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On the expanding configurations of viscous radiation gaseous stars: Thermodynamic model

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Abstract

In this work, we study the stability of the expanding configurations of radiation gaseous stars. Such expanding configurations exist for a thermodynamic model, given as a class of self-similar solutions to the associated dynamic system with viscosity coefficients satisfying $2\mu + 3\lambda = 0$ for the monatomic gas; that is, the bulk viscosity is vanishing. With respect to small perturbations, this work shows that the linearly expanding homogeneous solutions are stable for a large expanding rate. This is an extensive study of the result in [14] by Hadžić and Jang.

MSC: 35A01; 35Q30; 35Q35; 35Q85; 76E20

Keywords: Asymptote; Homogeneous solutions; Stability; Radiation gaseous stars

1. Introduction

The evolution of a viscous gaseous star can be described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{u}) = 0 & \text{in } \Omega(t), \\ \partial_t (\rho \vec{u}) + \operatorname{div}(\rho \vec{u} \otimes \vec{u}) + \nabla p + \rho \nabla \psi = \operatorname{div} \mathbb{S} & \text{in } \Omega(t), \\ \partial_t (c_\nu \rho \theta) + \operatorname{div}(c_\nu \rho \theta \vec{u}) + p_\theta \operatorname{div} \vec{u} - \Delta \theta = \epsilon \rho + \mathbb{S} : \nabla \vec{u} & \text{in } \Omega(t), \\ \Delta \psi = \rho & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

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where $\rho, \vec{u}, \psi, \Omega(t)$ denote the density, the velocity field, the self-gravitation potential and the evolving occupied domain. $\mathbb{S} = \mu(\nabla\vec{u} + \nabla\vec{u}^\top) + \lambda \mathrm{div}\vec{u}\, \mathbb{I}_3$ is the Newtonian viscous tensor, and $c_\nu > 0, \epsilon > 0$ denote constants for the specific heat coefficient and the rate of generation of energy. The viscosity coefficients are taken as the ones for the monatomic gas in this work; that is,

$$\mu = \text{constant} > 0, \ 2\mu + 3\lambda = 0. \tag{2}$$

This is indicated by the kinetic theory (see [24, pp 3, (1,11)]). Notice that this implies the flows are affected only by the shear viscosity ($\mu > 0$) but not by the bulk viscosity ($2\mu + 3\lambda = 0$). p is the pressure potential, which is taken as

$$p = K\rho\theta \ \ (K = \text{constant} > 0), \tag{3}$$

where θ is the temperature distribution.

System (1) is complemented with the boundary conditions,

$$\begin{aligned} \vec{u} \cdot \vec{n} &= \mathcal{V} & &\text{on } \Gamma(t) := \partial\Omega(t), \\ p\mathbb{I}_3\vec{n} - \mathbb{S}\vec{n} &= 0 & &\text{on } \Gamma(t), \\ \theta &= 0 & &\text{on } \Gamma(t), \end{aligned}$$

where $\mathcal{V}$ is the normal velocity of the moving surface $\Gamma(t)$. Such a system is referred to as **the thermodynamic model for radiation gaseous stars**. In fact, we will show the existence of expanding configurations when $c_\nu = 3K$ in the following. The stability of such expanding configurations is investigated as well.

The gaseous star problem has been studied in a huge number of literatures. To name a few, in [1], Auchmuty and Beals studied the variational solutions for the rotating gaseous stars; that is, to find the solutions (equilibrium states) to the following system:

$$\begin{cases} \mathrm{div}(\rho\vec{u}) = 0 & \text{in } \Omega := \{\rho > 0\} \subset \mathbb{R}^3, \\ \mathrm{div}(\rho\vec{u} \otimes \vec{u}) + \nabla p(\rho) = -\rho\nabla\psi & \text{in } \Omega, \\ \Delta\psi = \rho & \text{in } \mathbb{R}^3, \end{cases} \tag{4}$$

where $p(\rho)$ is a given function of the density ρ. Under some conditions on the angular momentum or the angular velocity, and the equation of state, it is shown that there exists at least one compactly supported solution (with $\mathrm{supp}\,\rho \subset B_R$ for some $0 < R < +\infty$). In [2], Caffarelli and Friedman showed that such solutions have at most a finite number of rings (i.e., the support of the density distribution is consisted of a finite number of connected components). Also, the regularity of the solutions was studied. See also in [4,5,9–11,21,27–29,35] for more discussions on the related problems.

On the other hand, when $\vec{u} \equiv 0$ in (4) (referred to as the non-rotating gaseous stars), the existence of equilibrium was established by Chandrasekhar in [3]. In particular, Lieb and Yau showed that the solution has to be spherically symmetric in [22]. In the case when $p = \rho^\gamma$, such solutions are called the Lane-Emden solutions for the non-rotating gaseous stars.

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The linear stability and instability theory of these solutions was studied in [23], where Lin studied the eigenvalue of the linearized operator of the associated evolutionary problem. A conditional nonlinear stability theory was given by Rein in [34].

The compactly supported solutions for the non-rotating gaseous star admit the physical vacuum boundary; that is, the sound speed is only $1/2$-Hölder continuous across the gas-vacuum interface. Such a singularity makes the study of the corresponding evolutionary problem challenging (see [25]). It is only recently that the local well-posedness in smooth functional spaces is studied by Coutand, Lindblad and Shkoller [6–8], Jang and Masmoudi [18,19], Luo, Xin and Zeng [30] in the settings of one spatial dimension, three spatial dimensions and spherical symmetry with or without self-gravitation. See [16] for a viscous flow. We refer the reader to [25,33, 36,37] and the references therein for other discussions on the vacuum-interface problems.

With the local well-posedness theory, it is possible to analyze the nonlinear stability and instability of the Lane-Emden solutions for the non-rotating gaseous stars. In particular, the works from Jang and Tice [15,17,20] show that for $6/5 \leq \gamma < 4/3$, the Lane-Emden solutions are unstable, and additional viscosities can not reduce such instability. When it comes to the case when $4/3 < \gamma < 2$, the asymptotic stability theory is first studied in [32] in the viscous case. See [31] for the model with degenerate viscosities.

Meanwhile, the isentropic model for radiation gaseous stars (that is, the gaseous star problem with $p = \rho^{4/3}$, see [3,9,14]) is rather complicated. On the one hand, only for a specific total mass $\int \rho \, dx = M_* > 0$, called the critical mass, the non-rotating gaseous star problem admits solutions with compact support. On the other hand, by studying a class of self-similar solutions in [12], the authors showed the existence of expanding and collapsing solutions with variant total mass. Such phenomena are originally studied in [13] and are far from fully understood. Recently, in [14], the authors show the asymptotic stability of the expanding solutions. The damping structure in the Lagrangian coordinates and the conservation of energy for smooth solutions are the two important ingredients in their work. When considering the heat generation and heat conductivity in the gaseous star, i.e. the thermodynamic model for radiation gaseous stars, in [26] the author shows that for $1/6 < \epsilon K < 1$, there exist infinitely many self-similar equilibrium states for the non-rotating radiation gaseous star problem. Also, there exist self-similar expanding and collapsing configurations just as in the isentropic model.

1.1. The expanding configurations in the spherically symmetric motion

We will study the aforementioned expanding solutions of the thermodynamic model of radiation gaseous stars. Considering the spherically symmetric motion, i.e., $\vec{u}(\vec{x},t) = u(r,t) \cdot \frac{\vec{x}}{r}$, $\rho(\vec{x},t) = \rho(r,t), \theta(\vec{x},t) = \theta(r,t), p(\vec{x},t) = p(r,t), \Omega(t) = B_{R(t)}$ where $r = |\vec{x}| \in [0, R(t))$, $\vec{x} \in \mathbb{R}^3$, system (1) can be written as

$$\begin{cases} \partial_t (r^2\rho) + \partial_r (r^2 \rho u) = 0, \\ \partial_t (r^2 \rho u) + \partial_r (r^2 \rho u^2) + r^2 \partial_r p + \rho \int_0^r s^2 \rho(t,s)\, ds \\ \qquad = (2\mu + \lambda) r^2 \Big[\dfrac{\partial_r (r^2 u)}{r^2}\Big]_r = \dfrac{4\mu}{3} r^2 \Big[\dfrac{\partial_r (r^2 u)}{r^2}\Big]_r, \\ \partial_t (c_v r^2 \rho\theta) + \partial_r (c_v r^2 \rho \theta u) + K\rho\theta \partial_r (r^2 u) - \partial_r (r^2 \partial_r \theta) - \varepsilon r^2 \rho \\ \qquad = 2\mu r^2 \big((\partial_r u)^2 + 2\big(\dfrac{u}{r}\big)^2\big) + \lambda r^2 \big(\partial_r u + 2\dfrac{u}{r}\big)^2 \\ \qquad = \dfrac{4\mu}{3} r^2 \big(\partial_r u - \dfrac{u}{r}\big)^2, \end{cases} \tag{5}$$

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where

$$p = K\rho\theta.$$

The boundary conditions then can be written as

$$\begin{aligned} u(R(t),t) &= \partial_t R(t),\ u(0,t)=0,\\ p - \mathfrak{B}\big|_{r=R(t)} &= 0,\\ \theta(R(t),t) &= 0, \end{aligned} \tag{6}$$

where $\mathfrak{B} = (2\mu+\lambda)\partial_r u + 2\lambda\dfrac{u}{r} = \dfrac{4\mu}{3}\big(\partial_r u - \dfrac{u}{r}\big)$.

The expanding configurations are obtained by searching solutions to (5) with the following self-similar ansatz:

$$\begin{aligned} r &= r_\alpha = \alpha(t)y,\\ \rho &= \rho_\alpha(t,r) = \alpha^{-3}(t)\bar\rho(y),\\ \theta &= \theta_\alpha(t,r) = \alpha^{-1}(t)\bar\theta(y),\\ u &= u_\alpha(t,r) = \alpha'(t)y, \end{aligned} \tag{7}$$

for $y \in [0, R_0)$ with some $0 < R_0 < +\infty$.

Notice, when the viscosity coefficients are assumed to be the ones for the monatomic gases as in (2), $\operatorname{div}\mathbb{S}$ automatically vanishes on the boundary for the self-similar ansatz (7). So does $\mathfrak{B}$.

1.1.1. The linearly homogeneous solutions

In [26], the following linearly homogeneous solution is given for system (5). Let $(\bar\rho, \bar\theta)$ be an equilibrium state for (5) with $1/6 < \epsilon K < 1$; that is, it satisfies the following

$$\begin{cases} y^2(K\bar\rho\bar\theta)_y + \bar\rho \int_0^y s^2\bar\rho(s)\,ds = 0,\\ -(y^2\bar\theta_y)_y = \epsilon y^2\bar\rho, \end{cases} \tag{8}$$

with $\bar\theta, \bar\rho > 0$ in $[0, R_0)$, and $\bar\theta(R_0) = \bar\rho(R_0) = 0$. Then after plugging the ansatz (7) into system (5), it holds,

$$\alpha''(t)\alpha(t) = 0,\ (3K - c_\nu)\alpha^{-2}(t)\alpha'(t) = 0.$$

Such a system admits non-trivial solutions if and only if

$$3K - c_\nu = 0.$$

Therefore the linearly expanding solution of form (7) is given when $3K = c_\nu$, and it admits

$$\alpha(t) = a_0 + a_1 t. \tag{9}$$

We will refer the constant a_1 as **the expanding rate** of the linearly expanding solutions.

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Also, near the boundary $y = R_0$, $\bar\rho, \bar\theta$ admit the following properties:

$$-\infty < \left(\bar\rho^{\frac{\epsilon K}{1-\epsilon K}}\right)_y, \left(\bar\theta\right)_y \le -C < 0,$$
$$\left|\partial_y^k \bar\rho\right| \le O(\bar\rho^{\frac{1-(1+k)\epsilon K}{1-\epsilon K}}), \ \left|\partial_y^k \bar\theta\right| \le O(1) + O(\bar\theta^{\frac{1-\epsilon K}{\epsilon K}-k+2}), \ k \in \mathbb{Z}^+.$$

Notice, this implies that near the boundary $y = R_0$, it holds,

$$\bar\rho^{\frac{\epsilon K}{1-\epsilon K}}(y), \ \bar\theta(y) \simeq R_0 - y. \tag{10}$$

In the inviscid case for the isentropic flow, the stability of the expanding solutions is given by Hadžić and Jang in [14]. The main observation in [14] is that in the Lagrangian coordinates, the perturbation variable admits a non-linear wave equation with damping. Such a structure provides the dissipation estimates and therefore controls the nonlinearity. Moreover, the conservation of the physical energy will give the energy coercivity together with the spectral gap of the associated linearized operator.

It should be emphasized that the conservation of energy plays important role in Hadžić and Jang's work. Roughly speaking, the spectral gap of the associated linearized operator gives a weighted Poincaré-type inequality in the space of functions with vanishing weighted mean value. The conservation of energy gives the control the remaining weighted mean value of the perturbation variable. Indeed, the difference of the perturbation variable and its mean value is of order $\mathcal{O}(2)$, and therefore can be treated as a nonlinearity term and controlled by the total energy.

The goal of current work is to analyze the effect of viscosities on the expanding solutions in the thermodynamics model. The benefit of conservation of physical energy no longer exists. This downside is expected to be complemented by the extra viscosity. However, the story is not so simple. In fact, while in the Lagrangian coordinates the damping structure (just as in Hadăzić and Jang's work) still exists, the extra integrability of the solution provided by it has different temporal weights with the one provided by the viscosity tensor. This will break the pattern of energy estimates as one will see. In particular, we fail to get the decay estimate of higher order energy, which will be explained later. In addition, the physical energy for the thermodynamic model is not monotone. In fact, it holds

$$\frac{d}{dt} E = R^2(t)\partial_r\theta(R(t),t) + \epsilon \int_0^{R(t)} r^2\rho \, dr,$$

where

$$E = \frac{1}{2}\int_0^{R(t)} r^2\rho u^2 \, dr + c_\nu \int_0^{R(t)} r^2 \rho\theta \, dr - \int_0^{R(t)} r\rho \int_0^r s^2\rho \, ds \, dr.$$

In the Lagrangian coordinates, this means that there are several linear forcing terms with no determined sign, which will cause troubles to close the energy estimates.

We perform delicate energy estimates with negative temporal weights in this work, which will control the growth of energy. While doing the temporal weighted estimates, we integrate the forcing terms in the temporal variable and it turns out that such calculations will give an extra denominator proportional to the expanding rate a_1. Therefore, by letting a_1 be large enough, we will be able to close the energy estimates. In the end, we make use of the imbalance of the temporal weight in the damping term and the viscosity tensor to recover the control of the perturbation and show the stability of the linear expanding solutions.

1.2. Main result

We will state the main result of this work in this section.

Theorem 1.1. *The linearly expanding solutions of the isentropic model for the radiation gaseous stars given by* (7) *to* (5) *with* (8) *and* (9) *are stable with respect to small perturbations if the expanding rate a_1 of the expanding solutions is large enough.*

2. Lagrangian formulations

In this section, we will write down system (5) in a fixed domain (i.e., in the Lagrangian coordinates). Denote the Lagrangian spatial and temporal variables as (x, t). Similarly as in [32], the Lagrangian unknown $r(x, t)$ is defined by

$$\begin{aligned} \int_0^{r(x,t)} s^2 \rho(s,t)\, ds = \int_0^{\alpha(t)x} s^2 \rho_\alpha(s,t)\, ds = \int_0^x s^2 \bar{\rho}(s)\, ds, \\ \text{and } r(x,0) = x,\ x \in [0, R_0), \end{aligned} \tag{11}$$

where $0 < R_0 < +\infty$ is the first zero of $\bar{\rho}$ (and $\bar{\theta}$). After applying spatial and temporal derivatives to (11), it follows,

$$\partial_t r(x,t) = u(r(x,t),t), \quad \rho(r(x,t),t) = \frac{x^2 \bar{\rho}(x)}{r^2 r_x}, \tag{12}$$

where the continuity equation $(5)_1$ is applied. Then the expanding solution (7) is given by

$$\begin{gathered} r_\alpha = \alpha(t)x,\ \rho_\alpha = \alpha^{-3}(t)\bar{\rho}(x),\ \theta_\alpha = \alpha^{-1}(t)\bar{\theta}(x), \\ u_\alpha = \alpha'(t)x, \end{gathered}$$

where $\alpha, \bar{\rho}, \bar{\theta}$ are given as in section 1.1. Moreover, in the Lagrangian coordinates, the unknowns $\{\rho, u, \theta\}$ in the moving domain $[0, R(t))$ are now replaced by $\{r, r_t, \theta\}$ in the fixed domain $[0, R_0)$.

Notice, we use the same notation θ for the temperature in the original moving coordinates and in the Lagrangian coordinates. System (5) can be written in terms of the Lagrangian variables in the Lagrangian coordinates as follows: for $x \in [0, R_0)$,

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$$\begin{cases} \left(\frac{x}{r}\right)^2 \bar\rho \partial_t^2 r + P_x + \left(\frac{x}{r}\right)^4 \frac{\bar\rho}{x^2} \int_0^x s^2 \bar\rho(s)\, ds = \frac{4}{3}\mu\left[\frac{(r^2 r_t)_x}{r^2 r_x}\right]_x, \\ 3Kx^2\bar\rho\partial_t\theta + K\frac{x^2\bar\rho\theta}{r^2 r_x}(r^2 r_t)_x - \left[\frac{r^2}{r_x}\theta_x\right]_x - \epsilon x^2 \bar\rho \\ \quad = \frac{4}{3}\mu r^2 r_x \left(\frac{r_{xt}}{r_x} - \frac{r_t}{r}\right)^2, \end{cases} \tag{13}$$

where

$$P = K\frac{x^2\bar\rho\theta}{r^2 r_x}.$$

Also, the boundary conditions (6) can be written as

$$r(0,t) = 0, \ \mathfrak{B} = \frac{4}{3}\mu\left(\frac{r_{xt}}{r_x} - \frac{r_t}{r}\right)\Big|_{x=R_0} = 0, \\ \theta(R_0, t) = 0. \tag{14}$$

What to do next is to define the perturbation unknowns and to write down the system satisfied by them. As one will see, the aforementioned damping structure will appear naturally, but with a temporal weight. In order to avoid dealing with the temporal weight, in [14], the authors introduce some new temporal variables for the self-similarly expanding solutions and the linearly expanding solutions, respectively. We will adopt the same strategy in the following and write down the corresponding system for the linearly expanding solution of the thermodynamic model.

2.1. Linearly expanding solutions in the perturbation variables

Considering the linearly expanding solution of the thermodynamic model, define the perturbation variables as

$$\eta := \frac{r(x,t)}{r_\alpha(x,t)} - 1 = \frac{r(x,t)}{\alpha(t)x} - 1, \ \varsigma := \alpha(\theta - \theta_\alpha) = \alpha(t)\theta - \bar\theta. \tag{15}$$

Then after employing (8), system (13) in terms of the perturbation variables $\{\eta, \varsigma\}$ is in the following form:

$$\begin{cases} \frac{\alpha^3(t)}{(1+\eta)^2} x\bar\rho\eta_{tt} + \frac{2\alpha^2(t)\alpha'(t)}{(1+\eta)^2} x\bar\rho\eta_t + \left[\frac{K\bar\rho(\varsigma+\bar\theta)}{(1+\eta)^2(1+\eta+x\eta_x)}\right]_x - \frac{(K\bar\rho\bar\theta)_x}{(1+\eta)^4} \\ \quad = \frac{4}{3}\mu\alpha^4(t)\left[\frac{(\alpha^3(t)x^3(1+\eta)^3)_{xt}}{(\alpha^3(t)x^3(1+\eta)^3)_x}\right]_x = \alpha^4(t)\mathfrak{B}_x + 4\mu\alpha^4(t)\left(\frac{\eta_t}{1+\eta}\right)_x, \\ 3Kx^2\bar\rho\varsigma_t + K\frac{\bar\rho(\varsigma+\bar\theta)(x^3(1+\eta)^2\eta_t)_x}{(1+\eta)^2(1+\eta+x\eta_x)} - \alpha(t)\left[\frac{(1+\eta)^2}{1+\eta+x\eta_x}x^2\varsigma_x \right. \\ \quad \left. + \left(\frac{(1+\eta)^2}{1+\eta+x\eta_x} - 1\right)x^2\bar\theta_x\right]_x = \alpha^2(t)x^2(1+\eta)^2(1+\eta+x\eta_x)\cdot\mathfrak{F}(\eta), \end{cases} \tag{16}$$

where

$$\mathfrak{B} := \frac{4}{3}\mu\Big(\frac{\eta_t + x\eta_{xt}}{1+\eta+x\eta_x} - \frac{\eta_t}{1+\eta}\Big),$$
$$\mathfrak{F}(\eta) := \frac{4}{3}\mu\Big[\frac{\eta_t + x\eta_{xt}}{1+\eta+x\eta_x} - \frac{\eta_t}{1+\eta}\Big]^2.$$

The associated boundary conditions are then given by

$$\mathfrak{B}(R_0, t) = 0, \ \varsigma(R_0, t) = 0.$$

Notice that α is growing linearly over time. It will be convenient to work with the linearly temporal variable τ, defined by

$$\tau = \tau(t) = \int_0^t \frac{1}{\alpha(\sigma)}\, d\sigma = \frac{\ln(1 + a_1 t/a_0)}{a_1}. \tag{17}$$

Consequently, we have

$$t(\tau) = \frac{a_0}{a_1}(e^{a_1\tau} - 1), \ \frac{d}{d\tau}t = a_0 e^{a_1\tau} = \tilde{\alpha}(\tau), \ \tilde{\alpha}(\tau(t)) := \alpha(t) = a_0 e^{a_1\tau}.$$

By denoting the unknowns in the linearly temporal variable as

$$\xi(x, \tau(t)) := \eta(x, t), \ \zeta(x, \tau(t)) := \varsigma(x, t), \tag{18}$$

system (16) takes the form

$$\begin{cases} \dfrac{1}{(1+\xi)^2}\big(\tilde{\alpha}x\bar{\rho}\xi_{\tau\tau} + \tilde{\alpha}_\tau x\bar{\rho}\xi_\tau\big) + \Big[\dfrac{K\bar{\rho}(\zeta+\bar{\theta})}{(1+\xi)^2(1+\xi+x\xi_x)}\Big]_x - \dfrac{(K\bar{\rho}\bar{\theta})_x}{(1+\xi)^4} \\ \quad = \tilde{\alpha}^3\big(\hat{\mathfrak{B}}_x + 4\mu\big(\dfrac{\xi_\tau}{1+\xi}\big)_x\big), \\ 3Kx^2\bar{\rho}\zeta_\tau + K\dfrac{\bar{\rho}(\zeta+\bar{\theta})(x^3(1+\xi)^2\xi_\tau)_x}{(1+\xi)^2(1+\xi+x\xi_x)} - \tilde{\alpha}^2\Big[\dfrac{(1+\xi)^2}{1+\xi+x\xi_x}x^2\zeta_x \\ \quad + \big(\dfrac{(1+\xi)^2}{1+\xi+x\xi_x} - 1\big)x^2\bar{\theta}_x\Big]_x = \tilde{\alpha}x^2(1+\xi)^2(1+\xi+x\xi_x)\cdot\hat{\mathfrak{F}}(\xi), \end{cases} \tag{19}$$

where

$$\begin{aligned} \hat{\mathfrak{B}} &:= \frac{4}{3}\mu\Big(\frac{\xi_\tau + x\xi_{x\tau}}{1+\xi+x\xi_x} - \frac{\xi_\tau}{1+\xi}\Big), \\ \hat{\mathfrak{F}}(\xi) &:= \frac{4}{3}\mu\Big[\frac{\xi_\tau + x\xi_{x\tau}}{1+\xi+x\xi_x} - \frac{\xi_\tau}{1+\xi}\Big]^2. \end{aligned} \tag{20}$$

Now the boundary conditions appear to be

$$\hat{\mathfrak{B}}(R_0, \tau) = 0, \ \zeta(R_0, \tau) = 0.$$

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2.2. Comments and methodology

To establish the appropriate energy estimates, it is worth looking at some linear model equations. First, consider the following linear equation associated with $(19)_1$:

$$e^{k\tau}(f_{\tau\tau} + kf_\tau) = e^{3k\tau} f_{xx\tau},$$

where $k > 0$ is a constant. Then the L^2-estimate of this model equation is of the form

$$\frac{e^{k\tau}}{2}\|f_\tau\|^2_{L^\infty_\tau L^2_x} + \int \frac{ke^{k\tau}}{2}\|f_\tau\|^2_{L^2_x}\,d\tau + \int e^{3k\tau}\|f_{x\tau}\|^2_{L^2_x}\,d\tau \leq \text{initial data}.$$

The damping structure in the above equation will imply a faster decay. On the other hand, the following linear equation is associated with the temporal derivative of $(19)_1$:

$$e^{k\tau}(f_{\tau\tau\tau} + kf_{\tau\tau}) = e^{3k\tau} f_{xx\tau\tau} + e^{3k\tau} f_\tau + \cdots.$$

We remark that such a structure is a consequence of the monoatomic gas viscous coefficients in (2). Then the L^2-estimate will yield

$$\begin{aligned}&\frac{e^{k\tau}}{2}\|f_{\tau\tau}\|^2_{L^\infty_\tau L^2_x} + \int \frac{ke^{k\tau}}{2}\|f_{\tau\tau}\|^2_{L^2_x}\,d\tau + \int \frac{e^{3k\tau}}{2}\|f_{x\tau\tau}\|^2_{L^2_x}\,d\tau \\ &\leq \int \frac{e^{3k\tau}}{2}\|f_\tau\|^2_{L^2_x}\,d\tau + \cdots,\end{aligned}$$

where we can not bound the first integral on the right since we only have the bound of $\int e^{k\tau}\|f_\tau\|^2_{L^2_x}\,d\tau$ from the previous analysis. Therefore we shall apply the negative temporal weight to manipulate the estimate. Such estimates will control the growth of the higher order norms. Eventually, as long as the growth is not too large, we can use the elliptic structure of the equation to recover the spatial regularity and estimates of ξ.

On the other hand, the equation $(19)_2$ has the form

$$g_\tau - e^{2k\tau} g_{xx} = e^{k\tau}(f_\tau^2 + f_{x\tau}^2) + e^{2k\tau}(f_{xx} + f_x) + \cdots.$$

Then similar L^2-estimate yields

$$\frac{1}{2}\|g\|^2_{L^\infty_\tau L^2_x} + \int \frac{e^{2k\tau}}{2}\|g_x\|^2_{L^2_x}\,d\tau \leq \int \frac{e^{2k\tau}}{2}\|f\|^2_{L^2_x}\,d\tau + \cdots,$$

where we have no estimate on the first integral on the right. Instead, we only know $\|f\|_{L^\infty_\tau L^2_x}$ will be bounded. Therefore by choosing the temporal weight $e^{-\kappa\tau}$ with $\kappa > 0$ large enough, we can obtain the temporal weighted estimate. Such a structure exists for all the higher order estimates. We carefully track the temporal weights throughout our analysis and eventually perform elliptic estimates to establish the spatial regularity of ζ.

Unless stated otherwise, we adopt the following notations hereafter:

$$\int \cdot\, dx = \int_0^{R_0} \cdot\, dx, \quad \int \cdot\, d\tau = \int_0^{T} \cdot\, d\tau,$$

where $0 < R_0, T < \infty$ are positive constants and our solutions live in the space-time domain $(0, R_0) \times (0, T)$. Also, $\|\cdot\|_{L^p_\tau L^q_x}$ denotes the standard space-time Sobolev norm in the space-time variable $(x, \tau) \in (0, R_0) \times (0, T)$. For any fixed time $\tau \in (0, T)$, $\|\cdot\|_{L^q_x}$ is the standard Sobolev norm in the space variable $x \in (0, R_0)$. Here $p, q \in (1, \infty]$. $\sup_\tau$ will be used to represent $\sup_{0<\tau<T}$. $A \lesssim B$ is used to denote that there exists some positive constant C such that $A \leq C \cdot B$. $A \simeq B$ will mean $A \lesssim B$ and $B \lesssim A$. $C = C(\cdot)$ is a constant which is different from line to line and depends on the arguments. The following form of Hardy's inequality will be employed in this work:

Lemma 1 *(Hardy's inequality, [17]). Let k be a given real number, and g be a function satisfying $\int_0^1 s^k(g^2 + g'^2)\, ds < \infty$.*

1. *If $k > 1$, then we have*

$$\int_0^1 s^{k-2} g^2\, ds \leq C \int_0^1 s^k (g^2 + g'^2)\, ds.$$

2. *If $k < 1$, then g has a trace at $x = 0$ and*

$$\int_0^1 s^{k-2} (g - g(0))^2\, ds \leq C \int_0^1 s^k g'^2\, ds.$$

In this work, inspired by [32], let us define the relative entropy functional as

$$\mathfrak{H}(h) := \log(1 + h)^2 (1 + h + x h_x), \tag{21}$$

where $h : (0, R_0) \times (0, T) \mapsto \mathbb{R}$ is any smooth function. We will have the following estimates on the relative entropy.

Lemma 2. *For h satisfying,*

$$\max\{\|h\|_{L^\infty_\tau L^\infty_x}, \|x h_x\|_{L^\infty_\tau L^\infty_x}, \|h_t\|_{L^\infty_\tau L^\infty_x}, \|x h_{xt}\|_{L^\infty_\tau L^\infty_x}\} < \varepsilon, \tag{22}$$

with some $0 < \varepsilon < 1$ small enough, the following estimates of the function $\mathfrak{H}(h)$ hold:

$$\int (h_x^2 + x^2 h_{xx}^2)\, dx \lesssim \int \mathfrak{H}(h)_x^2\, dx, \tag{23}$$

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$$\int (h_{x\tau}^2 + x^2 h_{xx\tau}^2)\,dx \lesssim \int \mathfrak{H}(h)_{x\tau}^2\,dx + \varepsilon \int (h_x^2 + x^2 h_{xx}^2)\,dx. \tag{24}$$

Proof. Notice that

$$\mathfrak{H}(h)_x = \frac{1}{(1+h)(1+h+xh_x)}\big\{2(1+h+xh_x)h_x + (1+h)(2h_x + xh_{xx})\big\}$$
$$= \frac{4h_x + xh_{xx}}{(1+h)(1+h+xh_x)} + \frac{2(h+xh_x)h_x + h(2h_x + xh_{xx})}{(1+h)(1+h+xh_x)}.$$

Under the assumption (22), we have

$$\int \mathfrak{H}(h)_x^2\,dx \gtrsim \frac{1}{1+\varepsilon} \underbrace{\int (4h_x + xh_{xx})^2\,dx}_{\mathfrak{A}} - \varepsilon \int h_x^2 + x^2 h_{xx}^2\,dx.$$

$\mathfrak{A}$ can be calculated as follows:

$$\mathfrak{A} = 16 \int h_x^2\,dx + \int x^2 h_{xx}^2\,dx + 8 \int x h_x h_{xx}\,dx = 16 \int h_x^2\,dx + \int x^2 h_{xx}^2\,dx$$
$$-4 \int h_x^2\,dx + 4xh_x^2\big|_{x=R_0} \geq 12 \int h_x^2\,dx + \int x^2 h_{xx}^2\,dx.$$

Therefore, (23) follows easily. Similarly,

$$\mathfrak{H}(h)_{x\tau} = \frac{4h_{x\tau} + xh_{xx\tau}}{(1+h)(1+h+xh_x)} + (4h_x + xh_{xx})\Big\{\frac{1}{(1+h)(1+h+xh_x)}\Big\}_\tau$$
$$+ \Big\{\frac{2(h+xh_x)h_x + h(2h_x + xh_{xx})}{(1+h)(1+h+xh_x)}\Big\}_\tau,$$

from which we have

$$\int \mathfrak{H}(h)_{x\tau}^2\,dx \gtrsim \frac{1}{1+\varepsilon} \int (4h_{x\tau} + xh_{xx\tau})^2\,dx - \varepsilon \int (h_x^2 + x^2 h_{xx}^2 + h_{x\tau}^2$$
$$+ x^2 h_{xx\tau}^2)\,dx \gtrsim \frac{1}{1+\varepsilon} \int (12h_{x\tau}^2 + x^2 h_{xx\tau})\,dx - \varepsilon \int (h_x^2 + x^2 h_{xx}^2$$
$$+ h_{x\tau}^2 + x^2 h_{xx\tau}^2)\,dx.$$

This finishes the proof.

3. Stability of the linearly expanding homogeneous solutions

In this section, we will study the linearly expanding homogeneous solutions of the thermodynamic model, which are given in section 1.1. More precisely, we will study the stability of such solutions. We work with the perturbation variables (ξ, ζ) (defined in (18)) in the Lagrangian coordinates. The equations satisfied by (ξ, ζ) are given in (19) together with the boundary conditions in (20). The initial condition is given as $(\xi(\cdot, 0), \xi_\tau(\cdot, 0), \zeta(\cdot, 0)) = (\xi_0(\cdot), \xi_1(\cdot), \zeta_0(\cdot))$. To start with, we denote the point-wise bounds of the perturbation variables as

$$\begin{aligned} \hat{\omega} := \sup_\tau \{ & \left\| x\xi_x(\tau) \right\|_{L^\infty_x}, \left\| \xi(\tau) \right\|_{L^\infty_x}, \left\| x\xi_{x\tau}(\tau) \right\|_{L^\infty_x}, \left\| \xi_\tau(\tau) \right\|_{L^\infty_x}, \\ & \left\| (\zeta/\sigma)(\tau) \right\|_{L^\infty_x} \}, \end{aligned} \tag{25}$$

where $\sigma := R_0 - x$ denotes the distance to the boundary. We analyze the perturbation variables (ξ, ζ) in three steps.

First, by assuming that

$$\hat{\omega} < \hat{\varepsilon}_0$$

with $\hat{\varepsilon}_0$ sufficiently small, we perform estimates of the perturbation in section 3.1. We will establish the lower and higher order energy estimates with temporal weights. The temporal weights will track the growth of the energy functionals.

In section 3.2, we will explore some interior estimates. In particular, these will recover the regularity of the solutions near the coordinate center $x = 0$. Moreover, the estimates will establish the interchange of temporal weights between the temporal derivatives and the spatial derivatives.

Notice, we will make use of the large enough growing rate of the expanding solution to manipulate the extra forcing terms. Therefore, while establishing the temporal weighted estimates, we shall track carefully the constant a_1 during the proof.

To summing up the analysis, in section 3.3, we show that $\hat{\omega}$ can be indeed bounded by the total energy functional defined in (26) and therefore bounded by the initial energy (29), below. Then by choosing the initial energy small enough, we will end up with a globally bounded and small $\hat{\omega}$. Then a continuity argument will demonstrate the asymptotic stability of the linearly expanding homogeneous solutions.

In the following, the polynomial of $\hat{\omega}$ will be also denoted as $\hat{\omega}$. The total energy and dissipation functionals are given by

$$\begin{aligned} \hat{\mathcal{E}}(t) := {} & e^{(1+r_1)a_1\tau} \int x^4 \bar{\rho} \xi_\tau^2 \, dx + e^{l_1 a_1 \tau} \int x^2 \bar{\rho} \zeta^2 \, dx + \int x^4 \bar{\rho} \xi^2 \, dx + \int x^4 \xi_x^2 \, dx \\ & + \int x^2 \xi^2 \, dx + e^{(r_2-2)a_1\tau} \int x^4 \bar{\rho} \xi_{\tau\tau}^2 \, dx + e^{(l_2-2)a_1\tau} \int x^2 \bar{\rho} \zeta_\tau^2 \, dx \\ & + e^{((l_1+l_2)/2+1)a_1\tau} \int x^2 \zeta_x^2 \, dx + e^{((r_1+r_2)/2+3/2)a_1\tau} \int x^2 [(1+\xi) x \xi_{x\tau} \\ & - x\xi_x \xi_\tau]^2 \, dx + \int \chi (\xi^2 + x^2 \xi_x^2) \, dx + e^{(r_2+2)a_1\tau} \int \chi (x^2 \xi_{x\tau}^2 + \xi_\tau^2) \, dx \\ & + e^{(r_3-2)a_1\tau} \int \chi x^2 \bar{\rho} \xi_{\tau\tau}^2 \, dx + \int \xi_x^2 \, dx + \int x^2 \xi_{xx}^2 \, dx + \int \zeta_x^2 \, dx \\ & + e^{(r_3+2)a_1\tau} \int (\xi_{x\tau}^2 + x^2 \xi_{xx\tau}^2) \, dx + \int x^2 \zeta_{xx}^2 \, dx, \end{aligned} \tag{26}$$

$$\hat{\mathcal{D}}(t) := \int a_1 e^{(1+r_1)a_1\tau} \int x^4 \bar{\rho} \xi_\tau^2 \, dx \, d\tau + \int e^{(3+r_1)a_1\tau} \int x^2 [(1+\xi) x \xi_{x\tau}$$

$$
\begin{aligned}
&- x\xi_x\xi_\tau]^2\,dx\,d\tau + \int a_1 e^{l_1 a_1 \tau}\int x^2\bar\rho\zeta^2\,dx\,d\tau + \int e^{(2+l_1)a_1\tau}\int x^2\zeta_x^2\,dx\,d\tau \\
&+ \int a_1 e^{(r_2-2)a_1\tau}\int x^4\bar\rho\xi_{\tau\tau}^2\,dx\,d\tau + \int e^{r_2 a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau\tau} \\
&\quad - x\xi_x\xi_{\tau\tau}]^2\,dx\,d\tau + \int e^{l_2 a_1\tau}\int x^2\zeta_{x\tau}^2\,dx\,d\tau \\
&+ \int e^{(3+\mathfrak{r})a_1\tau}\int \chi\big(x^2\xi_{x\tau}^2+\xi_\tau^2\big)\,dx\,d\tau + \int e^{r_3 a_1\tau}\int \chi(x^2\xi_{x\tau\tau}^2+\xi_{\tau\tau}^2)\,dx\,d\tau, \qquad (27)
\end{aligned}
$$

where r_1, l_1 satisfies the following:

$$
\begin{aligned}
-1 < r_1 < 1,\ r_1 - 3 \le l_1 < -2,\\
r_2 \le r_1 - 1,\ l_2 + 2 \le 0,\ 0 \le r_2 - l_2 \le 2, \qquad (28)\\
-3 < \mathfrak{r} \le r_2 - 1,\ r_3 \le r_2 - 2,\ l_2 + 2 \ge 0.
\end{aligned}
$$

The initial energy is given by

$$
\begin{aligned}
\hat{\mathcal{E}}_0 = \hat{\mathcal{E}}_0(\xi,\zeta) &:= \int x^4\bar\rho\xi_1^2\,dx + \int x^2\bar\rho\zeta_0^2\,dx + \int x^4\bar\rho\xi_0^2\,dx \\
&+ \int x^4\xi_{0,x}^2\,dx + \int x^4\bar\rho\xi_2^2\,dx + \int x^2\bar\rho\zeta_1^2\,dx + \int \chi(\xi_0^2 + x^2\xi_{0,x}^2)\,dx \qquad (29)\\
&+ \int \chi x^2\bar\rho\xi_2^2\,dx + \int (\xi_{0,x}^2 + x^2\xi_{0,xx}^2)\,dx,
\end{aligned}
$$

where χ is the cut-off function defined as

$$
\chi(x) := \begin{cases} 1 & 0 \le x \le R_0/2, \\ 0 & 3R_0/4 \le x \le R_0, \end{cases}
$$

and $-8/R_0 \le \chi'(x) \le 0$ and ξ_2, ζ_1 are the initial data corresponding to $\xi_{\tau\tau}, \zeta_\tau$ defined by the equations in (19); that is

$$
\begin{aligned}
&\frac{1}{(1+\xi_0)^2}\big(a_0 x\bar\rho\xi_2 + a_0 a_1 x\bar\rho\xi_1\big) + \Big[\frac{K\bar\rho(\zeta_0+\bar\theta)}{(1+\xi_0)^2(1+\xi_0+x\xi_{0,x})}\Big]_x - \frac{(K\bar\rho\bar\theta)_x}{(1+\xi_0)^4} \\
&\quad = \frac{4\mu}{3}a_0^3\Big(\frac{\xi_1 + x\xi_{1,x}}{1+\xi_0+x\xi_{0,x}} + \frac{2\xi_1}{1+\xi_0}\Big)_x, \\
&3Kx^2\bar\rho\zeta_1 + K\frac{\bar\rho(\zeta_0+\bar\theta)(x^3(1+\xi_0)^2\xi_1)_x}{(1+\xi_0)^2(1+\xi_0+x\xi_{0,x})} - a_0^2\Big[\frac{(1+\xi_0)^2}{1+\xi_0+x\xi_{0,x}}x^2\zeta_{0,x} \\
&\quad + \Big(\frac{(1+\xi_0)^2}{1+\xi_0+x\xi_{0,x}} - 1\Big)x^2\bar\theta_x\Big]_x = \frac{4\mu}{3}a_0 x^2(1+\xi_0)^2(1+\xi_0+x\xi_{0,x}) \\
&\quad \times\Big[\frac{\xi_1 + x\xi_{1,x}}{1+\xi_0+x\xi_{0,x}} - \frac{\xi_1}{1+\xi_0}\Big]^2.
\end{aligned}
$$

Remark. *Here we write down an example for all the parameters, which satisfies the constraints in* (28),

$$r_1 = 1/2,\ r_2 = -1/2,\ l_1 = -5/2,\ l_2 = -2,\ \mathfrak{r} = -3/2, r_3 = -5/2. \tag{30}$$

3.1. Energy estimates

We start with the L^2-estimate of (19).

Lemma 3. *Considering a smooth solution* (ξ, ζ) *to* (19) *with the corresponding boundary conditions* (20)*, define the following functionals:*

$$\begin{aligned}
&\hat{\mathcal{E}}_{\xi,1} := e^{(1+r_1)a_1\tau} \int x^4 \bar{\rho} \xi_\tau^2 \, dx, \quad \hat{\mathcal{E}}_{\zeta,1} := e^{l_1 a_1 \tau} \int x^2 \bar{\rho} \zeta^2 \, dx, \\
&\hat{\mathcal{D}}_{\xi,1} = \hat{\mathcal{D}}_{\xi,11} + \hat{\mathcal{D}}_{\xi,12} := \int a_1 e^{(1+r_1)a_1\tau} \int x^4 \bar{\rho} \xi_\tau^2 \, dx \, d\tau \\
&\qquad + \int e^{(3+r_1)a_1\tau} \int x^2 [(1+\xi) x \xi_{x\tau} - x \xi_x \xi_\tau]^2 \, dx \, d\tau, \\
&\hat{\mathcal{D}}_{\zeta,1} = \hat{\mathcal{D}}_{\zeta,11} + \hat{\mathcal{D}}_{\zeta,12} := \int a_1 e^{l_1 a_1 \tau} \int x^2 \bar{\rho} \zeta^2 \, dx \, d\tau \\
&\qquad + \int e^{(2+l_1)a_1\tau} \int x^2 \zeta_x^2 \, dx \, d\tau,
\end{aligned} \tag{31}$$

where r_1, l_1 *are some constants satisfying the following constraints:*

$$-1 < r_1 < 1,\ l_1 - r_1 \geq -3,\ l_1 + 2 < 0. \tag{32}$$

Suppose that $a_1 > 0$ *is large enough and* $\hat{\omega} < \hat{\varepsilon}_0$ *is small enough. Then we have*

$$\hat{\mathcal{E}}_{\xi,1} + \hat{\mathcal{E}}_{\zeta,1} + \hat{\mathcal{D}}_{\xi,1} + \hat{\mathcal{D}}_{\zeta,1} \lesssim C(\hat{\varepsilon}_0, a_1, r_1, l_1) \hat{\mathcal{E}}_0. \tag{33}$$

Also,

$$\| x^2 \bar{\rho}^{1/2} \xi \|_{L_\tau^\infty L_x^2}^2 + \| x^2 \xi_x \|_{L_\tau^\infty L_x^2}^2 + \| x \xi \|_{L_\tau^\infty L_x^2}^2 \lesssim C(\hat{\varepsilon}_0, a_1, r_1, l_1) \hat{\mathcal{E}}_0. \tag{34}$$

Proof. Multiply $(19)_1$ with $\tilde{\alpha}^{r_1} x^3 (1+\xi)^2 \xi_\tau$ and integrate the resulting equation over the spatial variable. It yields the following:

$$\begin{aligned}
\frac{d}{d\tau} \hat{E}_{\xi,1} + \hat{D}_{\xi,1} &= \tilde{\alpha}^{r_1} \int K x^2 \bar{\rho} \zeta \cdot \Big(\frac{2\xi_\tau}{1+\xi} + \frac{\xi_\tau + x\xi_{x\tau}}{1+\xi+x\xi_x}\Big) \, dx \\
&\quad + \tilde{\alpha}^{r_1} \int K x^2 \bar{\rho} \bar{\theta} \Big(\frac{1}{1+\xi+x\xi_x} - \frac{1}{(1+\xi)^2}\Big) \cdot (3\xi_\tau + x\xi_{x\tau}) \, dx \\
&\quad + \tilde{\alpha}^{r_1} \int 2K x^2 \bar{\rho} \bar{\theta} \Big(\frac{x\xi_x}{(1+\xi)^3} + \frac{x\xi_x}{(1+\xi)(1+\xi+x\xi_x)}\Big) \cdot \xi_\tau \, dx \\
&=: \hat{I}_1 + \hat{I}_2 + \hat{I}_3,
\end{aligned} \tag{35}$$

where

$$\hat{E}_{\xi,1} := \frac{\tilde{\alpha}^{1+r_1}}{2} \int x^4 \bar{\rho} \xi_\tau^2 \, dx,$$
$$\hat{D}_{\xi,1} := \frac{(1-r_1)\tilde{\alpha}^{r_1}\tilde{\alpha}_\tau}{2} \int x^4 \bar{\rho} \xi_\tau^2 \, dx + \frac{4\mu}{3}\tilde{\alpha}^{3+r_1} \int \frac{x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2}{1+\xi+x\xi_x} \, dx.$$

In the meantime, multiply (19)$_2$ with $\tilde{\alpha}^{l_1}\zeta$ and integrate the resulting in the spatial variable. We have

$$\begin{aligned} \frac{d}{d\tau}\hat{E}_{\zeta,1} + \hat{D}_{\zeta,1} = {} & -\tilde{\alpha}^{2+l_1} \int x^2 \bar{\theta}_x \Big(\frac{(1+\xi)^2}{1+\xi+x\xi_x} - 1 \Big) \cdot \zeta_x \, dx \\ & - \tilde{\alpha}^{l_1} \int \frac{K\bar{\rho}(\zeta+\bar{\theta})[x^3(1+\xi)^2\xi_\tau]_x}{(1+\xi)^2(1+\xi+x\xi_x)} \cdot \zeta \, dx + \tilde{\alpha}^{1+l_1} \int x^2(1+\xi)^2 \\ & \times (1+\xi+x\xi_x)\hat{\mathfrak{F}}(\xi) \cdot \zeta \, dx =: \hat{I}_4 + \hat{I}_5 + \hat{I}_6, \end{aligned} \tag{36}$$

where

$$\hat{E}_{\zeta,1} := \frac{3K\tilde{\alpha}^{l_1}}{2} \int x^2 \bar{\rho} \zeta^2 \, dx,$$
$$\hat{D}_{\zeta,1} := -l_1 \frac{3K\tilde{\alpha}^{l_1-1}\tilde{\alpha}_\tau}{2} \int x^2 \bar{\rho}\zeta^2 \, dx + \tilde{\alpha}^{2+l_1} \int \frac{(1+\xi)^2}{1+\xi+x\xi_x} \cdot x^2 \zeta_x^2 \, dx.$$

By choosing $1 - r_1 > 0$, $-l_1 > 0$, the following estimates on the energy and dissipation functionals $\hat{E}_{\xi,1}$, $\hat{E}_{\zeta,1}$, $\hat{D}_{\xi,1}$, $\hat{D}_{\zeta,1}$ hold:

$$\hat{E}_{\xi,1} \gtrsim \tilde{\alpha}^{1+r_1} \int x^4 \bar{\rho} \xi_\tau^2 \, dx \gtrsim e^{(1+r_1)a_1\tau} \int x^4 \bar{\rho} \xi_\tau^2 \, dx,$$
$$\hat{E}_{\zeta,1} \gtrsim \tilde{\alpha}^{l_1} \int x^2 \bar{\rho} \zeta^2 \, dx \gtrsim e^{l_1 a_1 \tau} \int x^2 \bar{\rho}\zeta^2 \, dx,$$
$$\begin{aligned} \hat{D}_{\xi,1} & \gtrsim (1-r_1)\tilde{\alpha}^{r_1}\tilde{\alpha}_\tau \int x^4 \bar{\rho}\xi_\tau^2 \, dx + (1-\hat{\omega})\tilde{\alpha}^{3+r_1} \int x^2[(1+\xi)x\xi_{x\tau} \\ & \quad - x\xi_x\xi_\tau]^2 \, dx \gtrsim (1-r_1)a_1 e^{(r_1+1)a_1\tau} \int x^4 \bar{\rho}\xi_\tau^2 \, dx \\ & \quad + (1-\hat{\omega})e^{(3+r_1)a_1\tau} \int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2 \, dx, \end{aligned}$$
$$\begin{aligned} \hat{D}_{\zeta,1} & \gtrsim -l_1\tilde{\alpha}^{l_1-1}\tilde{\alpha}_\tau \int x^2\bar{\rho}\zeta^2 \, dx + (1-\hat{\omega})\tilde{\alpha}^{2+l_1} \int x^2 \zeta_x^2 \, dx \\ & \gtrsim -l_1 a_1 e^{l_1 a_1 \tau} \int x^2 \bar{\rho}\zeta^2 \, dx + (1-\hat{\omega})e^{(2+l_1)a_1\tau} \int x^2 \zeta_x^2 \, dx. \end{aligned}$$

Integration in the temporal variable of (35) and (36) yields the following:

$$\begin{aligned}&\hat{\mathcal{E}}_{\xi,1}+(1-r_1)\hat{\mathcal{D}}_{\xi,11}+\hat{\mathcal{D}}_{\xi,12}\lesssim\int\hat{I}_1\,d\tau+\int\hat{I}_2\,d\tau+\int\hat{I}_3\,d\tau+\hat{\mathcal{E}}_0,\\&\hat{\mathcal{E}}_{\zeta,1}+(-l_1)\hat{\mathcal{D}}_{\zeta,11}+\hat{\mathcal{D}}_{\zeta,12}\lesssim\int\hat{I}_4\,d\tau+\int\hat{I}_5\,d\tau+\int\hat{I}_6\,d\tau+\hat{\mathcal{E}}_0.\end{aligned}\tag{37}$$

What is left is to perform estimates on the right of (37). In order to do so, we shall need the following estimates. Hölder's inequality implies

$$\begin{aligned}&\Big(\int x^4\bar{\rho}\xi^2\,dx\Big)^{1/2}\frac{d}{d\tau}\Big(\int x^4\bar{\rho}\xi^2\,dx\Big)^{1/2}=\frac{1}{2}\frac{d}{d\tau}\int x^4\bar{\rho}\xi^2\,dx\\&=\int x^4\bar{\rho}\xi\xi_\tau\,dx\le\Big(\int x^4\bar{\rho}\xi^2\,dx\Big)^{1/2}\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx\Big)^{1/2},\\&\Big(\int x^2\Big(\frac{x\xi_x}{1+\xi}\Big)^2\,dx\Big)^{1/2}\frac{d}{d\tau}\Big(\int x^2\Big(\frac{x\xi_x}{1+\xi}\Big)^2\,dx\Big)^{1/2}=\frac{1}{2}\frac{d}{d\tau}\int x^2\Big(\frac{x\xi_x}{1+\xi}\Big)^2\,dx\\&\le\Big(\int x^2\Big(\frac{x\xi_x}{1+\xi}\Big)^2\,dx\Big)^{1/2}\Big(\int x^2\Big[\Big(\frac{x\xi_x}{1+\xi}\Big)_\tau\Big]^2\,dx\Big)^{1/2},\end{aligned}$$

from which one can derive

$$\begin{aligned}&\frac{d}{d\tau}\Big(\int x^4\bar{\rho}\xi^2\,dx\Big)^{1/2}\le\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx\Big)^{1/2},\\&\frac{d}{d\tau}\Big(\int x^2\Big(\frac{x\xi_x}{1+\xi}\Big)^2\,dx\Big)^{1/2}\le\Big(\int x^2\Big[\Big(\frac{x\xi_x}{1+\xi}\Big)_\tau\Big]^2\,dx\Big)^{1/2},\end{aligned}$$

and therefore

$$\begin{aligned}&\int x^4\bar{\rho}\xi^2\,dx\lesssim\int x^4\bar{\rho}\xi_0^2\,dx+\Big[\int\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx\Big)^{1/2}\,d\tau\Big]^2\\&\qquad\lesssim\int x^4\bar{\rho}\xi_0^2\,dx+\int a_1^{-1}e^{-(1+r_1)a_1\tau}\,d\tau\cdot\hat{\mathcal{D}}_{\xi,11},\\&\int x^4\xi_x^2\,dx\lesssim\int x^4\xi_{0,x}^2\,dx+\Big[\int\Big(\int x^2[(1+\xi)x\xi_{x\tau}-x\xi_x\xi_\tau]^2\,dx\Big)^{1/2}\,d\tau\Big]^2\\&\qquad\lesssim\int x^4\xi_{0,x}^2\,dx+\int e^{-(3+r_1)a_1\tau}\,d\tau\cdot\hat{\mathcal{D}}_{\xi,12}.\end{aligned}\tag{38}$$

Moreover, by applying Hardy's inequality, one can derive

$$\begin{aligned}\int x^2(\xi_\tau^2+x^2\xi_{x\tau}^2)\,dx\lesssim(1+\hat{\omega})\int x^2\xi_\tau^2\,dx\\+\int x^2((1+\xi)x\xi_{x\tau}-x\xi_x\xi_\tau)^2\,dx\end{aligned}$$

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$$\lesssim (1+\hat{\omega})\int x^4(R_0-x)^2\xi_\tau^2 + x^4\xi_{x\tau}^2\,dx$$
$$+\int x^2((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)^2\,dx$$
$$\lesssim (1+\hat{\omega})\int x^4(R_0-x)^2\xi_\tau^2\,dx + \hat{\omega}\int x^2\xi_\tau^2\,dx$$
$$+(1+\hat{\omega})\int x^2((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)^2\,dx,$$

which implies, provided that $\hat{\omega}$ is small enough,

$$\int x^2(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx \lesssim (1+\hat{\omega})\int x^4(R_0-x)^2\xi_\tau^2\,dx$$
$$+(1+\hat{\omega})\int x^2((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)^2\,dx$$

Repeating the above arguments, one can get

$$\int x^2\xi_\tau^2\,dx + \int x^4\xi_{x\tau}^2\,dx \lesssim (1+\hat{\omega})\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx + \int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx\Big), \quad (39)$$

where we have used the fact that $\bar{\rho}^{\frac{\varepsilon K}{1-\varepsilon K}}(x) \simeq (R_0-x)$ from (10). Similarly, Hardy's and the Poincaré inequalities imply

$$\int x^2\xi^2\,dx \lesssim \int x^4\bar{\rho}\xi^2\,dx + \int x^4\xi_x^2\,dx, \quad \int x^2\zeta^2\,dx \lesssim \int x^2\zeta_x^2\,dx.$$

In the meantime, the right of (37) can be estimated as follows:

$$\int \hat{I}_1\,d\tau \lesssim (1+\hat{\omega})\int e^{r_1a_1\tau}\int x^2\bar{\rho}|\zeta|(|\xi_\tau| + |x\xi_{x\tau}|)\,dx\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,1}$$
$$+ C_\epsilon(1+\hat{\omega})a_1^{-1}\sup_\tau e^{(r_1-l_1-3)a_1\tau}\cdot\hat{\mathcal{D}}_{\zeta,12} + C_\epsilon(1+\hat{\omega})a_1^{-1}$$
$$\times\sup_\tau e^{(r_1-l_1-3)a_1\tau}\cdot\hat{\mathcal{D}}_{\zeta,11}, \quad (40)$$

$$\int \hat{I}_2\,d\tau + \int \hat{I}_3\,d\tau \lesssim (1+\hat{\omega})\int e^{r_1a_1\tau}\int x^2\bar{\rho}\bar{\theta}(|\xi| + |x\xi_x|)(|\xi_\tau|$$
$$+|x\xi_{x\tau}|)\,dx\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,1} + C_\epsilon(1+\hat{\omega})\big(a_1^{-1}\int e^{(r_1-1)a_1\tau}\,d\tau$$
$$+\int e^{(r_1-3)a_1\tau}\,d\tau\big)\cdot\sup_\tau\Big\{\int x^4\bar{\rho}\xi^2\,dx + \int x^4\xi_x^2\,dx\Big\}, \quad (41)$$

$$\int \hat{I}_4\,d\tau \lesssim (1+\hat{\omega})\int e^{(2+l_1)a_1\tau}\int x^2|\bar{\theta}_x|(|\xi| + |x\xi_x|)|\zeta_x|\,dx \lesssim \epsilon\hat{\mathcal{D}}_{\zeta,12}$$
$$+ C_\epsilon(1+\hat{\omega})\int e^{(2+l_1)a_1\tau}\,d\tau\cdot\sup_\tau\Big\{\int x^4\bar{\rho}\xi^2\,dx + \int x^4\xi_x^2\,dx\Big\}, \quad (42)$$

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$$\int \hat{I}_5 \, d\tau \lesssim (1+\hat{\omega}) \int e^{l_1 a_1 \tau} \int x^2 \bar{\rho}(|\zeta| + |\bar{\theta}|)(|\xi_\tau| + |x\xi_{x\tau}|) \cdot |\zeta| \, dx \, d\tau$$
$$\lesssim \epsilon \hat{\mathcal{D}}_{\zeta,1} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \sup_\tau e^{(l_1 - r_1 - 3)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11}$$
$$+ C_\epsilon (1+\hat{\omega}) a_1^{-1} \sup_\tau e^{(l_1 - r_1 - 3)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,12}, \tag{43}$$

$$\int \hat{I}_6 \, d\tau \lesssim (1+\hat{\omega}) \int e^{(1+l_1)a_1\tau} \int x^2 (\xi_\tau^2 + x^2 \xi_{x\tau}^2) \cdot \zeta \, dx \, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\zeta,12}$$
$$+ C_\epsilon \hat{\omega} a_1^{-1} \sup_\tau e^{(l_1 - r_1 - 1)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11} + C_\epsilon \hat{\omega} \sup_\tau e^{(l_1 - r_1 - 3)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,12}. \tag{44}$$

Therefore, for r_1, l_1 satisfying (32), (37) and (38) together with the inequalities from (40) to (44) imply the following energy estimates:

$$\hat{\mathcal{E}}_{\xi,1} + \hat{\mathcal{D}}_{\xi,1} \lesssim \epsilon \hat{\mathcal{D}}_{\xi,1} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \hat{\mathcal{D}}_{\zeta,1}$$
$$+ C_\epsilon (1+\hat{\omega})(a_1^{-2} + a_1^{-1}) \times (\hat{\mathcal{E}}_0 + a_1^{-2} \hat{\mathcal{D}}_{\xi,11} + a_1^{-1} \hat{\mathcal{D}}_{\xi,12}) + \hat{\mathcal{E}}_0,$$
$$\hat{\mathcal{E}}_{\zeta,1} + \hat{\mathcal{D}}_{\zeta,1} \lesssim \epsilon \hat{\mathcal{D}}_{\zeta,1} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \times (\hat{\mathcal{E}}_0 + a_1^{-2} \hat{\mathcal{D}}_{\xi,11} + a_1^{-1} \hat{\mathcal{D}}_{\xi,12})$$
$$+ C_\epsilon (1+\hat{\omega})(a_1^{-1} + \hat{\omega}) \hat{\mathcal{D}}_{\xi,1} + \hat{\mathcal{E}}_0.$$

Consequently, after choosing $\epsilon > 0$ small enough, for sufficiently large $a_1 > 0$ and sufficiently small $\hat{\varepsilon}_0 > 0$, the above estimates yield (33). In particular, (38) and (39) imply (34).

Next, we shall write down the corresponding temporal derivative version of system (19). Direct calculation of $\partial_\tau [\tilde{\alpha}^{-3} (1+\xi)^2 \cdot (19)_1]$ and $\partial_\tau [\tilde{\alpha}^{-2} \cdot (19)_2]$ yields the following system:

$$\begin{cases} \tilde{\alpha}^{-2} x \bar{\rho} \xi_{\tau\tau\tau} - \tilde{\alpha}^{-3} \tilde{\alpha}_\tau x \bar{\rho} \xi_{\tau\tau} + (\tilde{\alpha}^{-3} \tilde{\alpha}_{\tau\tau} - 3\tilde{\alpha}^{-4} \tilde{\alpha}_\tau^2) x \bar{\rho} \xi_\tau \\ \qquad + \Big\{ \tilde{\alpha}^{-3} (1+\xi)^2 \Big\{ \Big[\dfrac{K \bar{\rho}(\zeta + \bar{\theta})}{(1+\xi)^2 (1+\xi+x\xi_x)} \Big]_x - \dfrac{(K\bar{\rho}\bar{\theta})_x}{(1+\xi)^4} \Big\} \Big\}_\tau \\ \qquad = (1+\xi)^2 \Big(\hat{\mathfrak{B}}_{x\tau} + 4\mu \Big(\dfrac{\xi_\tau}{1+\xi} \Big)_{x\tau} \Big) + 2(1+\xi)\xi_\tau \Big(\hat{\mathfrak{B}}_x + 4\mu \Big(\dfrac{\xi_\tau}{1+\xi} \Big)_x \Big), \\ 3K\tilde{\alpha}^{-2} x^2 \bar{\rho} \zeta_{\tau\tau} - 6K\tilde{\alpha}^{-3} \tilde{\alpha}_\tau x^2 \bar{\rho} \zeta_\tau + \Big[K\tilde{\alpha}^{-2} \dfrac{\bar{\rho}(\zeta+\bar{\theta})[x^3(1+\xi)^2 \xi_\tau]_x}{(1+\xi)^2(1+\xi+x\xi_x)} \Big]_\tau \\ \qquad - \Big[\dfrac{(1+\xi)^2}{1+\xi+x\xi_x} x^2 \zeta_x + \Big(\dfrac{(1+\xi)^2}{1+\xi+x\xi_x} - 1 \Big) x^2 \bar{\theta}_x \Big]_{x\tau} \\ \qquad = [\tilde{\alpha}^{-1} x^2 (1+\xi)^2 (1+\xi+x\xi_x) \cdot \hat{\mathfrak{F}}(\xi)]_\tau, \end{cases} \tag{45}$$

with

$$\hat{\mathfrak{B}}_\tau(R_0, \tau) = 0, \;\; \zeta_\tau(R_0, \tau) = 0.$$

Similarly as before, we shall perform the L^2-based energy estimate in the following.

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Lemma 4. *Under the assumptions as in Lemma 3, define the functionals:*

$$\begin{aligned}
&\hat{\mathcal{E}}_{\xi,2} := e^{(r_2-2)a_1\tau}\int x^4\bar{\rho}\xi_{\tau\tau}^2\,dx,\quad \hat{\mathcal{E}}_{\zeta,2} := e^{(l_2-2)a_1\tau}\int x^2\bar{\rho}\zeta_\tau^2\,dx,\\
&\hat{\mathcal{D}}_{\xi,2} = \hat{\mathcal{D}}_{\xi,21}+\hat{\mathcal{D}}_{\xi,22} := \int a_1 e^{(r_2-2)a_1\tau}\int x^4\bar{\rho}\xi_{\tau\tau}^2\,dx\,d\tau\\
&\quad + \int e^{r_2a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau\tau}-x\xi_x\xi_{\tau\tau}]^2\,dx\,d\tau,\\
&\hat{\mathcal{D}}_{\zeta,2} := \int e^{l_2a_1\tau}\int x^2\zeta_{x\tau}^2\,dx\,d\tau,
\end{aligned}\tag{46}$$

where r_2, l_2 satisfy the constraints:

$$r_2 \le r_1 - 1,\ l_2+2\le 0,\ 0\le r_2-l_2\le 2. \tag{47}$$

For a_1 large enough and $\hat{\varepsilon}_0$ small enough, we have the following estimates:

$$\hat{\mathcal{E}}_{\xi,2}+\hat{\mathcal{E}}_{\zeta,2}+\hat{\mathcal{D}}_{\xi,2}+\hat{\mathcal{D}}_{\zeta,2}\lesssim C(\hat{\varepsilon}_0,a_1,r_1,l_1,r_2,l_2)\hat{\mathcal{E}}_0. \tag{48}$$

Proof. Multiply $(45)_1$ with $\tilde{\alpha}^{r_2}x^3\xi_{\tau\tau}$ and integrate the resulting in the spatial variable. It holds,

$$\frac{d}{d\tau}\hat{E}_{\xi,2}+\hat{D}_{\xi,2} = \hat{L}_1+\hat{L}_2+\hat{L}_3+\hat{L}_4, \tag{49}$$

where

$$\begin{aligned}
&\hat{E}_{\xi,2} := \frac{\tilde{\alpha}^{r_2-2}}{2}\int x^4\bar{\rho}\xi_{\tau\tau}^2\,dx,\\
&\hat{D}_{\xi,2} := -\frac{r_2}{2}\tilde{\alpha}^{r_2-3}\tilde{\alpha}_\tau\int x^4\bar{\rho}\xi_{\tau\tau}^2\,dx + \frac{4}{3}\mu\tilde{\alpha}^{r_2}\int\frac{x^2[(1+\xi)x\xi_{x\tau\tau}-x\xi_x\xi_{\tau\tau}]^2}{1+\xi+x\xi_x}\,dx,\\
&\hat{L}_1 := -(\tilde{\alpha}^{r_2-3}\tilde{\alpha}_{\tau\tau}-3\tilde{\alpha}^{r_2-4}\tilde{\alpha}_\tau^2)\int x^4\bar{\rho}\xi_\tau\xi_{\tau\tau}\,dx,\\
&\hat{L}_2 := -\frac{4}{3}\mu\tilde{\alpha}^{r_2}\int x^2\left\{-\frac{(1+\xi)^2(\xi_\tau+x\xi_{x\tau})^2(\xi_{\tau\tau}+x\xi_{x\tau\tau})}{(1+\xi+x\xi_x)^2}\right.\\
&\qquad\left.+2\frac{(1+\xi)\xi_\tau(\xi_\tau+x\xi_{x\tau})(\xi_{\tau\tau}+x\xi_{x\tau\tau})}{1+\xi+x\xi_x}-\xi_\tau^2(\xi_{\tau\tau}+x\xi_{x\tau\tau})\right\}dx,\\
&\hat{L}_3 := -3\tilde{\alpha}^{r_2-4}\tilde{\alpha}_\tau\int x^2\left\{\frac{2K\bar{\rho}\zeta x\xi_x\xi_{\tau\tau}}{(1+\xi)(1+\xi+x\xi_x)}+\frac{2K\bar{\rho}\bar{\theta}x\xi_x\xi_{\tau\tau}}{(1+\xi)(1+\xi+x\xi_x)}\right.\\
&\qquad+\frac{2K\bar{\rho}\bar{\theta}x\xi_x\xi_{\tau\tau}}{(1+\xi)^3}+\frac{K\bar{\rho}\zeta(3\xi_{\tau\tau}+x\xi_{x\tau\tau})}{1+\xi+x\xi_x}+\frac{K\bar{\rho}\bar{\theta}(3\xi_{\tau\tau}+x\xi_{x\tau\tau})}{1+\xi+x\xi_x}\\
&\qquad\left.-\frac{K\bar{\rho}\bar{\theta}(3\xi_{\tau\tau}+x\xi_{x\tau\tau})}{(1+\xi)^2}\right\}dx,
\end{aligned}$$

$$\hat{L}_4 := \tilde{\alpha}^{r_2-3} \int x^2 \Big\{ (3\xi_{\tau\tau} + x\xi_{x\tau\tau}) \Big[\frac{K\bar{\rho}\zeta}{1+\xi+x\xi_x} + \frac{K\bar{\rho}\bar{\theta}}{1+\xi+x\xi_x} - \frac{K\bar{\rho}\bar{\theta}}{(1+\xi)^2} \Big]_\tau \\ + \xi_{\tau\tau} \Big[\frac{2K\bar{\rho}\zeta x\xi_x}{(1+\xi)(1+\xi+x\xi_x)} + \frac{2K\bar{\rho}\bar{\theta}x\xi_x}{(1+\xi)(1+\xi+x\xi_x)} + \frac{2K\bar{\rho}\bar{\theta}x\xi_x}{(1+\xi)^3} \Big]_\tau \Big\} dx.$$

In the meantime, multiply (45)$_2$ with $\tilde{\alpha}^{l_2}\zeta_\tau$ and integrate the resulting in the spatial variable. It holds,

$$\frac{d}{d\tau}\hat{E}_{\zeta,2} + \hat{D}_{\zeta,2} = \hat{L}_5 + \hat{L}_6 + \hat{L}_7 + \hat{L}_8, \tag{50}$$

where

$$\hat{E}_{\zeta,2} := \frac{3K}{2}\tilde{\alpha}^{l_2-2} \int x^2 \bar{\rho}\zeta_\tau^2 \, dx,$$

$$\hat{D}_{\zeta,2} := -3K(l_2/2+1)\tilde{\alpha}^{l_2-3}\tilde{\alpha}_\tau \int x^2\bar{\rho}\zeta_\tau^2\, dx + \tilde{\alpha}^{l_2} \int \frac{(1+\xi)^2}{1+\xi+x\xi_x} \cdot x^2 \zeta_{x\tau}^2 \, dx,$$

$$\hat{L}_5 := -\tilde{\alpha}^{l_2} \int x^2 \zeta_{x\tau} \Big\{ \zeta_x \cdot \Big[\frac{(1+\xi)^2}{1+\xi+x\xi_x} \Big]_\tau + \Big(\frac{(1+\xi)^2}{1+\xi+x\xi_x} - 1 \Big)_\tau \bar{\theta}_x \Big\} dx,$$

$$\hat{L}_6 := 2K\tilde{\alpha}^{l_2-3}\tilde{\alpha}_\tau \int \zeta_\tau \cdot \frac{\bar{\rho}(\zeta+\bar{\theta})[x^3(1+\xi)^2\xi_\tau]_x}{(1+\xi)^2(1+\xi+x\xi_x)} \, dx,$$

$$\hat{L}_7 := -K\tilde{\alpha}^{l_2-2} \int \zeta_\tau \cdot \Big\{ \frac{\bar{\rho}(\zeta+\bar{\theta})[x^3(1+\xi)^2\xi_\tau]_x}{(1+\xi)^2(1+\xi+x\xi_x)} \Big\}_\tau \, dx,$$

$$\hat{L}_8 := -\tilde{\alpha}^{l_2-2}\tilde{\alpha}_\tau \int x^2(1+\xi)^2(1+\xi+x\xi_x)\hat{\mathfrak{F}}(\xi)\zeta_\tau \, dx \\ + \tilde{\alpha}^{l_2-1} \int x^2[(1+\xi)^2(1+\xi+x\xi_x)]_\tau \hat{\mathfrak{F}}(\xi) \cdot \zeta_\tau \, dx \\ + \tilde{\alpha}^{l_2-1} \int x^2(1+\xi)^2(1+\xi+x\xi_x)[\hat{\mathfrak{F}}(\xi)]_\tau \cdot \zeta_\tau \, dx.$$

Integration in the temporal variable of (49) and (50) yields the following inequalities, provided that $r_2 < 0, l_2 + 2 \leq 0$,

$$\begin{aligned} &\hat{\mathcal{E}}_{\xi,2} + (-r_2)\hat{\mathcal{D}}_{\xi,21} + (1-\hat{\omega})\hat{\mathcal{D}}_{\xi,22} \lesssim \int \hat{L}_1 \, d\tau + \int \hat{L}_2 \, d\tau + \int \hat{L}_3 \, d\tau \\ &\quad + \int \hat{L}_4 \, d\tau + \hat{\mathcal{E}}_0, \\ &\hat{\mathcal{E}}_{\zeta,2} + (-l_2/2-1) \int a_1 e^{(l_2-2)a_1\tau} \int x^2\bar{\rho}\zeta_\tau^2 \, dx\, d\tau + (1-\hat{\omega})\hat{\mathcal{D}}_{\zeta,2} \\ &\quad \lesssim \int \hat{L}_5 \, d\tau + \int \hat{L}_6 \, d\tau + \int \hat{L}_7 \, d\tau + \int \hat{L}_8 \, d\tau + \hat{\mathcal{E}}_0. \end{aligned} \tag{51}$$

Similarly as before, we establish the following estimates concerning the right of (51):

$$\int \hat{L}_1\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\xi,21} + C_\epsilon a_1^2 \sup_\tau e^{(r_2-r_1-3)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11},$$

$$\int \hat{L}_2\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\xi,2} + C_\epsilon \hat{\omega}(a_1^{-2} \sup_\tau e^{(r_2-r_1+1)a_1\tau} + a_1^{-1} \sup_\tau e^{(r_2-r_1-1)a_1\tau})$$
$$\times \hat{\mathcal{D}}_{\xi,11} + C_\epsilon \hat{\omega}(a_1^{-1} \sup_\tau e^{(r_2-r_1-1)a_1\tau} + \sup_\tau e^{(r_2-r_1-3)a_1\tau}) \cdot \hat{\mathcal{D}}_{\xi,12},$$

$$\int \hat{L}_3\, d\tau \lesssim (1+\hat{\omega}) \int a_1 e^{(r_2-3)a_1\tau} \int x^2 \bar{\rho}(|\zeta| + (|\xi| + |x\xi_x|))(|\xi_{\tau\tau}|$$
$$+ |x\xi_{x\tau\tau}|)\, dx\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\xi,2} + C_\epsilon (1+\hat{\omega})(a_1 \sup_\tau e^{(r_2-l_1-6)a_1\tau}$$
$$+ a_1^2 \sup_\tau e^{(r_2-l_1-8)a_1\tau}) \cdot \hat{\mathcal{D}}_{\zeta,12} + C_\epsilon (1+\hat{\omega}) \int (a_1 e^{(r_2-4)a_1\tau}$$
$$+ a_1^2 e^{(r_2-6)a_1\tau})\, d\tau \times (\int x^4 \bar{\rho}\xi^2\, dx + \int x^4 \xi_x^2\, dx),$$

$$\int \hat{L}_4\, d\tau \lesssim (1+\hat{\omega}) \int e^{(r_2-3)a_1\tau} \int x^2 \bar{\rho}(|\zeta_\tau| + |\xi_\tau| + |x\xi_{x\tau})(|\xi_{\tau\tau}|$$
$$+ |x\xi_{x\tau\tau}|)\, dx\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\xi,2} + C_\epsilon (1+\hat{\omega}) \int (a_1^{-1} e^{(r_2-l_2-2)a_1\tau} + e^{(r_2-l_2-4)a_1\tau})\, d\tau$$
$$\times \hat{\mathcal{E}}_{\zeta,2} + C_\epsilon (1+\hat{\omega}) \sup_\tau (a_1^{-2} e^{(r_2-r_1-5)a_1\tau} + a_1^{-1} e^{(r_2-r_1-7)a_1\tau}) \cdot \hat{\mathcal{D}}_{\xi,11}$$
$$+ C_\epsilon (1+\hat{\omega})(a_1^{-1} \sup_\tau e^{(r_2-r_1-7)a_1\tau} + \sup_\tau e^{(r_2-r_1-9)a_1\tau}) \cdot \hat{\mathcal{D}}_{\xi,12},$$

$$\int \hat{L}_5\, d\tau \lesssim (1+\hat{\omega}) \int e^{l_2 a_1\tau} \int x^2 |\zeta_{x\tau}| \cdot (|\zeta_x| + |\bar{\theta}_x|)(|\xi_\tau| + |x\xi_{x\tau}|)\, dx\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\zeta,2}$$
$$+ C_\epsilon \hat{\omega} \sup_\tau e^{(l_2-l_1-2)a_1\tau} \cdot \hat{\mathcal{D}}_{\zeta,12} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \sup_\tau e^{(l_2-r_1-1)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11}$$
$$+ C_\epsilon (1+\hat{\omega}) \sup_\tau e^{(l_2-r_1-3)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,12},$$

$$\int \hat{L}_6\, d\tau \lesssim (1+\hat{\omega}) \int a_1 e^{(l_2-2)a_1\tau} \int x^2 |\zeta_\tau|(|\zeta| + |\bar{\theta}|) \cdot (|\xi_\tau| + |x\xi_{x\tau}|)\, dx\, d\tau$$
$$\lesssim \epsilon \hat{\mathcal{D}}_{\zeta,2} + C_\epsilon (1+\hat{\omega})\hat{\omega} a_1^2 \sup_\tau e^{(l_2-l_1-6)a_1\tau} \cdot \hat{\mathcal{D}}_{\zeta,12} + C_\epsilon (1+\hat{\omega}) a_1$$
$$\times \sup_\tau e^{(l_2-r_1-5)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11} + C_\epsilon (1+\hat{\omega}) a_1^2 \sup_\tau e^{(l_2-r_1-7)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,12},$$

$$\int \hat{L}_7\, d\tau \lesssim (1+\hat{\omega}) \int e^{(l_2-2)a_1\tau} \int x^2 \bar{\rho} |\zeta_\tau| \Big[|\xi_\tau|^2 + |x\xi_{x\tau}|^2 + |\xi_{\tau\tau}| + |x\xi_{x\tau\tau}|$$
$$+ (|\xi_\tau| + |\xi_{x\tau}|)|\zeta_\tau| \Big]\, dx\, d\tau \lesssim \epsilon \hat{\mathcal{D}}_{\zeta,2} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \sup_\tau e^{(l_2-r_1-5)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,11}$$
$$+ C_\epsilon (1+\hat{\omega}) \sup_\tau e^{(l_2-r_1-7)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,12} + C_\epsilon (1+\hat{\omega}) a_1^{-1} \sup_\tau e^{(l_2-r_2-2)a_1\tau}$$
$$\times \hat{\mathcal{D}}_{\xi,21} + C_\epsilon (1+\hat{\omega}) \sup_\tau e^{(l_2-r_2-4)a_1\tau} \cdot \hat{\mathcal{D}}_{\xi,22} + C_\epsilon \hat{\omega} \int e^{-2a_1\tau}\, d\tau \cdot \hat{\mathcal{E}}_{\zeta,2},$$

$$\int \hat{L}_8\,d\tau \lesssim (1+\hat{\omega})\int e^{(l_2-1)a_1\tau}\int x^2|\zeta_\tau|\Big[a_1(|\xi_\tau|^2+|x\xi_{x\tau}|^2)+(|\xi_\tau|^3+|x\xi_{x\tau}|^3)$$
$$+(|\xi_\tau|+|x\xi_{x\tau}|)\cdot(|\xi_{\tau\tau}|+|x\xi_{x\tau\tau}|)\Big]\,dx\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\zeta,2}+C_\epsilon\hat{\omega}(a_1+a_1^{-1})$$
$$\times \sup_\tau e^{(l_2-r_1-3)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,11}+C_\epsilon\hat{\omega}(a_1^2+1)\sup_\tau e^{(l_2-r_1-5)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,12}$$
$$+C_\epsilon\hat{\omega}a_1^{-1}\sup_\tau e^{(l_2-r_2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,21}+C_\epsilon\hat{\omega}\sup_\tau e^{(l_2-r_2-2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,22}.$$

Now we shall choose r_2, l_2 satisfying (47). Then the above estimates yield

$$\hat{\mathcal{E}}_{\xi,2}+(1-\hat{\omega})\hat{\mathcal{D}}_{\xi,2}\lesssim \epsilon\hat{\mathcal{D}}_{\xi,2}+C_\epsilon(1+\hat{\omega})(a_1^{-2}+a_1^{-1})\hat{\mathcal{E}}_{\zeta,2}+C_\epsilon(1+\hat{\omega})(a_1$$
$$+a_1^2)\hat{\mathcal{D}}_{\zeta,1}+C_\epsilon(a_1^2+a_1^{-2}+1)\hat{\mathcal{D}}_{\xi,1}+(1+\hat{\omega})(a_1+1)(\int x^4\bar{\rho}\xi^2\,dx$$
$$+\int x^4\xi_x^2\,dx)+\hat{\mathcal{E}}_0,$$
$$\hat{\mathcal{E}}_{\zeta,2}+(1-\hat{\omega})\hat{\mathcal{D}}_{\zeta,2}\lesssim \epsilon\hat{\mathcal{D}}_{\zeta,2}+C_\omega(1+\hat{\omega})(1+a_1^{-1})\hat{\mathcal{D}}_{\xi,2}+C_\epsilon(1+\hat{\omega})(1$$
$$+a_1^{-1}+a_1^2)\hat{\mathcal{D}}_{\xi,1}+C_\epsilon(\hat{\omega}+\hat{\omega}a_1^2)\hat{\mathcal{D}}_{\zeta,1}+\hat{\omega}a_1^{-1}\hat{\mathcal{E}}_{\zeta,2}+\hat{\mathcal{E}}_0.$$

Therefore, after choosing $\epsilon>0$ small enough, for sufficiently small $\hat{\varepsilon}_0>0$ and sufficiently large a_1, these estimates imply (48), where we have applied (33) and (34).

3.2. Interior estimates

In the following, without loss of generality, we assume $0<\hat{\omega}<\hat{\varepsilon}_0<1$. We shall perform some interior estimates in this section. Multiply $(19)_1$ with $\chi x\xi_\tau$ and integrate the resulting equation in the spatial variable. After integration by parts, it holds,

$$\frac{4\mu}{3}\tilde{\alpha}^3\int\chi\left\{\frac{(\xi_\tau+x\xi_{x\tau})^2}{1+\xi+x\xi_x}+\frac{(1+\xi+x\xi_x)\xi_\tau^2}{(1+\xi)^2}\right\}dx=\hat{J}_1+\hat{J}_2+\hat{J}_3+\hat{J}_4, \tag{52}$$

where

$$\hat{J}_1:=-\frac{8\mu}{3}\tilde{\alpha}^3\int\chi'\frac{x\xi_\tau^2}{1+\xi}\,dx-\tilde{\alpha}^3\int\chi' x\hat{\mathfrak{B}}\xi_\tau\,dx$$
$$+\int\chi' K\bar{\rho}x\xi_\tau\Big(\frac{\zeta+\bar{\theta}}{(1+\xi)^2(1+\xi+x\xi_x)}-\frac{\bar{\theta}}{(1+\xi)^4}\Big)\,dx,$$
$$\hat{J}_2:=-\tilde{\alpha}\int\chi\frac{x^2\bar{\rho}\xi_{\tau\tau}\xi_\tau}{(1+\xi)^2}\,dx-\tilde{\alpha}_\tau\int\chi\frac{x^2\bar{\rho}\xi_\tau^2}{(1+\xi)^2}\,dx,$$
$$\hat{J}_3:=\int\chi K\bar{\rho}\Big(\frac{(\zeta+\bar{\theta})(\xi_\tau+x\xi_{x\tau})}{(1+\xi)^2(1+\xi+x\xi_x)}-\frac{\bar{\theta}(\xi_\tau+x\xi_{x\tau})}{(1+\xi)^4}+\frac{4\bar{\theta}x\xi_x\xi_\tau}{(1+\xi)^5}\Big)\,dx.$$

We have the following estimates on $\hat{J}_k$'s on the right of (52):

$$
\begin{aligned}
\hat{J}_1 \lesssim\, & e^{3a_1\tau}\int x^4\bar{\rho}\xi_\tau^2\,dx + e^{3a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx \\
& + \Big(\int x^4\bar{\rho}\xi_\tau^2\,dx\Big)^{1/2}\Big\{\Big(\int x^2\bar{\rho}\zeta^2\,dx\Big)^{1/2} + \Big(\int x^4\bar{\rho}\xi^2\,dx\Big)^{1/2} + \Big(\int x^4\xi_x^2\,dx\Big)^{1/2}\Big\}, \\
\hat{J}_2 \lesssim\, & e^{a_1\tau}\Big\{\Big(\int x^4\bar{\rho}\xi_{\tau\tau}^2\,dx\Big)^{1/2} + a_1\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx\Big)^{1/2}\Big\} \times \Big(\int \chi\xi_\tau^2\,dx\Big)^{1/2}, \\
\hat{J}_3 \lesssim\, & \Big\{\Big(\int \chi\zeta^2\,dx\Big)^{1/2} + \Big(\int \chi((\xi + x\xi_x)^2 + \xi^2)\,dx\Big)^{1/2}\Big\} \\
& \times \Big(\int \chi((\xi_\tau + x\xi_{x\tau})^2 + \xi_\tau^2)\,dx\Big)^{1/2}.
\end{aligned}
$$

We will derive the following lemma from (52):

Lemma 5. *Under the same assumptions as in Lemma 4, we have the following inequalities:*

$$
\int e^{(3+\mathfrak{r})a_1\tau}\int \chi((\xi_\tau + x\xi_{x\tau})^2 + \xi_\tau^2)\,dx\,d\tau \le C(\hat{\varepsilon}_0, a_1, r_1, r_2, l_1, l_2, \mathfrak{r})\hat{\mathcal{E}}_0, \tag{53}
$$

$$
\int \chi(\xi^2 + x^2\xi_x^2)\,dx \le C(\hat{\varepsilon}_0, a_1, r_1, r_2, l_1, l_2, \mathfrak{r})\hat{\mathcal{E}}_0, \tag{54}
$$

where

$$
-3 < \mathfrak{r} \le r_2 - 1 < 0. \tag{55}
$$

In addition, the following estimates hold:

$$
\begin{aligned}
\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx \le C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2)e^{\mathfrak{d}_1 a_1\tau}\cdot\hat{\mathcal{E}}_0, \\
\int x^2\zeta_x^2\,dx \le C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2)e^{\mathfrak{d}_2 a_1\tau}\cdot\hat{\mathcal{E}}_0,
\end{aligned} \tag{56}
$$

with

$$
\mathfrak{d}_1 := -(r_1 + r_2)/2 - 3/2 < 0, \quad \mathfrak{d}_2 := -(l_1 + l_2)/2 - 1 > 0. \tag{57}
$$

Moreover, as a corollary, we have

$$
\int \chi(x^2\xi_{x\tau}^2 + \xi_\tau^2)\,dx \lesssim C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2)e^{\mathfrak{d}_3 a_1\tau}\cdot\hat{\mathcal{E}}_0, \tag{58}
$$

with

$$
\mathfrak{d}_3 := -r_2 - 2 < 0. \tag{59}
$$

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Proof. Multiply (52) with $e^{\mathfrak{r}a_1\tau}$ and apply Cauchy's inequality to the resulting equation. It follows,

$$\begin{aligned}
&(1-\epsilon)e^{(3+\mathfrak{r})a_1\tau}\int \chi((\xi_\tau + x\xi_{x\tau})^2 + \xi_\tau^2)\,dx \lesssim C_\epsilon\Big\{e^{(\mathfrak{r}-1)a_1\tau}\int x^4\bar\rho\xi_{\tau\tau}^2\,dx \\
&\quad + a_1^2 e^{(\mathfrak{r}-1)a_1\tau}\int x^4\bar\rho\xi_\tau^2\,dx + e^{(\mathfrak{r}-3)a_1\tau}\int \chi((\xi+x\xi_x)^2+\xi^2)\,dx \\
&\quad + e^{(\mathfrak{r}-3)a_1\tau}\int \chi\zeta^2\,dx\Big\} + e^{(3+\mathfrak{r})a_1\tau}\int x^4\bar\rho\xi_\tau^2\,dx \\
&\quad + e^{(3+\mathfrak{r})a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx + e^{\mathfrak{r}a_1\tau}\big(\int x^4\bar\rho\xi_\tau^2\,dx\big)^{1/2} \\
&\quad \times\Big\{\big(\int x^2\bar\rho\zeta^2\,dx\big)^{1/2} + \big(\int x^4\bar\rho\xi^2\,dx\big)^{1/2} + \big(\int x^4\xi_x^2\,dx\big)^{1/2}\Big\}.
\end{aligned} \tag{60}$$

Similarly to (38), one has

$$\big(\int \chi((\xi+x\xi_x)^2+\xi^2)\,dx\big)^{1/2} \lesssim \hat{\mathcal{E}}_0^{1/2} + \int\big(\int \chi((\xi_\tau+x\xi_{x\tau})^2+\xi_\tau^2)\,dx\big)^{1/2}\,d\tau.$$

Therefore, we have the following inequality

$$\begin{aligned}
&e^{(3+\mathfrak{r})a_1\tau}\int \chi((\xi_\tau+x\xi_{x\tau})^2+\xi_\tau^2)\,dx \lesssim e^{(\mathfrak{r}-3)a_1\tau}\cdot\int e^{-(3+\mathfrak{r})a_1\tau}\,d\tau \\
&\quad\times\int e^{(3+\mathfrak{r})a_1\tau}\int \chi((\xi_\tau+x\xi_{x\tau})^2+\xi_\tau^2)\,dx\,d\tau + \hat R_1,
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
&\hat R_1 := \hat{\mathcal{E}}_0\cdot e^{(\mathfrak{r}-3)a_1\tau} + e^{(\mathfrak{r}-3)a_1\tau}\big(\int \chi\zeta^2\,dx + \int x^2\bar\rho\zeta^2\,dx\big) \\
&\quad + e^{(\mathfrak{r}-1)a_1\tau}\int x^4\bar\rho\xi_{\tau\tau}^2\,dx + (a_1^2 e^{(\mathfrak{r}-1)a_1\tau} + e^{(3+\mathfrak{r})a_1\tau})\int x^4\bar\rho\xi_\tau^2\,dx \\
&\quad + e^{(3+\mathfrak{r})a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx + e^{(\mathfrak{r}-3)a_1\tau}\cdot\sup_\tau\Big\{\int x^4\bar\rho\xi^2\,dx \\
&\quad + \int x^4\xi_x^2\,dx\Big\} \lesssim \hat{\mathcal{E}}_0\cdot e^{(\mathfrak{r}-3)a_1\tau} + a_1^{-1}e^{(\mathfrak{r}-l_1-3)a_1\tau}\cdot a_1 e^{l_1a_1\tau}\int x^2\bar\rho\zeta^2\,dx \\
&\quad + e^{(\mathfrak{r}-l_1-5)a_1\tau}\cdot e^{(2+l_1)a_1\tau}\int x^2\zeta_x^2\,dx + a_1^{-1}e^{(\mathfrak{r}-r_2+1)a_1\tau} \\
&\quad \times a_1 e^{(r_2-2)a_1\tau}\int x^4\bar\rho\xi_{\tau\tau}^2\,dx + (a_1 e^{(\mathfrak{r}-r_1-2)a_1\tau} + a_1^{-1}e^{(\mathfrak{r}-r_1+2)a_1\tau}) \\
&\quad \times a_1 e^{(1+r_1)a_1\tau}\int x^4\bar\rho\xi_\tau^2\,dx + e^{(\mathfrak{r}-r_1)a_1\tau}\cdot e^{(3+r_1)a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau}
\end{aligned}$$

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$$- x\xi_x\xi_\tau]^2\,dx + e^{(\mathfrak{r}-3)a_1\tau}\cdot\sup_\tau\Big\{\int x^4\bar\rho\xi^2\,dx + \int x^4\xi_x^2\,dx\Big\}.$$

Then after applying Grönwall's inequality together with (33), (34) and (48), it admits

$$\begin{aligned}&\int e^{(3+\mathfrak{r})a_1\tau}\int \chi((\xi_\tau + x\xi_{x\tau})^2 + \xi_\tau^2)\,dx\,d\tau\\ &\lesssim e^{C\int e^{(\mathfrak{r}-3)a_1\tau}\,d\tau\cdot\int e^{-(3+\mathfrak{r})a_1\tau}\,d\tau}\times\int \hat R_1\,d\tau\\ &\le C(\hat\varepsilon_0, a_1, r_1, r_2, l_1, l_2, \mathfrak{r})\hat{\mathcal{E}}_0,\end{aligned}\tag{53}$$

provided that (55) holds. Consequently,

$$\begin{aligned}&\int \chi(x^2\xi_x^2 + \xi^2)\,dx \lesssim \int \chi((\xi + x\xi_x)^2 + \xi^2)\,dx \lesssim \hat{\mathcal{E}}_0\\ &\quad + \int e^{-(3+\mathfrak{r})a_1\tau}\,d\tau\cdot\int e^{(3+\mathfrak{r})a_1\tau}\int\chi((\xi_\tau + x\xi_{x\tau})^2 + \xi_\tau^2)\,dx\,d\tau\\ &\le C(\hat\varepsilon_0, a_1, r_1, r_2, l_1, l_2, \mathfrak{r})\hat{\mathcal{E}}_0.\end{aligned}\tag{54}$$

On the other hand, from (35) and (36), one has

$$\begin{aligned}&e^{(3+r_1)a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx \lesssim a_1 e^{(1+r_1)a_1\tau}\int x^4\bar\rho\xi_\tau^2\,dx\\ &\quad + e^{(1+r_1)a_1\tau}\Big(\int x^4\bar\rho\xi_{\tau\tau}^2\,dx\Big)^{1/2}\Big(\int x^4\bar\rho\xi_\tau^2\,dx\Big)^{1/2} + \hat I_1 + \hat I_2 + \hat I_3,\\ &e^{(2+l_1)a_1\tau}\int x^2\zeta_x^2\,dx \le a_1 e^{l_1a_1\tau}\int x^2\bar\rho\zeta^2\,dx + e^{l_1a_1\tau}\Big(\int x^2\bar\rho\zeta_\tau^2\,dx\Big)^{1/2}\\ &\quad\times\Big(\int x^2\bar\rho\zeta^2\,dx\Big)^{1/2} + \hat I_4 + \hat I_5 + \hat I_6.\end{aligned}\tag{62}$$

The right of the above inequalities can be estimated as follows:

$$\begin{aligned}&\hat I_1 \lesssim e^{r_1a_1\tau}\Big(\int x^2\bar\rho\zeta^2\,dx\Big)^{1/2}\times\Big(\int x^4\bar\rho\xi_\tau^2\,dx + \int x^2[(1+\xi)x\xi_{x\tau}\\ &\quad - x\xi_x\xi_\tau]^2\,dx\Big)^{1/2} \lesssim \epsilon e^{(3+r_1)a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx\\ &\quad + C_\epsilon e^{(r_1-l_1-3)a_1\tau}\hat{\mathcal{E}}_{\zeta,1} + e^{(r_1/2-1/2-l_1/2)a_1\tau}\cdot\hat{\mathcal{E}}_{\xi,1}^{1/2}\hat{\mathcal{E}}_{\zeta,1}^{1/2},\\ &\hat I_2, \hat I_3 \lesssim \epsilon e^{(3+r_1)a_1\tau}\int x^2[(1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau]^2\,dx + C_\epsilon e^{(r_1-3)a_1\tau}\\ &\quad\times\sup_\tau\Big\{\int x^4\bar\rho\xi^2\,dx + \int x^4\xi_x^2\,dx\Big\} + e^{(r_1/2-1/2)a_1\tau}\sup_\tau\Big\{\Big(\int x^4\bar\rho\xi^2\,dx\Big)^{1/2}\\ &\quad + \Big(\int x^4\xi_x^2\,dx\Big)^{1/2}\Big\}\cdot\hat{\mathcal{E}}_{\xi,1}^{1/2},\end{aligned}$$

$$
\begin{aligned}
\hat{I}_4 &\lesssim e^{(2+l_1)a_1\tau}\Big(\int x^4\bar{\rho}\xi^2\,dx+\int x^4\xi_x^2\,dx\Big)^{1/2}\times\Big(\int x^2\zeta_x^2\,dx\Big)^{1/2}\\
&\lesssim \epsilon e^{(2+l_1)a_1\tau}\int x^2\zeta_x^2\,dx+C_\epsilon e^{(2+l_1)a_1\tau}\sup_\tau\Big\{\int x^4\bar{\rho}\xi^2\,dx+\int x^4\xi_x^2\,dx\Big\},\\
\hat{I}_5 &\lesssim e^{l_1a_1\tau}\Big(\int x^2\bar{\rho}\zeta^2\,dx\Big)^{1/2}\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx+\int x^2[(1+\xi)x\xi_{x\tau}-x\xi_x\xi_\tau]^2\,dx\Big)^{1/2}\\
&\lesssim e^{(l_1/2-r_1/2-1/2)a_1\tau}\hat{\mathcal{E}}_{\zeta,1}^{1/2}\hat{\mathcal{E}}_{\xi,1}^{1/2}+e^{l_1/2a_1\tau}\hat{\mathcal{E}}_{\zeta,1}^{1/2}\Big(\int x^2[(1+\xi)x\xi_{x\tau}\\
&\quad-x\xi_x\xi_\tau]^2\,dx\Big)^{1/2},\\
\hat{I}_6 &\lesssim e^{(1+l_1)a_1\tau}\hat{\omega}\Big(\int x^2\zeta_x^2\,dx\Big)^{1/2}\Big(\int x^4\bar{\rho}\xi_\tau^2\,dx+\int x^2[(1+\xi)x\xi_{x\tau}\\
&\quad-x\xi_x\xi_\tau]^2\,dx\Big)^{1/2}\lesssim \epsilon e^{(2+l_1)a_1\tau}\int x^2\zeta_x^2\,dx+C_\epsilon\hat{\omega}e^{(l_1-r_1-1)a_1\tau}\hat{\mathcal{E}}_{\xi,1}\\
&\quad+C_\epsilon\hat{\omega}e^{l_1a_1\tau}\cdot\int x^2[(1+\xi)x\xi_{x\tau}-x\xi_x\xi_\tau]^2\,dx.
\end{aligned}
$$

Therefore, from (62), together with (33), (34) and (48), we have the following estimates:

$$
\begin{aligned}
&\int x^2[(1+\xi)x\xi_{x\tau}-x\xi_x\xi_\tau]^2\,dx\le C(\hat{\varepsilon}_0,a_1,r_1,l_1,r_2,l_2)e^{\mathfrak{d}_1a_1\tau}\cdot\hat{\mathcal{E}}_0,\\
&\int x^2\zeta_x^2\,dx\le C(\hat{\varepsilon}_0,a_1,r_1,l_1,r_2,l_2)e^{\mathfrak{d}_2a_1\tau}\cdot\hat{\mathcal{E}}_0,
\end{aligned}
\tag{56}
$$

where $\mathfrak{d}_1,\mathfrak{d}_2$ are as in (57). Combining (61) and (56) yields the following,

$$
\begin{aligned}
\int\chi((\xi_\tau+x\xi_{x\tau})^2+\xi_\tau^2)\,dx &\lesssim e^{-6a_1\tau}\cdot\int e^{-(3+\mathfrak{r})a_1\tau}\,d\tau\\
&\quad\times\int e^{(3+\mathfrak{r})a_1\tau}\int\chi((\xi_\tau+x\xi_{x\tau})^2+\xi_\tau^2)\,dx\,d\tau+\hat{R}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{R}_2 &:= e^{-(3+\mathfrak{r})a_1\tau}\hat{R}_1\lesssim\hat{\mathcal{E}}_0\cdot e^{-6a_1\tau}+e^{-6a_1\tau}\int x^2\zeta_x^2\,dx+\int x^2[(1+\xi)x\xi_{x\tau}\\
&\quad-x\xi_x\xi_\tau]^2\,dx+e^{(-r_2-2)a_1\tau}\hat{\mathcal{E}}_{\xi,2}+(a_1^2+1)e^{(-r_1-1)a_1\tau}\hat{\mathcal{E}}_{\xi,1}\\
&\quad+e^{-6a_1\tau}\cdot\sup_\tau\Big\{\int x^4\bar{\rho}\xi^2\,dx+\int x^4\xi_x^2\,dx\Big\}.
\end{aligned}
$$

Consequently,

$$
\int\chi(x^2\xi_{x\tau}^2+\xi_\tau^2)\,dx\lesssim e^{\mathfrak{d}_3a_1\tau}\cdot\hat{\mathcal{E}}_0,
\tag{58}
$$

where

$$\mathfrak{d}_3 := \max\{-6, -6 + \mathfrak{d}_2, \mathfrak{d}_1, -r_2 - 2, -r_1 - 1\} = -r_2 - 2 < 0. \tag{59}$$

In the following two lemmas, we shall derive some regularity estimates of ξ, ζ around the symmetric center $x = 0$.

First, we will need the following lemma concerning the estimate of the interior energy functional.

Lemma 6. *Under the same assumptions as in Lemma 5, define the interior energy and dissipation functionals:*

$$\begin{aligned} \hat{\mathcal{E}}_{\xi,3} &:= e^{(r_3-2)a_1\tau} \int \chi x^2 \bar{\rho} \xi_{\tau\tau}^2 \, dx, \\ \hat{\mathcal{D}}_{\xi,3} &:= \int e^{r_3 a_1 \tau} \int \chi (x^2 \xi_{x\tau\tau}^2 + \xi_{\tau\tau}^2) \, dx \, d\tau, \end{aligned} \tag{63}$$

where r_3 satisfies

$$r_3 \leq r_2 - 2 < 0. \tag{64}$$

For $\hat{\omega} < \hat{\varepsilon}_0$ small enough, we have the following interior energy estimate:

$$\hat{\mathcal{E}}_{\xi,3} + \hat{\mathcal{D}}_{\xi,3} \leq C(\hat{\varepsilon}_0, a_1, r_1, r_2, l_1, l_2, \mathfrak{r}, r_3)\hat{\mathcal{E}}_0. \tag{65}$$

Proof. Multiply $(45)_1$ with $\tilde{\alpha}^{r_3} \chi x \xi_{\tau\tau}$ and integrate the resulting equation in the spatial variable. After integration by parts, it follows,

$$\frac{d}{d\tau} \hat{E}_{\xi,3} + \hat{D}_{\xi,3} = \hat{K}_1 + \hat{K}_2 + \hat{K}_3 + \hat{K}_4 + \hat{K}_5 + \hat{K}_6, \tag{66}$$

where

$$\begin{aligned} \hat{E}_{\xi,3} &:= \frac{\tilde{\alpha}^{r_3-2}}{2} \int \chi x^2 \bar{\rho} \xi_{\tau\tau}^2 \, dx, \\ \hat{D}_{\xi,3} &:= -\frac{r_3}{2} \tilde{\alpha}^{r_3-3} \tilde{\alpha}_\tau \int \chi x^2 \bar{\rho} \xi_{\tau\tau}^2 \, dx \\ &\quad + \frac{4}{3} \mu \tilde{\alpha}^{r_3} \int \chi \left\{ \frac{(1+\xi)^2 x^2 \xi_{x\tau\tau}^2}{1+\xi+x\xi_x} + \frac{[2(1+\xi+x\xi_x)^2 - (1+\xi)^2]\xi_{\tau\tau}^2}{1+\xi+x\xi_x} \right\} dx, \\ \hat{K}_1 &:= -(\tilde{\alpha}^{r_3-3}\tilde{\alpha}_{\tau\tau} - 3\tilde{\alpha}^{r_3-4}\tilde{\alpha}_\tau^2) \int \chi x^2 \bar{\rho} \xi_\tau \xi_{\tau\tau} \, dx, \\ \hat{K}_2 &:= \frac{4}{3} \mu \tilde{\alpha}^{r_3} \int \chi \Big\{ (x(1+\xi)^2 \xi_{\tau\tau})_x \Big(\frac{(\xi_\tau + x\xi_{x\tau})^2}{(1+\xi+x\xi_x)^2} - \frac{\xi_\tau^2}{(1+\xi)^2} \Big) \\ &\quad + 3(1+\xi)^2 \xi_{\tau\tau} \Big(-\frac{2x\xi_{x\tau}\xi_\tau}{(1+\xi)^2} + \frac{2x\xi_x\xi_\tau^2}{(1+\xi)^3} \Big) + 6x(1+\xi)\xi_\tau\xi_{\tau\tau} \Big(\frac{\xi_\tau}{1+\xi} \Big)_x \\ &\quad - 2(x(1+\xi)\xi_\tau\xi_{\tau\tau})_x \Big(\frac{\xi_\tau + x\xi_{x\tau}}{1+\xi+x\xi_x} - \frac{\xi_\tau}{1+\xi} \Big) \Big\} dx, \end{aligned}$$

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$$\hat{K}_3 := -\tilde{\alpha}^{r_3}\int \chi'\left\{x(1+\xi)^2\xi_{\tau\tau}\hat{\mathfrak{B}}_\tau + 2x(1+\xi)\xi_\tau\xi_{\tau\tau}\hat{\mathfrak{B}} + \frac{4}{3}\mu x(1+\xi)\xi_{\tau\tau}^2\right\}dx,$$

$$\hat{K}_4 := -3\tilde{\alpha}^{r_3-4}\tilde{\alpha}_\tau\int \chi\left\{\frac{K\bar{\rho}(\zeta+\bar{\theta})[x(1+\xi)^2\xi_{\tau\tau}]_x}{(1+\xi)^2(1+\xi+x\xi_x)} - K\bar{\rho}\bar{\theta}\Big(\frac{x\xi_{\tau\tau}}{(1+\xi)^2}\Big)_x\right\}dx,$$

$$\hat{K}_5 := \tilde{\alpha}^{r_3-3}\int \chi\left\{(\xi_{\tau\tau}+x\xi_{x\tau\tau})\Big[\frac{K\bar{\rho}(\zeta+\bar{\theta})}{1+\xi+x\xi_x} - \frac{K\bar{\rho}\bar{\theta}}{(1+\xi)^2}\Big]_\tau \right.$$
$$\left. + \xi_{\tau\tau}\Big[\frac{2K\bar{\rho}(\zeta+\bar{\theta})x\xi_x}{(1+\xi)(1+\xi+x\xi_x)} + \frac{2K\bar{\rho}\bar{\theta}x\xi_x}{(1+\xi)^3}\Big]_\tau\right\}dx,$$

$$\hat{K}_6 := -3\tilde{\alpha}^{r_3-4}\tilde{\alpha}_\tau\int \chi' x\left\{\frac{K\bar{\rho}(\zeta+\bar{\theta})\xi_{\tau\tau}}{1+\xi+x\xi_x} - \frac{K\bar{\rho}\bar{\theta}\xi_{\tau\tau}}{(1+\xi)^2}\right\}dx$$
$$+ \tilde{\alpha}^{r_3-3}\int \chi' x\xi_{\tau\tau}\left\{\frac{K\bar{\rho}(\zeta+\bar{\theta})}{1+\xi+x\xi_x} - \frac{K\bar{\rho}\bar{\theta}}{(1+\xi)^2}\right\}_\tau dx.$$

Then for $r_3 < 0$, integration in the temporal variable of (66) yields the following:

$$\begin{aligned}\hat{\mathcal{E}}_{\xi,3} + \hat{\mathcal{D}}_{\xi,3} \lesssim & \int \hat{K}_1\,d\tau + \int \hat{K}_2\,d\tau + \int \hat{K}_3\,d\tau + \int \hat{K}_4\,d\tau \\ & + \int \hat{K}_5\,d\tau + \int \hat{K}_6\,d\tau + \hat{\mathcal{E}}_0.\end{aligned} \tag{67}$$

The right of (67) are estimated as follows:

$$\int \hat{K}_1\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,3} + C_\epsilon a_1^3 \sup_\tau e^{(r_3-r_1-5)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,11},$$

$$\int \hat{K}_2\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,3} + C_\epsilon\hat{\omega}\int e^{r_3 a_1\tau}\int \chi(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx\,d\tau,$$

$$\begin{aligned}\int \hat{K}_3\,d\tau \lesssim{} & (1+\hat{\omega})\hat{\mathcal{D}}_{\xi,21}^{1/2}\Big\{a_1^{-1}\sup_\tau e^{(r_3-r_2+2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,21}^{1/2} + a_1^{-1/2} \\ & \times \sup_\tau e^{(r_3-r_2+1)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,22}^{1/2} + \hat{\omega}a_1^{-1}\sup_\tau e^{(r_3-r_2/2-r_1/2+1/2)a_1\tau} \\ & \times \hat{\mathcal{D}}_{\xi,11}^{1/2} + \hat{\omega}a_1^{-1/2}\sup_\tau e^{(r_3-r_2/2-r_1/2-1/2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,12}^{1/2}\Big\},\end{aligned}$$

$$\begin{aligned}\int \hat{K}_4\,d\tau \lesssim{} & (1+\hat{\omega})\int a_1 e^{(r_3-3)a_1\tau}\int \chi\bar{\rho}(|\zeta|+|\xi|+|x\xi_x|)(|\xi_{\tau\tau}| \\ & + |x\xi_{x\tau\tau}|)\,dx\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,3} + C_\epsilon(1+\hat{\omega})a_1^2\sup_\tau e^{(r_3-l_1-8)a_1\tau}\cdot\hat{\mathcal{D}}_{\zeta,12} \\ & + C_\epsilon(1+\hat{\omega})\int a_1^2 e^{(r_3-6)a_1\tau}\,d\tau\cdot\sup_\tau\int \chi(\xi^2+x^2\xi_x^2)\,dx,\end{aligned}$$

$$\begin{aligned}\int \hat{K}_5\,d\tau \lesssim{} & (1+\hat{\omega})\int e^{(r_3-3)a_1\tau}\int \chi\bar{\rho}(|\zeta_\tau|+|\xi_\tau|+|\xi_{x\tau}|)(|\xi_{\tau\tau}| \\ & + |x\xi_{x\tau\tau}|)\,dx\,d\tau \lesssim \epsilon\hat{\mathcal{D}}_{\xi,3} + C_\epsilon(1+\hat{\omega})\sup_\tau e^{(r_3-l_2-6)a_1\tau}\cdot\hat{\mathcal{D}}_{\zeta,2}\end{aligned}$$

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$$
\begin{aligned}
&\quad + C_\epsilon(1+\hat{\omega})\int e^{(r_3-6)a_1\tau}\int \chi(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx\,d\tau,\\
&\int \hat{K}_6\,d\tau \lesssim (1+\hat{\omega})\int e^{(r_3-3)a_1\tau}\int x^4\bar{\rho}(a_1|\zeta| + a_1|\xi| + a_1|x\xi_x| + |\zeta_\tau|\\
&\quad + |\xi_\tau| + |x\xi_{x\tau}|)|\xi_{\tau\tau}|\,dx\,d\tau \lesssim (1+\hat{\omega})\hat{\mathcal{D}}_{\xi,21}^{1/2}\\
&\quad \times \Big\{a_1^{1/2}\sup_\tau e^{(r_3-r_2/2-l_1/2-3)a_1\tau}\cdot\hat{\mathcal{D}}_{\zeta,12}^{1/2} + a_1^{1/2}\big(\int e^{(2r_3-r_2-4)a_1\tau}\,d\tau\big)^{1/2}\\
&\quad \times \sup_\tau\big(\int x^4\bar{\rho}\xi^2\,dx + \int x^4\xi_x^2\,dx\big)^{1/2} + a_1^{-1/2}\sup_\tau e^{(r_3-r_2/2-l_2/2-2)a_1\tau}\\
&\quad \times \hat{\mathcal{D}}_{\zeta,2}^{1/2} + a_1^{-1}\sup_\tau e^{(r_3-r_2/2-r_1/2-5/2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,11}^{1/2}\\
&\quad + a_1^{-1/2}\sup_\tau e^{(r_3-r_2/2-r_1/2-7/2)a_1\tau}\cdot\hat{\mathcal{D}}_{\xi,12}^{1/2}\Big\}.
\end{aligned}
$$

Therefore from (67), it holds,

$$
\begin{aligned}
&\hat{\mathcal{E}}_{\xi,3} + (1-\hat{\omega})\hat{\mathcal{D}}_{\xi,3} \lesssim C(\hat{\varepsilon}_0, a_1)\Big\{\hat{\mathcal{D}}_{\xi,1} + \hat{\mathcal{D}}_{\xi,2} + \hat{\mathcal{D}}_{\zeta,1} + \hat{\mathcal{D}}_{\zeta,2}\\
&\quad + \int e^{(r_3-\mathfrak{r}-3)a_1\tau}\cdot e^{(3+\mathfrak{r})a_1\tau}\int\chi(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx\,d\tau + \int e^{(r_3-6)a_1\tau}\,d\tau\\
&\quad \times \int \chi(\xi^2 + x^2\xi_x^2)\,dx + \int e^{(r_3-\mathfrak{r}-9)a_1\tau}\cdot e^{(3+\mathfrak{r})a_1\tau}\int\chi(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx\,d\tau\Big\}\\
&\quad + \int e^{(2r_3-r_2-4)a_1\tau}\,d\tau\cdot\sup_\tau\big(\int x^4\bar{\rho}\xi^2\,dx + \int x^4\xi_x^2\,dx\big) + \hat{\mathcal{E}}_0 \lesssim \hat{\mathcal{E}}_0,
\end{aligned}
$$

provided that (64) holds. Then with $\hat{\varepsilon}_0$ small enough, we finish the proof of the lemma.

Now we can consider the elliptic estimates of (19). We rewrite the system in the following in terms of the relative entropy functional $\hat{\mathcal{G}} = \mathfrak{H}(\xi) = \log(1+\xi)^2(1+\xi+x\xi_x)$; that is,

$$
\begin{cases}
\frac{4}{3}\mu\hat{\mathcal{G}}_{x\tau} + \tilde{\alpha}^{-3}\dfrac{K\bar{\rho}(\zeta+\bar{\theta})}{(1+\xi)^2(1+\xi+x\xi_x)}\hat{\mathcal{G}}_x = \tilde{\alpha}^{-3}\dfrac{(K\bar{\rho}\zeta)_x}{(1+\xi)^2(1+\xi+x\xi_x)}\\
\qquad + \tilde{\alpha}^{-3}(K\bar{\rho}\bar{\theta})_x\big(\dfrac{1}{(1+\xi)^2(1+\xi+x\xi_x)} - \dfrac{1}{(1+\xi)^4}\big) + \dfrac{1}{(1+\xi)^2}\\
\qquad \times\big(\tilde{\alpha}^{-2}x\bar{\rho}\xi_{\tau\tau} + \tilde{\alpha}^{-3}\tilde{\alpha}_\tau x\bar{\rho}\xi_\tau\big),\\
\big[\dfrac{x^2(1+\xi)^2}{1+\xi+x\xi_x}\zeta_x\big]_x = \big[\big(1 - \dfrac{(1+\xi)^2}{1+\xi+x\xi_x}\big)x^2\bar{\theta}_x\big]_x + K\tilde{\alpha}^{-2}\\
\qquad \times\dfrac{\bar{\rho}(\zeta+\bar{\theta})(x^3(1+\xi)^2\xi_\tau)_x}{(1+\xi)^2(1+\xi+x\xi_x)} + 3K\tilde{\alpha}^{-2}x^2\bar{\rho}\zeta_\tau - \tilde{\alpha}^{-1}x^2(1+\xi)^2\\
\qquad \times(1+\xi+x\xi_x)\cdot\hat{\mathfrak{F}}(\xi).
\end{cases}
\tag{68}
$$

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Lemma 7. *Under the same assumptions as in Lemma 6, supposed that*

$$l_2 + 2 \geq 0, \tag{69}$$

we have

$$\begin{aligned} &\|\xi_x\|^2_{L^\infty_\tau L^2_x} + \|x\xi_{xx}\|^2_{L^\infty_\tau L^2_x} + \|\zeta_x\|^2_{L^\infty_\tau L^2_x} \leq C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2, \mathfrak{r}, r_3)\hat{\mathcal{E}}_0, \\ &\|\xi_{x\tau}\|^2_{L^\infty_\tau L^2_x} + \|x\xi_{xx\tau}\|^2_{L^\infty_\tau L^2_x} \lesssim C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2, \mathfrak{r}, r_3)e^{(-r_3-2)a_1\tau}\hat{\mathcal{E}}_0. \end{aligned} \tag{70}$$

In the meantime, we have the following:

$$\|x\zeta_{xx}\|^2_{L^\infty_\tau L^2_x} \leq C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2, \mathfrak{r}, r_3)(\hat{\mathcal{E}}_0 + \hat{\mathcal{E}}_0^2). \tag{71}$$

Proof. From $(68)_2$, by noticing that

$$\Big(1 - \frac{(1+\xi)^2}{1+\xi+x\xi_x}\Big)_x = -\frac{4(1+\xi)^3\xi_x}{(1+\xi)^2(1+\xi+x\xi_x)} + \frac{(1+\xi)^4}{(1+\xi)^2(1+\xi+x\xi_x)}\hat{\mathcal{G}}_x,$$

it holds,

$$\begin{aligned} &\int \frac{1}{x^2(1+\xi)^2(1+\xi+x\xi_x)}\Big\{\Big[\frac{x^2(1+\xi)^2}{1+\xi+x\xi_x}\zeta_x\Big]_x\Big\}^2 dx \lesssim (1+\hat{\omega})\int x^2(\xi^2 \\ &\quad + x^2\xi_x^2)\,dx + (1+\hat{\omega})\int \hat{\mathcal{G}}_x^2\,dx + (1+\hat{\omega})e^{-4a_1\tau}\int x^2(\xi_\tau^2 + x^2\xi_{x\tau}^2)\,dx \\ &\quad + (1+\hat{\omega})e^{-4a_1\tau}\int x^2\bar{\rho}\zeta_\tau^2\,dx + \hat{\omega}e^{-2a_1\tau}\int x^2(\xi_\tau^2 + x^2\xi_{x\tau}^2) \\ &\lesssim \int \hat{\mathcal{G}}_x^2\,dx + (e^{(\mathfrak{d}_1-4)a_1\tau} + \hat{\omega}e^{(\mathfrak{d}_1-2)a_1\tau})\cdot\hat{\mathcal{E}}_0 + (e^{(-r_1-5)b\tau} + \hat{\omega}e^{(-r_1-3)b\tau}) \\ &\quad \times \hat{\mathcal{E}}_{\xi,1} + e^{(-l_2-2)a_1\tau}\cdot\hat{\mathcal{E}}_{\zeta,2} + \hat{\mathcal{E}}_0 \lesssim \int \hat{\mathcal{G}}_x^2\,dx + \hat{\mathcal{E}}_0, \end{aligned}$$

provided that (69) holds, where we have used (33) (34), (39), (48), (56) and the fact that $|(x^2\bar{\theta}_x)_x| \lesssim x^2\|\bar{\theta}_{xx}\|_{L^\infty_x}$, $|\bar{\theta}_x| \lesssim x\|\bar{\theta}_{xx}\|_{L^\infty_x}$ due to $\bar{\theta}_x(0) = 0$. Meanwhile, direct calculation shows,

$$\begin{aligned} &\int \frac{1}{x^2(1+\xi)^2(1+\xi+x\xi_x)}\Big\{\Big[\frac{x^2(1+\xi)^2}{1+\xi+x\xi_x}\zeta_x\Big]_x\Big\}^2 dx = \int\Big[\frac{x(1+\xi)}{\sqrt{1+\xi+x\xi_x}} \\ &\quad \times \Big(\frac{\zeta_x}{1+\xi+x\xi_x}\Big)_x + 2\sqrt{1+\xi+x\xi_x}\frac{\zeta_x}{1+\xi+x\xi_x}\Big]^2 dx = \int \frac{x^2(1+\xi)^2}{1+\xi+x\xi_x} \\ &\quad \times \Big[\Big(\frac{\zeta_x}{1+\xi+x\xi_x}\Big)_x\Big]^2 + 2(1+\xi+x\xi_x)\Big[\frac{\zeta_x}{1+\xi+x\xi_x}\Big]^2 dx \\ &\quad + 2x(1+\xi)\Big[\frac{\zeta_x}{1+\xi+x\xi_x}\Big]^2\Big|_{x=R_0}. \end{aligned}$$

Therefore, combining the above inequalities yields,

$$\int \zeta_x^2\, dx \lesssim \int \hat{\mathcal{G}}_x^2\, dx + \hat{\mathcal{E}}_0, \tag{72}$$

$$\int x^2 \zeta_{xx}^2\, dx \lesssim \int \zeta_x^2\, dx + (\|x\zeta_x\|_{L_x^\infty}^2 + 1) \int \hat{\mathcal{G}}_x^2\, dx + \hat{\mathcal{E}}_0. \tag{73}$$

Here we have applied the following identity:

$$(\frac{\zeta_x}{1+\xi+x\xi_x})_x = \frac{2(1+\xi)\xi_x\zeta_x + (1+\xi)^2\zeta_{xx}}{(1+\xi)^2(1+\xi+x\xi_x)} - \frac{(1+\xi)^2\zeta_x}{(1+\xi)^2(1+\xi+x\xi_x)}\hat{\mathcal{G}}_x.$$

On the other hand, after multiplying (68)$_1$ with $\hat{\mathcal{G}}_x$ and integrating the resultant, one can derive,

$$\begin{aligned}
&\frac{d}{d\tau}\frac{2\mu}{3}\int \hat{\mathcal{G}}_x^2\, dx + \tilde{\alpha}^{-3}\int \frac{K\bar{\rho}(\zeta+\bar{\theta})}{(1+\xi)^2(1+\xi+x\xi_x)}\hat{\mathcal{G}}_x^2\, dx \lesssim \epsilon e^{-3a_1\tau}\int \hat{\mathcal{G}}_x^2\, dx \\
&\quad + C_\epsilon\Big\{ e^{-3a_1\tau}\int \zeta_x^2\, dx + e^{-3a_1\tau}\int (\xi^2 + x^2\xi_x^2)\, dx + a_1 e^{(-2-r_1)a_1\tau} \\
&\quad \times a_1 e^{(1+r_1)a_1\tau}\int x^4\bar{\rho}\xi_\tau^2\, dx + a_1^2 e^{(-4-r_1)a_1\tau} \\
&\quad \times e^{(3+r_1)a_1\tau}\int x^2((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)^2\, dx\Big\} + (a_1^{-1}e^{(-r_2-2)a_1\tau} \\
&\quad + e^{(-r_2-4)a_1\tau})\cdot\int \hat{\mathcal{G}}_x^2\, dx + a_1 e^{(r_2-2)a_1\tau}\int x^4\bar{\rho}\xi_{\tau\tau}^2\, dx \\
&\quad + e^{r_2 a_1\tau}\int x^2((1+\xi)x\xi_{x\tau\tau} - x\xi_x\xi_{\tau\tau})^2\, dx.
\end{aligned}$$

Then by noticing (33), (34), (48), (54) (72), applying Grönwall's inequality then yields,

$$\int \hat{\mathcal{G}}_x^2\, dx + \int \zeta_x^2\, dx \lesssim \hat{\mathcal{E}}_0. \tag{74}$$

Meanwhile, from (68)$_1$,

$$\begin{aligned}
&\int \hat{\mathcal{G}}_{x\tau}^2\, dx \lesssim e^{-6a_1\tau}\int \hat{\mathcal{G}}_x^2\, dx + e^{-6a_1\tau}\int \zeta_x^2\, dx + e^{-6a_1\tau}\int (\xi^2 + x^2\xi_x^2)\, dx \\
&\quad + e^{-4a_1\tau}\int x^2\bar{\rho}\xi_{\tau\tau}^2\, dx + a_1^2 e^{-4a_1\tau}\int x^2\bar{\rho}\xi_\tau^2\, dx \lesssim (e^{(-r_3-2)a_1\tau} + e^{(-r_2-2)b\tau} \\
&\quad + a_1^2 e^{(-r_1-5)a_1\tau} + a_1^2 e^{(\mathfrak{d}_3-4)a_1\tau} + e^{-6a_1\tau})\cdot\hat{\mathcal{E}}_0,
\end{aligned}$$

where (33), (34) (48), (54), (58), (65), (74) are applied. Consequently, we have

$$\int \hat{\mathcal{G}}_{x\tau}^2\, dx \lesssim e^{(-r_3-2)a_1\tau}\hat{\mathcal{E}}_0. \tag{75}$$

Then together with Lemma 2, (74) and (75) imply (70). Consequently, from (73)

$$\|x\zeta_{xx}\|^2_{L^\infty_\tau L^2_x} \lesssim (1+\|x\zeta_x\|^2_{L^\infty_x})\hat{\mathcal{E}}_0 \lesssim \hat{\mathcal{E}}_0 + \hat{\mathcal{E}}_0^2 + \|\zeta_x\|^2_{L^\infty_\tau L^2_x} + \frac{1}{2}\|x\zeta_{xx}\|^2_{L^\infty_\tau L^2_x},$$

where the following pointwise estimate is used:

$$|x\zeta_x|^2 \lesssim \|x\zeta_x\|^2_{L^2_x} + \int |x\zeta_x(\zeta_x + x\zeta_{xx})|\, dx \lesssim (1+1/\epsilon)\|\zeta_x\|^2_{L^2_x} + \epsilon\|x\zeta_{xx}\|^2_{L^2_x}.$$

This finishes the proof.

3.3. Pointwise bounds and asymptotic stability

In this section, we shall derive the following:

$$\hat{\omega}^2 \le C\hat{\mathcal{E}}(\tau) + C\hat{\mathcal{E}}_0 + C\hat{\mathcal{E}}^2(\tau) + C\hat{\mathcal{E}}_0^2 \le C(\hat{\mathcal{E}}_0 + \hat{\mathcal{E}}_0^2). \tag{76}$$

This will be sufficient to establish the asymptotic stability theory of the linearly expanding homogeneous solution of the thermodynamic model for the radiation gaseous star. Indeed, applying the continuity arguments with small enough $\hat{\varepsilon}_0$ and initial energy $\hat{\mathcal{E}}_0$ will finish the proof of Theorem 1.1.

What is left is to show inequality (76). First of all, after applying Hardy's inequality repeatedly, and the Sobolev embedding theorem, it holds,

$$\|x\xi_x\|^2_{L^\infty_x} \lesssim \|x\xi_x\|^2_{L^2_x} + \|\xi_x + x\xi_{xx}\|^2_{L^2_x} \lesssim \|\xi_x\|^2_{L^2_x} + \|x\xi_{xx}\|^2_{L^2_x}, \tag{77}$$

$$\begin{aligned} \|\xi\|^2_{L^\infty_x} &\lesssim \|\xi\|^2_{L^2_x} + \|\xi_x\|^2_{L^2_x} \lesssim \|x\bar{\rho}^{\frac{\varepsilon K}{1-\varepsilon K}}\xi\|^2_{L^2_x} + \|\xi_x\|^2_{L^2_x} \\ &\lesssim \|x^2\bar{\rho}^{1/2}\xi\|^2_{L^2_x} + \|\xi_x\|^2_{L^2_x}, \end{aligned} \tag{78}$$

where we have used the fact that $\bar{\rho}^{\frac{\varepsilon K}{1-\varepsilon K}}(x) \simeq (R_0 - x)$ from (10). On the other hand, applying the fundamental theorem of calculus and Hölder's inequality yields

$$\begin{aligned} &\|x\xi_{x\tau}\|^2_{L^\infty_x} \lesssim \|x\xi_{x\tau}\|^2_{L^2_x(R_0/4,R_0/2)} + \int_0^{R_0} |(x^2\xi^2_{x\tau})_x|\, dx \\ &\lesssim \|x\xi_{x\tau}\|^2_{L^2_x(R_0/4,R_0/2)} + \|x\xi_{x\tau}\|_{L^2_x}\big(\|\xi_{x\tau}\|_{L^2_x} + \|x\xi_{xx\tau}\|_{L^2_x}\big) \\ &\lesssim (1+\hat{\omega})\big(\|x((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)\|^2_{L^2_x} + \|x^2\bar{\rho}^{1/2}\xi_\tau\|^2_{L^2_x}\big) \\ &\quad + \big(\|\chi^{1/2}x\xi_{x\tau}\|_{L^2_x} + \|x^2\xi_{x\tau}\|_{L^2_x}\big)\big(\|\xi_{x\tau}\|_{L^2_x} + \|x\xi_{xx\tau}\|_{L^2_x}\big). \end{aligned}$$

Here we have separated the norm of $\|x\xi_{x\tau}\|_{L^2_x}$ in the interior and boundary subdomains and used the following form of the mean value theorem

$$\exists x_0 \in (R_0/4, R_0/2), \text{ such that } \left|x\xi_{x\tau}(x_0,\tau)\right| \le \frac{4}{R_0}\left\|x\xi_{x\tau}\right\|_{L^2_x(R_0/4,R_0/2)}.$$

Then employing (39) yields

$$\begin{aligned}\left\|x\xi_{x\tau}\right\|^2_{L^\infty_x} &\lesssim (1+\hat{\omega})\big(\left\|x((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)\right\|^2_{L^2_x} + \left\|x^2\bar{\rho}^{1/2}\xi_\tau\right\|^2_{L^2_x}\big)\\ &+ (1+\hat{\omega})\big(\left\|\chi^{1/2}x\xi_{x\tau}\right\|_{L^2_x} + \left\|x^2\bar{\rho}^{1/2}\xi_\tau\right\|_{L^2_x}\\ &\quad + \left\|x((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)\right\|_{L^2_x}\big)\cdot\big(\left\|\xi_{x\tau}\right\|_{L^2_x} + \left\|x\xi_{xx\tau}\right\|_{L^2_x}\big).\end{aligned} \tag{79}$$

Similarly,

$$\begin{aligned}\left\|\xi_\tau\right\|^2_{L^\infty_x} &\lesssim \left\|x^2\bar{\rho}^{1/2}\xi_\tau\right\|^2_{L^2_x} + (1+\hat{\omega})\big(\left\|\chi^{1/2}\xi_\tau\right\|_{L^2_x} + \left\|x^2\bar{\rho}^{1/2}\xi_\tau\right\|_{L^2_x}\\ &\quad + \left\|x((1+\xi)x\xi_{x\tau} - x\xi_x\xi_\tau)\right\|_{L^2_x}\big)\left\|\xi_{x\tau}\right\|_{L^2_x}.\end{aligned} \tag{80}$$

(77), (78), (79), (80), together with (70), yield

$$\begin{aligned}&\left\|x\xi_x\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|\xi\right\|^2_{L^\infty_\tau L^\infty_x} \lesssim C\hat{\mathcal{E}}(\tau) + C\hat{\mathcal{E}}_0,\\ &\left\|x\xi_{x\tau}\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|\xi_\tau\right\|^2_{L^\infty_\tau L^\infty_x} \lesssim (e^{-(1+r_1)a_1\tau} + e^{-((r_1+r_2)/2+3/2)a_1\tau})\hat{\mathcal{E}}(\tau)\\ &\quad + (e^{-(r_2/2+1)a_1\tau} + e^{-((r_1+r_2)/4+3/4)a_1\tau} + e^{-(r_1/2+1/2)})e^{-(r_3/2+1)}\\ &\quad \times \hat{\mathcal{E}}(\tau)^{1/2}\cdot\hat{\mathcal{E}}_0^{1/2} \lesssim e^{-((r_2+r_3)/2+2)a_1\tau}(\hat{\mathcal{E}}(\tau)+\hat{\mathcal{E}}_0).\end{aligned} \tag{81}$$

Moreover, we have the following lemma:

Lemma 8. *Under the same assumptions as in Lemma 7, we have the following:*

$$\begin{aligned}&\left\|x\xi_x\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|\xi\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|x\xi_{x\tau}\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|\xi_\tau\right\|^2_{L^\infty_\tau L^\infty_x} + \left\|\zeta/\sigma\right\|^2_{L^\infty_\tau L^\infty_x}\\ &\quad \le C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2, \mathfrak{r}, r_3)(\hat{\mathcal{E}}(\tau) + \hat{\mathcal{E}}_0 + \hat{\mathcal{E}}^2(\tau) + \hat{\mathcal{E}}_0^2).\end{aligned} \tag{82}$$

This finishes the first inequality in (76).

Proof. The pointwise bound of $x\xi_x, \xi, x\xi_{x\tau}, \xi_\tau$ is a direct consequence of (81). What is left is the estimate of $\left\|\zeta/\sigma\right\|_{L^\infty_\tau L^\infty_x}$. This is a direct consequence of (70) and (71) by applying the Poincaré inequality, the Sobolev embedding theorem and the fundamental theorem of calculus.

The second part of the inequality (76) is a direct consequence of Lemma 3, Lemma 4, Lemma 5, Lemma 6 and Lemma 7.

Lemma 9. *Under the same assumptions as in Lemma 7, we have the total energy inequality* $\hat{\mathcal{E}}(\tau) + \hat{\mathcal{D}}(\tau) \le C(\hat{\varepsilon}_0, a_1, r_1, l_1, r_2, l_2, \mathfrak{r}, r_3)(\hat{\mathcal{E}}_0 + \hat{\mathcal{E}}_0^2)$.

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