}^2, \quad Y_3(t) = -2\Re\langle \mu_2(t, \cdot), v_2(t, \cdot) \rangle_{2, \delta};$$

$$Z_1(t) = 2\Re\langle \mu_1(t, \cdot), \mu_2(t, \cdot) \rangle_{2, \delta}, \quad Z_2(t) = -2\Re\langle \mu_1(t, \cdot), v_2(t, \cdot) \rangle_{2, \delta},$$

$$Z_3(t) = -2\Re\langle v_1(t, \cdot), \mu_2(t, \cdot) \rangle_{2, \delta}, \quad Z_4(t) = 2\Re\langle v_1(t, \cdot), v_2(t, \cdot) \rangle_{2, \delta}.$$

Furthermore it is apparent that $\|\hat{u}(t, \cdot)\|_{2, \delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$.

Theorem 1.1. *Let $N \geq 3$ be an integer and $K \in \mathbb{N}$. Let $u(t, x)$ be the weak solution of (1.1) under the Fourier transform with the initial data $u_0(x)$ and $u_1(x)$ satisfying $u_0 \in H^1(\mathbb{R}^N) \cap L^{1, 2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1, 2K}(\mathbb{R}^N)$. Then*

$$\|u(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_2^2 = t^{-N/2+1} \left(\sum_{j=0}^{K-1} {}_N W_j t^{-j} + O(t^{-K}) \right) \quad (t \rightarrow \infty),$$

where the ${}_N W_j$ ($j \in \{0, \dots, K-1\}$) are coefficients dependent on the space dimension N and the initial data $u_0(x)$ and $u_1(x)$.

Theorem 1.2. Let $N \geq 3$ be an integer and $K \in \mathbb{N}$. Let $u(t, x)$ be the weak solution of (1.1) under the Fourier transform with the initial data $u_0(x)$ and $u_1(x)$ satisfying $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$. Consider the asymptotic profile of $\hat{u}(t, \xi)$ found by Ikehata in [2]:

$$v(t, \xi) = P_1 e^{-t|\xi|^2/2} \frac{\sin(t|\xi|)}{|\xi|} + P_0 e^{-t|\xi|^2/2} \cos(t|\xi|),$$

where $P_0 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_0(x) dx$ and $P_1 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_1(x) dx$. Then as $t \rightarrow \infty$

$$\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 = t^{-N/2+1} \left(\sum_{j=1}^{K-1} {}_N V_j t^{-j} + O(t^{-K}) \right),$$

where the ${}_N V_j$ ($j \in \{1, \dots, K-1\}$) are coefficients dependent on the space dimension N and the initial data $u_0(x)$ and $u_1(x)$.

In the process of proving Theorem 1.2 we will find the asymptotic expansions of $X_2(t)$, $Y_2(t)$, and $Z_4(t)$. Since $\|v(t, \cdot)\|_2^2 = \|v(t, \cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = X_2(t) + Y_2(t) + Z_4(t) + O(e^{-t\delta^2/2})$, we obtain the expansion of the squared L^2 -norm of the asymptotic profile $v(t, \xi)$.

Corollary 1.3. Let $N \geq 3$ be an integer and $K \in \mathbb{N}$. Let $v(t, \xi)$ be the asymptotic profile of $\hat{u}(t, \xi)$ found by Ikehata in [2]. Then

$$\|v(t, \cdot)\|_2^2 = t^{-N/2+1} \left(\sum_{j=0}^{K-1} {}_N U_j t^{-j} + O(t^{-K}) \right) \quad (t \rightarrow \infty),$$

where the ${}_N U_j$ ($j \in \{0, \dots, K-1\}$) are coefficients dependent on the space dimension N and the initial data $u_0(x) \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ of (1.1).

Remark. With space dimension $N \geq 3$, the first $K = 2$ coefficients from Theorem 1.1 and Corollary 1.3 are given below.

$${}_N W_0 = \frac{|P_1|^2 \pi^{N/2}}{N-2}, \quad {}_N W_1 = \begin{cases} \frac{|P_1|^2 \pi^{N/2}}{8} + \frac{|P_0|^2 \pi^{N/2}}{2} + \frac{\pi^{N/2}}{2N} \sum_{j=1}^N d_{2e_j} & N \neq 4 \\ \frac{3|P_1|^2 \pi^2}{8} + \frac{|P_0|^2 \pi^2}{2} + \frac{\pi^2}{8} \sum_{j=1}^4 d_{2e_j} & N = 4, \end{cases} \quad (1.15)$$

$${}_N U_0 = \frac{|P_1|^2 \pi^{N/2}}{N-2}, \quad {}_N U_1 = \begin{cases} \frac{|P_0|^2 \pi^{N/2}}{2} & N \neq 4 \\ \frac{|P_1|^2 \pi^2}{4} + \frac{|P_0|^2 \pi^2}{2} & N = 4. \end{cases} \quad (1.16)$$

In the case $N \geq 3$ and $K = 2$, we may only determine the first non-zero coefficient from Theorem 1.2.

$${}_N V_1 = \frac{N(N+2)|P_1|^2 \pi^{N/2}}{512} + \frac{\pi^{N/2}}{2N} \sum_{j=1}^N d_{2e_j} - \frac{\pi^{N/2}}{N} \sum_{j=1}^N \Re(\bar{P}_1 c_{2e_j}). \quad (1.17)$$

The rest of this paper is divided into two more sections and an appendix. With Section 2 we prove Theorem 1.1, and with Section 3 we prove Theorem 1.2. Any necessary lemmas are collected before proceeding with the proof, and recalled as needed. The proof of Corollary 1.3 is part of the proof of Theorem 1.2 and is mentioned when complete. Finally, the Appendix contains *Mathematica* code used in computing some coefficients in the expansions.

2. Proof of Theorem 1.1

2.1. Auxiliary lemmas

For $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, and $t > 0$ define

$$G_{1,m}^\epsilon(t) := \int_0^\epsilon r^m \exp(-tr^2 - irt\sqrt{4-r^2})dr. \quad (2.1)$$

$G_{1,m}^\epsilon(t)$ will be treated using the small perturbation method found on p. 96 in Chapter 5 of de Bruijn's *Asymptotic Methods in Analysis* [8].

We define

$$g(t, r) := \exp(irt(2 - \sqrt{4-r^2})), \quad (2.2)$$

and rewrite (2.1) as

$$G_{1,m}^\epsilon(t) = \int_0^\epsilon r^m \exp(-tr^2 - 2irt)g(t, r)dr. \quad (2.3)$$

We consider the Taylor expansion of $g(t, r)$ at $r = 0$, $g(t, r) = \sum_{k=0}^\infty g_k(t)r^k$, where $g_k(t)$ is a polynomial in t of degree at most $k/3$ (to be shown). The first few terms are

$$g(t, r) = 1 + \frac{it}{4}r^3 + \frac{it}{64}r^5 - \frac{t^2}{32}r^6 + \frac{it}{512}r^7 - \frac{t^2}{256}r^8 + \left(\frac{5it}{16384} - \frac{it^3}{384}\right)r^9 + \dots$$

We need the following lemma.

Lemma 2.1. *Let $J \in \mathbb{N}_0$ and $0 < \epsilon < 2$. Then there exists $C_J > 0$ such that*

$$\tilde{g}_J(t, r) := g(t, r) - \sum_{k=0}^{3J-1} g_k(t)r^k \quad (2.4)$$

satisfies

$$|\tilde{g}_J(t, r)| \leq C_J t^J r^{3J}$$

for all $0 \leq r \leq \epsilon$ and $t \geq 1$.

Proof. Let $0 < \epsilon < 2$. If $J = 0$, then $\tilde{g}_J(t, r) = g(t, r)$ and the result holds since $|g(t, r)| \leq 1$ for all $0 \leq r \leq \epsilon$ and $t \in \mathbb{R}$. If $J \in \mathbb{N}$, we let $f(r) := r(2 - \sqrt{4 - r^2})$. Then $g(t, r) = e^{i(tf(r))}$. For every $x \geq 0$

$$\begin{aligned} \left| e^{ix} - \sum_{k=0}^{J-1} \frac{1}{k!} (ix)^k \right| &= \left| \int_0^x \frac{i^J e^{is}}{(J-1)!} (x-s)^{J-1} ds \right| \\ &\leq \frac{1}{(J-1)!} \int_0^x (x-s)^{J-1} ds \\ &= \frac{x^J}{J!}. \end{aligned}$$

Hence for all $t \geq 0$ and $0 \leq r \leq \epsilon$,

$$\left| g(t, r) - \sum_{k=0}^{J-1} \frac{1}{k!} (itf(r))^k \right| \leq \frac{t^J (f(r))^J}{J!}. \quad (2.5)$$

Now for every $k \in \{0, \dots, J-1\}$, use the Taylor expansion $(f(r))^k = f_k(r) + \tilde{f}_k(r)$, where $f_k(r)$ is a polynomial in r of degree at most $3J-1$ and $|\tilde{f}_k(r)| \leq C_k r^{3J}$ for $0 \leq r \leq \epsilon$. We substitute the Taylor expansion for $(f(r))^k$ into (2.5) and note that $0 \leq f(r) \leq r^3$ for $0 \leq r \leq \epsilon$ to obtain

$$\left| g(t, r) - \sum_{k=0}^{J-1} \frac{i^k t^k}{k!} (f_k(r) + \tilde{f}_k(r)) \right| \leq \frac{1}{J!} t^J r^{3J}.$$

By the reverse triangle inequality

$$\left| g(t, r) - \sum_{k=0}^{J-1} \frac{i^k t^k}{k!} f_k(r) \right| - \left| \sum_{k=0}^{J-1} \frac{i^k t^k}{k!} \tilde{f}_k(r) \right| \leq \frac{1}{J!} t^J r^{3J},$$

which implies

$$\left| g(t, r) - \sum_{k=0}^{J-1} \frac{i^k t^k}{k!} f_k(r) \right| \leq \sum_{k=0}^{J-1} \frac{t^k}{k!} C_k r^{3J} + \frac{1}{J!} t^J r^{3J}.$$

Letting $t \geq 1$ gives the desired result.

Since the Taylor expansion of $f(r)$ about $r = 0$ has lowest order term r^3 , each $f_k(r)$ will have lowest order term r^{3k} for every $k \in \mathbb{N}_0$. Since

$$\sum_{k=0}^{J-1} \frac{i^k t^k}{k!} f_k(r) = \sum_{k=0}^{3J-1} g_k(t) r^k$$

when the former is rearranged by powers of r , we find that $\deg(g_k(t)) \leq k/3$. \square

We now prove the following proposition.

Proposition 2.2. *Let $0 < \epsilon < 2$ and $m, Q \in \mathbb{N}_0$. Then as $t \rightarrow \infty$*

$$G_{1,m}^\epsilon(t) = \sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where, for $n \in \{0, \dots, Q-1\}$,

$$B_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m, Q \in \mathbb{N}_0$. We define for $k, J \in \mathbb{N}_0$

$$F_k^\epsilon(t) := \int_0^\epsilon r^k \exp(-tr^2 - 2irt) dr \quad (2.6)$$

and

$$\tilde{G}_{1,k,J}^\epsilon(t) := \int_0^\epsilon r^k \exp(-tr^2 - 2irt) \tilde{g}_J(t, r) dr.$$

Thus we obtain from (2.3) and (2.4)

$$G_{1,m}^\epsilon(t) = \sum_{k=0}^{3J-1} g_k(t) F_{m+k}^\epsilon(t) + \tilde{G}_{1,m,J}^\epsilon(t). \quad (2.7)$$

By Lemma 2.1, for $t \geq 1$

$$|\tilde{G}_{1,m,J}^\epsilon(t)| \leq C_J t^J \int_0^\infty r^{m+3J} e^{-tr^2} dr = \frac{1}{2} C_J \Gamma\left(\frac{m+3J+1}{2}\right) t^{-(m+J+1)/2}. \quad (2.8)$$

We now seek an asymptotic expansion of $F_k^\epsilon(t)$ for $k \in \mathbb{N}_0$. We note that

$$F_k^\epsilon(t) = \left(\int_0^\infty - \int_\epsilon^\infty \right) r^k \exp(-tr^2 - 2irt) dr;$$

the former term we denote by $\tilde{F}_k(t)$, and the latter term is $O(e^{-t\epsilon^2/2})$ as $t \rightarrow \infty$. So the asymptotic expansions of $F_k^\epsilon(t)$ and $\tilde{F}_k(t)$ will be the same if $\tilde{F}_k(t)$ exhibits slower-than-exponential decay.

For $\Re(\lambda) < 0$ and $z \in \mathbb{C}$, let $D_\lambda(z)$ denote the parabolic cylinder function

$$D_\lambda(z) := \frac{e^{-z^2/4}}{\Gamma(-\lambda)} \int_0^\infty s^{-\lambda-1} \exp\left(-\frac{s^2}{2} - zs\right) ds$$

as found on p. 328 in Chapter 8 of [9] by Magnus, Oberhettinger, and Soni. With the substitution $r = s/\sqrt{2t}$ in $\tilde{F}_k(t)$, we obtain

$$\begin{aligned} \tilde{F}_k(t) &= (2t)^{-(k+1)/2} \int_0^\infty s^k \exp\left(-\frac{s^2}{2} - is\sqrt{2t}\right) ds \\ &= (2t)^{-(k+1)/2} e^{-t/2} k! D_{-k-1}(i\sqrt{2t}). \end{aligned}$$

From p. 331 in Chapter 8 of [9], we know the asymptotic expansion of $D_\lambda(z)$ as $|z| \rightarrow \infty$ in the set $\{z \in \mathbb{C} : |\arg(z)| < 3\pi/4\} \subseteq \mathbb{C}$. Therefore

$$F_k^\epsilon(t) = \tilde{F}_k(t) + O(e^{-t\epsilon^2/2}) = \sum_{p=0}^{P-1} A_{k,p} t^{-k-p-1} + O(t^{-k-P-1}) \quad (t \rightarrow \infty), \quad (2.9)$$

where $P \in \mathbb{N}_0$ and

$$A_{k,p} = \frac{k!}{(2i)^{k+1}} \frac{\left(\frac{k+1}{2}\right)_p \left(\frac{k+2}{2}\right)_p}{p!} \quad (2.10)$$

with the Pochhammer symbol $(a)_p$ denoting the rising factorial

$$(a)_p = \begin{cases} 1 & p = 0 \\ a(a+1) \cdot \dots \cdot (a+p-1) & p \in \mathbb{N}. \end{cases}$$

Our goal is to find an asymptotic expansion of $G_{1,m}^\epsilon(t)$ with $Q \in \mathbb{N}_0$ explicit terms and then a O -term. Combining (2.7) and (2.9), we see that the leading term is of order t^{-m-1} , followed by order $t^{-m-2}, t^{-m-3}, \dots$. Thus, we want the error term to be $O(t^{-m-Q-1})$. By the estimate (2.8), we know the error term $\tilde{G}_{1,m,J}^\epsilon(t)$ of $G_{1,m}^\epsilon(t)$ is of order $t^{-(m+J+1)/2}$. So we want $-m-Q-1 = -(m+J+1)/2$, and thus choose $J = J_{m,Q} := m + 2Q + 1$.

By substituting (2.9) into (2.7), as $t \rightarrow \infty$ we have

$$G_{1,m}^\epsilon(t) = \sum_{k=0}^{3J-1} g_k(t) \left(\sum_{p=0}^{P-1} A_{m+k,p} t^{-m-k-p-1} + O(t^{-m-k-P-1}) \right) + O(t^{-m-Q-1}) \quad (2.11)$$

$$= \sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}). \quad (2.12)$$

To compute the $B_{m,n}$ we first recall $g(t, r) := \exp(irt(2 - \sqrt{4 - r^2})) = \sum_{k=0}^{\infty} g_k(t)r^k$, if we expand in a Taylor series at $r = 0$. We determined in Lemma 2.1 that for each $k \in \mathbb{N}_0$, $\deg(g_k(t)) \leq \lfloor k/3 \rfloor$.

Obtaining (2.12) from (2.11) is a matter of having the proper ranges on the indices k and p and then rearranging terms according to the powers of t . To determine the range of the index k , we observe that any monomial in $g_k(t)$ is of the form $a(it)^l$ for some $0 \leq l \leq k/3$ and $a \in \mathbb{R}$. For each $0 \leq p \leq P - 1$ the contribution of this monomial to (2.11) is a monomial of degree $l - m - k - p - 1$. So that this monomial is not absorbed into the $O(t^{-m-Q-1})$ -term of (2.11) we require $l - m - k - p - 1 \geq -m - Q$, which implies $-2k/3 - p - 1 \geq -Q$ since $0 \leq l \leq k/3$. This implies $k \leq 3(Q - 1)/2 - 3p/2 \leq 3(Q - 1)/2$. Thus the range on the index k is $0 \leq k \leq \lfloor 3(Q - 1)/2 \rfloor$.

Now let $0 \leq k \leq \lfloor 3(Q - 1)/2 \rfloor$. The fact that any monomial in $g_k(t)$ is of the form $a(it)^l$ for some $0 \leq l \leq k/3$ and $a \in \mathbb{R}$ can help us find the range on the index p as well. Start with the contribution of that monomial to (2.11) as before. With the same reasoning we require $-2k/3 - p - 1 \geq -Q$, which implies $p \leq Q - 2k/3 - 1$. Therefore the range on the index p is $0 \leq p \leq P - 1$ where $P = P_k := \lfloor Q - 2k/3 \rfloor$.

Using a computer algebra system such as *Mathematica*, one can write a program to compute as many $B_{m,n}$ as desired. For any $m \in \mathbb{N}_0$ we have

$$\begin{aligned} B_{m,0} &= A_{m,0} \\ B_{m,1} &= A_{m,1} \\ B_{m,2} &= A_{m,2} + \frac{i}{4} A_{m+3,0}. \end{aligned} \tag{2.13}$$

Lastly we verify that for $m, n \in \mathbb{N}_0$

$$B_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

For each k , $g_k(t)$ is an even or odd polynomial in t with k . (To see this, we use the fact that $g(-t, r) = g(t, -r)$ to compare the coefficients of the Taylor expansions of both sides in a small neighborhood of $r = 0$.) Using the definition of $A_{k,p}$ in (2.10) we see that $i^{m+k+1} A_{m+k,p} > 0$, and so any term of the inner sum of (2.11) is of the form $bi^{-m-k-1}t^{-m-k-p-1}$ where $b > 0$. We recall that any monomial of $g_k(t)$ is of the form $a(it)^l$, where $a \in \mathbb{R}$ and $0 \leq l \leq \lfloor k/3 \rfloor$ is even or odd with k (since $g_k(t)$ is even or odd with k). Thus the contribution of (2.11) to (2.12) is of the form

$$a(it)^l bi^{-m-k-1}t^{-m-k-p-1} = abi^{l-k}i^{-m-1}t^{-m-(k+p-l)-1}.$$

Since l and k are either both even or both odd, $l - k$ is even. The contribution is just

$$ci^{-m-1}t^{-m-(k+p-l)-1}$$

where $c \in \mathbb{R}$. The coefficient ci^{-m-1} is thus a contributing term of $B_{m,n}$ with $n = k + p - l$, and any other such term of $B_{m,n}$ will also be of the form ci^{-m-1} with varying $c \in \mathbb{R}$. Therefore for each $m \in \mathbb{N}_0$, $i^{m+1}B_{m,n} \in \mathbb{R}$ and we deduce what was claimed. \square

Lemma 2.3. For $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$ define

$$I_{1,m}^\epsilon(t) := \int_0^\epsilon r^m e^{-tr^2} \sin^2(tr\sqrt{1-r^2/4}) dr. \quad (2.14)$$

1. If $m \in \mathbb{N}_0$ is even, then as $t \rightarrow \infty$

$$I_{1,m}^\epsilon(t) = \frac{1}{4} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}$ is odd, then as $t \rightarrow \infty$

$$I_{1,m}^\epsilon(t) = \frac{1}{4} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + \sum_{n=0}^{Q-1} -\frac{1}{2} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

Proof. Using the identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$, we obtain

$$I_{1,m}^\epsilon(t) = \frac{1}{2} \int_0^\epsilon r^m e^{-tr^2} dr - \frac{1}{2} \int_0^\epsilon r^m e^{-tr^2} \cos(tr\sqrt{4-r^2}) dr. \quad (2.15)$$

We note

$$\int_0^\epsilon r^m e^{-tr^2} dr = \int_0^\infty r^m e^{-tr^2} dr - \int_\epsilon^\infty r^m e^{-tr^2} dr,$$

the latter term of which is $O(e^{-t\epsilon^2/2})$ for all $t \geq 1$. Further,

$$\int_0^\infty r^m e^{-tr^2} dr = \frac{1}{2} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2}.$$

We also observe that the second integral of (2.15) satisfies

$$\int_0^\epsilon r^m e^{-tr^2} \cos(tr\sqrt{4-r^2}) dr = \Re(G_{1,m}^\epsilon(t)).$$

By Proposition 2.2, for $t \geq 1$ we may rewrite (2.15) as

$$\begin{aligned} I_{1,m}^\epsilon(t) &= \frac{1}{2} \left(\int_0^\infty r^m e^{-tr^2} dr - \int_\epsilon^\infty r^m e^{-tr^2} dr \right) - \frac{1}{2} \Re(G_{1,m}^\epsilon(t)) \\ &= \frac{1}{2} \left(\frac{1}{2} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + O(e^{-t\epsilon^2/2}) \right) - \frac{1}{2} \Re \left(\sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}) \right) \\ &= \frac{1}{4} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + \sum_{n=0}^{Q-1} -\frac{1}{2} \Re(B_{m,n}) t^{-m-n-1} + O(t^{-m-Q-1}), \end{aligned} \quad (2.16)$$

for any $Q \in \mathbb{N}_0$. We complete the proof by using the fact that $B_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}$ is odd and $B_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}$ is even. \square

Lemma 2.4. For $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$ define

$$I_{2,m}^\epsilon(t) := \int_0^\epsilon r^m e^{-tr^2} \cos^2(tr\sqrt{1-r^2/4}) dr.$$

1. If $m \in \mathbb{N}_0$ is even, then as $t \rightarrow \infty$

$$I_{2,m}^\epsilon(t) = \frac{1}{4} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}$ is odd, then as $t \rightarrow \infty$

$$I_{2,m}^\epsilon(t) = \frac{1}{4} \Gamma\left(\frac{m+1}{2}\right) t^{-m/2-1/2} + \sum_{n=0}^{Q-1} \frac{1}{2} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

Proof. The proof is similar to the proof of Lemma 2.3, but instead we use the identity $\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$. \square

Lemma 2.5. For $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$ define

$$I_{3,m}^\epsilon(t) := \int_0^\epsilon r^m e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \cos(tr\sqrt{1-r^2/4}) dr.$$

1. If $m \in \mathbb{N}_0$ is even, then as $t \rightarrow \infty$

$$I_{3,m}^\epsilon(t) = \sum_{n=0}^{Q-1} -\frac{1}{2} \Im(B_{m,n}) t^{-m-n-1} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}$ is odd, then as $t \rightarrow \infty$

$$I_{3,m}^\epsilon(t) = O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

Proof. Using Proposition 2.2 and the identity $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$ we obtain

$$\begin{aligned} I_{3,m}^\epsilon(t) &= \frac{1}{2} \int_0^\epsilon r^m e^{-tr^2} \sin(tr\sqrt{4-r^2}) dr \\ &= -\frac{1}{2} \Im(G_{1,m}^\epsilon(t)) \\ &= -\frac{1}{2} \Im \left(\sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}) \right) \quad (t \geq 1) \\ &= \sum_{n=0}^{Q-1} -\frac{1}{2} \Im(B_{m,n}) t^{-m-n-1} + O(t^{-m-Q-1}) \quad (t \geq 1), \end{aligned}$$

for any $Q \in \mathbb{N}_0$. We complete the proof by using the fact that $B_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}$ is odd and $B_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}_0$ is even. \square

2.2. Intermediate computations

We are now in a position to find the asymptotic expansion of $\|u(t, \cdot)\|_2^2$ as $t \rightarrow \infty$. Let us first recall that since the Fourier transform as defined is an $L^2(\mathbb{R}^N)$ -isometry, $\|u(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_2^2$. Also recall that with the fixed $0 < \delta < 1$, $\|\hat{u}(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_{2,\delta}^2 + O(e^{-\eta t})$ as $t \rightarrow \infty$ for some $\eta > 0$. Since $\|\hat{u}(t, \cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$, we may determine the asymptotic expansion in three steps. Over the next few sections, we will do so and arrive at the proof of Theorem 1.1. Let us assume that, unless otherwise stated, $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0, u_1 \in L^{1,2K}(\mathbb{R}^N)$ in addition to the assumptions $u_0 \in H^1(\mathbb{R}^N)$, $u_1 \in L^2(\mathbb{R}^N)$ given in (1.1).

2.2.1. Expansion of $X_1(t)$

We observe

$$X_1(t) = \int_{|\xi| \leq \delta} \left| \hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right|^2 (h_1(t, \xi))^2 d\xi,$$

into which we substitute (1.6) and (1.10). After simplifying, we have a sum of integrals indexed by the multi-indices σ plus an integral of a $O(|\xi|^{2K})$ -term. We estimate the integral with the $O(|\xi|^{2K})$ -term once, as all others may be estimated similarly.

$$\begin{aligned} \left| \int_{|\xi| \leq \delta} O(|\xi|^{2K}) e^{-t|\xi|^2} \frac{\sin^2(t|\xi| \sqrt{1 - |\xi|^2/4})}{|\xi|^2(1 - |\xi|^2/4)} d\xi \right| &\leq \frac{M}{1 - \delta^2/4} \int_{|\xi| \leq \delta} |\xi|^{2K-2} e^{-t|\xi|^2} d\xi \\ &= \frac{M\omega_{N-1}}{1 - \delta^2/4} \int_0^\delta r^{2K+N-3} e^{-tr^2} dr \\ &= O(t^{-K-N/2+1}) \quad (t \rightarrow \infty) \end{aligned}$$

for some $M > 0$, where ω_{N-1} is the surface area of the $(N-1)$ -sphere in \mathbb{R}^N .

Therefore as $t \rightarrow \infty$

$$X_1(t) = \sum_{|\sigma| \leq 2K-1} d_\sigma \int_{|\xi| \leq \delta} \xi^\sigma e^{-t|\xi|^2} \frac{\sin^2(t|\xi| \sqrt{1 - |\xi|^2/4})}{|\xi|^2(1 - |\xi|^2/4)} d\xi + O(t^{-K-N/2+1}).$$

Observe that, aside from ξ^σ , the integrand of the preceding equation is radially defined in ξ . Hence if even one entry of the multi-index σ is odd, the integral equals 0. So the sum can in fact only range over $0 \leq |\sigma| \leq K-1$ if we replace σ by 2σ , after which we switch to spherical coordinates. Hence as $t \rightarrow \infty$

$$X_1(t) = \sum_{|\sigma| \leq K-1} d_{2\sigma} J_\sigma \int_0^\delta r^{2|\sigma|+N-3} e^{-tr^2} \frac{\sin^2(tr \sqrt{1 - r^2/4})}{1 - r^2/4} dr + O(t^{-K-N/2+1}), \quad (2.17)$$

where

$$J_\sigma = \frac{2 \cdot \Gamma(\sigma_1 + \frac{1}{2}) \cdots \Gamma(\sigma_N + \frac{1}{2})}{\Gamma(|\sigma| + \frac{N}{2})}.$$

Since $0 < \delta < 1 < 2$ and $0 \leq r \leq \delta$ we may write for $L \in \mathbb{N}_0$

$$\frac{1}{1 - r^2/4} = \sum_{k=0}^{L-1} \frac{r^{2k}}{4^k} + O(r^{2L}). \quad (2.18)$$

Then for each $0 \leq |\sigma| \leq K-1$ we define $L = L_\sigma := K - |\sigma| \in \mathbb{N}$. We substitute (2.18) into (2.17) and estimate the integrals with the $O(r^{2L_\sigma})$ -terms (by choice of L_σ , each integral is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$). By Lemma 2.3 with $Q = Q_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - N/2 + 1 \rceil\}$, we obtain as $t \rightarrow \infty$

$$\begin{aligned}
X_1(t) &= \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} \frac{d_{2\sigma} J_\sigma}{4^k} \int_0^\delta r^{2|\sigma|+2k+N-3} e^{-tr^2} \sin^2(tr\sqrt{1-r^2/4}) dr + O(t^{-K-N/2+1}) \\
&= \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} \frac{d_{2\sigma} J_\sigma}{4^k} I_{1,2|\sigma|+2k+N-3}^\delta(t) + O(t^{-K-N/2+1}) \\
&= \sum_{j=0}^{K-1} \left({}_N X_1^1(j) + {}_N X_1^2(j) \right) t^{-j-N/2+1} + O(t^{-K-N/2+1}), \tag{2.19}
\end{aligned}$$

where

$$\begin{aligned}
{}_N X_1^1(j) &= \sum_{|\sigma|+k=j} \frac{d_{2\sigma} J_\sigma}{4^{k+1}} \Gamma(j + \frac{N}{2} - 1), \\
{}_N X_1^2(j) &= \begin{cases} \sum_{2|\sigma|+2k+n=j-N/2+1} -\frac{d_{2\sigma} J_\sigma}{2^{2k+1}} B_{2|\sigma|+2k+N-3,n} & N \geq 4 \text{ even and } \frac{N}{2} - 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

2.2.2. Expansion of $Y_1(t)$

We first observe

$$Y_1(t) = \int_{|\xi| \leq \delta} |\hat{u}_0(\xi)|^2 (h_2(t, \xi))^2 d\xi. \tag{2.20}$$

In order to obtain an asymptotic expansion of $Y_1(t)$ that is not simply $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$, we must further assume that $K \geq 2$. With this additional assumption, in the neighborhood $\{\xi \in \mathbb{R}^N : |\xi| \leq \delta\}$ we now truncate (1.7) to

$$|\hat{u}_0(\xi)|^2 = \sum_{|\sigma| \leq 2K-3} f_\sigma \xi^\sigma + O(|\xi|^{2K-2}). \tag{2.21}$$

We substitute (1.11) and (2.21) into (2.20) and estimate the integral with the $O(|\xi|^{2K-2})$ -term we obtain as $t \rightarrow \infty$

$$Y_1(t) = \sum_{|\sigma| \leq 2K-3} f_\sigma \int_{|\xi| \leq \delta} \xi^\sigma e^{-t|\xi|^2} \cos^2(t|\xi|\sqrt{1-|\xi|^2/4}) d\xi + O(t^{-K-N/2+1}). \tag{2.22}$$

As we noted for $X_1(t)$, if some multi-index σ satisfying $0 \leq |\sigma| \leq 2K-3$ has an odd entry, the integral in (2.22) vanishes. So we consider multi-indices with only even entries and switch to spherical coordinates to obtain

$$Y_1(t) = \sum_{|\sigma| \leq K-2} f_{2\sigma} J_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2} \cos^2(tr\sqrt{1-r^2/4}) dr + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty).$$

By Lemma 2.4 with $Q = Q_\sigma := \max\{0, \lceil K - 2|\sigma| - N/2 - 1 \rceil\}$, we obtain as $t \rightarrow \infty$

$$\begin{aligned} Y_1(t) &= \sum_{|\sigma| \leq K-2} f_{2\sigma} J_\sigma I_{2,2|\sigma|+N-1}^\delta(t) + O(t^{-K-N/2+1}) \\ &= \sum_{j=0}^{K-1} \left({}_N Y_1^1(j) + {}_N Y_1^2(j) \right) t^{-j-N/2+1} + O(t^{-K-N/2+1}), \end{aligned} \quad (2.23)$$

where

$$\begin{aligned} {}_N Y_1^1(j) &= \begin{cases} \sum_{|\sigma|=j-1} \frac{f_{2\sigma} J_\sigma}{4} \Gamma(j + \frac{N}{2} - 1) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases} \\ {}_N Y_1^2(j) &= \begin{cases} \sum_{2|\sigma|+n=j-N/2-1} \frac{f_{2\sigma} J_\sigma}{2} B_{2|\sigma|+N-1,n} & N \geq 4 \text{ even and } \frac{N}{2} + 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2.2.3. Expansion of $Z_1(t)$

Again we need to require $K \geq 2$ in order to obtain more than just a $O(t^{-K-N/2+1})$ -term for the asymptotic expansion of $Z_1(t)$. Nonetheless we begin with

$$Z_1(t) = 2\Re \left(\int_{|\xi| \leq \delta} \left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) \bar{\hat{u}}_0(\xi) h_1(t, \xi) h_2(t, \xi) d\xi \right). \quad (2.24)$$

We substitute (1.8), (1.10), and (1.11) into (2.24) and estimate the resulting integral with the $O(|\xi|^{2K})$ -term; it is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Then switching to spherical coordinates and using the definitions of $h_1(t, \xi)$ and $h_2(t, \xi)$, we obtain

$$\begin{aligned} Z_1(t) &= \sum_{|\sigma| \leq K-1} 2\Re(l_{2\sigma}) J_\sigma \int_{|\xi| \leq \delta} r^{2|\sigma|+N-2} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4}) \cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}} dr \\ &\quad + O(t^{-K-N/2+1}). \end{aligned} \quad (2.25)$$

We observe the Taylor expansion of $(1-r^2/4)^{-1/2}$ in the δ -neighborhood of $r=0$ is

$$\frac{1}{\sqrt{1-r^2/4}} = \sum_{k=0}^{L-1} \alpha_k r^{2k} + O(r^{2L}), \quad (2.26)$$

where $L \in \mathbb{N}_0$ and

$$\alpha_k = \frac{(2k-1)!!}{8^k \cdot k!} \quad (k \in \mathbb{N}_0),$$

with the double factorial function defined

$$m!! = \begin{cases} 1 & m = -1, 0 \\ m(m-2) \cdot \dots \cdot (3)(1) & m \in \mathbb{N} \text{ odd} \\ m(m-2) \cdot \dots \cdot (4)(2) & m \in \mathbb{N} \text{ even.} \end{cases} \quad (2.27)$$

By substituting (2.26) in (2.25) with $L = L_\sigma := K - |\sigma|$, each resulting integral with a $O(r^{2L_\sigma})$ -term is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Therefore

$$Z_1(t) = \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} 2\Re(l_{2\sigma}) J_\sigma \alpha_k I_{3,2|\sigma|+2k+N-2}^\delta(t) + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty).$$

By Lemma 2.5 with $Q = Q_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - N/2 \rceil\}$

$$Z_1(t) = \sum_{j=0}^{K-1} {}_N Z_1^1(j) t^{-j-N/2+1} + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty), \quad (2.28)$$

where

$${}_N Z_1^1(j) = \begin{cases} \sum_{2|\sigma|+2k+n=j-N/2} -\Re(l_{2\sigma}) J_\sigma \alpha_k \Im(B_{2|\sigma|+2k+N-2,n}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

2.2.4. Asymptotic expansion of $\|u(t, \cdot)\|_2^2$ as $t \rightarrow \infty$

From the discussion at the beginning of section 2.2, $\|u(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_{2,\delta}^2 + O(e^{-\eta t})$ as $t \rightarrow \infty$ for some $\eta > 0$ dependent on the fixed $0 < \delta < 1$. Since $\|\hat{u}(t, \cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$, we combine results (2.19), (2.23), and (2.28) to obtain the asymptotic expansion

$$\|u(t, \cdot)\|_2^2 = t^{-N/2+1} \left(\sum_{j=0}^{K-1} {}_N W_j t^{-j} + O(t^{-K}) \right) \quad (t \rightarrow \infty),$$

where, for $j \in \{0, \dots, K-1\}$,

$${}_N W_j = {}_N X_1^1(j) + {}_N X_1^2(j) + {}_N Y_1^1(j) + {}_N Y_1^2(j) + {}_N Z_1^1(j).$$

This completes the proof of Theorem 1.1.

The first two coefficients for each $N \geq 3$. Let us assume that $N \geq 3$ is odd and $K = 2$. Then $\|u(t, \cdot)\|_2^2 = {}_N W_0 t^{-N/2+1} + {}_N W_1 t^{-N/2} + O(t^{-N/2-1})$ as $t \rightarrow \infty$, where

$${}_N W_0 = \frac{d_0 J_0}{4} \Gamma\left(\frac{N}{2} - 1\right), \quad {}_N W_1 = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} J_\sigma}{4^{k+1}} \Gamma\left(\frac{N}{2}\right) + \frac{f_0 J_0}{4} \Gamma\left(\frac{N}{2}\right).$$

It is a straightforward computation to verify that for $m \geq 1$ odd

$$J_{e_j} = \frac{(\sqrt{2})^{m+1} \pi^{(m-1)/2}}{m!!} \quad (j \in \{1, \dots, m\}), \quad (2.29)$$

where $e_j = (\delta_{ij})_{i=1}^m$ is an m -dimensional multi-index of magnitude one, and $m!!$ denotes the double factorial as given in (2.27). We combine the results (2.29) with the identities $d_0 = |P_1|^2$, $f_0 = |P_0|^2$, and $J_0 = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ to simplify each ${}_N W_j$ ($j \in \{0, 1\}$) and obtain (1.15) for dimension $N \geq 3$ odd.

Remark. It is again straightforward to verify that for $m \geq 2$ even

$$J_{e_j} = \frac{(2\pi)^{m/2}}{m!!} \quad (j \in \{1, \dots, m\}). \quad (2.30)$$

Then if we assume that $N \geq 6$ is even, we obtain the same expressions for the first $K = 2$ coefficients (${}_N W_0$ and ${}_N W_1$) as the odd $N \geq 3$ case. The coefficients are given in (1.15).

Let us now assume that $N = 4$ and $K = 2$. Then $\|u(t, \cdot)\|_2^2 = {}_4 W_0 t^{-1} + {}_4 W_1 t^{-2} + O(t^{-3})$ as $t \rightarrow \infty$, where

$${}_4 W_0 = \frac{d_0 J_0}{4} \Gamma(1), \quad {}_4 W_1 = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} J_\sigma}{4^{k+1}} \Gamma(2) - \frac{d_0 J_0}{2} B_{1,0} + \frac{f_0 J_0}{4} \Gamma(2).$$

Each ${}_4 W_j$ can be simplified by using the results for J_{e_j} from (2.30) and the expressions for the $B_{m,n}$ given in (2.13). We obtain the $N = 4$ case of (1.15).

3. Proof of Theorem 1.2

Now that we have found the asymptotic expansion of $\|u(t, \cdot)\|_2^2$ as $t \rightarrow \infty$ in dimension $N \geq 3$, we seek the full asymptotic expansion of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$ as $t \rightarrow \infty$ in dimension $N \geq 3$. To do this we need to finish computing the expansions of $X(t)$, $Y(t)$, and $Z(t)$, each given in (1.12), (1.13), and (1.14), respectively. But before we are able to find these expansions, we must first study the expansions of some more integrals that appear in the analysis.

3.1. Auxiliary lemmas

Lemma 3.1. For $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, and $t > 0$ define

$$G_{2,m}^\epsilon(t) := \int_0^\epsilon r^m \exp(-tr^2 - irt\sqrt{1-r^2/4} + irt) dr. \quad (3.1)$$

If $Q \in \mathbb{N}_0$, then as $t \rightarrow \infty$

$$G_{2,m}^\epsilon(t) = \sum_{n=0}^{2Q-1} C_{m,n} t^{-m/2-n/2-1/2} + O(t^{-m/2-Q-1/2}),$$

where, for $n \in \{0, \dots, 2Q-1\}$,

$$C_{m,n} \in \begin{cases} \mathbb{R} & n \in \mathbb{N}_0 \text{ even} \\ i\mathbb{R} & n \in \mathbb{N} \text{ odd.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. With $g(t, r) = \exp(irt(2 - \sqrt{4 - r^2}))$ as in (2.2), we rewrite (3.1) as

$$G_{2,m}^\epsilon(t) = \int_0^\epsilon r^m e^{-tr^2} g\left(\frac{t}{2}, r\right) dr. \quad (3.2)$$

Expanding $g(t/2, r)$ in a Taylor series about $r = 0$, we obtain $g(t/2, r) = \sum_{k=0}^\infty g_k(t/2)r^k$. By Lemma 2.1 with $t \geq 2$, we have that for any $J \in \mathbb{N}_0$ there exists $C_J > 0$ such that

$$|\tilde{g}_J(\frac{t}{2}, r)| \leq \frac{C_J}{2^J} t^J r^{3J}. \quad (3.3)$$

We then rewrite (3.2) as

$$G_{2,m}^\epsilon(t) = \sum_{k=0}^{3J-1} g_k\left(\frac{t}{2}\right) \int_0^\epsilon r^{m+k} e^{-tr^2} dr + \int_0^\epsilon r^m e^{-tr^2} \tilde{g}_J\left(\frac{t}{2}, r\right) dr.$$

By the estimate (3.3), for $t \geq 2$ the final term is $O(t^{-(m+J+1)/2})$. Thus as $t \rightarrow \infty$

$$\begin{aligned} G_{2,m}^\epsilon(t) &= \sum_{k=0}^{3J-1} g_k\left(\frac{t}{2}\right) \int_0^\epsilon r^{m+k} e^{-tr^2} dr + O(t^{-(m+J+1)/2}) \\ &= \sum_{k=0}^{3J-1} g_k\left(\frac{t}{2}\right) \left(\frac{1}{2} \Gamma\left(\frac{m+k+1}{2}\right) t^{-(m+k+1)/2} + O(e^{-t\epsilon^2/2}) \right) + O(t^{-m/2-J/2-1/2}) \\ &= \sum_{k=0}^{3J-1} \frac{1}{2} g_k\left(\frac{t}{2}\right) \Gamma\left(\frac{m+k+1}{2}\right) t^{-m/2-k/2-1/2} + O(t^{-m/2-J/2-1/2}). \end{aligned} \quad (3.4)$$

Let $Q \in \mathbb{N}_0$ and suppose that, after collecting terms according to powers of t , we want the first $2Q$ terms of the above expansion and then a O -term. Then as $t \rightarrow \infty$

$$G_{2,m}^\epsilon(t) = \sum_{n=0}^{2Q-1} C_{m,n} t^{-m/2-n/2-1/2} + O(t^{-m/2-Q-1/2}). \quad (3.5)$$

So that the error terms of (3.4) and (3.5) agree, we require that $-m/2 - Q - 1/2 = -(m + J + 1)/2$, which is equivalent to $J = 2Q$.

Observe that $C_{m,n}$ is the coefficient of $t^{-m/2-n/2-1/2}$ when the sum (3.4) is rearranged according to the powers of t . Each term $\frac{1}{2} g_k(\frac{t}{2}) \Gamma(\frac{m+k+1}{2}) t^{-m/2-k/2-1/2}$ has leading degree $j - m/2 - k/2 - 1/2$, where j is a non-negative integer at most $k/3$ (see proof of Lemma 2.1). Note that $-m/2 - n/2 - 1/2 = j - m/2 - k/2 - 1/2$ if and only if $n = k - 2j$. Since

$j \leq k/3$, this implies that $k/3 \leq n$, which in turn implies $0 \leq k \leq 3n$. So for fixed $m \in \mathbb{N}_0$ and $n \in \{0, \dots, 2Q - 1\}$, to compute $C_{m,n}$ we must find the first $3n + 1$ terms of (3.4) and look for the coefficient of $t^{-m/2-n/2-1/2}$ after collecting terms according to the powers of t .

For $m \in \mathbb{N}_0$, we may compute the first few $C_{m,n}$ using a computer algebra system like *Mathematica*,

$$\begin{aligned} C_{m,0} &= \frac{1}{2} \Gamma\left(\frac{m+1}{2}\right) \\ C_{m,1} &= \frac{i}{16} \Gamma\left(\frac{m+4}{2}\right) \\ C_{m,2} &= -\frac{1}{256} \Gamma\left(\frac{m+7}{2}\right) \\ C_{m,3} &= \frac{i}{256} \Gamma\left(\frac{m+6}{2}\right) - \frac{i}{6144} \Gamma\left(\frac{m+10}{2}\right) \\ C_{m,4} &= \frac{1}{196608} \Gamma\left(\frac{m+13}{2}\right) - \frac{1}{2048} \Gamma\left(\frac{m+9}{2}\right). \end{aligned} \quad (3.6)$$

We now determine if $C_{m,n}$ is real or purely imaginary depending only on n . Suppose $n \geq 0$ is even and fix $0 \leq k \leq 3n$. Then a generic term of $\frac{1}{2} g_k\left(\frac{t}{2}\right) \Gamma\left(\frac{m+k+1}{2}\right) t^{-m/2-k/2-1/2}$ is of the form $a(it)^l t^{-m/2-k/2-1/2}$, where $a \in \mathbb{R}$ and $0 \leq l \leq k/3$. In order for this term to contribute to $C_{m,n}$, it must hold that the powers of t are equal, i.e., $l - m/2 - k/2 - 1/2 = -m/2 - n/2 - 1/2$, which is equivalent to $n = k - 2l$. Since n is assumed to be even, it must hold that k is even. So in fact, only even $0 \leq k \leq 3n$ are able to contribute terms to $C_{m,n}$ if n is even. Since k is even, $g_k(t/2)$ is an even polynomial in t (see proof of Proposition 2.2), and thus l is even. Hence for even n , any contributing term to $C_{m,n}$ is real, and thus $C_{m,n} \in \mathbb{R}$. A similar argument shows that if n is odd, then $C_{m,n} \in i\mathbb{R}$. \square

Lemma 3.2. For $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, and $t > 0$ define

$$G_{3,m}^\epsilon(t) := \int_0^\epsilon r^m \exp(-tr^2 - irt\sqrt{1-r^2/4} - irt) dr.$$

If $Q \in \mathbb{N}_0$, then as $t \rightarrow \infty$

$$G_{3,m}^\epsilon(t) = \sum_{n=0}^{Q-1} E_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where, for $n \in \{0, \dots, Q - 1\}$,

$$E_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. We observe that $G_{3,m}^\epsilon(t)$ is closely related to $G_{1,m}^\epsilon(t)$:

$$G_{3,m}^\epsilon(t) = \int_0^\epsilon r^m \exp(-tr^2 - 2irt) g\left(\frac{t}{2}, r\right) dr,$$

where $g(t, r) = \exp(irt(2 - \sqrt{4 - r^2}))$ as in (2.2). The analysis of $G_{1,m}^\epsilon(t)$ carries over. Since we now have the argument $t/2$ instead of t in g , for $Q \in \mathbb{N}_0$ the expansion is

$$G_{3,m}^\epsilon(t) = \sum_{n=0}^{Q-1} E_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}) \quad (t \rightarrow \infty).$$

The $E_{m,n}$ satisfy $E_{m,n} \in i\mathbb{R}$ if m is even, and $E_{m,n} \in \mathbb{R}$ if m is odd, just like $B_{m,n}$. Further, the $E_{m,n}$ can be computed on a computer algebra system like *Mathematica*; the first three are

$$\begin{aligned} E_{m,0} &= A_{m,0} \\ E_{m,1} &= A_{m,1} \\ E_{m,2} &= A_{m,2} + \frac{i}{8} A_{m+3,0}. \quad \square \end{aligned} \tag{3.7}$$

Lemma 3.3. Let $m \in \mathbb{N}_0$, $0 < \epsilon < 2$, and $t > 0$. Define

$$\begin{aligned} H_{1,m}^\epsilon(t) &:= \int_0^\epsilon r^m e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \sin(tr) dr, \\ H_{2,m}^\epsilon(t) &:= \int_0^\epsilon r^m e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \cos(tr) dr. \end{aligned}$$

1. If $m \in \mathbb{N}_0$ is even, then as $t \rightarrow \infty$

$$H_{1,m}^\epsilon(t) = H_{2,m}^\epsilon(t) = \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-m/2-n-1/2} + O(t^{-m/2-Q-1/2}), \tag{3.8}$$

where $Q \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}_0$ is odd, then as $t \rightarrow \infty$

$$\begin{aligned} H_{1,m}^\epsilon(t) &= \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-m/2-n-1/2} + O(t^{-m/2-Q-1/2}) + \sum_{n=0}^{R-1} -\frac{1}{2} E_{m,n} t^{-m-n-1} \\ &\quad + O(t^{-m-R-1}), \end{aligned} \tag{3.9}$$

$$H_{2,m}^\epsilon(t) = \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-m/2-n-1/2} + O(t^{-m/2-Q-1/2}) + \sum_{n=0}^{R-1} \frac{1}{2} E_{m,n} t^{-m-n-1} + O(t^{-m-R-1}),$$

where $Q, R \in \mathbb{N}_0$.

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. We recall that $\sin(x) \sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$. Hence

$$H_{1,m}^\epsilon(t) = \frac{1}{2} \Re(G_{2,m}^\epsilon(t)) - \frac{1}{2} \Re(G_{3,m}^\epsilon(t)),$$

to which we apply Lemmas 3.1 and 3.2. Therefore as $t \rightarrow \infty$

$$H_{1,m}^\epsilon(t) = \frac{1}{2} \Re \left(\sum_{n=0}^{2Q-1} C_{m,n} t^{-(m+n+1)/2} + O(t^{-m/2-Q-1/2}) \right) - \frac{1}{2} \Re \left(\sum_{n=0}^{R-1} E_{m,n} t^{-m-n-1} + O(t^{-m-R-1}) \right),$$

where $Q, R \in \mathbb{N}_0$. We now recall the following: for any $m \in \mathbb{N}_0$, $C_{m,n} \in i\mathbb{R}$ if $n \in \mathbb{N}$ is odd and $C_{m,n} \in \mathbb{R}$ if $n \in \mathbb{N}_0$ is even; and $E_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}_0$ is even and $E_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}_0$ is odd. If $m \in \mathbb{N}_0$ is odd, (3.9) is immediate by linearity of $\Re(\cdot)$. If $m \in \mathbb{N}_0$ is even, then by setting $R \geq Q$ we obtain (3.8).

To prove the claims about $H_{2,m}^\epsilon(t)$, we use the fact that $\cos(x) \cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$ and repeat the previous argument. \square

Lemma 3.4. Let $m \in \mathbb{N}_0$, $0 < \epsilon < 2$, and $t > 0$. Define

$$H_{3,m}^\epsilon(t) := \int_0^\epsilon r^m e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \cos(tr) dr,$$

$$H_{4,m}^\epsilon(t) := \int_0^\epsilon r^m e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \sin(tr) dr.$$

1. If $m \in \mathbb{N}_0$ is even, then as $t \rightarrow \infty$

$$H_{3,m}^\epsilon(t) = \sum_{n=0}^{R-1} -\frac{1}{2} \Im(E_{m,n}) t^{-m-n-1} + O(t^{-m-R-1}) + \sum_{n=0}^{Q-1} -\frac{1}{2} \Im(C_{m,2n+1}) t^{-m/2-n-1} + O(t^{-m/2-Q-1/2}), \quad (3.10)$$

$$H_{4,m}^\epsilon(t) = \sum_{n=0}^{R-1} -\frac{1}{2} \Im(E_{m,n}) t^{-m-n-1} + O(t^{-m-R-1}) \\ + \sum_{n=0}^{Q-1} \frac{1}{2} \Im(C_{m,2n+1}) t^{-m/2-n-1} + O(t^{-m/2-Q-1/2}),$$

where $Q, R \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}_0$ is odd, then as $t \rightarrow \infty$

$$H_{3,m}^\epsilon(t) = \sum_{n=0}^{Q-1} -\frac{1}{2} \Im(C_{m,2n+1}) t^{-m/2-n-1} + O(t^{-m/2-Q-1/2}), \quad (3.11) \\ H_{4,m}^\epsilon(t) = \sum_{n=0}^{Q-1} \frac{1}{2} \Im(C_{m,2n+1}) t^{-m/2-n-1} + O(t^{-m/2-Q-1/2}),$$

where $Q \in \mathbb{N}_0$.

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. Since $\sin(x) \cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$, we have

$$H_{3,m}^\epsilon(t) = -\frac{1}{2} \Im(G_{3,m}^\epsilon(t)) - \frac{1}{2} \Im(G_{2,m}^\epsilon(t)). \quad (3.12)$$

We apply Lemmas 3.2 and 3.1 to (3.12) to find that as $t \rightarrow \infty$

$$H_{3,m}^\epsilon(t) = -\frac{1}{2} \Im \left(\sum_{n=0}^{R-1} E_{m,n} t^{-m-n-1} + O(t^{-m-R-1}) \right) \\ - \frac{1}{2} \Im \left(\sum_{n=0}^{2Q-1} C_{m,n} t^{-(m+n+1)/2} + O(t^{-m/2-Q-1/2}) \right),$$

where $Q, R \in \mathbb{N}_0$. By the facts about $E_{m,n}$ and $C_{m,n}$ we recalled in the proof of Lemma 3.3 we conclude that if $m \in \mathbb{N}_0$ is even, then (3.10) follows from linearity of $\Im(\cdot)$. If $m \in \mathbb{N}_0$ is odd, then by setting $R \geq Q$ we obtain (3.11).

To prove the claims about $H_{4,m}^\epsilon(t)$, we use the fact that $\cos(x) \sin(y) = \frac{1}{2}(\sin(x+y) - \sin(x-y))$ and repeat the previous argument. \square

3.2. Intermediate computations

We are now in a position to find the asymptotic expansions of the remaining parts of $X(t)$, $Y(t)$, and $Z(t)$. In doing so, we will be able to determine the asymptotic expansion of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$ as $t \rightarrow \infty$. Using results about $X_1(t)$, $Y_1(t)$, and $Z_1(t)$ from section 2.2 and the sections that follow, we will be able to prove Theorem 1.2. Let us assume that $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$.

3.2.1. Expansion of $X_2(t)$

$$X_2(t) = |P_1|^2 \int_{|\xi| \leq \delta} e^{-t|\xi|^2} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi.$$

We switch to spherical coordinates and use the fact that $\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$ to obtain

$$\begin{aligned} X_2(t) &= J_0 |P_1|^2 \int_0^\delta r^{N-3} e^{-tr^2} \sin^2(tr) dr \\ &= \frac{J_0 |P_1|^2}{2} \int_0^\delta r^{N-3} e^{-tr^2} dr - \frac{J_0 |P_0|^2}{2} \int_0^\delta r^{N-3} e^{-tr^2} \cos(2tr) dr. \end{aligned} \quad (3.13)$$

As $t \rightarrow \infty$ the integral in the first term of (3.13) is $\frac{1}{2} \Gamma(\frac{N}{2} - 1) t^{-N/2+1} + O(e^{-t\delta^2/2})$. The integral in the second term of (3.13) equals $\Re(F_{N-3}^\delta(t))$, where $F_k^\delta(t)$ is as in (2.6). Thus by (3.13) and (2.9), as $t \rightarrow \infty$

$$X_2(t) = \frac{J_0 |P_1|^2}{4} \Gamma(\frac{N}{2} - 1) t^{-N/2+1} - \sum_{p=0}^{P-1} \frac{J_0 |P_1|^2}{2} \Re(A_{N-3,p}) t^{-N+2-p} + O(t^{-N+2-P}). \quad (3.14)$$

By definition of $A_{k,p}$ in (2.10), the sum in (3.14) vanishes unless $N \geq 4$ is even. And if $N \geq 4$ is even, we require $K \geq N/2$ else each term of the sum in (3.14) is $O(t^{-K-N/2+1})$. In any case we choose $P := \max\{0, \lceil K - N/2 + 1 \rceil\}$ so that $O(t^{-N+2-P}) = O(t^{-K-N/2+1})$. Thus

$$X_2(t) = \sum_{j=0}^{K-1} {}_N X_2^1(j) t^{-N/2+1-j} + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty), \quad (3.15)$$

where

$${}_N X_2^1(j) = \begin{cases} \frac{J_0 |P_1|^2}{4} \Gamma(\frac{N}{2} - 1) & j = 0 \\ -\frac{J_0 |P_1|^2}{2} A_{N-3, j-N/2+1} & N \geq 4 \text{ even and } \frac{N}{2} - 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

3.2.2. Expansion of $X_3(t)$

We observe

$$X_3(t) = -2\Re \left(\bar{P}_1 \int_{|\xi| \leq \delta} e^{-\frac{t|\xi|^2}{2}} \left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) \frac{\sin(t|\xi|)}{|\xi|} h_1(t, \xi) d\xi \right).$$

We substitute (1.5) and (1.10) in to the above expression for $X_3(t)$ and estimate the resulting integral with the $O(|\xi|^{2K})$ -term to find it is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Thus as $t \rightarrow \infty$

$$\begin{aligned}
X_3(t) &= -2\Re \left(\sum_{|\sigma| \leq 2K-1} \bar{P}_1 c_\sigma \int_{|\xi| \leq \delta} \xi^\sigma e^{-t|\xi|^2} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4}) \sin(t|\xi|)}{|\xi|^2 \sqrt{1-|\xi|^2/4}} d\xi \right) \\
&\quad + O(t^{-K-N/2+1}) \\
&= \sum_{|\sigma| \leq K-1} -2\Re(\bar{P}_1 c_{2\sigma}) \int_{|\xi| \leq \delta} \xi^{2\sigma} e^{-t|\xi|^2} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4}) \sin(t|\xi|)}{|\xi|^2 \sqrt{1-|\xi|^2/4}} d\xi \\
&\quad + O(t^{-K-N/2+1}) \\
&= \sum_{|\sigma| \leq K-1} -2\Re(\bar{P}_1 c_{2\sigma}) J_\sigma \int_0^\delta r^{2|\sigma|+N-3} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4}) \sin(tr)}{\sqrt{1-r^2/4}} dr \\
&\quad + O(t^{-K-N/2+1}). \tag{3.16}
\end{aligned}$$

The second equality follows from the fact that if any σ satisfying $0 \leq |\sigma| \leq K-1$ has an odd entry, then the integral evaluates to zero.

We use (2.26) in (3.16) with $L = L_\sigma := K - |\sigma|$ to get rid of the denominator in the integrals. By choice of L_σ , the integrals with the $O(r^{2L})$ -terms are all $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Thus

$$X_3(t) = \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} -2\Re(\bar{P}_1 c_{2\sigma}) J_\sigma \alpha_k H_{1,2|\sigma|+2k+N-3}^\delta(t) + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty). \tag{3.17}$$

Since $N \geq 3$ we may use Lemma 3.3 in (3.17) with $Q = Q_{\sigma,k} := K - |\sigma| - k$ and $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - N/2 + 1 \rceil\}$ to obtain that as $t \rightarrow \infty$

$$X_3(t) = \sum_{j=0}^{K-1} \left({}_N X_3^1(j) + {}_N X_3^2(j) \right) t^{-j-N/2+1} + O(t^{-K-N/2+1}), \tag{3.18}$$

where

$$\begin{aligned}
{}_N X_3^1(j) &= \sum_{|\sigma|+k+n=j} -\Re(\bar{P}_1 c_{2\sigma}) J_\sigma \alpha_k C_{2|\sigma|+2k+N-3,2n}, \\
{}_N X_3^2(j) &= \begin{cases} \sum_{2|\sigma|+2k+n=j-N/2+1} \Re(\bar{P}_1 c_{2\sigma}) J_\sigma \alpha_k E_{2|\sigma|+2k+N-3,n} \\ N \geq 4 \text{ even and } \frac{N}{2} - 1 \leq j \leq K-1 \\ 0 \quad \text{otherwise.} \end{cases}
\end{aligned}$$

3.2.3. Asymptotic expansion of $X(t)$ as $t \rightarrow \infty$

Combining results (2.19), (3.15), and (3.18) for $X_1(t)$, $X_2(t)$, and $X_3(t)$, respectively, with the fact that $X(t) = X_1(t) + X_2(t) + X_3(t)$, we obtain the expansion as $t \rightarrow \infty$

$$X(t) = \sum_{j=0}^{K-1} {}_N V_j^X t^{-j-N/2+1} + O(t^{-K-N/2+1}),$$

where, for $j \in \{0, \dots, K-1\}$,

$${}_N V_j^X = {}_N X_1^1(j) + {}_N X_1^2(j) + {}_N X_2^1(j) + {}_N X_3^1(j) + {}_N X_3^2(j).$$

The first two coefficients for each $N \geq 3$. Let us first assume that $N \geq 3$ is odd and $K = 2$. Then $X(t) = {}_N V_0^X t^{-N/2+1} + {}_N V_1^X t^{-N/2} + O(t^{-N/2-1})$ as $t \rightarrow \infty$, where

$$\begin{aligned} {}_N V_0^X &= \frac{d_0 J_0}{4} \Gamma\left(\frac{N}{2} - 1\right) + \frac{J_0 |P_1|^2}{4} \Gamma\left(\frac{N}{2} - 1\right) - \Re(\bar{P}_1 c_0) J_0 \alpha_0 C_{N-3,0}, \\ {}_N V_1^X &= \sum_{|\sigma|+k=1} \frac{d_{2\sigma} J_\sigma}{4^{k+1}} \Gamma\left(\frac{N}{2}\right) + \sum_{|\sigma|+k+n=1} -\Re(\bar{P}_1 c_{2\sigma}) J_\sigma \alpha_k C_{2|\sigma|+2k+N-3,2n}. \end{aligned}$$

We recall facts (2.29) and (2.30) from section 2.2.4 about the J_σ , the identities for the $C_{m,n}$ given in (3.6), and that $d_0 = |P_1|^2$ and $c_0 = P_1$ in order to simplify each ${}_N V_j^X$ ($j \in \{0, 1\}$).

$${}_N V_0^X = 0, \quad {}_N V_1^X = \frac{N(N+2)|P_1|^2 \pi^{N/2}}{512} + \frac{\pi^{N/2}}{2N} \sum_{j=1}^N d_{2e_j} - \frac{\pi^{N/2}}{N} \sum_{j=1}^N \Re(\bar{P}_1 c_{2e_j}).$$

Remark. Through similar computations it can be shown that the preceding quantities for ${}_N V_j^X$ ($j \in \{0, 1\}$) are also valid for the cases $N \geq 4$ even with $K = 2$. It is also worth noting that in all cases the $t^{-N/2+1}$ -term vanishes.

3.2.4. Expansion of $Y_2(t)$

We use the fact that $\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$ and then switch to spherical coordinates to obtain

$$\begin{aligned} Y_2(t) &= |P_0|^2 \int_{|\xi| \leq \delta} e^{-t|\xi|^2} \cos^2(t|\xi|) d\xi \\ &= \frac{J_0 |P_0|^2}{2} \int_0^\delta r^{N-1} e^{-tr^2} dr + \frac{J_0 |P_0|^2}{2} \int_0^\delta r^{N-1} e^{-tr^2} \cos(2tr) dr. \end{aligned} \quad (3.19)$$

The asymptotics of the integral in the first term of (3.19) are $\int_0^\delta r^{N-1} e^{-tr^2} dr = \frac{1}{2} \Gamma\left(\frac{N}{2}\right) t^{-N/2} + O(e^{-t\delta^2/2})$ as $t \rightarrow \infty$. The integral in the second term of (3.19) equals $\Re(F_{N-1}^\delta(t))$. Thus, combining (2.9) and (3.19) we obtain as $t \rightarrow \infty$

$$Y_2(t) = \frac{J_0 |P_0|^2}{4} \Gamma\left(\frac{N}{2}\right) t^{-N/2} + \sum_{p=0}^{P-1} \frac{J_0 |P_0|^2}{2} \Re(A_{N-1,p}) t^{-N-p} + O(t^{-N-P}). \quad (3.20)$$

Much the same as with $X_2(t)$, the sum in (3.20) vanishes unless $N \geq 4$ is even, and even so, every term is $O(t^{-K-N/2+1})$ unless $K \geq N/2 + 2$. Regardless, we choose $P := \max\{0, \lceil K - N/2 - 1 \rceil\}$ so that $O(t^{-N-P}) = O(t^{-K-N/2+1})$. Therefore

$$Y_2(t) = \sum_{j=0}^{K-1} {}_N Y_2^1(j) t^{-N/2+1-j} + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty), \quad (3.21)$$

where

$${}_N Y_2^1(j) = \begin{cases} \frac{J_0 |P_0|^2}{4} \Gamma(\frac{N}{2}) & j = 1 \\ \frac{J_0 |P_0|^2}{2} A_{N-1, j-N/2-1} & N \geq 4 \text{ even and } \frac{N}{2} + 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

3.2.5. Expansion of $Y_3(t)$

$$Y_3(t) = -2\Re \left(\bar{P}_0 \int_{|\xi| \leq \delta} \hat{u}_0(\xi) e^{-\frac{t|\xi|^2}{2}} h_2(t, \xi) \cos(t|\xi|) d\xi \right).$$

Using (1.11) we obtain

$$Y_3(t) = -2\Re \left(\bar{P}_0 \int_{|\xi| \leq \delta} \hat{u}_0(\xi) e^{-t|\xi|^2} \cos(t|\xi| \sqrt{1 - |\xi|^2/4}) \cos(t|\xi|) d\xi \right) \quad (3.22)$$

Let us assume that $K \geq 2$. Then into (3.22) we substitute (1.4) truncated to $\sum_{|\sigma| \leq 2K-3} b_\sigma \xi^\sigma + O(|\xi|^{2K-2})$. The estimate of the integral with the $O(|\xi|^{2K-2})$ -term is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Therefore as $t \rightarrow \infty$

$$\begin{aligned} Y_3(t) &= -2\Re \left(\sum_{|\sigma| \leq 2K-3} \bar{P}_0 b_\sigma \int_{|\xi| \leq \delta} \xi^\sigma e^{-t|\xi|^2} \cos(t|\xi| \sqrt{1 - |\xi|^2/4}) \cos(t|\xi|) d\xi \right) \\ &\quad + O(t^{-K-N/2+1}) \\ &= \sum_{|\sigma| \leq K-2} -2\Re(\bar{P}_0 b_{2\sigma}) J_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2} \cos(tr \sqrt{1 - r^2/4}) \cos(tr) dr \\ &\quad + O(t^{-K-N/2+1}) \\ &= \sum_{|\sigma| \leq K-2} -2\Re(\bar{P}_0 b_{2\sigma}) J_\sigma H_{2,2|\sigma|+N-1}^\delta(t) + O(t^{-K-N/2+1}). \end{aligned} \quad (3.23)$$

The second equality is obtained by noting that multi-indices σ satisfying $0 \leq |\sigma| \leq 2K - 3$ with an odd entry yield an integral that evaluates to zero, then switching to spherical coordinates and using linearity of $\Re(\cdot)$.

We now apply Lemma 3.3 to equality (3.23) with $Q = Q_\sigma := K - |\sigma| - 1$ and $R = R_\sigma := \max\{0, \lceil K - 2|\sigma| - N/2 - 1 \rceil\}$. Thus as $t \rightarrow \infty$

$$Y_3(t) = \sum_{j=0}^{K-1} \left({}_N Y_3^1(j) + {}_N Y_3^2(j) \right) t^{-N/2+1-j} + O(t^{-K-N/2+1}), \quad (3.24)$$

where

$${}_N Y_3^1(j) = \begin{cases} \sum_{|\sigma|+n=j-1} -\Re(\bar{P}_0 b_{2\sigma}) J_\sigma C_{2|\sigma|+N-1,2n} & 1 \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$${}_N Y_3^2(j) = \begin{cases} \sum_{2|\sigma|+n=j-N/2-1} -\Re(\bar{P}_0 b_{2\sigma}) J_\sigma E_{2|\sigma|+N-1,n} & N \geq 4 \text{ even and } \frac{N}{2} + 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

3.2.6. Asymptotic expansion of $Y(t)$ as $t \rightarrow \infty$

We combine results (2.23), (3.21), and (3.24) for $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$, respectively, with the fact that $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$ to obtain the expansion as $t \rightarrow \infty$

$$Y(t) = \sum_{j=0}^{K-1} {}_N V_j^Y t^{-j-N/2+1} + O(t^{-K-N/2+1}),$$

where, for $j \in \{0, \dots, K-1\}$,

$${}_N V_j^Y = {}_N Y_1^1(j) + {}_N Y_1^2(j) + {}_N Y_2^1(j) + {}_N Y_3^1(j) + {}_N Y_3^2(j).$$

The first two coefficients for each $N \geq 3$. We first assume that $N \geq 3$ and $K = 2$. Then $Y(t) = {}_N V_0^Y t^{-N/2+1} + {}_N V_1^Y t^{-N/2} + O(t^{-N/2-1})$ as $t \rightarrow \infty$, where

$${}_N V_0^Y = 0, \quad {}_N V_1^Y = \frac{f_0 J_0}{4} \Gamma\left(\frac{N}{2}\right) + \frac{J_0 |P_0|^2}{4} \Gamma\left(\frac{N}{2}\right) - \Re(\bar{P}_0 b_0) J_0 C_{N-1,0}.$$

We may simplify the ${}_N V_j^Y$ ($j \in \{0, 1\}$) with the identities for the $C_{m,n}$ given in (3.6). We find

$${}_N V_0^Y = 0, \quad {}_N V_1^Y = 0.$$

3.2.7. Expansion of $Z_2(t)$

We begin with

$$Z_2(t) = -2\Re \left(\bar{P}_0 \int_{|\xi| \leq \delta} \left(\hat{u}_1(\xi) + \frac{|\xi|^2}{2} \hat{u}_0(\xi) \right) e^{-\frac{t|\xi|^2}{2}} h_1(t, \xi) \cos(t|\xi|) d\xi \right). \quad (3.25)$$

We then substitute (1.5) and (1.10) into (3.25), and estimate the resulting integral with the $O(|\xi|^{2K})$ -term and find it is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. After eliminating the integrals that evaluate to zero, and then switching to spherical coordinates we obtain as $t \rightarrow \infty$

$$Z_2(t) = \sum_{|\sigma| \leq K-1} -2\Re(\bar{P}_0 c_{2\sigma}) J_\sigma \int_0^\delta r^{2|\sigma|+N-2} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4}) \cos(tr)}{\sqrt{1-r^2/4}} dr + O(t^{-K-N/2+1}). \quad (3.26)$$

Into (3.26) we substitute (2.26) with $L = L_\sigma := K - |\sigma|$. The resulting integrals with the $O(r^{2L_\sigma})$ -terms are all $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. Thus as $t \rightarrow \infty$

$$\begin{aligned} Z_2(t) &= \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} -2\Re(\bar{P}_0 c_{2\sigma}) J_\sigma \alpha_k \int_0^\delta r^{2|\sigma|+2k+N-2} e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \cos(tr) dr \\ &\quad + O(t^{-K-N/2+1}) \\ &= \sum_{|\sigma| \leq K-1} \sum_{k=0}^{L_\sigma-1} -2\Re(\bar{P}_0 c_{2\sigma}) J_\sigma \alpha_k H_{3,2|\sigma|+2k+N-2}^\delta(t) + O(t^{-K-N/2+1}). \end{aligned} \quad (3.27)$$

Let us now apply Lemma 3.4 to (3.27) with $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - N/2 \rceil\}$ and $Q = Q_{\sigma,k} := K - |\sigma| - k$. Then as $t \rightarrow \infty$

$$Z_2(t) = \sum_{j=0}^{K-1} \left({}_N Z_2^1(j) + {}_N Z_2^2(j) \right) t^{-j-N/2+1} + O(t^{-K-N/2+1}), \quad (3.28)$$

where

$$\begin{aligned} {}_N Z_2^1(j) &= \begin{cases} \sum_{2|\sigma|+2k+n=j-N/2} \Re(\bar{P}_0 c_{2\sigma}) J_\sigma \alpha_k \Im(E_{2|\sigma|+2k+N-2,n}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases} \\ {}_N Z_2^2(j) &= \begin{cases} \sum_{|\sigma|+k+n=j-1} \Re(\bar{P}_0 c_{2\sigma}) J_\sigma \alpha_k \Im(C_{2|\sigma|+2k+N-2,2n+1}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3.2.8. Expansion of $Z_3(t)$

We begin with

$$Z_3(t) = -2\Re \left(P_1 \int_{|\xi| \leq \delta} \bar{u}_0(\xi) e^{-\frac{t|\xi|^2}{2}} h_2(t, \xi) \frac{\sin(t|\xi|)}{|\xi|} d\xi \right).$$

We use the definition of $h_2(t, \xi)$ as given in (1.11), then substitute the conjugate of (1.4) and estimate the integral with the $O(|\xi|^{2K})$ -term; it is $O(t^{-K-N/2+1})$ as $t \rightarrow \infty$. We then get rid of

the terms which evaluate to zero ($0 \leq |\sigma| \leq 2K - 1$ with an odd entry) and switch to spherical coordinates to find

$$\begin{aligned} Z_3(t) &= \sum_{|\sigma| \leq K-1} -2\Re(P_1 \bar{b}_{2\sigma}) J_\sigma \int_0^\delta r^{2|\sigma|+N-2} e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \sin(tr) dr \\ &\quad + O(t^{-K-N/2+1}) \\ &= \sum_{|\sigma| \leq K-1} -2\Re(P_1 \bar{b}_{2\sigma}) J_\sigma H_{4,2|\sigma|+N-2}^\delta(t) + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty). \end{aligned} \quad (3.29)$$

Therefore by applying Lemma 3.4 to (3.29) with $R = R_\sigma := \max\{0, \lceil K - 2|\sigma| - N/2 \rceil\}$ and $Q = Q_\sigma := K - |\sigma|$, we obtain

$$Z_3(t) = \sum_{j=0}^{K-1} \left({}_N Z_3^1(j) + {}_N Z_3^2(j) \right) t^{-j-N/2+1} + O(t^{-K-N/2+1}) \quad (3.30)$$

as $t \rightarrow \infty$, where

$$\begin{aligned} {}_N Z_3^1(j) &= \begin{cases} \sum_{2|\sigma|+n=j-N/2} \Re(P_1 \bar{b}_{2\sigma}) J_\sigma \Im(E_{2|\sigma|+N-2,n}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases} \\ {}_N Z_3^2(j) &= \begin{cases} \sum_{|\sigma|+n=j-1} -\Re(P_1 \bar{b}_{2\sigma}) J_\sigma \Im(C_{2|\sigma|+N-2,2n+1}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3.2.9. Expansion of $Z_4(t)$

Finally we seek the expansion of

$$Z_4(t) = 2\Re \left(P_1 \bar{P}_0 \int_{|\xi| \leq \delta} e^{-t|\xi|^2} \frac{\sin(t|\xi|) \cos(t|\xi|)}{|\xi|} d\xi \right).$$

Switching to spherical coordinates and using the identity $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$ we obtain

$$\begin{aligned} Z_4(t) &= \Re(P_1 \bar{P}_0) J_0 \int_0^\delta r^{N-2} e^{-tr^2} \sin(2tr) dr \\ &= -\Re(P_1 \bar{P}_0) J_0 \Im \left(\int_0^\delta r^{N-2} \exp(-tr^2 - 2itr) dr \right) \\ &= -\Re(P_1 \bar{P}_0) J_0 \Im(F_{N-2}^\delta(t)), \end{aligned} \quad (3.31)$$

where $F_k^\delta(t)$ is as in (2.6). We now use (2.9) in (3.31) with $P := \max\{0, \lceil K - N/2 \rceil\}$ to obtain as $t \rightarrow \infty$

$$\begin{aligned} Z_4(t) &= \sum_{p=0}^{P-1} -\Re(P_1 \bar{P}_0) J_0 \Im(A_{N-2,p}) t^{-N-p+1} + O(t^{-K-N/2+1}) \\ &= \sum_{j=0}^{K-1} {}_N Z_4^1(j) t^{-j-N/2+1} + O(t^{-K-N/2+1}), \end{aligned} \quad (3.32)$$

where

$${}_N Z_4^1(j) = \begin{cases} -\Re(P_1 \bar{P}_0) J_0 \Im(A_{N-2,j-N/2}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

3.2.10. Asymptotic expansion of $Z(t)$ as $t \rightarrow \infty$

Let us combine results (2.28), (3.28), (3.30), and (3.32) for $Z_1(t)$, $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$, respectively, with the fact that $Z(t) = Z_1(t) + Z_2(t) + Z_3(t) + Z_4(t)$ to obtain the expansion as $t \rightarrow \infty$

$$Z(t) = \sum_{j=0}^{K-1} {}_N V_j^Z t^{-j-N/2+1} + O(t^{-K-N/2+1}),$$

where, for $j \in \{0, \dots, K-1\}$,

$${}_N V_j^Z = {}_N Z_1^1(j) + {}_N Z_2^1(j) + {}_N Z_2^2(j) + {}_N Z_3^1(j) + {}_N Z_3^2(j) + {}_N Z_4^1(j).$$

The first two coefficients for each $N \geq 3$. We assume that $N \geq 3$ and $K = 2$. Then $Z(t) = {}_N V_0^Z t^{-N/2+1} + {}_N V_1^Z t^{-N/2} + O(t^{-N/2-1})$ as $t \rightarrow \infty$, where

$${}_N V_0^Z = 0, \quad {}_N V_1^Z = \Re(\bar{P}_0 c_0) J_0 \alpha_0 \Im(C_{N-2,1}) - \Re(P_1 \bar{b}_0) J_0 \Im(C_{N-2,1}).$$

We simplify the ${}_N V_j^Z$ ($j \in \{0, 1\}$) using (3.6). Hence

$${}_N V_0^Z = 0, \quad {}_N V_1^Z = 0.$$

3.3. Asymptotic expansion of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$ as $t \rightarrow \infty$

Let us recall that for the fixed $0 < \delta < 1$ there exists $\eta > 0$ such that $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = X(t) + Y(t) + Z(t) + O(e^{-\eta t})$ as $t \rightarrow \infty$. We will now combine the results from the prior analysis to obtain the full asymptotic expansion of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$ as $t \rightarrow \infty$ in a single closed form for $N \geq 3$.

Let $N \geq 3$ and $K \geq 1$ be integers as given in the statement of Theorem 1.2. Then as $t \rightarrow \infty$

$$\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot) - v(t, \cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = \sum_{j=0}^{K-1} {}_N V_j t^{-j-N/2+1} + O(t^{-K-N/2+1}), \quad (3.33)$$

where ${}_N V_j = {}_N V_j^X + {}_N V_j^Y + {}_N V_j^Z$ for every $j \in \{0, \dots, K-1\}$. We determine the first two coefficients ${}_N V_0$ and ${}_N V_1$.

Assume that $N \geq 3$ and $K = 2$. Since for $j \in \{0, 1\}$ the ${}_N V_j^X$, ${}_N V_j^Y$, and ${}_N V_j^Z$ are given in a common form for $N \geq 3$, we have $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2 = {}_N V_0 t^{-N/2+1} + {}_N V_1 t^{-N/2} + O(t^{-N/2-1})$ as $t \rightarrow \infty$, where ${}_N V_0 = 0$ and ${}_N V_1$ is as in (1.17). Since ${}_N V_0 = 0$, the proof of Theorem 1.2 is complete.

3.4. Asymptotic expansion of $\|v(t, \cdot)\|_2^2$ as $t \rightarrow \infty$

Let us again recall that for the fixed $0 < \delta < 1$, $\|v(t, \cdot)\|_2^2 = \|v(t, \cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = X_2(t) + Y_2(t) + Z_4(t) + O(e^{-t\delta^2/2})$ as $t \rightarrow \infty$. We then combine results (3.15), (3.21), and (3.32) to obtain the proof of Corollary 1.3 and the expansion

$$\|v(t, \cdot)\|_2^2 = \|v(t, \cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = \sum_{j=0}^{K-1} {}_N U_j t^{-j-N/2+1} + O(t^{-K-N/2+1}) \quad (t \rightarrow \infty),$$

where, for $j \in \{0, \dots, K-1\}$,

$${}_N U_j = {}_N X_2^1(j) + {}_N Y_2^1(j) + {}_N Z_4^1(j).$$

The first $K = 2$ coefficients for dimension $N \geq 3$ are given in (1.16).

Remark. For each $N \geq 3$ the first coefficient ${}_N U_0$ in the expansion is equal to that of $\|u(t, \cdot)\|_2^2 = \|\hat{u}(t, \cdot)\|_2^2$. This leads to the cancellation of the leading terms when considering the expansion of $\|\hat{u}(t, \cdot) - v(t, \cdot)\|_2^2$.

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Appendix A. Computing $B_{m,n}$, $E_{m,n}$, and $C_{m,n}$ with *Mathematica*

In the proofs of Proposition 2.2 and Lemmas 3.1 and 3.2, we claimed the coefficients in the expansions of $G_{1,m}^\epsilon(t)$, $G_{2,m}^\epsilon(t)$, and $G_{3,m}^\epsilon(t)$ may be computed with a computer algebra system like *Mathematica*. The following lines of code will return the first few terms in each of the expansions.

These first four commands allow for us to set the desired accuracy. Since the accuracy is currently set to 5, we are able to correctly determine, for any $m \in \mathbb{N}_0$, $B_{m,n}$ and $E_{m,n}$ for $0 \leq n \leq 5 - 1 = 4$, and $C_{m,n}$ for $0 \leq n \leq 2(5) - 1 = 9$.

```
acc = 5;
Bacc = acc;
Cacc = 2*acc;
Eacc = acc;
```

The following sequence of commands returns the coefficients in the expansion of $G_{1,m}^\epsilon(t)$, where $0 < \epsilon < 2$.

```
g1 = Exp[I*r*t*(2 - Sqrt[4 - r^2])];
glcoeff[k_Integer] := SeriesCoefficient[g1, {r, 0, k}];
G1 = Sum[glcoeff[k]*Aa[m + k, p]*t^(-m - k - p - 1), {k, 0, 3/2*(Bacc - 1)}, {p, 0, Floor[Bacc - 2/3*k] - 1}];
Bb[m, n_Integer] := Coefficient[G1, t, -m - n - 1];
For[i = 0, i <= Bacc - 1, i++, Print["Bb[m,", i, "]", "=", Bb[m, i]]]
```

The following sequence of commands returns the coefficients in the expansions of $G_{3,m}^\epsilon(t)$ and $G_{2,m}^\epsilon(t)$, respectively, where $0 < \epsilon < 2$.

```
g2 = Exp[I*r*(t/2)*(2 - Sqrt[4 - r^2])];
g2coeff[k_Integer] := SeriesCoefficient[g2, {r, 0, k}];
G3 = Sum[g2coeff[k]*Aa[m + k, p]*t^(-m - k - p - 1), {k, 0, 3/2*(Bacc - 1)}, {p, 0, Floor[Bacc - 2/3*k] - 1}];
Ee[m, n_Integer] := Coefficient[G3, t, -m - n - 1];
For[i = 0, i <= Eacc - 1, i++, Print["Ee[m,", i, "]", "=", Ee[m, i]]];
G2 = Apart[Sum[1/2*g2coeff[k]*Gamma[(m + k + 1)/2]*t^(-(m + k + 1)/2), {k, 0, 3*Cacc}], t];
Cc[m, n_Integer] := Coefficient[Apart[G2*t^(m/2 + n/2 + 1/2), t], t, 0];
For[i = 0, i <= Cacc - 1, i++, Print["Cc[m,", i, "]", "=", Cc[m, i]]]
```

To obtain numerical values for the coefficients, we need only insert the following line of code defining $A_{k,p}$ for $k, p \in \mathbb{N}_0$ as in (2.10) after the first four lines given above.

```
Aa[k_, p_] = k!*Pochhammer[(k + 1)/2, p]*Pochhammer[(k + 2)/2, p]/((2*I)^(k + 1)*p!)
```

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