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On a shallow-water model with the Coriolis effect

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Abstract

In the present study an asymptotic model for wave propagation in shallow water with the effect of the Coriolis force is derived from the governing equation in two dimensional flows. The transport equation theory is then applied to investigate the local well-posedness and wave breaking phenomena for this model. The nonexistence of the Camassa-Holm-type peaked solution and classification of various traveling-wave solutions to the new system are also established. Moreover it is shown that all the symmetric waves to this model are traveling waves.

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1. Introduction

The study of water waves has been a source of intriguing – and often difficult – mathematical subject due to the familiar phenomena and various mathematical models [46]. The water wave

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problem for gravity waves is described by Euler's equations, which is the result of applying Newton's second law to an inviscid fluid [32]. Their complexity led physicists and mathematicians to derive simpler set of equations likely to describe the dynamics of the water-waves equations in some specific physical regimes.

To simplify matters, attention is restricted to model equations with only two unknowns here instead of the governing equations for two dimensional flows with four unknowns: the horizontal velocity, the vertical velocity, the pressure and the free surface. Inspired by Constantin and Ivanov's modeling approach in [13], it enables us to derive a new system with the Coriolis effects from the rotation-Green-Naghdi equations by using the asymptotic expansion in the Camassa-Holm (CH) scaling, i.e. $\mu \ll 1$, $\varepsilon = O(\sqrt{\mu})$ (here ε and μ are the amplitude parameter and the shallowness (long wavelength) parameter respectively), namely,

$$\begin{cases} \eta_t + ((1+\eta)u)_x = 0, & t>0, \quad x \in \mathbb{R}, \\ u_t - u_{txx} + uu_x + 4u_x u_{xx} + uu_{xxx} + \eta_x + 2\Omega\eta_t = 0, & t>0, \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where u is connected with the average of horizontal velocity, η is related to free surface elevation from equilibrium with the boundary condition $u \to 0$ and $\eta \to 0$ when $|x| \to \infty$, Ω is a dimensionless parameter describing the strength of the Coriolis effect ($\Omega > 0$).

Such equations are derived by adopting the technique involving the construction of asymptotic expansions with respect to two dimensionless parameters ε and μ. The significance of introducing the dimensionless parameters is that it is often possible to deduce from their values some insight on the behavior of the flow. It is worth introducing some terminology at this point [36]:

1) Small/large amplitude regimes. It is said that the flow under consideration is in a small amplitude regime if $\varepsilon \ll 1$. If no smallness assumption is made on ε (i.e. if $\varepsilon = O(1)$), then the flow is said to be in a large amplitude regime.
2) Shallow/deep water regime. The shallow water regime corresponds to $\mu \ll 1$. If this condition is not satisfied, then the situation is in deep water (the situation $\mu \approx 1$ is sometimes referred to as intermediate depth).

Throughout this paper, we consider only shallow water waves, i.e. $\mu \ll 1$. Then, simpler models under additional assumptions on ε can be derived. That is, one may find shallow water waves in particular asymptotic regimes by relating ε and μ.

Before we commence our motivation to study the new model of shallow water waves, we shall focus on three specific physical regimes by connecting ε and μ. First of all, consider the shallow water wave with small-amplitude, i.e. long wave regime.

$$\mu \ll 1, \quad \varepsilon = O(\mu). \tag{1.2}$$

The most well-known model within this regime is the Korteweg-de Vries (KdV) equation [10,35]

$$\eta_t + \eta_x + \frac{2}{3}\varepsilon\eta\eta_x + \frac{1}{6}\mu\eta_{xxx} = 0,$$

where η is related to the free surface as well as the horizontal component of the velocity. That's why this regime is also called KdV regime. The KdV equation describes the propagation of unidirectional, one-dimensional waves on the surface of water over a flat bottom. It is a representative as an integrable equation which provides an explanation of the phenomenon observed by Russell possessing smooth solitary waves that such waves can propagate in a liquid medium with out change of form. Actually, the derivation of the KdV equation allows great flexibility and the approach naturally allows the various alternatives. Indeed, it can be derived from the following Boussinesq's equations [2,3,15] within the same regime by specializing to a wave moving to the right.

$$\begin{cases} \eta_t + [(1+\varepsilon\eta)w]_x - \frac{1}{6}\mu w_{xxx} = 0, \\ w_t + \varepsilon w w_x + \eta_x - \frac{1}{2}\mu w_{xxt} = 0, \end{cases}$$

where η is related to the free surface and w is connected to the horizontal velocity. Moreover, the KdV equation belongs to a wider class of equations – the Benjamin-Bona-Mahoney (BBM) equations [4,6]

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x + \mu(\beta + \frac{1}{6})\eta_{xxx} + \mu\beta\eta_{xxt} = 0, \tag{1.3}$$

where $\beta \le 0$. As mentioned by Whitham [46] that the nonlinear shallow water equations which neglect dispersive altogether lead to breaking of the typical hyperbolic kind, with the development of a vertical slope and a multivalued profile. Notice that among these models (such as KdV) the balance between the nonlinear effects and the dispersive effects under the scaling (1.2) is provided and the wave breaking phenomena (i.e. the solution remains bounded while its slope becomes unbounded in finite time) is not captured even though they provide good asymptotic approximations to the full water wave problem [1,34]. So in order to investigate shallow water waves with wave breaking phenomena, one may consider new models, whose behavior is more nonlinear than dispersive. Then a possible method to derive such equations directly from the governing equation is to adjust the relation between ε and μ.

In light of the foregoing discuss, we now introduce the shallow water wave equation with moderate amplitude, characterized by large values of ε, i.e.

$$\mu \ll 1, \quad \varepsilon = O(\sqrt{\mu}). \tag{1.4}$$

Unlike the case of the BBM equations (including KdV) in the regime (1.2), in the present setting (1.2) with the formal asymptotic procedures, the correct generalization of the BBM equations (1.3) under the scaling (1.4) is provided in the following class of equations [14]:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon\mu(\gamma u u_{xxx} + \delta u_x u_{xx}),$$

where u is related to the horizontal velocity u^θ ($\theta \in [0,1]$) evaluated at the level line θ of the fluid domain, α, γ, δ and $\beta < 0$ are constants in terms of two parameters. As shown in [14], this two-parameter family of equations is consistent with the Green-Naghdi equation, which implies that it is a good approximation to the governing equations for water waves. Moreover, with proper conditions on $\alpha, \beta, \gamma, \delta$ and transformations on u, one recovers two prominent equations which are completely integrable. One is the Camassa-Holm (CH) equation [7]

$$u_t + \kappa u_x + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}. \tag{1.5}$$

The CH equation was first obtained by Fokas and Fuchssteiner [22] as a bi-Hamiltonian generalization of the KdV equation. It also arises in the study of a certain non-Newtonian fluids [5] and models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [16]. The novelty of Camassa and Holm's work was the physical derivation and the discovery that the solitary wave solutions to this equation are soliton. Another one is the Degasperis-Procesi (DP) equation [18],

$$u_t + \kappa u_x + 4uu_x - u_{txx} = 3u_x u_{xx} + uu_{xxx}. \tag{1.6}$$

Much attention is intrigued by the equation (1.5) and (1.6) in more than two decades because they inherit some properties of KdV equation [33,35,37,42] but not limit to the smooth solitary wave, which exactly answered the question raised by Whitham [46]. A breakthrough of the CH equation and DP equation is that, when $\kappa = 0$, they admit peaked solitary waves [7,8,17,38]. The wave profile of so-called "peakon" is shaped like $u(t,x) = ce^{-|x-ct|}$, $c \in \mathbb{R}$, which has largest amplitude c and speed c. The first derivative u_x is smooth except at the peak, where it has a jump discontinuity. Another important feature is that the CH equation and DP equation accommodate wave breaking phenomena [7,12,14,41], i.e. the solution remains bounded while its slope becomes unbounded in finite time, due to the capture of stronger nonlinear effects than the classical nonlinear dispersive BBM and KdV equations.

In addition, the shallow water wave equation with large amplitude gives another physical regime

$$\mu \ll 1, \quad \varepsilon = O(1). \tag{1.7}$$

The standard asymptotic procedure gives the classical Green-Naghdi (GN) equations [28] (also known as the Serre [44] or Su-Gardner equations [45])

$$\begin{cases} \eta_t + [(1+\varepsilon\eta)]_x = 0, \\ u_t + \eta_x + \varepsilon uu_x = \frac{\mu}{3}\frac{1}{1+\varepsilon\eta}\Big[(1+\varepsilon\eta)^3(u_{xt} + \varepsilon uu_{xx} - \varepsilon u_x^2)\Big]_x, \end{cases} \tag{1.8}$$

which takes into account the dispersive effects neglected by the shallow-water equations. These equations coupled the free surface elevation η, to the vertically averaged horizontal component of the velocity u. A rigorous justification of the GN model can be found in [40] for the 1D water waves with a flat bottom; the general case was handled in [1,19] based on a well-posedness theory.

Following discovering of these models mentioned above, we are motivated to search some models for shallow water wave equations with moderate amplitude exhibiting the Coriolis effect. One of our purposes in the present paper is to derive this kind of model equation (1.1) with the Coriolis effect under the CH regime and understand how the Coriolis forcing due to the Earth rotation affects the wave propagation, wave breaking mechanics as well as formulation of those peaked traveling-wave solutions. It is noted that in the process of the derivation of the asymptotic model equation in the present paper, the rotation parameter Ω is considered as a fixed $O(1)$ constant relevant to the amplitude parameter ε and the shallowness parameter μ. The motivation with such a fixed Ω in the asymptotic expansion is that such a model could retain more mixed terms of the free surface component and horizontal velocity component in the asymptotic system of equations so that we can perform the analytical study, with an emphasis of investigating whether or not these rotation can defer or enhance the formation of singularity or phenomena of wave breaking by interaction between those two components free surface and horizontal velocity. Of course, it is interesting to know how the interaction of those two components caused by the rotation parameter will affect symmetry property of traveling waves, existence or non-existence of certain solitary waves or surface waves, and stability or instability of those waves, for example. Moreover, a small parameter Ω is introduced in [23], where the Coriolis parameter Ω and the amplitude parameter ε are of the same order of magnitude. For some other models including the Coriolis force in the shallow-water wave propagation regime, we refer the reader to recent work in [21,30,31].

It is found that the consideration of the Coriolis effect gives rise to $-[(1+\eta)u]_x$ or η_t into the second equation of system in (1.1). For the model in question, we first concern with its local well-posedness in suitably function space. The two equations for u and η possess the transport structure. Due to transport equation theory applied in [24], the local well-posedness for system (1.1) will be established. Moreover, the asymptotic expansion in the Camassa-Holm regime gives the fluid convection between nonlinear steeping and amplification, which has interesting implications for the fluid motion, particular in the relation to the wave breaking phenomena and the permanent waves. On the other hand, using Moser-type estimates and Littlewood-Paley analysis for the transport equation, we obtain the blow-up criterion. Our approach is to trace the dynamics of the solution and its gradient along the characteristics. Then the dynamics of the wave-breaking quantity is established by a refined analysis on evolution of the solution u and η.

Another interesting issue investigated here is the traveling-wave solutions of (1.1) in the form of $(u(t,x),\eta(t,x)) = (\varphi_\sigma(x-\sigma t), \psi_\sigma(x-\sigma t))$, $\sigma \in \mathbb{R}$ for the function $\varphi_\sigma, \psi_\sigma : \mathbb{R} \to \mathbb{R}$ such that $\varphi_\sigma \to 0$, $\psi_\sigma \to 0$ as $|x| \to \infty$. It is known that the traveling-wave solution of the classical CH equation appears to be a weakly peaked soliton [7], which is one of the interesting features for the CH-type equation. It was also found that the mCH equation, as the dual equation of the mKdV equation, admits peaked solitons [27]. As one can see that the system (1.1) has the same nonlinear terms in u with the CH equation except for the coefficients. A natural question remains that how the Coriolis forcing and the convection between nonlinear steeping and amplification affects the propagation of the traveling waves, in particular, the peaked solitons. To this end, it is of interest to study and classify the traveling-wave solutions of (1.1) and the existence of the CH-type peakon solution. Actually, we use a suitable framework for weak solutions to classify all weak traveling waves of equation (1.1) as [38,39]. It is unclear whether the system (1.1) with the weak Coriolis effect supports traveling waves with singularities. Using a natural weak formulation of (1.1), we can define exactly in what sense the peaked and cusped traveling waves are solutions. In fact, it turns out that the equation for φ takes the form $\varphi_x^2 = F(\varphi)$, where F is no longer a rational function comparing with the CH equation. A standard phase-plane analysis determines the behavior of solution near the zeros and poles of F. In fact, peaked traveling waves exist when F has a removable pole and cusped traveling waves exists correspond to when F has a non-removable pole. Due to the added Earth rotation term, the numerator of F is a combination of polynomial and nature logarithmic function whose root are not explicit. By analyzing each possible case carefully, we show here only cusped and smooth traveling waves exist for (1.1), and the CH-type peakon does not exist.

From the classification of the traveling waves for system (1.1), it admits both cusped and smooth traveling solutions. Both of these two solutions are symmetric around the crest. Conversely, this raises the interesting question whether symmetry is a priori guaranteed for traveling waves. Adopting the idea in [20], we are able to give an affirmative answer to the case of system (1.1).

The remainder of this paper is organized as follows. The derivation of the new model and the Green-Naghdi equations with Coriolis force are presented in Section 2. The local well-posedness of the new model and the wave-breaking criteria for solutions to (1.1) by using transport equation theory are contained in Section 3. Section 4 is devoted to the investigation of existence and nonexistence of Camassa-Holm-type peaked solution and classification of the traveling-wave solution of the model. Moreover, it is demonstrated that all the horizontal symmetric waves should be traveling waves.

Notation. Throughout this paper, we denote the norm of the Lebesgue space $L^p(\mathbb{R})$ by $\|\cdot\|_{L^p}$, $1 \le p \le \infty$ and the norm in Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$ by $\|\cdot\|_{H^s}$. The spatial convolution on $\mathbb{R}$ is denoted by '$*$'.

2. Derivation of the model

It is observed that in certain ranges of scales in the geophysical water waves fluid dynamics is primarily influenced by the interaction of gravity and the Earth's rotation. Consider now that water flows are incompressible and inviscid with a constant density ρ and no surface tension, and the interface between the air and the water is a free surface. Then such a motion of water flow occupying a dynamic domain $\mathcal{D}_t$ in $\mathbb{R}^3$ under the influence of the gravity g and the Coriolis force due to the Earth's rotation can be described by the Euler-Coriolis equations [29], namely,

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla P + \vec{g}, \quad x\in\mathcal{D}_t,\ t>0, \\ \nabla\cdot\vec{u} = 0, \quad x\in\mathcal{D}_t,\ t>0, \\ \vec{u}|_{t=0} = \vec{u}_0, \quad x\in\mathcal{D}_0, \end{cases} \tag{2.1}$$

where $\vec{u} = (u, v, w)^T$ is the fluid velocity, $P(t, x, y, z)$ is the pressure in the fluid, $\vec{g} = (0, 0, -g)^T$ with $g \approx 9.8$ m/s^2 the constant gravitational acceleration at the Earth's surface, and $\vec{\Omega} = (0,\ \Omega_0\cos\phi, \Omega_0\sin\phi)^T$, with the rotational frequency $\Omega_0 \approx 73\cdot 10^{-6}$ rad/s and the local latitude ϕ, is the angular velocity vector which is directed along the axis of rotation of the rotating reference frame. We adopt a rotating framework with the origin located at a point on the Earth's surface, with the x-axis chosen horizontally due east, the y-axis horizontally due north and the z-axis upward. We now focus on two-dimensional flows, moving in the zonal direction along the equator independent of the y-coordinate, in other words, $v \equiv 0$ throughout the flow. In 2D case, consider here waves at the surface of water with a flat bed, and assume that $\mathcal{D}_t = \{(x, z) : 0 < z < h_0 + \eta(t, x)\}$, where h_0 is the typical depth of the water and $\eta(t, x)$ measures the deviation from the average level. Under the f-plane approximation ($\sin\phi \approx 0,\ \phi \ll 1$), the motion of inviscid irrotational fluid near the Equator in the region $\mathcal{D}_t$ with a constant density ρ is described by the Euler's equations in two dimensions [11,29],

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega_0 w = -\frac{1}{\rho}P_x, \\ w_t + uw_x + ww_z - 2\Omega_0 u = -\frac{1}{\rho}P_z - g, \end{cases} \tag{2.2}$$

the incompressibility of the fluid,

$$u_x + w_z = 0, \tag{2.3}$$

and the irrotational condition,

$$u_z - w_x = 0. \tag{2.4}$$

The pressure is written as

$$P(t, x, z) = P_a + \rho g(h_0 - z) + p(t, x, z),$$

where P_a is the constant atmosphere pressure, and p is a pressure variable measuring the hydrostatic pressure distribution.

The dynamic condition posed on the surface $\Sigma_t = \{(x, z) : z = h_0 + \eta(t, x)\}$ yields $P = P_a$. Then there appears that

$$p(t, x, h_0 + \eta(t, x)) = \rho g\eta(t, x). \tag{2.5}$$

Meanwhile, the kinematic condition on the surface is given by

$$w = \eta_t + u\eta_x, \quad \text{when} \quad z = h_0 + \eta(t, x). \tag{2.6}$$

Finally, we pose "no-flow" condition at the flat bottom $\{z = 0\}$, that is,

$$w|_{z=0} = 0. \tag{2.7}$$

There are many shallow water models as appropriate approximations to the full Euler dynamics when the water depth is small compared to the horizontal wavelength scale [1,14]. We denote the amplitude parameter ε and the shallowness parameter μ respectively by

$$\varepsilon = a/h_0, \qquad \mu = h_0^2/\lambda^2, \tag{2.8}$$

where a is the typical amplitude of the wave and λ is the typical wavelength. According to the magnitude of the physical quantities, we introduce dimensionless quantities as follows

$$x \to \lambda x, \quad z \to h_0 z, \quad \eta \to a\eta, \quad t \to \frac{\lambda}{\sqrt{gh_0}} t,$$

and

$$u \to \sqrt{gh_0} u, \quad w \to \sqrt{\mu g h_0} w, \quad p \to \rho g h_0 p.$$

And under the influence of the Earth rotation, we introduce

$$\Omega = \sqrt{h_0/g}\, \Omega_0. \tag{2.9}$$

Furthermore, considering whenever $\varepsilon \to 0$,

$$u \to 0, \quad w \to 0, \quad p \to 0,$$

that is, u, w and p are proportional to the wave amplitude. In this case, we choose a scaling

$$u \to \varepsilon u, \quad w \to \varepsilon w, \quad p \to \varepsilon p. \tag{2.10}$$

Therefore the governing equations become

$$
\begin{aligned}
&u_t + \varepsilon(uu_x + wu_z) + 2\Omega w = -p_x && \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), && (2.11a)\\
&\mu\,(w_t + \varepsilon(uw_x + ww_z)) - 2\Omega u = -p_z && \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), && (2.11b)\\
&u_x + w_z = 0 && \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), && (2.11c)\\
&u_z - \mu w_x = 0 && \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), && (2.11d)\\
&p = \eta && \text{on} \quad z = 1 + \varepsilon\eta(t,x), && (2.11e)\\
&w = \eta_t + \varepsilon u\eta_x && \text{on} \quad z = 1 + \varepsilon\eta(t,x), && (2.11f)\\
&w = 0 && \text{on} \quad z = 0. && (2.11g)
\end{aligned}
$$

It is our purpose here to establish new model with the effect of the Earth rotation in shallow water wave with moderate amplitude. Similar to the classical Green-Naghdi (GN) equations [28]. Let $\bar{u}$ be the average horizontal velocity,

$$\bar{u}(t,x) \stackrel{\text{def}}{=} \frac{1}{h}\int_0^h u(t,x,z)\,dz, \tag{2.12}$$

where $h = h(t,x) = 1 + \varepsilon\eta(t,x)$. We multiply (2.12) by h and differentiate it with respect to x to find

$$(h\bar{u})_x = \int_0^h u_x\,dz + \varepsilon\eta_x u_h,$$

where $u_h = u(t,x,z)|_{z=h}$. Then the above equation combining with (2.11c), (2.11f) and (2.11g) gives rise to the first equation of the Green-Naghdi equations

$$\eta_t + (h\bar{u})_x = 0. \tag{2.13}$$

To derive the second equation of the Green-Naghdi equations, let

$$u(t,x,z) = u_0(t,x,z) + \mu u_1(t,x,z) + O(\mu^2). \tag{2.14}$$

For the linear problem ($\mu \to 0$), the expression in (2.11d) implies $u_{0,z} = 0$. Hence, u_0 is a function independent of z, i.e. $u_0 = u_0(x,t)$. From (2.11c) and (2.11d), we have

$$\mu u_{xx} + u_{zz} = 0,$$

which implies

$$\begin{aligned} &\mu^0: \ u_{0,zz}=0, \\ &\mu^1: \ u_{0,xx}=-u_{1,zz}. \end{aligned} \tag{2.15}$$

Considering $u_0=u_0(t,x)$, the equation of order μ^1 implies

$$u_1=-\frac{z^2}{2}u_{0,xx}+z\Psi(t,x),$$

where $\Psi(t,x)$ is an arbitrary function. Therefore,

$$u=u_0-\mu\frac{z^2}{2}u_{0,xx}+\mu z\Psi(t,x)+O(\mu^2). \tag{2.16}$$

Taking advantage of the governing equations, one has

$$w=-zu_{0,x}+\mu\frac{z^3}{6}u_{0,xxx}-\mu\frac{z^2}{2}\Psi_x(t,x)+O(\mu^2), \tag{2.17}$$

and

$$\begin{aligned} p=\eta-\frac{\mu}{2}(h^2-z^2)(u_{0,xt}+\varepsilon u_0u_{0,xx}-\varepsilon u_{0,x}^2)-2\Omega(h-z)u_0 \\ +\frac{\mu}{3}\Omega(h^3-z^3)u_{0,xx}-\mu\Omega(h^2-z^2)\Psi+O(\mu^2). \end{aligned}$$

Then differentiating the expression of p with respect to x, combining with the expressions of u, w and integrating with respect to z from 0 to h give rise to the equation related to u_0 and h only, namely,

$$\begin{aligned} u_{0,t}+\mu\frac{h}{2}\Psi_t+\varepsilon u_0u_{0,x}+\varepsilon\mu\frac{h}{2}u_0\Psi_x+\eta_x=\frac{\mu}{2}h^2(u_{0,xt}+\varepsilon u_0u_{0,xx}-\varepsilon u_{0,x}^2)_x \\ +\mu hh_x(u_{0,xt}+\varepsilon u_0u_{0,xx}-\varepsilon u_{0,x}^2)+2\Omega(h_xu_0+hu_{0,x})-\mu\Omega h^2h_xu_{0,xx} \\ -\frac{\mu}{3}\Omega h^3u_{0,xxx}+2\mu\Omega hh_x\Psi+\mu\Omega h^2\Psi_x+O(\mu^2). \end{aligned} \tag{2.18}$$

Notice that the relation between $\bar{u}$ and u_0 is

$$\bar{u}=u_0-\mu\frac{h^2}{6}u_{0,xx}+\mu\frac{h}{2}\Psi+O(\mu^2), \tag{2.19}$$

which can be obtained by using the function u's expression (2.16) in the definition of $\bar{u}$ (2.12). Finally, the following system called the rotation-Green-Naghdi (R-GN) equations are revealed by the relation between u_0 and $\bar{u}$, which performs as

$$\begin{cases} \eta_t + ((1+\varepsilon\eta)\bar{u})_x = 0, \\ \bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x + 2\Omega\eta_t = \frac{\mu}{3(1+\varepsilon\eta)}\left((1+\varepsilon\eta)^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2)\right)_x + O(\mu^2). \end{cases} \tag{2.20}$$

The detailed derivation of the R-GN equations can be found in [26]. It is noted that the R-GN model in (2.20) (for the solution $(\eta, \bar{u})$) is locally well-posed in the Sobolev space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ with $s > \frac{3}{2}$ [9], while the case of without the Coliolis effect was studied in [40].

We now consider the rotation parameter Ω as a fixed $O(1)$ constant relevant to the amplitude parameter ε and the shallowness parameter μ. As is pointed out in the introduction, it is our purpose with a fixed Ω in the asymptotic expansion to investigate how symmetry, existence and non-existence of traveling waves, and stability issues of those solitary waves can be affected by interaction between free surface and velocity components in the asymptotic system.

The model equations in (1.1) could be derived from the R-CH equations with moderate amplitude. For simplicity, $\bar{u}$ is replaced by u in (2.20). Let us start with the linear terms in (2.20) in terms of ϵ and μ given by

$$\begin{cases} \eta_t + u_x = O(\epsilon, \mu), \\ u_t + \eta_x + 2\Omega\eta_t = O(\epsilon, \mu). \end{cases} \tag{2.21}$$

This formula then yields

$$\begin{cases} \eta_{tt} - \eta_{xx} - 2\Omega\eta_{xt} = O(\epsilon, \mu), \\ u_{tt} - u_{xx} - 2\Omega u_{xt} = O(\epsilon, \mu). \end{cases} \tag{2.22}$$

Solving the second order linear partial differential equation mentioned above, we obtain

$$\begin{cases} u = u_1\left(x - (\sqrt{\Omega^2+1} - \Omega)t\right) + u_2\left(x + (\sqrt{\Omega^2+1} + \Omega)t\right) + O(\epsilon, \mu), \\ \eta = \eta_1\left(x - (\sqrt{\Omega^2+1} - \Omega)t\right) + \eta_2\left(x + (\sqrt{\Omega^2+1} + \Omega)t\right) + O(\epsilon, \mu). \end{cases} \tag{2.23}$$

For the simplicity, we only consider the waves move towards to the right side, i.e.

$$\begin{cases} u = u(x - ct) + O(\epsilon, \mu), \\ \eta = \eta(x - ct) + O(\epsilon, \mu), \end{cases} \tag{2.24}$$

where $c = \sqrt{\Omega^2+1} - \Omega$, and Ω is a dimensionless parameter describing the strength of the Coriolis effect, which implies

$$\begin{cases} u_t = -cu_x + O(\epsilon, \mu), \\ \eta_t = -c\eta_x + O(\epsilon, \mu). \end{cases} \tag{2.25}$$

Hence, it is deduced that

$$\eta_t + u_x = O(\epsilon, \mu), \tag{2.26}$$

$$\eta_x + \frac{1}{c^2} u_t = O(\epsilon, \mu), \tag{2.27}$$

$$\eta - \frac{1}{c} u = O(\epsilon, \mu). \tag{2.28}$$

The equation (2.28) and the expansion of equation (2.20) with a remain term of order $O(\epsilon^2\mu, \mu^2)$ thus imply the following system

$$\begin{cases} \eta_t + [(1+\epsilon\eta)u]_x = 0, \\ (u - \frac{\mu}{3}u_{xx})_t + \epsilon u u_x + \eta_x + 2\Omega\eta_t = -\frac{\epsilon\mu}{3}\left(u u_{xx} + \frac{3}{2}u_x^2\right)_x + O(\epsilon^2\mu, \mu^2). \end{cases} \tag{2.29}$$

Applying the transformation $u(t,x) = \epsilon u(\sqrt{\mu}t, \sqrt{\mu}x), \eta(t,x) = \epsilon\eta(\sqrt{\mu}t, \sqrt{\mu}x)$ the above equation (2.29) and ignoring terms of order $O(\epsilon^2\mu, \mu^2)$ yield the following system

$$\begin{cases} \eta_t + ((1+\eta)u)_x = 0, \\ u_t - u_{txx} + u u_x + 4u_x u_{xx} + u u_{xxx} + \eta_x + 2\Omega\eta_t = 0. \end{cases} \tag{2.30}$$

3. Local well-posedness and wave-breaking criteria

In this section, we consider local well-posedness of the Cauchy problem to the system (1.1) and present the wave-breaking criteria for solutions to (1.1) by using transport equation theory.

Now we are in a position to state the local well-posedness result of the following Cauchy problem, which may be similarly obtained as in [25] (up to a slight modification).

Theorem 3.1. *Given any* $X_0 = \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, *there exist a maximal* $T = T(\Omega, \|X_0\|_{H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})}) > 0$, *and a unique solution* $X = \begin{pmatrix} u \\ \eta \end{pmatrix}$ *to* (1.1) *such that*

$$X = X(\cdot, X_0) \in C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping $X_0 \mapsto X(\cdot, X_0) : H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$ *is continuous and the maximal existence time* T *can be chosen independently of the Sobolev order* s.

Lemma 3.1. *Let* $X_0 = \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s \geq 3/2$, *and let* T *be the maximal existence time of the solution* $X = \begin{pmatrix} u \\ \eta \end{pmatrix}$ *to* (1.1) *with initial data* X_0. *Then we have for all* $t \in [0,T)$,

$$\int_{\mathbb{R}} u^2(t,x) + u_x^2(t,x) + \eta^2(t,x)dx \leq e^{2(\Omega+1)\int_0^t \|u_x\|_{L^\infty} d\tau}\Big(\int_{\mathbb{R}} u_0^2(x) + u_{0,x}^2(x) + \eta_0^2(x)dx\Big).$$

A direct computation combining with the Gronwall's inequality gives rise to the above lemma, so the detailed proof is omitted.

To consider the wave-breaking criteria, we first recall the following propositions.

Proposition 3.1. *[24] (1-D Moser-type estimates). The following estimates hold:*

(i) *For $s \geq 0$,*

$$\|fg\|_{H^s(\mathbb{R})} \leq C(\|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})} + \|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}). \tag{3.1}$$

(ii) *For $s > 0$,*

$$\|f\partial_x g\|_{H^s(\mathbb{R})} \leq C(\|f\|_{L^\infty(\mathbb{R})}\|\partial_x g\|_{H^s(\mathbb{R})} + \|f\|_{H^{s+1}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}). \tag{3.2}$$

(iii) *For $s_1 \leq \frac{1}{2}$, $s_2 > \frac{1}{2}$ and $s_1 + s_2 > 0$,*

$$\|fg\|_{H^{s_1}(\mathbb{R})} \leq C\|f\|_{H^{s_1}(\mathbb{R})}\|g\|_{H^{s_2}(\mathbb{R})}, \tag{3.3}$$

where C's are constants independent of f and g.

Proposition 3.2. *[24] Suppose that $s > -\frac{d}{2}$. Let v be a vector field such that ∇v belongs to $L^1([0,T]; H^{s-1})$ if $s > 1 + \frac{d}{2}$ or to $L^1([0,T]; H^{\frac{d}{2}} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in H^s$, $F \in L^1([0,T]; H^s)$ and that $f \in L^\infty([0,T]; H^s) \cap C([0,T]; S')$ solves the d-dimensional linear transport equations*

$$(T) \qquad \begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only on s, p and d such that the following statements hold:

(1) *If $s \neq 1 + \frac{d}{2}$,*

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + \int_0^t \|F(\tau)\|_{H^s} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^s} d\tau, \tag{3.4}$$

or,

$$\|f\|_{H^s} \leq e^{CV(t)}\Big(\|f_0\|_{H^s} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{H^s} d\tau\Big), \tag{3.5}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}} \cap L^\infty} d\tau$ if $s < 1 + \frac{d}{2}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}} d\tau$. Else,

(2) *If $f = v$, then for all $s > 0$, the estimates (3.4) and (3.5) hold with $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau$.*

Proposition 3.3. *[24] Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0,T]; H^s)$, v, $\partial_x v \in L^1([0,T]; L^\infty)$ and that $f \in L^\infty([0,T]; H^s) \cap C([0,T]; S')$ solves the 1-dimensional linear transport equation*

$$(T) \qquad \begin{cases} \partial_t f + v \cdot \partial_x f = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only on s such that the following statements hold:

$$\|f\|_{H^s} \le \|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau + C\int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau, \tag{3.6}$$

or,

$$\|f\|_{H^s} \le e^{CV(t)}\Big(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\Big), \tag{3.7}$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty}) d\tau$.

The above proposition was proved in [43] using Littlewood-Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument, we can obtain the following blow-up criterion.

Theorem 3.2. *Let $X_0 = \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$, and $X = \begin{pmatrix} u \\ \eta \end{pmatrix}$ be the corresponding solution to (1.1). Assume $T > 0$ is the maximal time of existence. Then*

$$T < \infty \qquad \Longrightarrow \qquad \int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.8}$$

Proof. We shall prove this theorem by an inductive argument with respect to the index s. To this end, let us first give a control on $\|\eta(t)\|_{L^\infty}$ and $\|u(t)\|_{L^\infty}$.

In fact, applying the maximal principle to the transport equation about $\rho := \eta + 1$,

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{3.9}$$

we have

$$\|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty} + C\int_0^t \|\partial_x u\|_{L^\infty} \|\rho\|_{L^\infty} d\tau.$$

A simple application of Gronwall's inequality implies

$$\|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty} e^{C\int_0^t \|\partial_x u\|_{L^\infty} d\tau}, \tag{3.10}$$

which gives rise to

$$\|\eta(t)\|_{L^\infty} \le \|\rho(t)\|_{L^\infty} + 1 \le 1 + (1 + \|\eta_0\|_{L^\infty}) e^{C\int_0^t \|\partial_x u\|_{L^\infty} d\tau}. \tag{3.11}$$

Now let us concentrate our attention to the proof of Theorem 3.2. This can be achieved as follows.

Step 1. For $\frac{3}{2} < s < 3$, applying Proposition 3.2 to the transport equation with respect to η,

$$\eta_t + u\eta_x + \eta u_x + u_x = 0, \tag{3.12}$$

we have (for every $1 < s < 2$, indeed)

$$\|\eta(t)\|_{H^{s-1}} \le \|\eta_0\|_{H^{s-1}} + C\int_0^t \|\eta\partial_x u + \partial_x u\|_{H^{s-1}} d\tau + C\int_0^t \|\eta\|_{H^{s-1}}(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}) d\tau.$$

Using (3.1), one has

$$\|\eta\partial_x u + \partial_x u\|_{H^{s-1}} \le \|\partial_x u\|_{H^{s-1}} + C(\|\eta\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{H^{s-1}}\|\eta\|_{L^\infty}). \tag{3.13}$$

Therefore, we have

$$\begin{aligned}\|\eta(t)\|_{H^{s-1}} \le \|\eta_0\|_{H^{s-1}} &+ C\int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}(1 + \|\eta(\tau)\|_{L^\infty}) d\tau \\ &+ C\int_0^t \|\eta(\tau)\|_{H^{s-1}}(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}) d\tau. \end{aligned} \tag{3.14}$$

On the other hand, Proposition 3.1 applied to the equation about u,

$$u_t - uu_x + \partial_x p * \Big(u^2 + \frac{1}{2}u_x^2 + \eta - 2\Omega((1+\eta)u)\Big) = 0,$$

implies (for every $s > 1$, indeed)

$$\begin{aligned}\|u(t)\|_{H^s} \le \|u_0\|_{H^s} &+ C\int_0^t \Big\|\partial_x p * \Big(u^2 + \frac{1}{2}u_x^2 + \eta - 2\Omega((1+\eta)u)\Big)(\tau)\Big\|_{H^s} d\tau \\ &+ C\int_0^t \|u(\tau)\|_{H^s}\|\partial_x u(\tau)\|_{L^\infty} d\tau.\end{aligned}$$

Thanks to the Moser-type estimate (3.1) and Proposition 2.3 in [24], one has

$$\begin{aligned}
&\left\|\partial_x p*\left(u^2+\frac{1}{2}u_x^2+\eta-2\Omega((1+\eta)u)\right)\right\|_{H^s}\\
&\le C\left\|u^2+\frac{1}{2}u_x^2+\eta-2\Omega((1+\eta)u)\right\|_{H^{s-1}}\\
&\le C\left(\|u\|_{H^{s-1}}\|u\|_{L^\infty}+\|\partial_x u\|_{H^{s-1}}\|\partial_x u\|_{L^\infty}\right.\\
&\quad\left.+\|\eta\|_{H^{s-1}}+\|u\|_{H^{s-1}}+\|u\|_{H^{s-1}}\|\eta\|_{L^\infty}+\|\eta\|_{H^{s-1}}\|u\|_{L^\infty}\right)
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\|u(t)\|_{H^s}\le \|u_0\|_{H^s}&+C\int_0^t\|u(\tau)\|_{H^s}(\|u(\tau)\|_{L^\infty}+\|\partial_x u(\tau)\|_{L^\infty}+\|\eta(\tau)\|_{L^\infty}+1)d\tau\\
&+C\int_0^t\|\eta(\tau)\|_{H^{s-1}}(\|\eta(\tau)\|_{L^\infty}+\|u(\tau)\|_{L^\infty}+1)d\tau,
\end{aligned}\tag{3.15}$$

which together with (3.14) ensures that

$$\begin{aligned}
\|u(t)\|_{H^s}+\|\eta(t)\|_{H^{s-1}}\le \|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}+C\int_0^t\Big(\|u(\tau)\|_{H^s}+\|\eta(\tau)\|_{H^{s-1}}\Big)\\
\cdot\Big(\|u(\tau)\|_{L^\infty}+\|\partial_x u(\tau)\|_{L^\infty}+\|\eta(\tau)\|_{L^\infty}+1\Big)d\tau.
\end{aligned}\tag{3.16}$$

Using the Gronwall inequality yields

$$\|u(t)\|_{H^s}+\|\eta(t)\|_{H^{s-1}}\le\Big(\|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}\Big)e^{C\int_0^t(\|u(\tau)\|_{L^\infty}+\|\partial_x u(\tau)\|_{L^\infty}+\|\eta(\tau)\|_{L^\infty}+1)d\tau}.\tag{3.17}$$

In view of the Sobolev embedding theorem $H^s\hookrightarrow L^\infty$ (for $s>1/2$), it is then found from Lemma 3.1 that

$$\begin{aligned}
\|u(t)\|_{L^\infty}\le C\|u\|_{H^1}&\le C\Big(\int_{\mathbb{R}}u^2+u_x^2+\eta^2dx\Big)^{\frac{1}{2}}\\
&\le e^{(\Omega+1)\int_0^t\|u_x\|_{L^\infty}d\tau}\Big(\int_{\mathbb{R}}u_0^2(x)+u_{0,x}^2(x)+\eta_0^2(x)dx\Big)^{\frac{1}{2}}\\
&\le e^{(\Omega+1)\int_0^t\|u_x\|_{L^\infty}d\tau}\Big(\|u_0\|_{H^1}+\|\eta_0\|_{L^2}\Big),
\end{aligned}\tag{3.18}$$

which together with (3.11) and (3.17) implies that

$$\begin{aligned}
&\|u(t)\|_{H^s}+\|\eta(t)\|_{H^{s-1}}\\
&\le\Big(\|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}\Big)e^{C\int_0^t\|\partial_x u(\tau)\|_{L^\infty}d\tau+C_1(t+1)\exp\{(\Omega+1+C)\int_0^t\|\partial_x u(\tau)\|_{L^\infty}d\tau\}},
\end{aligned}\tag{3.19}$$

where $C_1 = C_1(\|u_0\|_{H^1}, \|\eta_0\|_{L^2}, \|\eta_0\|_{L^\infty})$. Therefore, if the maximal existence time $T < \infty$ satisfies $\int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty$, we get from (3.19) that

$$\limsup_{t \to T} (\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}}) < \infty \tag{3.20}$$

contradicts the assumption on the maximal existence time $T < \infty$. This completes the proof of Theorem 3.2 for $s \in (\frac{3}{2}, 2)$.

Step 2. For $s \in [2, \frac{5}{2})$, applying Proposition 3.1 to the transport equation (3.12), we have

$$\|\eta(t)\|_{H^{s-1}} \leq \|\eta_0\|_{H^{s-1}} + C\int_0^t \|(\eta\partial_x u + \partial_x u)(\tau)\|_{H^{s-1}} d\tau + C\int_0^t \|\eta\|_{H^{s-1}} \|\partial_x u\|_{L^\infty \cap H^{\frac{1}{2}}} d\tau.$$

(3.13) applied implies that

$$\|\eta(t)\|_{H^{s-1}} \leq \|\eta_0\|_{H^{s-1}} + C\int_0^t \|\partial_x u\|_{H^{s-1}}(1 + \|\eta\|_{L^\infty}) d\tau + C\int_0^t \|\eta\|_{H^{s-1}} \|\partial_x u\|_{L^\infty \cap H^{\frac{1}{2}}} d\tau,$$

which together with (3.15) yields

$$\begin{aligned}\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}} \leq &\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}} \\ &+ C\int_0^t \big(\|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}}\big)\big(\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \|\eta(\tau)\|_{L^\infty} + 1\big) d\tau,\end{aligned}$$

with $0 < \epsilon_0 < \frac{1}{2}$, where we used the fact $H^{\frac{1}{2}+\epsilon_0} \hookrightarrow L^\infty \cap H^{\frac{1}{2}}$. Applying Gronwall's inequality gives

$$\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}} \leq \big(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}} + K\big) e^{C\int_0^t (\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \|\eta(\tau)\|_{L^\infty} + 1) d\tau}. \tag{3.21}$$

Therefore, using the uniqueness of the solution in Theorem 3.1, (2.24) and (3.20), we get that: if the maximal existence time $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty$, then (3.21) implies that

$$\limsup_{t \to T} (\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}}) < \infty \tag{3.22}$$

which contradicts the assumption on the maximal existence time $T < \infty$. This completes the proof of Theorem 3.2 for $s \in [2, \frac{5}{2})$.

Step 3. For $2 < s < 3$, by differentiating once (3.12) with respect to x, we get

$$\partial_t \eta_x + u\partial_x(\eta_x) + 2u_x\eta_x + \eta u_{xx} + u_{xx} = 0. \tag{3.23}$$

Proposition 3.3 applied to (3.23) implies that

$$
\begin{aligned}
\|\eta_x(t)\|_{H^{s-2}} \leq \|\eta_{0,x}\|_{H^{s-2}} &+ C\int_0^t \|(2\eta_x u_x + \eta u_{xx} + u_{xx})(\tau)\|_{H^{s-2}} d\tau \\
&+ C\int_0^t \|\eta_x(\tau)\|_{H^{s-2}}(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}) d\tau \\
&\leq \|\eta_{0,x}\|_{H^{s-2}} + C\int_0^t \Big(\|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}}\Big) \\
&\quad \cdot \Big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\eta(\tau)\|_{L^\infty} + 1\Big) d\tau,
\end{aligned}
\tag{3.24}
$$

where we used the following estimates from (3.2):

$$
\|\eta_x u_x\|_{H^{s-2}} \leq C\left(\|\partial_x u\|_{H^{s-1}}\|\eta\|_{L^\infty} + \|\partial_x \eta\|_{H^{s-2}}\|u_x\|_{L^\infty}\right)
$$

and

$$
\|\eta u_{xx}\|_{H^{s-2}} \leq C\left(\|\eta\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|u_{xx}\|_{H^{s-2}}\|\eta\|_{L^\infty}\right).
$$

The estimate (3.24) together with (3.14) and (3.15) (where $s-1$ is replaced by $s-2$), thus imply that

$$
\begin{aligned}
\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}} \leq \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}} &+ C\int_0^t \Big(\|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}}\Big) \\
&\cdot \Big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\eta(\tau)\|_{L^\infty} + 1\Big) d\tau.
\end{aligned}
\tag{3.25}
$$

Again, applying Gronwall's inequality gives (3.17). Therefore, using arguments as in Step 1, it completes the proof of Theorem 3.2 for $s \in (2,3)$.

Step 4. For $s = k \in \mathbb{N}$, $k \geq 3$, by differentiating (3.12) $k-2$ times with respect to x, we have

$$
\partial_t \partial_x^{k-2}\eta + u\partial_x(\partial_x^{k-2}\eta) + \sum_{l_1+l_2=k-3,\ l_1,\ l_2\geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\eta + \eta\partial_x(\partial_x^{k-2}u) + \partial_x^{k-1}u = 0.
\tag{3.26}
$$

Applying Proposition 3.2 to the transport equation (3.26), we have

$$
\begin{aligned}
&\|\partial_x^{k-2}\eta(t)\|_{H^1} \\
&\leq \|\partial_x^{k-2}\eta_0\|_{H^1} + C\int_0^t \|\partial_x^{k-2}\eta(\tau)\|_{H^1}\|\partial_x u(\tau)\|_{L^\infty\cap H^{\frac{1}{2}}} d\tau
\end{aligned}
$$

$$+C\int_0^t \Big\|\Big(\sum_{l_1+l_2=k-3,\ l_1,\ l_2\ge 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\eta+\eta\partial_x(\partial_x^{k-2}u)+\partial_x^{k-1}u\Big)(\tau)\Big\|_{H^1}d\tau.$$

Since H^1 is an algebra, we have

$$\|\eta\partial_x(\partial_x^{k-2}u)\|_{H^1}\le C\|\eta\|_{H^1}\|\partial_x^{k-1}u\|_{H^1}\le C\|\eta\|_{H^1}\|u\|_{H^s}$$

and

$$\begin{aligned}&\Big\|\sum_{l_1+l_2=k-3,\ l_1,\ l_2\ge 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\eta\Big\|_{H^1}\\ &\le C\sum_{l_1+l_2=k-3,\ l_1,\ l_2\ge 0} C_{l_1,l_2}\|\partial_x^{l_1+1}u\|_{H^1}\|\partial_x^{l_2+1}\eta\|_{H^1}\le C\|u\|_{H^{s-1}}\|\eta\|_{H^{s-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\partial_x^{k-2}\eta(t)\|_{H^1}\\ &\le\|\partial_x^{k-2}\eta_0\|_{H^1}+C\int_0^t\big(\|u\|_{H^s}+\|\eta\|_{H^{s-1}}\big)\big(\|u\|_{H^{s-1}}+\|\eta\|_{H^1}+1\big)d\tau.\end{aligned}\tag{3.27}$$

(3.27), together with (3.14) and (3.15) (where $s-1$ is replaced by 1), implies that

$$\begin{aligned}\|u(t)\|_{H^s}+\|\eta(t)\|_{H^{s-1}}&\le\|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}\\ &\quad+C\int_0^t\big(\|u(\tau)\|_{H^s}+\|\eta(\tau)\|_{H^{s-1}}\big)\big(\|u(\tau)\|_{H^{s-1}}+\|\eta(\tau)\|_{H^1}+1\big)d\tau.\end{aligned}$$

Applying Gronwall's inequality yields

$$\|u(t)\|_{H^s}+\|\eta(t)\|_{H^{s-1}}\le\big(\|u_0\|_{H^s}+\|\eta_0\|_{H^{s-1}}\big)e^{C\int_0^t(\|u\|_{H^{s-1}}+\|\eta\|_{H^1}+1)d\tau}.\tag{3.28}$$

Therefore, if the maximal existence time $T<\infty$ satisfies $\int_0^T\|\partial_x u(\tau)\|_{L^\infty}d\tau<\infty$, using the uniqueness of the solution in Theorem 3.1, we get that

$$\|u(t)\|_{H^{s-1}}+\|\rho(t)\|_{H^1}$$

is uniformly bounded by the induction assumption, which together with (3.28) implies

$$\limsup_{t\to T}\,(\|u(t)\|_{H^s}+\|\rho(t)\|_{H^{s-1}})<\infty.$$

This leads to a contradiction.

Step 5. For $k < s < k+1$ with $k \in \mathbb{N}$, $k \geq 3$, by differentiating (3.12) $k-1$ times with respect to x, we have

$$\partial_t \partial_x^{k-1}\eta + u\partial_x(\partial_x^{k-1}\eta) + \sum_{l_1+l_2=k-2,\ l_1,\ l_2\geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\eta + \eta\partial_x(\partial_x^{k-1}u) + \partial_x^k u = 0. \tag{3.29}$$

Proposition 3.3 applied again implies that

$$\begin{aligned}
&\|\partial_x^{k-1}\eta(t)\|_{H^{s-k}} \\
&\leq \|\partial_x^{k-1}\eta_0\|_{H^{s-k}} + C\int_0^t \|\partial_x^{k-1}\eta(\tau)\|_{H^{s-k}}\big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty}\big)d\tau \\
&\quad + C\int_0^t \Big\|\Big(\sum_{l_1+l_2=k-2,\ l_1,\ l_2\geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\eta + \eta\partial_x(\partial_x^{k-1}u + \partial_x^k u)\Big)(\tau)\Big\|_{H^{s-k}} d\tau.
\end{aligned}$$

Using (3.2) and the Sobolev embedding inequality, we have $\forall\epsilon_0 \in (0, \frac{1}{2})$

$$\begin{aligned}
\|\eta\partial_x(\partial_x^{k-1}u)\|_{H^{s-k}} &\leq C\big(\|\eta\|_{L^\infty}\|\partial_x^k u\|_{H^{s-k}} + \|\eta\|_{H^{s-k+1}}\|\partial_x^{k-1}u\|_{L^\infty}\big) \\
&\leq C\big(\|\eta\|_{L^\infty}\|u\|_{H^s} + \|\eta\|_{H^{s-k+1}}\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\sum_{l_1+l_2=k-2,\ l_1,\ l_2\geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\Big\|_{H^{s-k}} \\
&\leq C\sum_{l_1+l_2=k-2,\ l_1,\ l_2\geq 0} C_{l_1,l_2}\big(\|\partial_x^{l_1+1}u\|_{L^\infty}\|\partial_x^{l_2+1}\eta\|_{H^{s-k}} + \|\partial_x^{l_2}\eta\|_{L^\infty}\|\partial_x^{l_1+1}u\|_{H^{s-k+1}}\big) \\
&\leq C\big(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\|\eta\|_{H^{s-1}} + \|\rho\|_{H^{k-\frac{3}{2}+\epsilon_0}}\|u\|_{H^s}\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\partial_x^{k-1}\eta(t)\|_{H^{s-k}} \leq& \|\partial_x^{k-1}\eta_0\|_{H^{s-k}} + C\int_0^t (\|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}}) \\
&\cdot(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1)d\tau.
\end{aligned} \tag{3.30}$$

(3.30), together with (3.15) and (3.14) (where $s-1$ is replaced by $s-k$), implies that

$$\|u(t)\|_{H^s} + \|\eta(t)\|_{H^{s-1}} \leq \|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}} + C\int_0^t \big(\|u(\tau)\|_{H^s} + \|\eta(\tau)\|_{H^{s-1}}\big)$$

$$\cdot \left(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1\right) d\tau.$$

Again applying Gronwall's inequality gives

$$\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \leq \left(\|u_0\|_{H^s} + \|\eta_0\|_{H^{s-1}}\right) e^{C\int_0^t (\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta\|_{H^{k-\frac{3}{2}+\epsilon_0}} + 1) d\tau}. \tag{3.31}$$

In consequence, if the maximal existence time $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty$, using the uniqueness of the solution in Theorem 3.1, we get that

$$\|u(t)\|_{H^{k-\frac{1}{2}+\epsilon_0}} + \|\eta(t)\|_{H^{k-\frac{3}{2}+\epsilon_0}}$$

is uniformly bounded by the induction assumption, which implies

$$\limsup_{t\to T} (\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}) < \infty.$$

This leads to a contradiction. Therefore, in view of Step 1 to Step 5, this completes the proof of Theorem 3.2. □

4. Traveling waves

In this section, we consider the traveling-wave solutions of (1.1), i.e. solutions of the form

$$(u(t,x), \eta(t,x)) = (\varphi(x - \sigma t), \psi(x - \sigma t)), \quad \sigma \in \mathbb{R},$$

for some functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \to 0$, $\psi \to 0$ as $|x| \to \infty$.

The system (1.1) can be written in a weak form as

$$\begin{cases} u_t - uu_x = -\partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \eta - 2\Omega((1+\eta)u)\right), & t > 0, \quad x \in \mathbb{R}, \\ \eta_t + u\eta_x = -(1+\eta)u_x, & t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{4.1}$$

A weak solution to (1.1) is defined as follows.

Definition 4.1. Let $0 < T \leq \infty$. A function $\vec{u} = (u_0, \eta_0) \in C([0,T); H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ is called a weak solution of

$$\begin{cases} u_t - u_{xxt} + uu_x + 4u_x u_{xx} + uu_{xxx} + \eta_x + 2\Omega\eta_t = 0, \\ \eta_t + ((1+\eta)u)_x = 0, \end{cases} \tag{4.2}$$

on $[0, T)$ if it satisfies the following identity:

$$\begin{cases} \int_0^T \int_{\mathbb{R}} [u\phi_t - \frac{1}{2}u^2\phi_x + p * \left(u^2 + \frac{1}{2}u_x^2 + \eta - 2\Omega((1+\eta)u)\right) \cdot \phi_x] dx dt = 0, \\ \int_0^T \int_{\mathbb{R}} [\eta\phi_t + (1+\eta)u\phi_x] dx dt = 0, \end{cases} \tag{4.3}$$

for any smooth test function $\phi(t,x) \in C_0^\infty([0,T)\times\mathbb{R})$, where $p(x)$ is the corresponding kernel of the convolution operator $(1-\partial_x^2)^{-1}$. If u is a weak solution on $[0,T)$ for every $T>0$, then it is called a global weak solution.

We now give the definition of a traveling-wave solution of (1.1).

Definition 4.2. A traveling-wave solution of (1.1) is a nontrivial solution of (1.1) of the form $\vec{\varphi}_\sigma(t,x) = (\varphi_\sigma(x-\sigma t), \psi_\sigma(x-\sigma t)) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with the constant wave speed $\sigma \in \mathbb{R}$ and φ_σ, ψ_σ vanishing at infinity.

For a traveling-wave solution $\vec{\varphi} = (\varphi, \psi)$ with speed $\sigma \in \mathbb{R}$, it satisfies

$$\begin{cases} [-\sigma\varphi - \frac{1}{2}\varphi^2 + p*(\varphi^2 + \frac{1}{2}\varphi_x^2 + \psi - 2\Omega((1+\psi)\varphi))]_x = 0, \\ [-\sigma\psi + (1+\psi)\varphi]_x = 0, \qquad\qquad \text{in } \mathscr{D}'(\mathbb{R}). \end{cases}$$

Integrating the above system and applying $(1-\partial_x^2)$ to the first equation we get

$$\begin{cases} -\sigma\varphi + \sigma\varphi_{xx} + \frac{1}{2}\varphi^2 + \varphi\varphi_{xx} + \frac{3}{2}\varphi_x^2 + (1-2\sigma\Omega)\psi = 0, \\ -\sigma\psi + (1+\psi)\varphi = 0, \qquad\qquad \text{in } \mathscr{D}'(\mathbb{R}). \end{cases} \tag{4.4}$$

In fact, the second equation of the above holds in a strong sense coming from the regularity of φ and ψ.

4.1. Nonexistence of the CH-type peakon

To investigate the traveling-wave solution of (1.1), we start with the demonstration of the nonexistence of the CH-type peakon.

Theorem 4.1. *For $1-2\sigma\Omega = 0$, there is no any nonzero weak solution of system (1.1) in the form of $u(t,x) = a(\sigma,t)e^{-|x-\sigma t|}$, where $\sigma \in \mathbb{R}$ and $a(\sigma,t) \in C(\mathbb{R}\times[0,T))$.*

Proof. Consider the traveling wave solution of the system (1.1), when $1-2\sigma\Omega = 0$, it reduces to be

$$u_t - u_{xxt} + uu_x + 4u_xu_{xx} + uu_{xxx} = 0, \quad t>0, \quad x\in\mathbb{R}. \tag{4.5}$$

Suppose the above equation (4.5) possesses the peaked traveling-wave solution in the form of

$$u_a(t,x) = a(\sigma,t)e^{-|x-\sigma t|}, \quad \sigma\in\mathbb{R} \text{ and } a(\sigma,t)\neq 0. \tag{4.6}$$

Then, for all $t\in\mathbb{R}_+$, in the sense of distribution and $\partial_x u_a(t,x) = -\operatorname{sign}(x-\sigma t)u_a(t,x)$ belongs to $L^\infty(\mathbb{R})$. For any test function $\phi(\cdot)\in C_c^\infty(\mathbb{R})$, by using integration by parts, we have

$$\int_{\mathbb{R}} \operatorname{sign}(y)e^{-|y|}\phi(y)dy = \int_{-\infty}^{0} -e^{y}\phi(y)dy + \int_{0}^{+\infty} e^{-y}\phi(y)dy = \int_{\mathbb{R}} e^{-|y|}\phi'(y)dy.$$

Note that

$$\partial_t u_a(t,x) = \partial_t a(\sigma,t)e^{-|x-\sigma t|} + \sigma\,\mathrm{sign}(x-\sigma t)u_a(t,x) \in L^\infty(\mathbb{R}) \text{ for all } t \geq 0. \tag{4.7}$$

Hence, using integration by parts, we deduce that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} \Big(u_a\phi_t - \frac{1}{2}u_a^2\phi_x\Big)dxdt + \int_{\mathbb{R}} u_a(0,x)\phi(0,x)dx \\
&\quad = -\int_0^\infty\int_{\mathbb{R}} \phi\Big[\partial_t u_a - u_a\cdot\partial_x u_a\Big]dxdt \\
&\quad = -\int_0^\infty\int_{\mathbb{R}} \phi\Big[\partial_t a(\sigma,t)e^{-|x-\sigma t|} + \sigma\,\mathrm{sign}(x-\sigma t)u_a + \mathrm{sign}(x-\sigma t)u_a^2\Big]dxdt
\end{aligned} \tag{4.8}$$

On the other hand, we know

$$u = (1-\partial_x^2)^{-1}m = p * m, \text{ where } p(x) = \frac{1}{2}e^{-|x|},$$

and the notation "$*$" denotes the convolution product on $\mathbb{R}$, defined by

$$\Big(f*g\Big)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

Hence,

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} \Big[(1-\partial_x^2)^{-1}\Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)\cdot\partial_x\phi\Big]dxdt \\
&\quad = -\int_0^\infty\int_{\mathbb{R}} \Big[\phi\cdot\partial_x p * \Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)\Big]dxdt.
\end{aligned} \tag{4.9}$$

It is noted that $\partial_x p(x) = -\frac{1}{2}\,\mathrm{sign}(x)e^{-|x|}$ for $x \in \mathbb{R}$. A simple computation reveals that

$$\begin{aligned}
\partial_x p * \Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)(t,x) = -\frac{1}{2}\int_{-\infty}^{+\infty} &\mathrm{sign}(x-y)e^{-|x-y|}\cdot\Big[a^2(\sigma,t)e^{-2|y-\sigma t|} \\
&+ \frac{1}{2}\,\mathrm{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|}\Big]dy.
\end{aligned} \tag{4.10}$$

When $x > \sigma t$, we split the right hand side of (4.10) into the following three parts.

$$\partial_x p * \Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)(t,x) = -\frac{1}{2}\Big(\int_{-\infty}^{\sigma t} + \int_{\sigma t}^{x} + \int_{x}^{+\infty}\Big) \operatorname{sign}(x-y)e^{-|x-y|} \cdot \Big[a^2(\sigma,t)e^{-2|y-\sigma t|}$$
$$+ \frac{1}{2}\operatorname{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|}\Big]dy$$
$$=: I_1 + I_2 + I_3.$$

In the case that $-\infty < y < \sigma t < x$, it follows that

$$\begin{aligned}
I_1 =& -\frac{1}{2}\int_{-\infty}^{\sigma t} e^{-x+y}\Big[a^2(\sigma,t)e^{2(y-\sigma t)} + \frac{1}{2}a^2(\sigma,t)e^{2(y-\sigma t)}\Big]dy \\
=& -\frac{1}{2}\int_{-\infty}^{\sigma t} e^{-x+y}\Big[\frac{3}{2}a^2(\sigma,t)e^{2(y-\sigma t)}\Big]dy = -\frac{3}{4}a^2(\sigma,t)e^{-x-2\sigma t}\int_{-\infty}^{\sigma t} e^{3y}dy \\
=& -\frac{3}{4}a^2(\sigma,t)e^{-x-2\sigma t}\Big[\frac{1}{3}e^{3y}\Big|_{-\infty}^{\sigma t}\Big] = -\frac{1}{4}a^2(\sigma,t)e^{-x-2\sigma t}\Big(e^{3\sigma t} - 0\Big) \\
=& -\frac{1}{4}a^2(\sigma,t)e^{-x+\sigma t}.
\end{aligned}$$

For $\sigma t < y < x$, a direct computation gives that

$$\begin{aligned}
I_2 =& -\frac{1}{2}\int_{\sigma t}^{x} e^{-x+y}\Big[a^2(\sigma,t)e^{2(-y+\sigma t)} + \frac{1}{2}a^2(\sigma,t)e^{2(-y+\sigma t)}\Big]dy \\
=& -\frac{1}{2}\int_{\sigma t}^{x} e^{-x+y}\Big[\frac{3}{2}a^2(\sigma,t)e^{-2y+2\sigma t)}\Big]dy = -\frac{3}{4}a^2(\sigma,t)e^{-x+2\sigma t}\int_{\sigma t}^{x} e^{-y}dy \\
=&\frac{3}{4}a^2(\sigma,t)e^{-x+2\sigma t}\Big[e^{-y}\Big|_{\sigma t}^{x}\Big] = \frac{3}{4}a^2(\sigma,t)e^{-x+2\sigma t}\Big(e^{-x} - e^{-\sigma t}\Big) \\
=&\frac{3}{4}a^2(\sigma,t)e^{-2x+2\sigma t} - \frac{3}{4}a^2(\sigma,t)e^{-x+\sigma t}.
\end{aligned}$$

For $\sigma t < x < y < +\infty$, we have

$$\begin{aligned}
I_3 =& \frac{1}{2}\int_{x}^{+\infty} e^{x-y}\Big[a^2(\sigma,t)e^{2(-y+\sigma t)} + \frac{1}{2}a^2(\sigma,t)e^{2(-y+\sigma t)}\Big]dy \\
=& \frac{1}{2}\int_{x}^{+\infty} \frac{3}{2}a^2(\sigma,t)e^{x+2\sigma t-3y}dy = \frac{3}{4}a^2(\sigma,t)e^{x+2\sigma t}\int_{x}^{+\infty} e^{-3y}dy
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}a^2(\sigma,t)e^{x+2\sigma t}\left[e^{-3y}\Big|_x^{+\infty}\right] = -\frac{1}{4}a^2(\sigma,t)e^{x+2\sigma t}\left(0 - e^{-3x}\right) \\
&= \frac{1}{4}a^2(\sigma,t)e^{-2x+2\sigma t}.
\end{aligned}
$$

Combining I_1, I_2 and I_3, for $x > \sigma t$, we have

$$
\begin{aligned}
\partial_x p * \left(u_a^2 + \frac{1}{2}u_{a,x}^2\right)(t,x) = &- \frac{1}{4}a^2(\sigma,t)e^{-x+\sigma t} + \frac{3}{4}a^2(\sigma,t)e^{-2x+2\sigma t} - \frac{3}{4}a^2(\sigma,t)e^{-x+\sigma t} \\
&+ \frac{1}{4}a^2(\sigma,t)e^{-2x+2\sigma t} \\
= &- a^2(\sigma,t)e^{-x+\sigma t} + a^2(\sigma,t)e^{-2x+2\sigma t}.
\end{aligned}
$$

When $x \le \sigma t$, we split the right hand side of (4.10) into the following three parts.

$$
\begin{aligned}
\partial_x p * \left(u_a^2 + \frac{1}{2}u_{a,x}^2\right)(t,x) = &- \frac{1}{2}\Big(\int_{-\infty}^{x} + \int_{x}^{\sigma t} + \int_{\sigma t}^{+\infty}\Big)\,\mathrm{sign}(x-y)e^{-|x-y|}\cdot\Big[a^2(\sigma,t)e^{-2|y-\sigma t|} \\
&+ \frac{1}{2}\,\mathrm{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|}\Big]dy \\
=:&\, II_1 + II_2 + II_3.
\end{aligned}
$$

For $-\infty < y < x \le \sigma t$, a simple computation shows that

$$
\begin{aligned}
II_1 &= -\frac{1}{2}\int_{-\infty}^{x} e^{-x+y}\left[a^2(\sigma,t)e^{2y-2\sigma t} + \frac{1}{2}a^2(\sigma,t)e^{2y-2\sigma t}\right]dy \\
&= -\frac{1}{2}\int_{-\infty}^{x} e^{-x+y}\left[\frac{3}{2}a^2(\sigma,t)e^{2y-2\sigma t}\right]dy = -\frac{3}{4}a^2(\sigma,t)e^{-x-2\sigma t}\int_{-\infty}^{x} e^{3y}dy \\
&= -\frac{1}{4}a^2(\sigma,t)e^{-x-2\sigma t}e^{3y}\Big|_{-\infty}^{x} = -\frac{1}{4}a^2(\sigma,t)e^{-x-2\sigma t}\left(e^{3x} - 0\right) \\
&= -\frac{1}{4}a^2(\sigma,t)e^{2x-2\sigma t}.
\end{aligned}
$$

For $x < y < \sigma t$, it is found that

$$
\begin{aligned}
II_2 &= -\frac{1}{2}\int_{x}^{\sigma t} -e^{x-y}\left[\frac{3}{2}a^2(\sigma,t)e^{2y-2\sigma t}\right]dy = \frac{3}{4}a^2(\sigma,t)e^{x-2\sigma t}\int_{x}^{\sigma t} e^{y}dy \\
&= \frac{3}{4}a^2(\sigma,t)e^{x-2\sigma t}\left(e^{y}\Big|_{x}^{\sigma t}\right) = \frac{3}{4}a^2(\sigma,t)e^{x-2\sigma t}\left(e^{\sigma t} - e^{x}\right) \\
&= \frac{3}{4}a^2(\sigma,t)e^{x-\sigma t} - \frac{3}{4}a^2(\sigma,t)e^{2x-2\sigma t}.
\end{aligned}
$$

For $x \le \sigma t < y < +\infty$, it is easy to check that

$$\begin{aligned}
II_3 =& -\frac{1}{2}\int_{\sigma t}^{+\infty} -e^{x-y}\Big[a^2(\sigma,t)e^{-2y+2\sigma t} + \frac{1}{2}a^2(\sigma,t)e^{-2y+2\sigma t}\Big]dy \\
=& \frac{3}{4}a^2(\sigma,t)e^{x+2\sigma t}\int_{\sigma t}^{+\infty} e^{-3y}dy = -\frac{1}{4}a^2(\sigma,t)e^{x+2\sigma t}\Big(e^{-3y}\Big|_{\sigma t}^{+\infty}\Big) \\
=& -\frac{1}{4}a^2(\sigma,t)e^{x+2\sigma t}\Big(0 - e^{-3\sigma t}\Big) = \frac{1}{4}a^2(\sigma,t)e^{x-\sigma t}.
\end{aligned}$$

Combining II_1, II_2 and II_3, in the case $x \le \sigma t$ gives

$$\begin{aligned}
\partial_x p * \Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)(t,x) =& -\frac{1}{4}a^2(\sigma,t)e^{2x-2\sigma t} + \frac{3}{4}a^2(\sigma,t)e^{x-\sigma t} - \frac{3}{4}a^2(\sigma,t)e^{2x-2\sigma t} \\
&+ \frac{1}{4}a^2(\sigma,t)e^{x-\sigma t} \\
=& -a^2(\sigma,t)e^{2x-2\sigma t} + a^2(\sigma,t)e^{x-\sigma t}.
\end{aligned}$$

According to (4.8), it is then inferred from these two cases mentioned above that

$$\partial_x p * \Big(u_a^2 + \frac{1}{2}u_{a,x}^2\Big)(t,x) = \begin{cases} -a^2(\sigma,t)e^{-x+\sigma t} + a^2(\sigma,t)e^{-2x+2\sigma t}, & \text{if } x > \sigma t, \\ -a^2(\sigma,t)e^{2x-2\sigma t} + a^2(\sigma,t)e^{x-\sigma t}, & \text{if } x \le \sigma t. \end{cases} \tag{4.11}$$

If the function in the form of (4.6) is a weak solution of equation (4.5), then combining (4.8), (4.9) and (4.11) yields that

$$\begin{cases} \partial_t a(\sigma,t)e^{-x+\sigma t} + \sigma u_a + u_a^2 = a^2(\sigma,t)e^{-x+\sigma t} - a^2(\sigma,t)e^{-2x+2\sigma t}, & \text{if } x > \sigma t, \\ \partial_t a(\sigma,t)e^{x-\sigma t} - \sigma u_a - u_a^2 = a^2(\sigma,t)e^{2x-2\sigma t} - a^2(\sigma,t)e^{x-\sigma t}, & \text{if } x \le \sigma t, \end{cases}$$

which implies that

$$\begin{cases} \Big[\partial_t a(\sigma,t) + \sigma a(\sigma,t) - a^2(\sigma,t)\Big]e^{-x+\sigma t} + 2a^2(\sigma,t)e^{-2x+2\sigma t} = 0, & \text{if } x > \sigma t, \\ \Big[\partial_t a(\sigma,t) - \sigma a(\sigma,t) + a^2(\sigma,t)\Big]e^{x-\sigma t} - 2a^2(\sigma,t)e^{2x-2\sigma t} = 0, & \text{if } x \le \sigma t. \end{cases}$$

By the linear independence of the functions $e^{-x+\sigma t}$, $e^{x-\sigma t}$, $e^{-2x+2\sigma t}$ and $e^{2x-2\sigma t}$, the above condition holds if and only if

$$a(\sigma,t) = 0,$$

which provides a trivial solution of equation (1.1), $u_a(t,x) = 0$, thereby concluding the proof of Theorem 4.1. □

4.2. Classification of traveling-wave solutions

Our attention in this subsection is turn to the classification of traveling-wave solutions based on the argument in [38].

Proposition 4.1. *If (φ, ψ) is a traveling-wave solution of (1.1) for some $\sigma \in \mathbb{R}$, then $\sigma \neq 0$ and $\varphi(x) \neq \sigma$ for any $x \in \mathbb{R}$.*

Proof. From the definition of traveling-wave solutions and the embedding theorem, we know that φ and ψ are both continuous. If $\sigma = 0$, then (4.4) becomes

$$\begin{cases} \frac{1}{2}\varphi^2 + \varphi\varphi_{xx} + \frac{3}{2}\varphi_x^2 + \psi = 0, \\ (1+\psi)\varphi = 0. \end{cases} \tag{4.12}$$

Since ψ vanishes at infinity, the second equation of the above system indicates that $\varphi(x) = 0$ for $|x|$ large enough. Denote $x_0 = \max\{x : \varphi(x) \neq 0\}$. Hence $\varphi(x) = 0$ on $[x_0, \infty)$ and $\varphi \not\equiv 0$ on $(x_0 - \delta, x_0)$ for any $\delta > 0$. Consider now the first equation of (4.12) on $[x_0, \infty)$ we see that $\psi \equiv 0$ on $[x_0, \infty)$. Then the continuity of ψ implies that there exists a $\delta_1 > 0$ such that $1 + \psi(x) > 0$ on $(x_0 - \delta_1, x_0)$. This together with the second equation of (4.12) leads to $\varphi \equiv 0$ on $(x_0 - \delta_1, x_0)$, which is a contradiction. Therefore $\sigma \neq 0$.

Next we show $\varphi \neq \sigma$. If not and there is some $x_1 \in \mathbb{R}$ such that $\varphi(x_1) = \sigma$. Then the second equation of (4.4) infers that

$$\varphi(x_1) = (\sigma - \varphi(x_1))\psi(x_1) = 0,$$

so $\sigma = 0$, which is a contradiction. This proves Proposition 4.1. □

According to the above proposition we obtain from the second equation of (4.4) that

$$\psi = \frac{\varphi}{\sigma - \varphi}. \tag{4.13}$$

Substituting this into the first equation (4.4) we obtain an equation for the unknown φ only

$$-\sigma\varphi + \sigma\varphi_{xx} + \frac{1}{2}\varphi^2 + \varphi\varphi_{xx} + \frac{3}{2}\varphi_x^2 + (1 - 2\sigma\Omega)\frac{\varphi}{\sigma - \varphi} = 0. \tag{4.14}$$

We can rewrite (4.14) as

$$((\varphi + \sigma)^2)_{xx} = -\varphi_x^2 + 2\sigma\varphi - \varphi^2 + 2(1 - 2\sigma\Omega) - 2\sigma(1 - 2\sigma\Omega)\frac{1}{\sigma - \varphi}, \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{4.15}$$

The following lemma deals with the regularity of the traveling waves.

Lemma 4.1. *Let $\vec{\varphi} = (\varphi, \psi)$ be a traveling-wave solution of (1.1). Then*

$$(\varphi + \sigma)^k \in C^j(\mathbb{R} \setminus \varphi^{-1}(-\sigma)), \ \text{for } k \geq 2^j. \tag{4.16}$$

Furthermore,

$$\varphi \in C^{\infty}(\mathbb{R} \setminus \varphi^{-1}(-\sigma)). \tag{4.17}$$

Proof. From Proposition 4.1 we know that $\sigma \neq 0$ and $\varphi \neq \sigma$ and thus φ satisfies (4.15). Let $v = \varphi + \sigma$ and denote

$$r(v) = 2\sigma(v-\sigma) - (v-\sigma)^2 + 2(1-2\sigma\Omega).$$

So $r(v)$ is a polynomial in v. From the fact that $\varphi - \sigma \neq 0$ we know that

$$2\sigma - v \neq 0. \tag{4.18}$$

Thus v satisfies

$$(v^2)_{xx} = -v_x^2 + r(v) - 2\sigma(1-2\sigma\Omega)(2\sigma - v)^{-1}.$$

From the definition of traveling-wave solutions that $\varphi \in H^1(\mathbb{R})$, we know $(v^2)_{xx} \in L^1_{loc}(\mathbb{R})$. Hence $(v^2)_x$ is absolutely continuous and hence

$$v^2 \in C^1(\mathbb{R}), \text{ and then } v \in C^1(\mathbb{R} \setminus v^{-1}(0)).$$

So from (4.18) and that $v - \sigma \in H^1(\mathbb{R}) \subset C(\mathbb{R})$ we know

$$(2\sigma - v)^{-1} \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus v^{-1}(0)).$$

Moreover,

$$\begin{aligned}
(v^k)_{xx} =& [(v^k)_x]_x = [kv^{k-1}v_x]_x = [\frac{k}{2}v^{k-2}(2vv_x)]_x \\
=& \frac{k}{2}(v^{k-2})_x(v^2)_x + \frac{k}{2}v^{k-2}(v^2)_{xx} \\
=& k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}[-v_x^2 + r(v) - 2\sigma(1-2\sigma\Omega)(2\sigma - v)^{-1}] \\
=& k(k-\frac{5}{2})v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}r(v) - kv^{k-2}\sigma(1-2\sigma\Omega)(2\sigma - v)^{-1}.
\end{aligned} \tag{4.19}$$

For $k = 3$, the right-hand side of the above equation is in $L^1_{loc}(\mathbb{R})$. Therefore, we have $v^3 \in C^1(\mathbb{R})$. Similarly, for $k \geq 4$, it is deduced from (4.19) that

$$(v^k)_{xx} = \frac{k}{4}(k-\frac{5}{2})v^{k-4}[(v^2)_x]^2 + \frac{k}{2}v^{k-2}r(v) - kv^{k-2}\sigma(1-2\sigma\Omega)(2\sigma - v)^{-1}.$$

Therefore $v^k \in C^2(\mathbb{R})$ for $k \geq 4$.

For $k \geq 8$, we know from the above that

$$v^4, v^{k-4}, v^{k-2}, v^{k-2}r(v) \in C^2(\mathbb{R}), \text{ and } v^{k-2}(2\sigma - v)^{-1} \in C^2(\mathbb{R} \setminus v^{-1}(0)).$$

Moreover, we have

$$v^{k-2}v_x^2 = \frac{1}{4}(v^4)_x \frac{1}{k-4}(v^{k-4})_x \in C^1(\mathbb{R}).$$

Hence from (4.19) we conclude that

$$v^k \in C^3(\mathbb{R} \setminus v^{-1}(0)), \quad k \geq 8.$$

Applying the same argument to higher values of k we prove that

$$v^k \in C^j(\mathbb{R} \setminus v^{-1}(0)), \quad \text{for} \quad k \geq 2^j.$$

Since $\sigma - \varphi \neq 0$, the above infers that $\varphi \in C^\infty(\mathbb{R} \setminus \varphi^{-1}(-\sigma))$ and so is ψ by (4.13). This completes the proof of Lemma 4.1. $\square$

Denote $\bar{x} = \min\{x : \varphi(x) = -\sigma\}$ (if $\varphi(x) \neq -\sigma$ for all x then let $\bar{x} = +\infty$), then $\bar{x} \leq +\infty$. From Lemma 4.1, a traveling-wave solution φ is smooth on $(-\infty, \bar{x})$ and hence (4.14) holds pointwise on $(-\infty, \bar{x})$. Therefore we may multiply by $(\varphi + \sigma)^2\varphi_x$ and integrate on $(-\infty, x]$ for $x < \bar{x}$ to get

$$\begin{aligned}\varphi_x^2 =& \frac{-\frac{1}{5}\varphi^5 + \frac{1}{3}[3\sigma^2 + 2(1-2\sigma\Omega)]\varphi^3 + [\sigma^3 + 3\sigma(1-2\sigma\Omega)]\varphi^2}{(\varphi+\sigma)^3} \\ &+ \frac{8\sigma^2(1-2\sigma\Omega)\varphi + 8\sigma^3(1-2\sigma\Omega)\ln|\sigma - \varphi|}{(\varphi+\sigma)^3} \\ :=& F(\varphi).\end{aligned} \tag{4.20}$$

Applying the similar arguments as introduced in [38], we have the following conclusions.

1. When φ approaches a simple zero m of $F(\varphi)$ so that $F(m) = 0$ and $F'(m) \neq 0$. The solution φ of (4.20) satisfies

$$\varphi_x^2 = (\varphi - m)F'(m) + O((\varphi - m)^2), \text{ as } \varphi \to m, \tag{4.21}$$

   where $f = O(g)$ as $x \to a$ means $|f(x)/g(x)|$ is bounded in some interval $[a-\varepsilon, a+\varepsilon]$ with $\varepsilon > 0$. Then, we have

$$\varphi(x) = m + \frac{1}{4}(x - x_0)^2F'(m) + O((x - x_0)^4), \text{ as } x \to x_0, \tag{4.22}$$

   where $\varphi(x_0) = m$.
2. If $F(\varphi)$ has a double zero at $\varphi = 0$ such that $F'(0) = 0$ and $F''(0) > 0$, then

$$\varphi_x^2 = \varphi^2F''(0) + O(\varphi^3), \text{ as } \varphi \to 0. \tag{4.23}$$

Hence,

$$\varphi = O\Big(\exp(-\sqrt{F''(0)}|x|)\Big), \text{ as } |x| \to \infty, \tag{4.24}$$

which implies $\varphi \to 0$ exponentially as $x \to \infty$.

3. If φ approaches a simple pole $\varphi(x_0) = -\sigma$ of $F(\varphi)$, then

$$\varphi(x) + \sigma = \lambda|x - x_0|^{2/3} + O((x - x_0)^{4/3}), \quad \text{as} \quad x \to x_0, \tag{4.25}$$

$$\varphi_x = \begin{cases} \frac{2}{3}\lambda|x - x_0|^{-1/3} + O((x - x_0)^{1/3}), & \text{as} \quad x \downarrow x_0, \\ -\frac{2}{3}\lambda|x - x_0|^{-1/3} + O((x - x_0)^{1/3}), & \text{as} \quad x \uparrow x_0, \end{cases} \tag{4.26}$$

for some constant λ. In particular, when $F(\varphi)$ has a pole, the solution φ has a cusp.

4. Peaked traveling waves occur when φ suddenly changes direction: $\varphi_x \mapsto -\varphi_x$ according to equation (4.20). By all the information mentioned above, it enables us to classify the traveling-wave solutions of (1.1) in two cases based on the form of (4.20).

4.2.1. The case when $1 - 2\sigma\Omega = 0$

In this case system (4.4) becomes an ODE for φ solely and an algebraic equation for ψ. More specifically, the equation for φ is

$$-\sigma\varphi + \sigma\varphi_{xx} + \frac{1}{2}\varphi^2 + \varphi\varphi_{xx} + \frac{3}{2}\varphi_x^2 = 0. \tag{4.27}$$

In this case, equation (4.20) becomes

$$\varphi_x^2 = \frac{-\frac{1}{5}\varphi^5 + \sigma^2\varphi^3 + \sigma^3\varphi^2}{(\varphi + \sigma)^3} \tag{4.28}$$

Let us start with

$$f_1(\varphi) = -\frac{1}{5}\varphi^3 + \sigma^2\varphi + \sigma^3.$$

Then, by the property of cubic polynomial, we can rewrite

$$f_1(\varphi) = -\frac{1}{5}[\varphi^3 - 5\sigma^2\varphi - 5\sigma^3]. \tag{4.29}$$

Let

$$m = -\frac{5}{3}\sigma^2, n = -\frac{5}{2}\sigma^3.$$

The determinator of equation $f_1(\varphi) = 0$ is defined by

$$D = n^2 + m^3 = (-\frac{5}{2}\sigma^3)^2 + (-\frac{5}{3}\sigma^2)^3 = \frac{25}{4}\sigma^6 - \frac{125}{27}\sigma^6 = \frac{175}{108}\sigma^6 > 0. \tag{4.30}$$

So f_1 has only one real root λ_1, which is

$$\lambda_1 = \left(\sqrt[3]{\frac{5}{2} + \sqrt{\frac{175}{108}}} + \sqrt[3]{\frac{5}{2} - \sqrt{\frac{175}{108}}} \right) \sigma. \tag{4.31}$$

Theorem 4.2. *Let $1 - 2\sigma\Omega = 0$. For system (1.1), there is an anticusped traveling-wave solution (φ, ψ) with $\min_{x \in \mathbb{R}} \varphi(x) = -\sigma$ and $\min_{x \in \mathbb{R}} \psi(x) = -\frac{1}{2}$.*

Proof. Due to $\sigma = \frac{1}{2\Omega} \geq 0$, equation (4.28) has the following form

$$\varphi_x^2 = \frac{\frac{1}{5}\varphi^2(\lambda_1 - \varphi)Q(\varphi)}{(\varphi + \sigma)^3} := F_1(\varphi), \tag{4.32}$$

where $Q(\varphi) > 0$ is a quadratic polynomial, and $\lambda_1 > 0$. From (4.32) we know that φ can not oscillate around zero near infinity. Let us consider the following two cases.

(1.1) If $\varphi(x) > 0$ near $-\infty$, then there is some x_0 sufficiently large negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$. $\sqrt{F_1(\varphi)}$ is locally Lipschitz continuous in φ for $0 < \varphi < \lambda_1$. Hence, there is a local solution to

$$\begin{cases} \varphi_x = \sqrt{F_1(\varphi)}, \\ \varphi(x_0) = \varepsilon, \end{cases}$$

on $[x_0 - L, x_0 + L]$ for some $L > 0$. Therefore by (4.22), (4.23) and (4.24), the smooth solution can be constructed with the maximum height $\varphi = \lambda_1$ and decay to zero at infinity. However, since $\sigma < \lambda_1$, φ may take σ, which contradicts with Proposition 4.1. Hence, smooth traveling waves are excluded.

(1.2) If $\varphi(x) < 0$ near $-\infty$. Then there is some x_0 sufficiently large negative so that $\varphi(x_0) = -\varepsilon < 0$, with ε sufficiently small, and $\varphi_x(x_0) < 0$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $-\sigma < \varphi < 0$, then $-\sigma$ becomes a pole of $F_1(\varphi)$. Then we may obtain a traveling-wave solution with a cusp at $\varphi = -\sigma$ by (4.25) and (4.26). Hence, by (4.13) we know that when φ exhibits a cusp singularity then ψ exhibits a cusp as well. Moreover, $-\frac{1}{2} < \psi < 0$ with $\min_{x \in \mathbb{R}} \psi(x) = -\frac{1}{2}$. This completes the proof of Theorem 4.2. $\square$

4.2.2. The case when $1 - 2\sigma\Omega \neq 0$

In this case, according to the Definition 4.2, suppose there exists a traveling-wave solution of (1.1) in the form $\vec{\varphi} = (\varphi, \psi)$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, then (4.15) admits a solution $\varphi \in H^1(\mathbb{R})$ such that φ decays to zero at infinity, i.e. $F(\varphi)$ in (4.20) has a double zero at $\varphi = 0$. This infers that $F'(0) = 0$ and $F''(0) > 0$. A direct computation shows that, if $1 - 2\sigma\Omega \neq 0$ and $\sigma \neq 0$,

$$F'(0) = -\frac{24}{\sigma}(1 - 2\sigma\Omega)\ln|\sigma| = 0, \text{ when } \sigma = 1. \tag{4.33}$$

Thus, for $\sigma = 1$, $F(\varphi)$ can be simplified as

$$\begin{aligned} F_2(\varphi) &= \frac{-\frac{1}{5}\varphi^5 + \frac{1}{3}(5-4\Omega)\varphi^3 + (4-6\Omega)\varphi^2 + 8(1-2\Omega)\varphi + 8(1-2\Omega)\ln|1-\varphi|}{(\varphi+1)^3} \\ &:= \frac{f_2(\varphi)}{(\varphi+1)^3}, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} F_2''(\varphi) = {} & \frac{\Big[-2\varphi^4 - 4\varphi^3 + 2(1+2\Omega)\varphi + (-8+20\Omega) - \frac{16(1-2\Omega)}{(\varphi-1)^2}\Big](\varphi+1)}{(\varphi+1)^5} - 4\cdot \\ & \frac{\Big[-\frac{2}{5}\varphi^5 - \varphi^4 + (1+2\Omega)\varphi^2 + (-8+20\Omega)\varphi + 16(1-2\Omega) + 16(1-2\Omega)\frac{1}{\varphi-1}\Big]}{(\varphi+1)^5}. \end{aligned} \tag{4.35}$$

Therefore, $F_2''(0) = -24 + 52\Omega > 0$ requires

$$\Omega > \frac{6}{13}. \tag{4.36}$$

Let $F_2(\varphi) = 0$. Then, it is deduced from (4.34) that

$$\Omega = \frac{-\frac{1}{5}\varphi^5 + \frac{5}{3}\varphi^3 + 4\varphi^2 + 8\varphi + 8\ln|1-\varphi|}{\frac{4}{3}\varphi^3 + 6\varphi^2 + 16\varphi + 16\ln|1-\varphi|}. \tag{4.37}$$

Lemma 4.2. *For $-1 < \varphi < 0$ or $0 < \varphi < 1$, consider Ω as a function of φ, then it is monotonic increasing with respect to φ.*

Proof. To show that Ω is monotonic increasing with respect to φ, we will start with the consideration of the first order derivative of Ω. If $\Omega' \geq 0$, then Ω is monotonic increasing. In fact, Ω' can be written in the following form

$$\Omega' = \frac{\varphi(\varphi+1)^2}{(1-\varphi)\Big(\frac{4}{3}\varphi^3 + 6\varphi^2 + 16\varphi + 16\ln|1-\varphi|\Big)^2} \cdot g(\varphi), \tag{4.38}$$

where $g(\varphi) = \Big[\varphi(\varphi-3)\Big(\frac{4}{3}\varphi^3 + 6\varphi^2 + 16\varphi + 16\ln|1-\varphi|\Big) + 4\Big(-\frac{1}{5}\varphi^5 + \frac{5}{3}\varphi^3 + 4\varphi^2 + 8\varphi + 8\ln|1-\varphi|\Big)\Big]$ is an increasing function on $(-1, 1)$. Indeed, $g'(\varphi) = \frac{2}{3}(2\varphi-3)h(\varphi)$, where $h(\varphi) = [\varphi(2\varphi^2 + 9\varphi + 24) + 24\ln|1-\varphi|]$. Since $h'(\varphi) = -\frac{6\varphi(\varphi+1)^2}{1-\varphi}$ and $h(0) = 0$, this implies that $h(\varphi)$ is increasing and $h(\varphi) < 0$ on $(-1, 0)$, as well as $h(\varphi)$ is decreasing and $h(\varphi) < 0$ on $(0, 1)$. Hence, $g'(\varphi) \geq 0$ on $(-1, 1)$, i.e. $g(\varphi)$ is an increasing function on $(-1, 1)$, which get along with $g(0) = 0$ gives that $g(\varphi) < 0$ on $(-1, 0)$ and $g(\varphi) > 0$ on $(0, 1)$. Thus, by (4.38), it is concluded that $\Omega' > 0$ on $(-1, 0) \cup (0, 1)$, which demonstrates Lemma 4.2. □

Let us discuss the maximum and minimum of a traveling-wave solution when $1-2\sigma\Omega \neq 0$. On the one hand, (4.33) and Proposition 4.1 infer $\varphi \neq 1$, which implies $\max_{x\in\mathbb{R}} \varphi < 1$. On the other hand, in view of (4.34), φ exhibits singularity at $\varphi = -1$, which implies $\min_{x\in\mathbb{R}} \varphi \leq -1$. Then, by the restrictions on the extremum of φ and Lemma 4.2, it is deduced that

$$-0.323 \approx \frac{8\ln 2 - 82/15}{16\ln 2 - 34/3} \leq \Omega < \frac{1}{2}, \tag{4.39}$$

which combining with (4.36) gives that the traveling-wave solution of (1.1) may exist when

$$\frac{6}{13} < \Omega < \frac{1}{2}. \tag{4.40}$$

Hence, we focus on $\frac{6}{13} < \Omega < \frac{1}{2}$ in the following.

Lemma 4.3. *Let $\frac{6}{13} < \Omega < \frac{1}{2}$. There is no root $\lambda_2 < 0$ such that $f_2(\lambda_2) = 0$ and there is a λ_3, where $0 < \lambda_3 < 1$, such that $f_2(\lambda_3) = 0$.*

Proof. We will begin with the proof of the monotonicity of $f_2(\varphi)$ and $f_2(\varphi) > 0$ for $\varphi < 0$. From (4.34), we know

$$\begin{aligned} f_2'(\varphi) &= -\varphi^4 + (5-4\Omega)\varphi^2 + 2(4-6\Omega)\varphi + 8(1-2\Omega) + 8(1-2\Omega)/(\varphi-1) \\ &= -\varphi^4 + 5\varphi^2 + \frac{8\varphi^2}{\varphi-1} - [4\varphi^2 + \frac{12\varphi^2+4\varphi}{\varphi-1}] \\ &= \frac{-\varphi^2(\varphi-3)(\varphi+1)^2}{\varphi-1} - \Big(\frac{4\varphi(\varphi+1)^2}{\varphi-1}\Big)\Omega. \end{aligned} \tag{4.41}$$

Then, it is deduced from $\frac{6}{13} < \Omega < \frac{1}{2}$ that $f_2'(\varphi) < 0$ if $\varphi < 0$. This implies f_2 is monotonic decreasing on $(-\infty, 0)$, which get along with $f_2(0) = 0$ infers $f_2(\varphi) > 0$ for $\varphi < 0$. Hence, we conclude that there is no root $\lambda_2 < 0$ such that $f_2(\lambda_2) = 0$.

Next, for $0 < \varphi < 1$, we know (4.41) can be rewritten as

$$f_2'(\varphi) = -\frac{\varphi(\varphi+1)^2}{\varphi-1}\hat{h}(\varphi), \text{ where } \hat{h}(\varphi) = \Big[\varphi^2 - 3\varphi + 4\Omega\Big].$$

Consider $\frac{6}{13} < \Omega < \frac{1}{2}$, then $\hat{h}(\varphi)$ has two real roots r_1 and r_2, where

$$\begin{aligned} r_1 &= \frac{3-\sqrt{9-16\Omega}}{2}, \text{ where } 0.69 \approx \frac{3-\sqrt{\frac{21}{13}}}{2} < r_1 < 1, \\ r_2 &= \frac{3+\sqrt{9-16\Omega}}{2}, \text{ where } r_2 > 1. \end{aligned} \tag{4.42}$$

Thus $\hat{h}(\varphi) > 0$ on $(0, r_1)$ and $\hat{h}(\varphi) < 0$ on $(r_1, 1)$, which implies that f_2 is increasing on $(0, r_1)$ and is decreasing on $(r_1, 1)$. Moreover, for $\frac{6}{13} < \Omega < \frac{1}{2}$, a simple computation shows that,

$$
\begin{aligned}
&f_2(0)=0, \qquad\qquad f_2(1)=-\infty,\\
&f_2(\tfrac{1}{2})=\Big(-\frac{1}{160}+\frac{5}{24}+1+4+8\ln\frac{1}{2}\Big)-\Big(\frac{1}{6}+\frac{3}{2}+8+16\ln\frac{1}{2}\Big)\Omega>0.
\end{aligned}
$$

Hence, according to the continuity of $f_2(\varphi)$, there exists a λ_3, such that $f_2(\lambda_3)=0$, where $0<\lambda_3<1$. Consequently, the proof of Lemma 4.3 is complete. □

Theorem 4.3. *For $\sigma=1$, when $\frac{6}{13}<\Omega<\frac{1}{2}$, system (1.1) possesses a smooth traveling-wave solution $\varphi>0$ with $\max\limits_{x\in\mathbb{R}}\varphi(x)=\lambda_3$, and an anticusped traveling-wave solution $\varphi<0$ with $\min\limits_{x\in\mathbb{R}}\varphi(x)=-1$.*

Proof. In the case $1-2\sigma\Omega\neq 0$, equation (4.28) has the following form

$$
\begin{aligned}
\varphi_x^2&=\frac{-\frac{1}{5}\varphi^5+\frac{1}{3}(5-4\Omega)\varphi^3+(4-6\Omega)\varphi^2+8(1-2\Omega)\varphi+8(1-2\Omega)\ln|1-\varphi|}{(\varphi+1)^3}\\
&:=\frac{f_2(\varphi)}{(\varphi+1)^3}=F_2(\varphi).
\end{aligned}
\tag{4.43}
$$

By (4.43) we know that φ can not oscillate around zero near infinity. Let us consider the following two cases.

(1.1) If $\varphi(x)>0$ near $-\infty$, then there is some x_0 sufficiently large negative so that $\varphi(x_0)=\varepsilon>0$, with ε sufficiently small, and $\varphi_x(x_0)>0$. By Lemma 4.3, $\sqrt{F_2(\varphi)}$ is locally Lipschitz continuous in φ for $0<\varphi<\lambda_3$. Hence, there is a local solution to

$$
\begin{cases}
\varphi_x=\sqrt{F_2(\varphi)},\\
\varphi(x_0)=\varepsilon,
\end{cases}
$$

on $[x_0-L,x_0+L]$ for some $L>0$. Therefore by (4.22), (4.23) and (4.24), the smooth solution can be constructed with the maximum height $\varphi=\lambda_3$ and exponential decay to zero at infinity

$$
\varphi(x)=O\Big(\exp(-\sqrt{52\Omega-24}|x|)\Big),\ \text{as } |x|\to\infty.
$$

(1.2) If $\varphi(x)<0$ near $-\infty$. In this case, we are solving

$$
\begin{cases}
\varphi_x=-\sqrt{F_2(\varphi)},\\
\varphi(x_0)=-\varepsilon,
\end{cases}
$$

for some x_0 sufficiently large negative and $\varepsilon>0$ sufficiently small. Since $\sqrt{F_2(\varphi)}$ is locally Lipschitz in φ for $-1<\varphi<0$, then -1 becomes a pole of $F_2(\varphi)$. Then we may obtain a traveling-wave solution with a anticusp at $\varphi=-1$ by (4.25) and (4.26). This completes the proof of Theorem 4.3. □

4.3. Symmetry of traveling-wave solutions

In this section, attention is turned to the unique x-symmetric weak solution of system (1.1). Following [20] and the general principle deduced therein, we will prove that such a solution must be a traveling wave. To achieve this goal, one starts with the definition for x-symmetric solution.

Definition 4.3. A solution $\vec{u}(t,x) = (u(t,x), \eta(t,x))$ is x-symmetric if there exists a function $b(t) \in C^1(\mathbb{R}_+)$ such that for every $t > 0$,

$$\vec{u}(t,x) = (u(t, 2b(t) - x), \eta(t, 2b(t) - x)) \tag{4.44}$$

for almost every $x \in \mathbb{R}$. We say that $b(t)$ is the symmetric axis of $\vec{u}(t,x)$.

To accommodate for weak solution, we will use $\langle , \rangle$ for distribution in the subsequent discussion and we rewrite (4.3) in Definition 4.1 as follows:

$$\begin{cases} \langle u, (1-\partial_x^2)\phi_t\rangle + 2\Omega\langle \eta, \phi_t\rangle + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, \phi_x\rangle + \langle \frac{1}{2}u^2, \phi_{xxx}\rangle = 0, \\ \langle \eta, \phi_t\rangle + \langle (1+\eta)u, \phi_x\rangle = 0. \end{cases} \tag{4.45}$$

A result related to traveling-wave solution may now be enunciated.

Theorem 4.4. *If $\vec{u}(t,x)$ is a unique weak solution of system (1.1) and is x-symmetric, then $\vec{u}(t,x)$ is a traveling wave.*

The proof of the main result is approached via a series of lemmas. The following lemma gives the form of a weak solution of (1.1), which implies the main theorem in this section.

Lemma 4.4. *Assume that $\vec{U} = (U(x), V(x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and satisfies*

$$\begin{cases} \int_{\mathbb{R}} [-cU(1-\partial_x^2)\phi_x - 2\Omega c V\phi_x + (\frac{1}{2}U^2 + \frac{1}{2}U_x^2 + V)\phi_x + \frac{1}{2}U^2\phi_{xxx}]\, dx = 0, \\ \int_{\mathbb{R}} [-cV\phi_x + (1+V)U\phi_x]\, dx = 0, \end{cases} \tag{4.46}$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Then $\vec{u}$ given by

$$\vec{u}(t,x) = \vec{U}(x - c(t-t_0)) \tag{4.47}$$

is a weak solution of system (1.1) for any fixed $t_0 \in \mathbb{R}$.

Proof. Without loss of generality, we can assume $t_0 = 0$. Following the arguments in [20], we get that $\vec{u}(t,x)$ belongs to $C(\mathbb{R}, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. For any $\zeta \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, letting $\zeta_c = \zeta(t, x + ct)$, it follows that

$$\begin{cases} \partial_x(\zeta_c) = (\zeta_x)_c, \\ \partial_t(\zeta_c) = (\zeta_t)_c + c(\zeta_x)_c. \end{cases} \tag{4.48}$$

Assume $\vec{u}(t,x) = \vec{U}(x-ct)$. One can easily check that

$$\begin{cases} \langle u, \zeta\rangle = \langle U, \zeta_c\rangle, \qquad \langle u^2, \zeta\rangle = \langle U^2, \zeta_c\rangle, \\ \langle u_x^2, \zeta\rangle = \langle U_x^2, \zeta_c\rangle, \qquad \langle \eta, \zeta\rangle = \langle V, \zeta_c\rangle, \\ \langle (1+\eta)u, \zeta\rangle = \langle (1+V)U, \zeta_c\rangle, \end{cases} \tag{4.49}$$

where $\vec{U} = (U, V) = (U(x), V(x))$. In view of (4.48) and (4.49), we obtain

$$\begin{cases} \langle u, (1-\partial_x^2)\zeta_t\rangle = \langle U, ((1-\partial_x^2)\partial_t\zeta)_c\rangle = \langle U, (1-\partial_x^2)(\partial_t\zeta_c - c\partial_x\zeta_c)\rangle, \\ \langle \eta, \zeta_t\rangle = \langle V, \partial_t\zeta_c - c\partial_x\zeta_c\rangle, \\ \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, \zeta_x\rangle = \langle \frac{1}{2}U^2 + \frac{1}{2}U_x^2 + V, \partial_x\zeta_c\rangle, \\ \langle \frac{1}{2}u^2, \zeta_{xxx}\rangle = \langle \frac{1}{2}U^2, \partial_x^3\zeta_c\rangle, \\ \langle (1+\eta)u, \zeta_x\rangle = \langle (1+V)U, \partial_x\zeta_c\rangle. \end{cases} \tag{4.50}$$

Notice that $\vec{U}$ is independent of time, for T large enough such that it does not belong to the support of ζ_c, which implies

$$\begin{aligned} \langle U, (1-\partial_x^2)\partial_t\zeta_c\rangle &= \int_{\mathbb{R}} U(x) \int_{\mathbb{R}_+} \partial_t(1-\partial_{x^2})\zeta_c \, dtdx \\ &= \int_{\mathbb{R}} U(x)[(1-\partial_{x^2})\zeta_c(T,x) - (1-\partial_x^2)\zeta_c(0,x)] \, dx = 0, \\ \langle U, \partial_t\zeta_c\rangle &= \int_{\mathbb{R}} U(x) \int_{\mathbb{R}_+} \partial_t\zeta_c \, dtdx \int_{\mathbb{R}} U(x)[\zeta_c(T,x) - \zeta_c(0,x)] \, dx = 0, \\ \langle V, \partial_t\zeta_c\rangle &= 0. \end{aligned} \tag{4.51}$$

Combining (4.50) with (4.51) gives that

$$\begin{aligned} &\langle u, (1-\partial_x^2)\phi_t\rangle + 2\Omega\langle \eta, \phi_t\rangle + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, \phi_x\rangle + \langle \frac{1}{2}u^2, \phi_{xxx}\rangle \\ &= \langle U, -c(1-\partial_x^2)\partial_x\zeta_c)\rangle + 2\Omega\langle V, -c\partial_x\zeta_c\rangle + \langle \frac{1}{2}U^2 + \frac{1}{2}U_x^2 + V, \partial_x\zeta_c\rangle + \langle \frac{1}{2}U^2, \partial_x^3\zeta_c\rangle \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big[-cU(1-\partial_x^2)\partial_x\zeta_c - 2\Omega cV\partial_x\zeta_c + (\frac{1}{2}U^2 + \frac{1}{2}U_x^2 + V)\partial_x\zeta_c + \frac{1}{2}U^2\partial_x^3\zeta_c \Big] \, dxdt \\ &= 0, \end{aligned}$$

and

$$\langle \eta, \phi_t \rangle + \langle (1+\eta)u, \phi_x \rangle = \langle V, -c\partial_x \zeta_c \rangle + \langle (1+V)U, \partial_x \zeta_c \rangle$$
$$= \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}} \Big[-cV\partial_x \zeta_c + (1+V)U\partial_x \zeta_c \Big] \, dxdt = 0,$$

where we applied (4.46) with $\phi(x) = \zeta_c(t,x)$, which belongs to $C_0^\infty(\mathbb{R})$, for every given $t \geq 0$. This completes the proof of Lemma 4.4. □

Proof. Recall Definition 4.3 and noting that $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ is dense in $C_0^1(\mathbb{R}_+ \times C_0^3(\mathbb{R}))$, we can only consider the test function ϕ belonging to $C_0^1(\mathbb{R}_+ \times C_0^3(\mathbb{R}))$. Let

$$\phi_b(t,x) = \phi(t, 2b(t) - x), \quad b(t) \in C^1(\mathbb{R}). \tag{4.52}$$

Then we obtain that $(\phi_b)_b = \phi$ and

$$\begin{cases} \partial_x u_b = -(\partial_x u)_b, \quad \partial_x \phi_b = -(\partial_x \phi)_b, \\ \partial_x \phi_b = (\partial_t \phi)_b + 2\dot{b}(\partial_x \phi)_b, \end{cases} \tag{4.53}$$

where $\dot{b}$ denotes the derivative of b with respect to time. Moreover,

$$\begin{cases} \langle u_b, \phi \rangle = \langle u, \phi_b \rangle, \quad \langle u_b^2, \phi \rangle = \langle u^2, \phi_b \rangle, \\ \langle (\partial_x u_b)^2, \phi \rangle = \langle (\partial_x u)^2, \phi_b \rangle, \quad \langle \eta_b, \phi \rangle = \langle \eta, \phi_b \rangle. \end{cases} \tag{4.54}$$

Since $\vec{u}$ is x-symmetric, which get long with (4.53) and (4.54) gives that

$$\begin{cases} \langle u, (1-\partial_x^2)\phi_t \rangle = \langle u, ((1-\partial_x^2)\partial_t \phi)_b \rangle = \langle u, (1-\partial_x^2)(\partial_t \phi_b + 2\dot{b}\partial_x \phi_b) \rangle, \\ \langle \eta, \phi_t \rangle = \langle \eta, \partial_t \phi_b + 2\dot{b}\partial_x \phi_b \rangle, \\ \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, \phi_x \rangle = \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, -\partial_x \phi_b \rangle, \\ \langle \frac{1}{2}u^2, \phi_{xxx} \rangle = \langle \frac{1}{2}u^2, -\partial_x^3 \phi_b \rangle, \\ \langle (1+\eta)u, \phi_x \rangle = \langle (1+\eta)u, -\partial_x \phi_b \rangle. \end{cases} \tag{4.55}$$

In view of (4.45), we get

$$\begin{cases} \langle u, (1-\partial_x^2)\phi_t \rangle + 2\Omega \langle \eta, \phi_t \rangle + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, \phi_x \rangle + \langle \frac{1}{2}u^2, \phi_{xxx} \rangle \\ \quad = \langle u, (1-\partial_x^2)(\partial_t \phi_b + 2\dot{b}\partial_x \phi_b) \rangle + 2\Omega \langle \eta, \partial_t \phi_b + 2\dot{b}\partial_x \phi_b \rangle \\ \qquad + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, -\partial_x \phi_b \rangle + \langle \frac{1}{2}u^2, -\partial_x^3 \phi_b \rangle \\ \quad = 0, \\ \langle \eta, \phi_t \rangle + \langle (1+\eta)u, \phi_x \rangle = \langle \eta, \partial_t \phi_b + 2\dot{b}\partial_x \phi_b \rangle + \langle (1+\eta)u, -\partial_x \phi_b \rangle = 0. \end{cases} \tag{4.56}$$

Notice that $(\phi_b)_b = \phi$. Substituting ϕ_b in the system (4.56) for ϕ, we have

$$\begin{cases} \langle u, (1-\partial_x^2)(\partial_t\phi + 2\dot{b}\partial_x\phi)\rangle + 2\Omega\langle \eta, \partial_t\phi + 2\dot{b}\partial_x\phi\rangle + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, -\partial_x\phi\rangle \\ \quad + \langle \frac{1}{2}u^2, -\partial_x^3\phi\rangle = 0, \\ \langle \eta, \partial_t\phi + 2\dot{b}\partial_x\phi\rangle + \langle (1+\eta)u, -\partial_x\phi\rangle = 0. \end{cases} \tag{4.57}$$

Combining (4.45) and (4.57), we have

$$\begin{cases} \langle u, 2\dot{b}(1-\partial_x^2)\partial_x\phi)\rangle + 2\Omega\langle \eta, 2\dot{b}\partial_x\phi\rangle + \langle \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta, -2\partial_x\phi\rangle \\ \quad + \langle \frac{1}{2}u^2, -2\partial_x^3\phi\rangle = 0, \\ \langle \eta, 2\dot{b}\partial_x\phi\rangle + \langle (1+\eta)u, -2\partial_x\phi\rangle = 0. \end{cases} \tag{4.58}$$

We consider a fixed but arbitrary $t_0 > 0$. For any $\phi \in C_0^\infty(\mathbb{R})$, let $\phi_\varepsilon(t,x) = \phi(x)\rho_\varepsilon(t)$, where $\rho_\varepsilon \in C_0^\infty(\mathbb{R}_+)$ is a mollifier with the property that $\rho_\varepsilon \to \delta(t-t_0)$, the Dirac mass at t_0, as $\varepsilon \to 0$. From (4.56), by using the test function $\phi_\varepsilon(t,x)$, we have

$$\begin{cases} \int_{\mathbb{R}} \left(2(1-\partial_x^2)\phi \int_{\mathbb{R}_+} \dot{b}u\rho_\varepsilon(t)dt\right) dx + 2\Omega \int_{\mathbb{R}} \left(2\partial_x\phi \int_{\mathbb{R}_+} \dot{b}\eta\rho_\varepsilon(t)dt\right) dx \\ \quad - \int_{\mathbb{R}} \left(2\partial_x\phi \int_{\mathbb{R}_+} (\frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta)\rho_\varepsilon(t)dt\right) dx - \int_{\mathbb{R}} \left(\partial_x^3\phi \int_{\mathbb{R}_+} u^2\rho_\varepsilon(t)\, dt\right) dx = 0, \\ \int_{\mathbb{R}} \left(2\partial_x\phi \int_{\mathbb{R}_+} \dot{b}u\rho_\varepsilon(t)dt\right) dx - \int_{\mathbb{R}} \left(2\partial_x\phi \int_{\mathbb{R}_+} \dot{b}(1+\eta)u\rho_\varepsilon(t)dt\right) dx = 0. \end{cases} \tag{4.59}$$

Note that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} \dot{b}u\rho_\varepsilon(t)\, dt = \dot{b}(t_0)\cdot u(t_0,x), \quad \lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} \dot{b}\eta\rho_\varepsilon(t)\, dt = \dot{b}(t_0)\cdot \eta(t_0,x), \text{ in } L^2(\mathbb{R}),$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} (\frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \eta)\rho_\varepsilon(t)\, dt = \frac{1}{2}u^2(t_0,x) + \frac{1}{2}u_x^2(t_0,x) + \eta(t_0,x),$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} u^2\cdot\rho_\varepsilon(t)\, dt = u^2(t_0,x),$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} \dot{b}(1+\eta)u\cdot\rho_\varepsilon(t)\, dt = \dot{b}(t_0)(1+\eta(t_0,x))u(t_0,x), \text{ in } L^1(\mathbb{R}).$$

Therefore, letting $\varepsilon \to 0$, (4.59) infers that

$$\begin{cases} \int_{\mathbb{R}} \dot{b}(t_0)\cdot u(t_0,x)(1-\partial_x^2)\partial_x\phi\, dx + 2\Omega \int_{\mathbb{R}} \dot{b}(t_0)\cdot\eta(t_0,x)\cdot\partial_x\phi\, dx \\ \quad - \int_{\mathbb{R}} \left(\frac{1}{2}u^2(t_0,x) + \frac{1}{2}u_x^2(t_0,x) + \eta(t_0,x)\right)\cdot\partial_x\phi\, dx - \int_{\mathbb{R}} \frac{1}{2}u^2(t_0,x)\partial_x^3\phi\, dx = 0, \\ \int_{\mathbb{R}} \dot{b}(t_0)\cdot\eta(t_0,x)\cdot\partial_x\phi\, dx - \int_{\mathbb{R}} \dot{b}(t_0)(1+\eta(t_0,x))u(t_0,x)\cdot\partial_x\phi\, dx = 0. \end{cases}$$

Thus, we deduce that $u(t_0, x)$ satisfies (4.46) for $c = \dot{b}(t_0)$. Applying Lemma 4.4, we get that $\tilde{u}(t, x) = u(t_0, x - \dot{b}(t_0)(t - t_0))$ is a traveling-wave solution of system (1.1). Since $\tilde{u}(t_0, x) = u(t_0, x)$, by the uniqueness assumption of the solution of system (1.1), we obtain $\tilde{u}(t, x) = u(t, x)$ for all time t. This completes the proof of Theorem 4.4. □

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