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Global classical solutions in a Keller-Segel(-Navier)-Stokes system modeling coral fertilization

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Abstract

This paper is devoted to the coupled Keller-Segel(-Navier)-Stokes system describing coral fertilization:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\nabla c) - nm, & x\in\Omega,\ t>0, \\ m_t + \mathbf{u}\cdot\nabla m = \Delta m - nm, & x\in\Omega,\ t>0, \\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - c + m, & x\in\Omega,\ t>0, \\ u_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p + (n+m)\nabla\phi, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases}$$

where $\kappa\in\{0,\ 1\}$, $\Omega\subset\mathbb{R}^N\,(N=2,\ 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and outward normal vector ν, the chemotactic sensitivity $S(x,n,c)$ is a tensor valued function satisfying $|S(x,n,c)|\le\frac{S_0(c)}{(1+n)^\theta}$ with a non-decreasing function $S_0\in C^2([0,+\infty))$ and $\theta\ge 0$. Under the specified boundary conditions $\nabla c\cdot\nu=\nabla m\cdot\nu=(\nabla n-nS(x,n,c)\nabla c)\cdot\nu=0$, $\mathbf{u}=0$ and some mild assumptions on the initial data $(n_0,\ m_0,\ c_0,\ \mathbf{u}_0)$, the global-in-time classical solutions are constructed. More precisely, if $\theta>0$, then for any large initial data there admits globally bounded solution; and if $\theta=0$, under some explicit smallness conditions on $\max\{\|c_0\|_{L^\infty(\Omega)},\ \|m_0\|_{L^\infty(\Omega)}\}$ the global-in-time classical solutions are also constructed.

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1. Introduction and results

In this paper, we consider the following Keller-Segel-(Navier-)Stokes system with rotational flux:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\nabla c) - nm, & x\in\Omega,\ t>0,\\ m_t + \mathbf{u}\cdot\nabla m = \Delta m - nm, & x\in\Omega,\ t>0,\\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - c + m, & x\in\Omega,\ t>0,\\ u_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p + (n+m)\nabla\phi, \quad \nabla\cdot\mathbf{u}=0, & x\in\Omega,\ t>0,\\ \nabla c\cdot\nu = \nabla m\cdot\nu = (\nabla n - nS(x,n,c)\nabla c)\cdot\nu = 0,\ \mathbf{u}=0 & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x),\ m(x,0)=m_0(x),\ c(x,0)=c_0(x),\ \mathbf{u}(x,0)=\mathbf{u}_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N (N=2,\ 3)$ is a bounded domain with smooth boundary $\partial\Omega$; ν denotes the outward normal vector of $\partial\Omega$; ϕ stands for the potential of the gravitational field; the chemotactic sensitivity function $S(x,n,c)=(s_{ij}(x,n,c))_{i,j\in\{1,\cdots,N\}}$ is tensor-valued. System (1.1) models the spatio-temporal dynamics of the coral fertilization in the fluid with velocity field $\mathbf{u}$ satisfying the incompressible (Navier-)Stokes equations with associated pressure P and external force $(n+m)\nabla\phi$. Here n represents the density of the sperms and m denotes the density of eggs which release the chemical signal with concentration c to attract the sperms.

To motivate our study, we first recall some related progresses on system (1.1). We begin with the chemotaxis-(Navier-)Stokes model proposed by Tuvel, et al.

Chemotaxis-(Navier-)Stokes model To describe the dynamics of swimming bacteria, *Bacillus subtilis*, in a water drop sitting on a glass surface, Tuval et al. proposed the following chemotaxis-(Navier-)Stokes model [24]:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot\big(n\chi(c)\nabla c\big),\\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - k(c)n,\\ \mathbf{u}_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \Delta\mathbf{u} + n\nabla\phi,\\ \nabla\cdot\mathbf{u}=0, \end{cases} \tag{1.2}$$

where the scalar function $\chi(c)$, $k(c)$ represent the chemotactic sensitivity and the oxygen consumption rate, respectively; the coefficient κ is related to the strength of nonlinear fluid convection. In particular, when $\kappa=0$, the corresponding system (1.2) is an incompressible chemotaxis-Stokes system which describes the cells' dynamics in the fluids flowing slowly. When $\kappa>0$, the corresponding system (1.2) is an incompressible chemotaxis-Navier-Stokes system, wherein the fluid velocity $\mathbf{u}$ is subjected to an incompressible Navier-Stokes system, the existence of the classical solutions of which, as we know, is still an open problem. There have been amounts of results on the global well-posedness of the Cauchy problem and the initial boundary value problem in the two or three dimensional case, under some structure conditions between $\chi(c)$ and $f(c)$, or some smallness conditions on the initial data, see [3], [12], [4], [19], [32], [31], [36], [35], etc; moreover, the small-convection limit problem studied in [27] shows that the solutions of the chemotaxis-Navier-Stokes system in the two dimensional bounded domain converge uniformly in time to the solution of its chemotaxis-Stokes counterpart. We also refer to [5], [18], [37] and references therein for the nonlinear diffusion models with porous medium type diffusion Δn^m instead of the linear one Δn.

Keller-Segel-(Navier-)Stokes model In contrast to the chemotaxis-(Navier-)Stokes model (1.2) involving the chemical signal consumption and thus effectively preventing the occurrence of cells aggregation, the following Keller-Segel-(Navier-)Stokes model with chemical signal secreted by the cells themselves is somewhat more difficult:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot\big(n\chi(c)\nabla c\big) + f(n), \\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - c + n, \\ \mathbf{u}_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + n\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.3}$$

The main difficulty differs from system (1.2) is that the uniform boundedness of c in (1.3) cannot be obtained by the comparison argument. Existing results on (1.3) have shown that such Keller-Segel-(Navier-)Stokes model has many properties similar to the fluid-free counterparts. For instance, in our previous work [17], it is shown that the corresponding Keller-Segel-Stokes system (1.3) with $f(n) = 0$ in two dimensional bounded domain admits unique classical solution when $\|n_0\|_{L^1(\Omega)}$ is appropriately small, such result is similar to the fluid-free Keller-Segel system; as we know, the solutions of the fluid-free Keller-Segel system may blow up in finite time when $\|n_0\|_{L^1(\Omega)}$ is larger than some thresholds, we thus guess that such phenomenon may occur in the Keller-Segel-(Navier-)Stokes model (1.3); however, as far as we know, relevant result is still a vacancy. Another similarity to the fluid-free Keller-Segel system is that, the quadratic source term may prevent effectively the finite-time blowup of solutions [23], [22]. The model (1.3) and some other relevant models studied in [6], [11], [10] describe the corral broadcast spawning, wherein the sperms and the eggs are viewed as one population. Very recently, Espejo and Winkler proposed a more realistic model in [7], therein the sperms and the eggs are treated as two different populations and only the sperms response to the chemical signal released by the eggs. Actually, the model proposed by them is the special version of system (1.1) with $\kappa = 1$ and $S(x,n,c) = I$, by establishing an entropy energy estimate on $\int_\Omega n(\cdot,t)\ln n(\cdot,t) + a\int_\Omega|\nabla c|^2 + b\int_\Omega|u|^2$ they established the global existence and large time behaviors of solutions in two dimensional bounded domains.

Chemotaxis model involving rotational flux Based on the experimental observations that the bacterial motion near surfaces of their surrounding fluid may involve rotational components (see [33], [34]), chemotaxis models with tensor-valued sensitivity function instead of the scalar one have been proposed and widely studied, see [9], [1], [25], [26], [16] etc. In particular, only very recently, in [13], the authors considered the three dimensional Keller-Segel-Stokes system (1.1) with matrix-valued sensitivity $S(x,n,c)$ satisfying

$$|S(x,n,c)| \le \frac{C_S}{(1+n)^\theta}$$

with $C_S > 0$, $\theta \geq 0$; wherein, the global boundedness and large time behavior of classical solution were established under the condition $\theta \geq \frac{1}{3}$ or $\theta \geq 0$ and the initial data satisfy certain smallness conditions. Evidently there is a gap $(0, \frac{1}{3})$ of θ, and it is natural therefore to ask how about $\theta \in (0, \frac{1}{3})$?

Motivated by the above works, in this paper, we try to study the system (1.1) with more general rotational flux. More precisely, throughout the sequel, we assume that the chemotactic sensitivity $S(x,n,c) \in C^2(\bar{\Omega} \times [0,+\infty)^2; \mathbb{R}^N \times \mathbb{R}^N)(N = 2,3)$ satisfies

$$|S(x,n,c)| \leq \frac{S_0(c)}{(1+n)^\theta} \text{ for all } (x,n,c) \in \Omega \times [0,+\infty)^2, \tag{1.4}$$

where $S_0(\cdot) : [0,+\infty) \to \mathbb{R}$ is a nondecreasing and nonnegative function and $\theta \geq 0$ is a parameter. We also assume that $\phi \in W^{2,\infty}(\Omega)$ and the initial data are given functions satisfying

$$\begin{cases} n_0 \in C(\overline{\Omega}),\ n_0 \geq 0 \text{ and } n_0 \not\equiv 0, \\ m_0 \in C(\overline{\Omega}),\ m_0 \geq 0 \text{ and } m_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega),\ c_0 \geq 0 \text{ and } c_0 \not\equiv 0, \\ \mathbf{u}_0 \in D(A^\alpha),\ \alpha \in (\frac{N}{4}, 1), \end{cases} \tag{1.5}$$

where A is the realization of the Stokes operator in $L^2(\Omega)$. Under these assumptions, we try to construct the global existence and uniform boundedness of solutions. As aforementioned, in [7], the authors have studied the boundedness and large time behavior of the two dimensional Keller-Segel-Navier-Stokes system (1.1) with scaler sensitivity $S(x,n,c) \equiv const.$, wherein the energy functional of $\int_\Omega n(\cdot,t)\ln n(\cdot,t) + a\int_\Omega |\nabla c|^2 + b\int_\Omega |u|^2$ plays a great role. As we know, such functional structure will be broken down by the tensor-valued sensitivity and thus will take great challenges for our situations. It is fortunately that, after careful observations, it is not hard to find that m and thus c are bounded by the maximum principle (see Lemma 2.2); therefore, the weighted estimate technique originally developed in [28] may be applicable for our case. Along this way, we shall establish the following results on the global existence and uniform boundedness of solutions to system (1.1).

Theorem 1.1. *Let $\kappa = 0$, $\alpha \in (\frac{3}{4}, 1)$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that $\phi \in W^{2,\infty}(\Omega)$ and $S(x,n,c) \in C^2(\bar{\Omega} \times [0,+\infty)^2; \mathbb{R}^3 \times \mathbb{R}^3)$ satisfies (1.4) with $\theta > 0$. Then for every initial data satisfying (1.5), the system (1.1) admits a global-in-time classical solution $(n, m, c, \mathbf{u}, P)$, which is uniformly bounded in the sense that there admits a positive constant C such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \leq C$$
$$\text{for all } t \in (0,+\infty).$$

Remark 1.1. Theorem 1.1 indicates that $\theta > 0$ is enough to ensure the global existence and uniform boundedness of solution of the three-dimensional Keller-Segel-Stokes system (1.1), which improves the result obtained in [13], therein $\theta \geq \frac{1}{3}$ is required.

Similar to the chemotaxis(-Navier)-Stokes system with rotational flux term considered in [1], the global-in-time classical solution to the Keller-Segel-Stokes system (1.1) with $|S(x,n,c)| \le S_0(c)$ (i.e., $\theta = 0$ in (1.4)) can also be constructed just under some smallness conditions on the initial data.

Proposition 1.1. *Let $\kappa = 0$, $p > \frac{3}{2}$, $\alpha \in (\frac{3}{4}, 1)$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and assume that $\phi \in W^{2,\infty}(\Omega)$ and $S(x,n,c) \in C^2(\bar{\Omega} \times [0,+\infty)^2; \mathbb{R}^3 \times \mathbb{R}^3)$ satisfies (1.4) with $\theta = 0$, and apart from (1.5), the initial data also satisfy*

$$M S_0(M) \le \frac{1}{2\sqrt{p(3p+1)} + p}, \tag{1.6}$$

where $M := \max\{\|c_0\|_{L^\infty(\Omega)}, \|m_0\|_{L^\infty(\Omega)}\}$. Then the system (1.1) admits a global-in-time classical solution (n, m, c, u, P), which is uniformly bounded in the sense that there exists a positive constant C such that

$$\begin{aligned}&\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \le C \\ &\quad \textit{for all } \ t \in (0,+\infty).\end{aligned}$$

Remark 1.2. Let $p = \frac{3}{2}$, from the inequality (1.6) we get an explicit upper bound $\frac{2}{2\sqrt{33}+3}$ of $M S_0(M)$ such that the global-in-time classical solution to the Keller-Segel-Stokes system (1.1) exists when $M S_0(M) < \frac{2}{2\sqrt{33}+3}$. Particularly, let $S_0(c) \equiv C_S$ with $C_S > 0$, then we have $M < \frac{2}{C_S(2\sqrt{33}+3)}$. Such smallness condition on the initial data is somewhat simpler than those in [13], wherein the initial data are required to satisfy

$$\|n_0 - n_\infty\|_{L^{p_0}(\Omega)} \le \epsilon, \;\; \|m_0 - m_\infty\|_{L^{q_0}(\Omega)} \le \epsilon, \;\; \|\nabla c_0\|_{L^3(\Omega)} \le \epsilon, \;\; \|\mathbf{u}_0\|_{L^3(\Omega)} \le \epsilon$$

for some $\epsilon > 0$, $q_0 > 3$, and $p_0 > \frac{3}{2}$ if $\int_\Omega n_0 > \int_\Omega m_0$, $p_0 > \frac{4}{3}$ if $\int_\Omega n_0 < \int_\Omega m_0$.

In the two dimensional case, in [7], by establishing an entropy energy estimate on $\int_\Omega n \ln n dx + a\int_\Omega |\nabla c|^2 dx + b\int_\Omega |\mathbf{u}|^2 dx$, the global existence of large-data classical solution to the corresponding Keller-Segel-Navier-Stokes system, i.e., system (1.1) with $\kappa = 1$ and $S(x,n,c) \equiv \mathcal{I}$, was already obtained. However, for the system (1.1) with general tensor-valued sensitivity such functional structure does not exist and thus all the estimates based on it are no longer available. The last Proposition in this paper shows that, similar to the three dimensional Keller-Segel-Stokes system (1.1), either $\theta > 0$ or some smallness condition on the initial data is also sufficient to ensure the existence of the global-in-time classical solution to the two dimensional Keller-Segel-Navier-Stokes system (1.1).

Proposition 1.2. *Let $\kappa = 1$, $p > 1$, $\alpha \in (\frac{1}{2}, 1)$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $\phi \in W^{2,\infty}(\Omega)$, $S(x,n,c) \in C^2(\bar{\Omega} \times [0,+\infty)^2; \mathbb{R}^2 \times \mathbb{R}^2)$ satisfies (1.4), the initial data satisfy (1.5). Then if $\theta > 0$ in (1.4) or if $\theta = 0$ in (1.4) and besides (1.5), the initial data also satisfy (1.6), the system (1.1) admits a global-in-time classical solution (n, m, c, u, P), which is uniformly bounded in the sense that there exists a positive constant C such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \le C$$
$$\textit{for all } t \in (0,+\infty).$$

Remark 1.3. Let $p = 1$, from (1.6) we see that the smallness condition on the initial data $MS_0(M) < \frac{1}{5}$ can ensure the global existence of classical solution. Similar result is obtained in [1] for the chemotaxis-Navier-Stokes system; however, the explicit upper bound of $\|c_0\|_{L^\infty(\Omega)}$ is not given in [1].

2. Global existence and boundedness for $S = 0$ on $\partial\Omega$

In this section, we consider the case that besides (1.4), the tensor-valued function S satisfies

$$S(x,n,c) = 0, \quad (x,n,c) \in \partial\Omega \times [0,\infty) \times [0,\infty), \tag{2.1}$$

which reduces the boundary condition for n in (1.1) to the homogeneous Neumann boundary condition, i.e.,

$$\nabla n \cdot \nu = 0, \ x \in \partial\Omega, \ t > 0.$$

2.1. Local existence of classical solutions

We first state the local existence of classical solutions, which can be proved in precisely the same manner as [21, Lemma 2.1], see also [15], etc.

Lemma 2.1. *Let $N \in \{2, 3\}$, $\phi \in W^{2,\infty}(\Omega)$ and assume that $S(x,n,c) \in C^2(\bar\Omega \times [0,+\infty)^2; \mathbb{R}^N \times \mathbb{R}^N)$ satisfies (1.4) and (2.1). Then for any initial data satisfying (1.5), the initial boundary value problem (1.1) admits a unique local-in-time classical solution $(n, m, c, \mathbf{u}, P)$ in $\Omega \times (0, T^*)$, up to addition of constants to P, satisfying*

$$\begin{aligned}
&n \in C^0(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\overline{\Omega} \times (0,T^*))\\
&m \in C^0(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\overline{\Omega} \times (0,T^*))\\
&c \in C^0(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\overline{\Omega} \times (0,T^*))\\
&\mathbf{u} \in C^0(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\overline{\Omega} \times (0,T^*))\\
&P \in C^{1,0}(\overline{\Omega} \times [0,T^*)).
\end{aligned}$$

Here, T^* denotes the maximal existence time. Moreover, we have n, m, c are nonnegative in $\Omega \times (0, T^*)$, and

$$\begin{aligned}
&\textit{if } T^* < \infty,\\
&\textit{then } \limsup_{t\to T^*}\{\,|n(\cdot,t)\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha \mathbf{u}(\cdot,t)\|_{L^2(\Omega)}\} = \infty.
\end{aligned} \tag{2.2}$$

Invoking the divergence free of the fluid and the homogeneous Neumann boundary conditions on n, m, c, some basic but important estimates can be established.

Lemma 2.2. *Under the conditions of Lemma 2.1, for all $t \in (0, T^*)$, the solution of (1.1) satisfies*

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)}, \ \|m(\cdot,t)\|_{L^1(\Omega)} \le \|m_0\|_{L^1(\Omega)}, \tag{2.3}$$

$$\int_0^t \int_\Omega n(\cdot,t)m(\cdot,s)dxds \le \min\{\|n_0\|_{L^1(\Omega)}, \ \|m_0\|_{L^1(\Omega)}\}, \tag{2.4}$$

$$\|n(\cdot,t)\|_{L^1(\Omega)} - \|m(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} - \|m_0\|_{L^1(\Omega)}, \tag{2.5}$$

$$\|m(\cdot,t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla m(\cdot,s)\|^2_{L^2(\Omega)} ds \le \|m_0\|^2_{L^2(\Omega)}, \tag{2.6}$$

$$\|m(\cdot,t)\|_{L^\infty(\Omega)} \le \|m_0\|_{L^\infty(\Omega)}, \tag{2.7}$$

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|c_0\|_{L^\infty(\Omega)}, \ \|m_0\|_{L^\infty(\Omega)}\} := M. \tag{2.8}$$

Proof. The proof is similar to Lemma 2.2 of [7], we omit the details here. □

2.2. L^p estimate for n

As we know, the uniform boundedness of L^p norm of n is crucial for the global existence of classical solution. Inspired by the weighted estimate argument developed in [28], see also [1], [20], etc., in this section, we shall invoke the weighted estimate of $\int_\Omega n^p \psi dx$ with appropriate choice of ψ to obtain the upper bound of n in $L^p(\Omega)$. We emphasize that such argument does not depend on the spatial dimension N and the value of κ.

Lemma 2.3. *Let $\theta \ge 0$ and the assumptions in Lemma 2.1 hold. Then for any $p > 1$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n+1)^p \psi(c)dx + \frac{p(p-1)}{2}\int_\Omega (n+1)^{p-2}|\nabla n|^2 \psi(c)dx + \int_\Omega (n+1)^p \psi''(c)|\nabla c|^2 dx \\
&\le \frac{4p}{p-1}\int_\Omega (n+1)^p |\nabla c|^2 \frac{\psi'^2(c)}{\psi(c)}dx - p\int_\Omega (n+1)^p \psi(c)mdx + p\int_\Omega (n+1)^{p-1}\psi(c)mdx \\
&\quad + p(p-1)S_0^2(M)\int_\Omega (n+1)^{p-2\theta}\psi(c)|\nabla c|^2 dx + pS_0(M)\int_\Omega (n+1)^{p-\theta}\psi'(c)|\nabla c|^2 dx \\
&\quad + \int_\Omega (n+1)^p \psi'(c)mdx - \int_\Omega (n+1)^p \psi'(c)cdx \quad \text{for all } t \in (0,T^*),
\end{aligned} \tag{2.9}$$

where $\psi(s) \in C^2([0,+\infty))$ is nonnegative and nondecreasing.

Proof. Using the first and the third equations in (1.1), by integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx &= p\int_\Omega (n+1)^{p-1}\psi(c)(\Delta n - \mathbf{u}\cdot\nabla n - \nabla\cdot(nS(x,n,c)\nabla c) - nm)dx\\ &\quad + \int_\Omega (n+1)^p\psi'(c)(\Delta c - \mathbf{u}\cdot\nabla c - c + m)\\ &= p\int_\Omega (n+1)^{p-1}\psi(c)(\Delta n - \nabla\cdot(nS(x,n,c)\nabla c) - nm)dx\\ &\quad + \int_\Omega (n+1)^p\psi'(c)(\Delta c - c + m)dx := I_1 + I_2\end{aligned} \tag{2.10}$$

for all $t \in (0, T^*)$, where we have used

$$p\int_\Omega (n+1)^{p-1}\psi(c)\mathbf{u}\cdot\nabla n dx + \int_\Omega (n+1)^p\psi'(c)\mathbf{u}\cdot\nabla c dx = \int_\Omega \mathbf{u}\cdot\nabla((n+1)^p\psi(c))dx = 0.$$

For I_1, by integration by parts, we have

$$\begin{aligned} I_1 = &-p(p-1)\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx - p\int_\Omega (n+1)^{p-1}\psi'(c)\nabla c\cdot\nabla n dx\\ &- p\int_\Omega (n+1)^{p-1}\psi(c)nm dx + p(p-1)\int_\Omega (n+1)^{p-2}n\psi(c)\nabla n\cdot(S(x,n,c)\nabla c)dx\\ &+ p\int_\Omega (n+1)^{p-1}n\psi'(c)\nabla c\cdot(S(x,n,c)\nabla c)dx.\end{aligned} \tag{2.11}$$

Using Young's inequality, we have

$$\begin{aligned} -p\int_\Omega (n+1)^{p-1}\psi'(c)\nabla c\cdot\nabla n dx \le{}& \frac{p(p-1)}{8}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx\\ &+ \frac{2p}{p-1}\int_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx.\end{aligned} \tag{2.12}$$

In light of (1.4) and $\|c\|_{L^\infty(\Omega)} \le M$ stated in (2.8), and using once more Young's inequality, we get

$$\begin{aligned}
&p(p-1)\int_\Omega (n+1)^{p-2} n\psi(c)\nabla n\cdot(S(x,n,c)\nabla c)dx\\
&\le p(p-1)S_0(M)\int_\Omega (n+1)^{p-1-\theta}\psi(c)|\nabla n||\nabla c|dx\\
&\le \frac{p(p-1)}{4}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx\\
&\quad + p(p-1)S_0^2(M)\int_\Omega (n+1)^{p-2\theta}\psi(c)|\nabla c|^2dx,
\end{aligned}\tag{2.13}$$

and

$$p\int_\Omega (n+1)^{p-1} n\psi'(c)\nabla c\cdot(S(x,n,c)\nabla c)dx \le pS_0(M)\int_\Omega (n+1)^{p-\theta}\psi'(c)|\nabla c|^2dx. \tag{2.14}$$

Substituting (2.12)-(2.14) into (2.11), we have

$$\begin{aligned}
I_1 \le &-\frac{5p(p-1)}{8}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx + \frac{2p}{p-1}\int_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx\\
&- p\int_\Omega (n+1)^{p-1}\psi(c)nmdx + p(p-1)S_0^2(M)\int_\Omega (n+1)^{p-2\theta}\psi(c)|\nabla c|^2dx\\
&+ pS_0(M)\int_\Omega (n+1)^{p-\theta}\psi'(c)|\nabla c|^2dx.
\end{aligned}\tag{2.15}$$

For I_2, we deduce using (2.12) and integration by parts that

$$\begin{aligned}
I_2 = &-p\int_\Omega (n+1)^{p-1}\psi'(c)\nabla n\cdot\nabla c dx\\
&- \int_\Omega (n+1)^p\psi''(c)|\nabla c|^2dx - \int_\Omega (n+1)^p\psi'(c)cdx + \int_\Omega (n+1)^p\psi'(c)mdx\\
\le &\frac{p(p-1)}{8}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx + \frac{2p}{p-1}\int_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx\\
&- \int_\Omega (n+1)^p\psi''(c)|\nabla c|^2dx - \int_\Omega (n+1)^p\psi'(c)cdx + \int_\Omega (n+1)^p\psi'(c)mdx.
\end{aligned}\tag{2.16}$$

Substituting (2.15) and (2.16) into (2.10), we have

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega (n+1)^p\psi(c)dx + \frac{p(p-1)}{2}\int\limits_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx + \int\limits_\Omega (n+1)^p\psi''(c)|\nabla c|^2dx \\
&\le \frac{4p}{p-1}\int\limits_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx - p\int\limits_\Omega (n+1)^{p-1}\psi(c)nmdx \\
&\quad + p(p-1)S_0^2(M)\int\limits_\Omega (n+1)^{p-2\theta}\psi(c)|\nabla c|^2dx + pS_0(M)\int\limits_\Omega (n+1)^{p-\theta}\psi'(c)|\nabla c|^2dx \\
&\quad + \int\limits_\Omega (n+1)^p\psi'(c)mdx - \int\limits_\Omega (n+1)^p\psi'(c)cdx,
\end{aligned}
\tag{2.17}
$$

from which one can easily get (2.9). $\square$

By constructing a proper function $\psi(s)$, from (2.9) we can get the uniform boundedness of n in $L^p(\Omega)$.

Lemma 2.4. *Let $\theta > 0$ and the assumptions in Lemma 2.1 hold. Then for any $p > 1$, there admits a positive constant C depending on p such that*

$$
\int\limits_\Omega n^p(\cdot,t)dx \le C(p) \text{ for all } t \in (0,T^*),
\tag{2.18}
$$

and

$$
\int\limits_t^{t+\tau}\int\limits_\Omega (n(\cdot,s)+1)^{p-2}|\nabla n(\cdot,s)|^2dxds \le C
\tag{2.19}
$$

for all $t \in (0, T^*-\tau)$ with $\tau = \min\{1, \frac{1}{2}T^*\}$.

Proof. Without loss of generality, we first assume that $p > \max\{2\theta, \frac{2}{7M}\}$. Define $\psi(s) := e^{\beta s^2}$ for $s \in [0,\infty)$ with $\beta > 0$ to be specified later. Simple computations show that

$$
\psi'(s) = 2\beta s\psi(s),\ \psi''(s) = (2\beta + 4\beta^2 s^2)\psi(s).
$$

To get the upper bound of n in $L^p(\Omega)$ from (2.9), we now need to control the terms on the right side of (2.9) by the terms on the left side. For this purpose, let first that $\beta \in (0, \frac{p-1}{(14p+2)M^2})$. Then we have

$$
\begin{aligned}
\frac{4p}{p-1}\int\limits_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx &= \int\limits_\Omega (n+1)^p|\nabla c|^2\frac{4p}{p-1}4\beta^2c^2\psi(c)dx \\
&\le \frac{1}{2}\int\limits_\Omega (n+1)^p|\nabla c|^2(2\beta+4\beta^2c^2)\psi(c)dx
\end{aligned}
\tag{2.20}
$$

$$= \frac{1}{2} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx.$$

By Young's inequality, we have

$$\begin{aligned}
& p(p-1) S_0^2(M) \int_{\Omega} (n+1)^{p-2\theta} \psi(c) |\nabla c|^2 dx \\
& \leq \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_1 \int_{\Omega} (\psi''(c))^{\frac{2\theta-p}{2\theta}} (\psi(c))^{\frac{p}{2\theta}} |\nabla c|^2 dx \\
& = \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_1 \int_{\Omega} (2\beta + 4\beta^2 c^2)^{\frac{2\theta-p}{2\theta}} \psi(c) |\nabla c|^2 dx \\
& \leq \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_2 \int_{\Omega} |\nabla c|^2 dx
\end{aligned} \tag{2.21}$$

with $C_1 = [p(p-1)S_0^2(M)]^{\frac{p}{2\theta}} (\frac{p}{4(p-2\theta)})^{\frac{2\theta-p}{2\theta}} \frac{2\theta}{p}$ and $C_2 = C_1 (2\beta)^{\frac{2\theta-p}{2\theta}} e^{\beta M^2}$. Here in the last inequality, we have used $\theta > 0$, $p > 2\theta$ and $\|c\|_{L^\infty(\Omega)} < M$. Following the similar procedure, we have

$$\begin{aligned}
& p S_0(M) \int_{\Omega} (n+1)^{p-\theta} \psi'(c) |\nabla c|^2 dx \\
& \leq \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_3 \int_{\Omega} (\psi''(c))^{\frac{\theta-p}{\theta}} (\psi'(c))^{\frac{p}{\theta}} |\nabla c|^2 dx \\
& = \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_3 \int_{\Omega} (2\beta + 4\beta^2 c^2)^{\frac{\theta-p}{\theta}} (2\beta c)^{\frac{p}{\theta}} \psi(c) |\nabla c|^2 dx \\
& \leq \frac{1}{4} \int_{\Omega} (n+1)^p \psi''(c) |\nabla c|^2 dx + C_4 \int_{\Omega} |\nabla c|^2 dx
\end{aligned} \tag{2.22}$$

with $C_3 = (p S_0^2(M))^{\frac{p}{\theta}} (\frac{p}{4(p-\theta)})^{\frac{\theta-p}{\theta}} \frac{\theta}{p}$ and $C_4 = 2C_3 \beta M^{\frac{p}{\theta}} e^{\beta M^2}$. By Young's inequality, we also have

$$p \int_{\Omega} (n+1)^{p-1} \psi(c) m dx \leq \frac{p}{2} \int_{\Omega} (n+1)^p \psi(c) m dx + (\frac{p}{2(p-1)})^{1-p} \int_{\Omega} \psi(c) m dx. \tag{2.23}$$

Substituting (2.20)-(2.23) into (2.9), and adding $\int_{\Omega} (n+1)^p \psi(c) dx$ to the resulted inequality, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx+\frac{p(p-1)}{2}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx+\int_\Omega (n+1)^p\psi(c)dx\\
&\le (C_2+C_4)\int_\Omega |\nabla c|^2dx-\frac{p}{2}\int_\Omega (n+1)^p\psi(c)mdx+(\frac{p}{2(p-1)})^{1-p}\int_\Omega \psi(c)mdx\\
&\quad+\int_\Omega (n+1)^p\psi'(c)mdx-\int_\Omega (n+1)^p\psi'(c)cdx+\int_\Omega (n+1)^p\psi(c)dx\\
&=(C_2+C_4)\int_\Omega |\nabla c|^2dx+\int_\Omega (2\beta c-\frac{p}{2})(n+1)^p\psi(c)mdx+\int_\Omega (n+1)^p\psi(c)dx\\
&\quad+(\frac{p}{2(p-1)})^{1-p}\int_\Omega \psi(c)mdx.
\end{aligned}
\tag{2.24}
$$

Since $\beta\in(0,\frac{p-1}{(14p+2)M^2})$ and $p>\frac{2}{7M}$, we have

$$
2\beta c-\frac{p}{2}\le\frac{p-1}{(7p+1)M}-\frac{2}{7M}<0.
$$

(2.24) then leads to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx+\frac{p(p-1)}{2}\int_\Omega (n+1)^{p-2}|\nabla n|^2dx+\int_\Omega (n+1)^p\psi(c)dx\\
&\le (C_2+C_4)\int_\Omega |\nabla c|^2dx+e^{\beta M^2}\int_\Omega (n+1)^pdx+(\frac{p}{2(p-1)})^{1-p}e^{\beta M^2}\|m_0\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{2.25}
$$

Using Galiardo-Nirenberg's inequality, we have

$$
\begin{aligned}
\int_\Omega (n+1)^pdx&=\|(n+1)^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\
&\le C_{GN}\|(n+1)^{\frac{p}{2}}\|^{2(1-\delta)}_{L^{\frac{2}{p}}(\Omega)}\|\nabla(n+1)^{\frac{p}{2}}\|^{2\delta}_{L^2(\Omega)}+C_{GN}\|(n+1)^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\\
&\le C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{p(1-\delta)}\|\nabla(n+1)^{\frac{p}{2}}\|^{2\delta}_{L^2(\Omega)}+C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^p
\end{aligned}
\tag{2.26}
$$

with $\delta=\frac{N(p-1)}{N(p-1)+2}\in(0,1)$. Whereupon, we apply Young's inequality, to deduce

$$
e^{\beta M^2}\int_\Omega (n+1)^pdx\le\frac{p-1}{p}\|\nabla(n+1)^{\frac{p}{2}}\|^2_{L^2(\Omega)}+C_5
$$

with $C_5=[e^{\beta M^2}C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{p(1-\delta)}]^{\frac{1}{1-\delta}}(1-\delta)(\frac{p-1}{p\delta})^{\frac{\delta}{\delta-1}}+e^{\beta M^2}C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^p$. Substituting this into (2.27), we then have

$$\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx + \frac{p(p-1)}{4}\int_\Omega (n+1)^{p-2}|\nabla n|^2 dx + \int_\Omega (n+1)^p\psi(c)dx$$
$$\le (C_2+C_4)\int_\Omega |\nabla c|^2 dx + (\frac{p}{2(p-1)})^{1-p} e^{\beta M^2}\|m_0\|_{L^\infty(\Omega)} + C_5 \qquad \text{for all } t\in(0,T^*). \tag{2.27}$$

Testing the third equation in (1.1) against c, and using Young's inequality to the term $\int_\Omega cm dx$, we obtain

$$\frac{d}{dt}\int_\Omega c^2 dx + \int_\Omega c^2 dx + 2\int_\Omega |\nabla c|^2 dx \le \|m_0\|^2_{L^\infty(\Omega)}|\Omega|. \tag{2.28}$$

A linear combination (2.27) + $\frac{(C_2+C_4)}{2}$ × (2.28) then yields

$$\frac{d}{dt}\Big(\int_\Omega (n+1)^p\psi(c)dx + \frac{(C_2+C_4)}{2}\int_\Omega c^2dx\Big) + \frac{p(p-1)}{4}\int_\Omega (n+1)^{p-2}|\nabla n|^2dx$$
$$+\Big(\int_\Omega (n+1)^p\psi(c)dx + \frac{(C_2+C_4)}{2}\int_\Omega c^2dx\Big)$$
$$\le (\frac{p}{2(p-1)})^{1-p}e^{\beta M^2}\|m_0\|_{L^\infty(\Omega)} + C_5 + \frac{(C_2+C_4)}{2}\|m_0\|^2_{L^\infty(\Omega)}|\Omega| \qquad \text{for all } t\in(0,T^*). \tag{2.29}$$

Define $y(t) := \int_\Omega (n+1)^p\psi(c)dx + \frac{(C_2+C_4)}{2}\int_\Omega c^2dx$, by ODE comparison argument, we conclude that

$$y(t) \le \max\left\{(\frac{p}{2(p-1)})^{1-p}e^{\beta M^2}\|m_0\|_{L^\infty(\Omega)} + C_5 + \frac{(C_2+C_4)}{2}\|m_0\|^2_{L^\infty(\Omega)}|\Omega|,\ y(0)\right\},$$

which combining with $\psi > 1$ then yields (2.18). Furthermore, integrating the inequality (2.29) over $(t, t+\tau)$ with $\tau = \min\{1, \frac{1}{2}T^*\}$, we then obtain (2.19). □

We remark that the derivation in Lemma 2.3 does not work in the case of $\theta = 0$. However, in this case, we also can obtain the bound of L^p norm of n under some smallness assumptions on the initial data.

Lemma 2.5. *Let $p > 1$, and $\theta = 0$ in (1.4). There exists a positive constant $C(p)$ with the property: If besides (1.5), the initial data additionally satisfies*

$$MS_0(M) \le \frac{1}{2\sqrt{p(3p+1)}+p} \tag{2.30}$$

with $M = \max\{\|m_0\|_{L^\infty(\Omega)}, \|c_0\|_{L^\infty(\Omega)}\}$, then we have

$$\|n(\cdot,t)\|_{L^p(\Omega)} \le C(p) \text{ for all } t\in(0,T^*), \tag{2.31}$$

and

$$\int_t^{t+\tau}\int_\Omega (n+1)^{p-2}|\nabla n|^2 dx \le C(p) \tag{2.32}$$

for all $t \in (0, T^* - \tau)$ *with* $\tau = \min\{1, \frac{1}{2}T^*\}$.

Proof. Let $\theta = 0$ in (2.10), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n+1)^p \psi(c)dx + \frac{p(p-1)}{2}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx + \int_\Omega (n+1)^p\psi''(c)|\nabla c|^2 dx \\
&\le \frac{4p}{p-1}\int_\Omega (n+1)^p|\nabla c|^2\frac{\psi'^2(c)}{\psi(c)}dx - p\int_\Omega (n+1)^p\psi(c)m dx + p\int_\Omega (n+1)^{p-1}\psi(c)m dx \\
&\quad + p(p-1)S_0^2(M)\int_\Omega (n+1)^p\psi(c)|\nabla c|^2 dx + pS_0(M)\int_\Omega (n+1)^p\psi'(c)|\nabla c)|^2 dx \\
&\quad + \int_\Omega (n+1)^p\psi'(c)m dx - \int_\Omega (n+1)^p\psi'(c)c dx \quad \text{for all } t \in (0, T^*).
\end{aligned} \tag{2.33}$$

Let $\psi = e^{\beta s^2}$ for $s \in [0, +\infty)$ with β to be specified later. Straight computations then yield

$$\begin{aligned}
&\psi''(c) - \frac{4p}{p-1}\frac{\psi'^2(c)}{\psi(c)} - p(p-1)S_0^2(M)\psi(c) - pS_0(M)\psi'(c) \\
&= \Big(2\beta + 4\beta^2c^2 - \frac{4p}{p-1}4\beta^2c^2 - p(p-1)S_0^2(M) - 2p\beta S_0(M)c\Big)\psi(c) \\
&\ge \Big(2\beta - \frac{4(3p+1)\beta^2M^2}{p-1} - p(p-1)S_0^2(M) - 2p\beta S_0(M)M\Big)\psi(c).
\end{aligned} \tag{2.34}$$

Here we have used $\|c\|_{L^\infty(\Omega)} \le M$ provided by (2.8) and the nondecreasing monotonicity of S_0. Let us define

$$F(\beta) := -\frac{4(3p+1)M^2\beta^2}{p-1} + \big(2 - 2pS_0(M)M\big)\beta - p(p-1)S_0^2(M).$$

Simple computations show that if $MS_0(M) \le \frac{1}{2\sqrt{p(3p+1)}+p}$, then there admits

$$\beta = \frac{\Big(2 - 2pS_0(M)M + \sqrt{(2pS_0(M)M-2)^2 - 16(3p+1)pS_0(M)^2M^2}\Big)(p-1)}{8(3p+1)M^2} > 0 \tag{2.35}$$

such that $F(\beta)=0$, and thus $\psi''(c)-\frac{4p}{p-1}\frac{\psi'^2(c)}{\psi(c)}-p(p-1)S_0^2(M)\psi(c)-pS_0(M)\psi'(c)\geq 0$. Whereupon, from (2.33) we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx+\int_\Omega (n+1)^p\psi(c)dx+\frac{p(p-1)}{2}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx\\
&\leq \int_\Omega (n+1)^p\psi'(c)mdx+\int_\Omega (n+1)^p\psi(c)dx \\
&\leq (2\beta M+1)e^{\beta M^2}\int_\Omega (n+1)^p dx \qquad \text{for all } t\in(0,T^*),
\end{aligned} \tag{2.36}$$

where we have used $\psi'(c)=2\beta ce^{\beta c^2}\geq 0$ and $\|c\|_{L^\infty(0,T^*;L^\infty(\Omega))}\leq M$. Similar to Lemma 2.4, by the Gagliardo-Nirenberg inequality (2.26) and the Young inequality, we have

$$\begin{aligned}
(2\beta M+1)e^{\beta M^2}\int_\Omega (n+1)^p dx &\leq \frac{p-1}{p}\|\nabla(n+1)^{\frac{p}{2}}\|^2_{L^2(\Omega)}+C_6\\
&\leq \frac{p(p-1)}{4}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx+C_6,
\end{aligned} \tag{2.37}$$

with

$$\begin{aligned}
C_6=&[(2\beta M+1)e^{\beta M^2}C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^{p(1-\delta)}]^{\frac{1}{1-\delta}}(1-\delta)(\frac{p-1}{p\delta})^{\frac{\delta}{\delta-1}}\\
&+(2\beta M+1)e^{\beta M^2}C_{GN}(\|n_0\|_{L^1(\Omega)}+|\Omega|)^p,
\end{aligned}$$

and $\delta=\frac{N(p-1)}{N(p-1)+2}\in(0,1)$. Combining (2.37) and (2.36), we then have

$$\frac{d}{dt}\int_\Omega (n+1)^p\psi(c)dx+\int_\Omega (n+1)^p\psi(c)dx+\frac{p(p-1)}{4}\int_\Omega (n+1)^{p-2}|\nabla n|^2\psi(c)dx\leq C_6 \tag{2.38}$$

for all $t\in(0,T^*)$. Therefore, we have

$$\int_\Omega (n+1)^p\psi(c)dx\leq \max\left\{\int_\Omega (n_0+1)^p e^{\beta c_0^2}dx, C_6\right\} \qquad \text{for all } t\in(0,T^*), \tag{2.39}$$

which implies (2.31), since $\psi(c)\geq 1$. Whereupon, integrating the inequality (2.38) over $(t,t+\tau)$ with $\tau=\min\{1,\frac{1}{2}T^*\}$ we get (2.32). $\square$

2.3. Proof of Theorem 1.1

In light of the extensibility criterion (2.2), to establish the global existence of solutions to system (1.1), we only need to show the uniform boundedness of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$, $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$, as well as $\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$. Actually, with the uniform boundedness of $\|n(\cdot,t)\|_{L^p(\Omega)}$ stated in (2.9), one can firstly apply the estimate of Stokes operator (see Corollary 3.4 of [29]) to get the following regularity property of u.

Lemma 2.6. *There exists a positive constant C such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \text{ for all } t \in (0,T^*). \tag{2.40}$$

Proof. Let $p > 3$ in (2.18), then we can apply the estimate of Stokes operator to the equation $(1.1)_4$ (see Corollary 3.4 of [29]) to obtain

$$\|Du(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all} \quad t \in (0,T^*),$$

and so by the Poincaré inequality we obtain the inequality (2.40). $\square$

Lemma 2.7. *There exists a positive constant C such that*

$$\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} \le C \text{ for all } t \in (0,T^*), \tag{2.41}$$

and

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \text{ for all } t \in (0,T^*). \tag{2.42}$$

Proof. Testing the third equation in (1.1) against Δc, we have (one can refer to Lemma 3.2 of [7] for the proof):

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla c(\cdot,t)|^2 dx + \int_\Omega |\Delta c(\cdot,t)|^2 dx + 2\int_\Omega |\nabla c(\cdot,t)|^2 dx \\ &\le C\left\{\int_\Omega |\nabla u(\cdot,t)|^2 dx + 1\right\} \text{ for all } t \in (0,T^*).\end{aligned} \tag{2.43}$$

In light of (2.40), $\int_\Omega |\nabla u(\cdot,t)|^2 dx$ is uniformly bounded in time; whereupon, by the ODE comparison argument we have

$$\|\nabla c(\cdot,t)\|_{L^2(\Omega)} \le C \text{ for all } t \in (0,T^*). \tag{2.44}$$

By the well-known estimates for the solution of the inhomogeneous linear heat equations with homogeneous Neumann boundary condition (see Lemma 2.1 of [8], or [14]), we further obtain that

$$
\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^\infty(0,T^*;L^\infty(\Omega)} &\le \|\nabla c_0\|_{L^\infty(\Omega)} + C\|m - u\cdot\nabla c\|_{L^\infty(0,T;L^4(\Omega))} \\
&\le \|\nabla c_0\|_{L^\infty(\Omega)} + C(\|m_0\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{4}} \\
&\quad + \|u\|_{L^\infty(0,T^*;L^\infty(\Omega))}\|\nabla c\|_{L^\infty(0,T;L^4(\Omega))}) \\
&\le C + C\|u\|_{L^\infty(0,T^*;L^\infty(\Omega))}\|\nabla c\|^{\frac{1}{2}}_{L^\infty(0,T^*;L^2(\Omega))}\|\nabla c\|^{\frac{1}{2}}_{L^\infty(0,T;L^\infty(\Omega))} \\
&\le C + C\|\nabla c\|^{\frac{1}{2}}_{L^\infty(0,T^*;L^\infty(\Omega))},
\end{aligned}
\tag{2.45}
$$

where we have used (2.40) and (2.44). Applying Young's inequality to the last term in (2.45) we obtain (2.41), which together with (2.8) then yields (2.42). □

Lemma 2.8. *There exists a positive constant C such that*

$$
\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \ \textit{for all} \ \ t \in (0,T^*). \tag{2.46}
$$

Proof. The variation-of-constants representation for n provides

$$
\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s) \\
+ n(\cdot,s)u(\cdot,s))\|_{L^\infty(\Omega)}ds
\end{aligned}
\tag{2.47}
$$

for all $t \in (0, T^*)$. The first term can be estimated as

$$
\|e^{t\Delta}n_0\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)} \text{ for all } t \in (0,T^*). \tag{2.48}
$$

Moreover, using the well-known $L^p - L^q$ estimates for Neumann heat semigroup (see Lemma 1.3 of [30]), there exists a positive constant C_0 such that

$$
\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s))\|_{L^\infty(\Omega)}ds \\
&\le C_0\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{8}})e^{-\lambda_1(t-s)}\|n(\cdot,s)S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\|_{L^4(\Omega)}ds \\
&\le C_0\int_0^t (1+(t-s)^{-\frac{7}{8}})e^{-\lambda_1(t-s)}\|S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s) + u(\cdot,s)\|_{L^\infty(\Omega)}\|n\|_{L^4(\Omega)}ds \\
&\le C_0\Big(S_0(M)\|\nabla c\|_{L^\infty(0,T^*;L^\infty(\Omega))} + \|u\|_{L^\infty(0,T^*;L^\infty(\Omega))}\Big)\|n\|_{L^\infty(0,T^*;L^4(\Omega))}
\end{aligned}
\tag{2.49}
$$

$$\times \int_0^\infty (1+t^{-\frac{7}{8}})e^{-\lambda_1 t}dt \text{ for all } t \in (0, T^*)$$

with $\lambda_1 > 0$ denoting the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Since $\int_0^\infty (1+t^{-\frac{7}{8}})e^{-\lambda_1 t}dt$ is finite, $\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(nS(x,n,c)\nabla c + nu)\|_{L^\infty(\Omega)}ds$ is bounded by (2.41), (2.40) and (2.9) (or (2.31)). This bound, together with (2.48) and (2.47), completes the proof of (2.46). □

Lemma 2.9. *There exists a positive constant C such that*

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C \text{ for all } t \in (0, T^*). \tag{2.50}$$

Proof. Applying the Helmholtz projection $\mathcal{P}$ to the fourth equation in (1.1), and then applying A^α to the variation-of-constants representation of u, we have

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le \|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}[(n(\cdot,s)+m(\cdot,s))\nabla\phi]\|_{L^2(\Omega)}ds. \tag{2.51}$$

For $\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)}$, we can see from (4.29) in [1] that

$$\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} \le Ce^{-\lambda_1(t-1)}\|u_0\|_{L^2(\Omega)} \tag{2.52}$$

for all $t \in (0, T^*)$, where λ_1 represents the eigenvalue of A. For $\int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}[(n(\cdot,s)+m(\cdot,s))\nabla\phi]\|_{L^2(\Omega)}ds$, similar to (4.31) in [1], we have

$$\begin{aligned}
&\int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}[(n(\cdot,s)+m(\cdot,s))\nabla\phi]\|_{L^2(\Omega)}ds \\
&\le C_\alpha\|\nabla\phi\|_{L^\infty(\Omega)}\|n(\cdot,t)+m(\cdot,t)\|_{L^\infty(0,T^*;L^2(\Omega))}\int_0^t (t-s)^{-\alpha}e^{-\lambda_1(t-s)}ds \\
&\le CC_\alpha\|\nabla\phi\|_{L^\infty(\Omega)}(\|n(\cdot,t)\|_{L^\infty(0,T^*;L^2(\Omega))} + \|m(\cdot,t)\|_{L^\infty(0,T^*;L^2(\Omega))})
\end{aligned} \tag{2.53}$$

for all $t \in (0, T^*)$. Above together with (2.7) and (2.8) then yields (2.50). □

Proof of Theorem 1.1. Actually, we can see from (2.7), (2.46), (2.42), (2.50) that there exists a positive constant C such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C$$

holds for all $t \in (0, T^*)$. From this and the extensibility criterion (2.2), we infer that $T^* = \infty$, which also implies the global existence and uniform boundedness of the classical solution to system (1.1). We thus complete the proof of Theorem 1.1. □

2.4. Proof of Proposition 1.1

It's easily to find that Lemma 2.6, Lemma 2.7, Lemma 2.8, Lemma 2.9 are all valid if $\|n(\cdot,t)\|_{L^p(\Omega)}$ is uniformly bounded for some $p>3$; and then Proposition 1.1 can be proved. However, in the case that $p\in(\frac{3}{2},3]$, we can no longer apply the estimate of Stokes operator (see Corollary 3.4 of [29]) to obtain the uniform bound of $\|u\|_{W^{1,\infty}(\Omega)}$ sated in Lemma 2.6; whereupon, the proof of Proposition 1.1 in this case should be modified slightly.

Lemma 2.10. *Let $p\in(\frac{3}{2},3]$, $\alpha\in(\frac{3}{4},\min\{1-\frac{3}{2p}+\frac{3}{4},1\}$. Then there exists a positive constant C such that*

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}\le C \text{ for all } t\in(0,T^*), \tag{2.54}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le C \text{ for all } t\in(0,T^*). \tag{2.55}$$

Proof. The proof is similar to Lemma 2.9. We only need to make small modifications on (2.53). In fact, we can see from Lemma 5.1 in [1] that

$$\begin{aligned}
&\int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}[(n(\cdot,s)+m(\cdot,s))\nabla\phi]\|_{L^2(\Omega)}ds\\
&\le C_\alpha\|\nabla\phi\|_{L^\infty(\Omega)}\|n(\cdot,t)+m(\cdot,t)\|_{L^\infty(0,T^*;L^p(\Omega))}\int_0^t (t-s)^{-\alpha-\frac{3}{2p}+\frac{3}{4}}e^{-\frac{\lambda_1(t-s)}{2}}ds\\
&\le CC_\alpha\|\nabla\phi\|_{L^\infty(\Omega)}(\|n(\cdot,t)\|_{L^\infty(0,T^*;L^p(\Omega))}+\|m(\cdot,t)\|_{L^\infty(0,T^*;L^p(\Omega))})\\
&\le C \quad \text{ for all } t\in(0,T^*),
\end{aligned} \tag{2.56}$$

where we have used Lemma 2.5 and the inequality (2.7). Upon combining (2.51) and (2.52) then yields (2.54). Since $4\alpha>3$, Soblev embedding then leads to (2.55). □

With the uniform bound of u at hand, we next show that (2.41) and then (2.42) also hold.

Lemma 2.11. *There exists a positive constant C such that*

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C$$

for all $t\in(0,T^*)$.

Proof. We first infer that for any $q_0>3$, there exists a positive constant c_1 such that

$$\|\nabla c(\cdot,t)\|_{L^{q_0}(\Omega)}\le c_1 \qquad \text{for all} \qquad t\in(0,T^*). \tag{2.57}$$

Let $T_1 := \min\{\frac{T^*}{2}, 1\}$, it's obviously that (2.57) holds for all $t \in (0, T_1)$. Therefore, we only need to show it holds also for $t \in [T_1, T^*)$. To this end, we first use the constant-variation representation of c to get

$$\begin{aligned}\|\nabla c(\cdot,t)\|_{L^{q_0}(\Omega)} \le \|\nabla e^{(\Delta-1)t}c_0\|_{L^{q_0}(\Omega)} + \int_0^t \|\nabla e^{(\Delta-1)(t-s)}(u(\cdot,s)\cdot\nabla c(\cdot,s))\|_{L^{q_0}(\Omega)}ds \\ + \int_0^t \|\nabla e^{(\Delta-1)(t-s)}m(\cdot,s)\|_{L^{q_0}(\Omega)}ds\end{aligned} \tag{2.58}$$

for all $t \in (T_1, T^*)$. For any $T \in (T_1, T^*)$, we define $M(T) := \sup_{t\in[T_1,T]} \|\nabla c(\cdot,t)\|_{L^{q_0}(\Omega)}$, and fix $q \in (3, q_0)$ satisfying $\frac{1}{q} - \frac{1}{q_0} < \frac{1}{3}$. By the $L^p - L^q$ estimates for the Neumann heat semigroup, we deduce

$$\begin{aligned}&\int_0^t \|\nabla e^{(\Delta-1)(t-s)}(u(\cdot,s)\cdot\nabla c(\cdot,s))\|_{L^{q_0}(\Omega)}ds \\
&\le c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})})e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)}ds \\
&\le c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})})e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^q(\Omega)}ds \\
&= c_1\int_0^{T_1} (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})})e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^q(\Omega)}ds \\
&\quad + c_1\int_T^{T_1} (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})})e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^q(\Omega)}ds \\
&\le c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})})e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\big(c_2\|c(\cdot,s)\|^{1-a}_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|^a_{L^{q_0}(\Omega)} \\
&\qquad + c_2\|c(\cdot,s)\|_{L^\infty(\Omega)}\big)ds \\
&\le c_3 M(T)^a + c_3\end{aligned} \tag{2.59}$$

for all $t \in (T_1, T)$, where $a = \frac{1-\frac{3}{q}}{1-\frac{3}{q_0}} \in (0, 1)$ and c_1, c_2, c_3 are positive constants independent of t. Similarly, we have

$$\begin{aligned}
&\int_0^t \|\nabla e^{(\Delta-1)(t-s)} m(\cdot,s)\|_{L^{q_0}(\Omega)} ds \\
&\le c_1 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})}) e^{-(\lambda_1+1)(t-s)} \|m(\cdot,s)\|_{L^q(\Omega)} ds \\
&\le c_1 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{q_0})}) e^{-(\lambda_1+1)(t-s)} \|m(\cdot,s)\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q}} ds \\
&\le c_4 \qquad \text{for all } t \in (T_1, T).
\end{aligned} \tag{2.60}$$

and

$$\|\nabla e^{(\Delta-1)t} c_0\|_{L^{q_0}(\Omega)} \le c_5 T_1^{-\frac{1}{2}} \|c_0\|_{L^{q_0}(\Omega)} \qquad \text{for all } t \in (T_1, T), \tag{2.61}$$

where c_4, c_5 are positive constants independent of t. Combining (2.58)-(2.61), we have

$$M(T) \le c_3 M(T)^a + c_3 + c_4 + c_5 T_1^{-\frac{1}{2}} \|c_0\|_{L^{q_0}(\Omega)}.$$

Since $a \in (0, 1)$, we have by Young's inequality

$$M(T) \le C$$

with $C > 0$ independent of T, and so (2.57) is obtained. With this at hand, we can further deduce the uniform boundedness of $\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)}$. Indeed, by small modifications of (2.45), we have

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^\infty(0,T^*;L^\infty(\Omega)} &\le \|\nabla c_0\|_{L^\infty(\Omega)} + C\|m - u\cdot\nabla c\|_{L^\infty(0,T;L^{q_0}(\Omega))} \\
&\le \|\nabla c_0\|_{L^\infty(\Omega)} + C(\|m_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q_0}} \\
&\quad + \|u\|_{L^\infty(0,T^*;L^\infty(\Omega))} \|\nabla c\|_{L^\infty(0,T;L^{q_0}(\Omega))}) \\
&\le C,
\end{aligned} \tag{2.62}$$

upon combining (2.8) then yields the desired result. □

Proof of Proposition 1.1. Since we have obtained the uniform boundedness of $\|m(\cdot,t)\|_{L^\infty(\Omega)}$, $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$, as well as $\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$, in light of the extensibility criterion (2.2), to prove the global existence of solution to system (1.1), we next only need to show that $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ is uniformly bounded in $(0, T^*)$. In fact, for any $T \in (0, T^*)$, we can get from the proof of Lemma 2.8 that

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &\le \|n_0\|_{L^\infty(\Omega)} \\
&\quad + \int_0^t \|e^{(t-s)\Delta} \nabla \cdot \Big(n(\cdot,s)\big(S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s) + u(\cdot,s)\big)\Big)\|_{L^\infty(\Omega)} ds
\end{aligned} \tag{2.63}$$

for all $t \in (0, T)$. For $p \in (\frac{3}{2}, 3]$, we have

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla \cdot \Big(n(\cdot,s)\big(S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s)+u(\cdot,s)\big)\Big)\|_{L^\infty(\Omega)}ds \\
&\le C_0\int_0^t (1+(t-s)^{-\frac{7}{8}})e^{-\lambda_1(t-s)}\|n(\cdot,s)\big(S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s)+u(\cdot,s)\big)\|_{L^4(\Omega)}ds \\
&\le C_0\int_0^t (1+(t-s)^{-\frac{7}{8}})e^{-\lambda_1(t-s)}\|S(x,n(\cdot,s),c(\cdot,s))\nabla c(\cdot,s)+u(\cdot,s)\|_{L^\infty(\Omega)}\|n(\cdot,s)\|_{L^4(\Omega)}ds \\
&\le C_0\Big(S_0(M)\|\nabla c\|_{L^\infty(0,T^*;L^\infty(\Omega))}+\|u\|_{L^\infty(0,T^*;L^\infty(\Omega))}\Big)\|n\|^{\frac{p}{4}}_{L^\infty(0,T^*;L^p(\Omega))}\|n\|^{1-\frac{p}{4}}_{L^\infty(0,T;L^\infty(\Omega))} \\
&\quad\times\int_0^\infty (1+t^{-\frac{7}{8}})e^{-\lambda_1 t}dt
\end{aligned} \tag{2.64}$$

for all $t \in (0, T)$. Applying Young's inequality and combining (2.63), we deduce

$$\|n\|_{L^\infty(0,T;L^\infty(\Omega))} \le C$$

with C independent of T. Therefore, $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ is uniformly bounded in $(0, T^*)$. We thus complete the proof of Proposition 1.1. □

2.5. Global existence and boundedness in the case: $N = 2$, $\kappa = 1$

To prove the uniform boundedness of classical solution, we only need to show that the uniform boundedness of $\|n(\cdot,t)\|_{L^p(\Omega)}$ and $\int_t^{t+\tau}\int_\Omega (n(\cdot,s)+1)^{p-2}|\nabla n(\cdot,s)|^2dxds$ for some $p \in (1,+\infty)$ can ensure the uniform boundedness of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$, $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ as well as $\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$. We first assert that

Lemma 2.12. *There admits a positive constant C such that*

$$\int_t^{t+\tau}\int_\Omega n^2(x,s)dxds \le C \tag{2.65}$$

for all $t \in (0,\ T^* - \tau)$ with $\tau = \min\{1,\ \frac{1}{2}T^*\}$.

Proof. Let $p \in (1, +\infty)$, by Gagliardo-Nirenberg's inequality and the uniform bound of $\|n(\cdot,t)\|_{L^1(\Omega)}$ stated in (2.3), we have

$$
\begin{aligned}
\int_\Omega (n(\cdot,t)+1)^2 dx &= \|\sqrt{(n(\cdot,t)+1)}\|^4_{L^4(\Omega)} \\
&\le C_{GN}\|(n(\cdot,t)+1)\|_{L^1(\Omega)}\|\nabla\sqrt{(n(\cdot,t)+1)}\|^2_{L^2(\Omega)} + C_{GN}\|(n(\cdot,t)+1)\|^2_{L^1(\Omega)} \\
&\le c_1 \int_\Omega (n(\cdot,t)+1)^{-1}|\nabla n(\cdot,t)|^2 dx + c_1 \\
&\le c_1 \int_\Omega (n(\cdot,t)+1)^{p-2}|\nabla n(\cdot,t)|^2 dx + c_1 \qquad \text{for all } t \in (0,T^*),
\end{aligned} \tag{2.66}
$$

where $c_1 = C_{GN}[(\|n_0\|_{L^1(\Omega)}+1)^2 + |\Omega|]$. Upon integrating over $(t, t+\tau)$ we have

$$
\begin{aligned}
\int_t^{t+\tau}\int_\Omega n^2(x,s)dxds &< \int_t^{t+\tau}\int_\Omega (n(x,s)+1)^2 dxds \\
&\le c_1 \int_t^{t+\tau}\int_\Omega (n(x,s)+1)^{p-2}|\nabla n(x,s)|^2 dxds + c_1\tau
\end{aligned} \tag{2.67}
$$

for all $t \in (0, T^*-\tau)$ with $\tau = \min\{1, \frac{1}{2}T^*\}$. Combining (2.67) and (2.19)(or (2.32)) then we get (2.65). □

With the time-space L^2 estimate of n, we then can obtain the uniform bound of $\|\nabla u(\cdot, t\|_{L^2(\Omega)}$.

Lemma 2.13. *There exists a positive constant C such that*

$$\|\nabla u(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t \in (0,T^*)$$

Proof. We get from [7, Lemma 3.3] that there exists $c_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega |u(\cdot,t)|^2 dx + \int_\Omega |\nabla u(\cdot,t)|^2 dx \le c_1(\|n(\cdot,t)\|^2_{L^p(\Omega)} + 1) \qquad \text{for all } t \in (0,T^*). \tag{2.68}$$

From which and the Poincaré inequality $\|u\|_{L^2(\Omega} \le C_P\|\nabla u\|_{L^2(\Omega)}$ as well as the uniform bound of $\|n(\cdot,t)\|_{L^p(\Omega)}$ obtained in (2.31) and (2.9), we conclude

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in (0,\ T^*) \tag{2.69}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u(x,s)|^2 dxds \le C \quad \text{for all } t \in (0,\ T^*-\tau) \tag{2.70}$$

with $\tau = \min\{1, \frac{1}{2}T^*\}$. With the above two estimates at hand, by the same procedure of the proof of [7, Lemma 3.6], we can further obtain the uniform bound of $\|\nabla u(\cdot,t)\|_{L^2(\Omega)}$. We omit the details here. □

Lemma 2.14. *There exists a positive constant C such that*

$$\|\nabla c(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \text{for all } t \in (0, T^*).$$

Proof. In fact, we can get from [7, Lemma 3.2] that

$$\frac{d}{dt}\int_\Omega |\nabla c(\cdot,t)|^2 dx + \int_\Omega |\Delta c(\cdot,t)|^2 dx + 2\int_\Omega |\nabla c(\cdot,t)|^2 dx \le C\Big(\int_\Omega |\nabla u(\cdot,t)|^2 dx + 1\Big) \quad (2.71)$$

for all $t \in (0, T^*)$. From this and Lemma 2.13 we can obtain the desired result. □

Proof of Proposition 1.2. With the uniform bound of $\|\nabla u(\cdot,t)\|_{L^2(\Omega)}$ at hand, by Poincaré inequality we can further obtain the uniform bound of $\|u(\cdot,t)\|_{L^q(\Omega)}$ for any $q > 1$. Then following the same procedure as Lemma 3.8, Lemma 3.9, Lemma 3.10 of [7], we can obtain the uniform bound of $\|n(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$, which in turn yields the uniform bound of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$; and thus the uniform bound of $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ can be obtained from Lemma 2.11. Since $\|m(\cdot,t)\|_{L^\infty(\Omega)}$ is also uniformly bounded, in view of the extensibility criterion (2.2), we then can assert that $T^* = \infty$ and the solution is uniformly bounded. We thus complete the proof of Proposition 1.2. □

3. Global existence and boundedness for general S

In this last section, we give the proof of our main results for the general tensor-valued S. In this case, the no-flux boundary condition for n i.e.,

$$\nabla n \cdot \nu = 0, \ x \in \partial\Omega, \ t > 0,$$

is invalid. To deal with this difficulty, we can use the standard approximation procedure in [29], see also [9]. In fact, we can first introduce a family of smooth functions $\rho_\epsilon \in C_0^\infty(\Omega)$ satisfying $\rho_\epsilon \in [0,1]$ in Ω and $\rho_\epsilon \to 1$ as $\epsilon \to 0$, and define $S_\epsilon(x, n_\epsilon, c_\epsilon) = \rho_\epsilon S(x, n_\epsilon, c_\epsilon)$. Then we regularize system (1.1) as follows:

$$\begin{cases} n_{\epsilon t} + \mathbf{u}_\epsilon \cdot \nabla n_\epsilon = \Delta n_\epsilon - \nabla\cdot(n_\epsilon S_\epsilon(x, n_\epsilon, c_\epsilon)\nabla c_\epsilon) - n_\epsilon m_\epsilon, & x \in \Omega, \ t > 0, \\ m_{\epsilon t} + \mathbf{u}_\epsilon \cdot \nabla m_\epsilon = \Delta m_\epsilon - n_\epsilon m_\epsilon, & x \in \Omega, \ t > 0, \\ c_{\epsilon t} + \mathbf{u}_\epsilon \cdot \nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + m_\epsilon, & x \in \Omega, \ t > 0, \\ u_{\epsilon t} + \kappa(\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - \nabla p_\epsilon + (n_\epsilon + m_\epsilon)\nabla\phi, \quad \nabla\cdot\mathbf{u}_\epsilon = 0, & x \in \Omega, \ t > 0, \\ \nabla c_\epsilon \cdot \nu = \nabla m_\epsilon \cdot \nu = \nabla n_\epsilon \cdot \nu = 0, \ \mathbf{u}_\epsilon = 0 & x \in \partial\Omega, \ t > 0, \\ n_\epsilon(x,0) = n_0(x), \ m_\epsilon(x,0) = m_0(x), \ c_\epsilon(x,0) = c_0(x), \ \mathbf{u}_\epsilon(x,0) = \mathbf{u}_0(x), & x \in \partial\Omega. \end{cases} \quad (3.1)$$

Following the same procedure as Section 2, we can show, under the same assumptions as Theorem 1.1 and Proposition 1.1 as well as 1.2, the above system admits a global classical solution $(n_\epsilon, m_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ such that

$$\|n_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|m_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u_\epsilon(\cdot,t)\|_{L^2(\Omega)} \le C$$
$$\text{for all } t \in (0,+\infty),\ \epsilon \in (0,1).$$

Then by the standard approximation procedure as that in [1, Section 6], see also [2, Section 5], we can show that Theorem 1.1 and Proposition 1.1 as well as Proposition 1.2 hold for general tensor-valued S. We thus complete our proofs of Theorem 1.1 and Proposition 1.1 as well as Proposition 1.2. □

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