

# $L^2$ Global Well-Posedness of the Initial Value Problem Associated to the Benjamin Equation

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We study well-posedness for the initial value problem associated to the Benjamin equation and the periodic Benjamin equation. Global results are established for data in  $L^2(\mathbb{R})$  and  $L^2(\mathbb{T})$ , respectively. We apply the recent theory, developed by Kenig, Ponce, and Vega and Bourgain, to deal with low-regularity data for the initial value problem associated to the Korteweg-de Vries equation. © 1999

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## 1. INTRODUCTION

In this paper we first consider the initial value problem (IVP) associated to the Benjamin equation, that is,

$$\begin{cases} \partial_t u - l\mathcal{H}\partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = 0, & x, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases} \quad (1.1)$$

where  $\mathcal{H}$  denotes the Hilbert transform

$$\mathcal{H}f(x) = \text{p.v.} \frac{1}{\pi} \int \frac{f(x-y)}{y} dy$$

and  $l$  is a positive real number.

This integro-differential equation models the unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin [B] to study gravity-capillary surface waves of

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solitary type on deep water. He also showed that solutions of the equation above satisfy the conserved quantities

$$I_1(u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x, t) dx$$

and

$$I_2(u) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} (\partial_x u)^2(x, t) - \frac{l}{2} u(x, t) \mathcal{H} \partial_x u(x, t) - \frac{1}{3} u^3(x, t) \right\} dx.$$

Several works have been devoted to the study of existence, stability and asymptotics of solitary waves solutions of (1.1), see for instance [B, ABR, A, CB]. Here we are interested in the study of well-posedness for the IVP with low regularity data.

The motivation to study well-posedness for low regularity data comes from the results on stability of solitary-wave solutions recently obtain in [A] for the system (1.1). Angulo shows that the solitary-wave solutions of (1.1) are stable in  $H^1(\mathbb{R})$  using the notion of stability introduced by Cazenave and Lions in [CL] (see also [Al]). In general, local (global) well-posedness for the systems in consideration is not known in the space, say  $H^\rho(\mathbb{R})$ , where stability of solitary-wave solutions is realized. To justify the stability result, the IVP might be solved for data in  $H^s(\mathbb{R})$ ,  $s > \rho$ , requiring the data being close to the solitary wave in the  $H^\rho$ -norm and thus establishing that the solutions will remain close to translations of the solitary wave in the same norm throughout their existence time. For the particular case of Benjamin equation, if one shows local well-posedness for data in  $H^1$ , the conservation law  $I_2(u)$  will give an *a priori* estimate in this space and therefore the global well-posedness for the IVP (1.1). This will allow to have the stability result in its total strength. Here we not only do that but we go further in the study of local well-posedness in spaces of low regularity mainly due to the presence of the conservation law in  $L^2(\mathbb{R})$ .

More precisely, we establish global well-posedness for the initial data in  $L^2(\mathbb{R})$ . So we first show that the IVP is locally well posed in  $L^2(\mathbb{R})$  and using the fact that smooth solutions of the Eq. (1.1) satisfy the conservation law  $I_1(u)$  we will extend the local solutions for any time.

Notice that the dispersive term of the Eq. (1.1), i.e.,  $\partial_x^3 + l\mathcal{H}\partial_x^2$ , is a combination of the dispersive terms of the Korteweg–de Vries (KdV) equation,

$$\partial_t v + \partial_x^3 v + v \partial_x v = 0$$

and the Benjamin–Ono (BO) equation

$$\partial_t w + \mathcal{H} \partial_x^2 w + w \partial_x w = 0.$$

Regarding the well-posedness of the IVP associated to the equations above we have that the best result concerning local well-posedness for the KdV equation was proved by Kenig, Ponce, and Vega [KPV1] for data in  $H^s(\mathbb{R})$ ,  $s > -3/4$  and the global result by Bourgain [Bo2] for data in  $L^2(\mathbb{R})$ . For the BO equation, Ponce [P] showed global well-posedness for data in  $H^s(\mathbb{R})$ ,  $s \geq 3/2$ . For further results concerning the BO equation see also [I]. We can observe the gap between the order of the Sobolev spaces in each case. This is due to the form of the operator modeling the dispersion relation of the terms  $\partial_x^3$  and  $\mathcal{H}\partial_x^2$ . Here the key to obtain better well-posedness results is the fact that for the Benjamin equation the dispersion property is stronger than that for the BO equation.

To study the IVP (1.1) we use its integral equivalent formulation

$$u(t) = V(t) u_0 + \int_0^t V(t-t') \partial_x(u^2(t')) dt',$$

where  $V(t) = e^{it(\partial_x^3 + \mathcal{H}\partial_x^2)}$  is the unitary operator associated to the linear equation.

The method of proof will be a combination of estimates and the contraction mapping principle. We follow the ideas used by Kenig, Ponce, and Vega used to establish their results for the KdV equation [KPV1, KPV2]. The main ingredient is the use of space time weight norm introduced by Bourgain [Bo1, Bo2] to study the KdV and nonlinear Schrödinger equations in the periodic case (see also [B]).

In order to give the statements of our main results regarding the IVP (1.1) we first shall introduce some notation.

We denote by  $\hat{f}$  the Fourier transform of  $f$  in both  $x$  and  $t$  variables and by  $\hat{f}^{(\cdot)}$  the Fourier transform in the  $(\cdot)$  variable.

For  $-1 < b < 1$ , let  $X_{s,b}$  denote the Hilbert space with norm

$$\|f\|_{X_{s,b}} = \left( \iint (1 + |\tau - |\zeta| \zeta + \zeta^3|)^{2b} (1 + |\zeta|)^{2s} |\hat{f}(\zeta, \tau)|^2 d\zeta d\tau \right)^{1/2}.$$

**THEOREM 1.1.** *If  $u_0 \in L^2(\mathbb{R})$  and  $b \in (1/2, 5/6)$ , then there exist  $T = T(\|u_0\|_2)$  and a unique solution  $u(t)$  of the IVP (1.1) such that*

$$u \in C([0, T]; L^2(\mathbb{R}))$$

$$u \in X_{0,b} \quad \text{with}$$

$$\partial_x(u^2) \in X_{0,b-1}.$$

Moreover, given  $T' \in (0, T)$  the map  $u_0 \rightarrow u(t)$  is Lipschitz continuous from  $L^2(\mathbb{R})$  to  $C([0, T]; L^2(\mathbb{R}))$ .

**COROLLARY 1.2.** *For any  $s > 0$  and  $u_0 \in H^s(\mathbb{R})$ , the same conclusions in Theorem 1.1 hold for  $u \in C([0, T]; H^s(\mathbb{R}))$ .*

Since the  $L^2$  conservation law is established for smooth solutions, the solution above also satisfies the  $L^2$  conservation law, then we have the global well-posedness result for data in  $L^2$ , that is,

**THEOREM 1.3.** *The solution obtained in Theorem 1.1 can be extended for any  $T > 0$ .*

Some remarks on the  $X_{s,b}$ , regarding regularity, follow. First observe that for  $f \in X_{s,b}$  the identity

$$\|f\|_{X_{s,b}} = \|V(\cdot) f(\cdot)\|_{H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))}$$

holds, here  $V(\cdot)$  is the unitary operator defined above.

On the other hand, if  $b > 1/2$ , the previous remark and Sobolev lemma imply

$$X_{s,b} \subset C(\mathbb{R}; H_x^s(\mathbb{R})).$$

We also notice that due to the method used in this paper and the considerations above one could expect to obtain a similar result as the one for the KdV equation, i.e., local well-posedness for  $s > -3/4$ . But at this point it is not clear what the best result concerning the local well-posedness might be.

The second part of this paper concerns the periodic initial value problem (PIVP) associated to the Benjamin equation, i.e.,

$$\begin{cases} \partial_t u - l\mathcal{H}\partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = 0, & x \in \mathbb{T}, \quad t \in \mathbb{R} \\ u(x, 0) = u_0(x), \end{cases} \quad (1.2)$$

where  $\mathcal{H}$  denotes the Hilbert transform and  $l$  is a positive number.

As in the previous case we will use the methods introduced by Bourgain [Bo2] and Kenig, Ponce, and Vega [KPV2] to study the well-posedness of the PIVP for the KdV equation.

We first observe that solutions of the Benjamin equation also satisfy the conserved quantity

$$\int u(x, t) dx = \int u_0(x).$$

We can assume, as in the KdV equation case, that  $\hat{u}(0, t) = 0$  to prove the following theorem.

**THEOREM 1.4.** *Let  $u_0 \in L^2(\mathbb{T})$  with  $\hat{u}_0(0) = 0$ . Then there exists  $T = T(\|u_0\|_2) > 0$  and a unique solution  $u$  of the PIVP (1.2) such that*

$$u \in C([0, T]; L^2(\mathbb{R})) \cup Y_{0,b} \quad \text{with}$$

$$\partial_x(u^2) \in Y_{0,b-1}.$$

Moreover, given  $T' \in (0, T)$  the map  $u_0 \rightarrow u(t)$  is Lipschitz continuous from  $L^2(\mathbb{R})$  to  $C([0, T']; L^2(\mathbb{R})) \cup Y_{0,b}$ .

Here, for  $s, b \in \mathbb{R}$ ,  $Y_{s,b}$  denotes the completion of the space  $\mathcal{Y}$  with respect to the norm

$$\|f\|_{Y_{s,b}} = \left( \sum_{n \neq 0} \int_{-\infty}^{\infty} (1 + |\tau - |n||n + n^3|)^{2b} |n|^{2s} |\hat{f}(n, \tau)|^2 d\tau \right)^{1/2}.$$

where  $\mathcal{Y}$  is the space of all  $f$  such that

- (1)  $f: \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$ ;
- (2)  $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$  for each  $x \in \mathbb{T}$ ;
- (3)  $x \rightarrow f(x, \cdot)$  is  $C^\infty$ ;
- (5)  $\hat{f}(0, \tau) = 0$  for all  $\tau \in \mathbb{R}$ .

**Remark 1.5.** The global result follows as a corollary of Theorem 1.4.

**Remark 1.6.** The general case,  $\hat{u}(0, t) \neq 0$ , can also be proved following the arguments given by Bourgain [Bo2].

The plan of this paper is as follows. In Section 2 the estimates needed to establish Theorem 1.1 will be proven and its proof will be given in Section 3. In Section 4 we treat the periodic case. We will only give the needed elements to prove Theorem 1.4 since its proof follows the same argument as the proof of Theorem 1.1.

Before leaving this section we will introduce some additional notation.

Let  $\psi \in C_0^\infty(\mathbb{R})$ , with  $\psi = 1$  on  $[-1/2, 1/2]$  and  $\text{supp } \psi \subset [-1, 1]$ . We denote  $\psi_\delta(\cdot) = \psi(\delta^{-1} \cdot)$  for some  $\delta \in \mathbb{R}$

## 2. ESTIMATES FOR THE PROOF OF THEOREM 1.1

In this section we will establish most of the needed estimates in the proof of Theorem 1.1. We begin given the main estimates in the  $X_{s,b}$  spaces. These estimates were proven by Kenig, Ponce, and Vega in [KPV1, KPV2].

LEMMA 2.1. Let  $s \in \mathbb{R}$ ,  $b \in (1/2, 1)$  and  $\delta \in (0, 1)$ , then for  $F \in X_{s,b}$

$$\|\psi_\delta F\|_{X_{s,b}} \leq C\delta^{(1-2b)/2} \|F\|_{X_{s,b}}. \quad (2.1)$$

Let  $a, b \in (0, 1/2)$  with  $a < b$  and  $\delta \in (0, 1)$ , then for  $F \in X_{s,-a}$  we have

$$\|\psi_\delta F\|_{X_{s,-b}} \leq C\delta^{(b-a)/4(1-a)} \|F\|_{X_{s,-a}}. \quad (2.2)$$

Next estimates are the analogous ones established by Kenig, Ponce, and Vega for the Korteweg–de Vries equation. We will sketch the proof of them just for the sake of clearness but the argument is essentially that in [KPV1].

LEMMA 2.2. Let  $1/2 < b < 1$ ,  $-1/2 < s < 1$ , and  $\delta \in (0, 1)$ . Then

$$\|\psi_\delta V(t) u_0\|_{X_{s,b}} \leq C\delta^{(1-2b)/2} \|u_0\|_{H^s}. \quad (2.3)$$

and

$$\left\| \psi_\delta \int_0^t V(t-t') F(t') dt' \right\|_{X_{s,b}} \leq C\delta^{(1-2b)/2} \|F\|_{X_{s,b-1}}. \quad (2.4)$$

*Proof.* Let  $\phi(\xi) = \xi^3 - l\xi |\xi|$ . To prove (2.3) we first notice that

$$\psi_\delta(t) V(t) u_0 = \psi(\delta^{-1}t) \iint e^{ix\xi + i\tau\delta(\tau + \phi(\xi))} \hat{u}_0(\xi) d\xi d\tau.$$

So

$$(\widehat{\psi_\delta(t) V(t) u_0})(\xi, \tau) = \delta \hat{\psi}(\delta(\tau + \phi(\xi))) \hat{u}_0(\xi).$$

From Plancherel's theorem we have

$$\begin{aligned} \|\psi_\delta V(t) u_0\|_{X_{s,b}}^2 &= C\delta^2 \iint (1 + |t + \phi(\xi)|)^{2b} (1 + |\xi|)^s \\ &\quad \times |\hat{\psi}(\delta(\tau + \phi(\xi)))|^2 |\hat{u}_0(\xi)|^2 d\tau d\xi \\ &= C \|u_0\|_{H^s}^2 \left( \int (1 + |\eta\delta|)^{2b} \hat{\psi}(\eta) d\eta \right) \end{aligned}$$

From this last inequality, estimate (2.3) follows.

To bound the left hand side of (2.4) we write it as

$$\begin{aligned}
 & \psi_{\delta} \int_0^t V(t-t') F(t') dt' \\
 &= \psi_{\delta}(t) \int_0^t \int e^{ix\xi - i(t-t')\phi(\xi)} \hat{F}^{(x)}(\xi, t') d\xi dt' \\
 &= \psi_{\delta}(t) \iint e^{ix\xi - it\phi(\xi)} \left( \int_0^t e^{it'\tau + it'\phi(\xi)} dt' \right) \hat{F}(\xi, \tau) d\xi d\tau \\
 &= \psi_{\delta}(t) \iint e^{ix\xi + it\tau} \frac{1 - e^{-it(\tau + \phi(\xi))}}{i(\tau + \phi(\xi))} \psi(\tau + \phi(\xi)) \hat{F}(\xi, \tau) d\xi d\tau \\
 &\quad + \psi_{\delta}(t) \iint e^{ix\xi + it\tau} \frac{1 - \psi(\tau + \phi(\xi))}{i(\tau + \phi(\xi))} \hat{F}(\xi, \tau) d\xi d\tau \\
 &\quad + \psi_{\delta}(t) \iint e^{ix\xi - it\phi(\xi)} \frac{1 - \psi(\tau + \phi(\xi))}{i(\tau + \phi(\xi))} \hat{F}(\xi, \tau) d\xi d\tau \\
 &= I_1 + I_2 + I_3.
 \end{aligned} \tag{2.5}$$

Now we estimate each term on the right hand side of (2.5). First we use the inequality (2.3) to bound  $I_3$ . So

$$\begin{aligned}
 \|I_3\|_{X_{s,b}} &\leq C\delta^{(1-2)/2} \left\| \left( \int \frac{1 - \psi(\tau + \phi(\xi))}{\tau + \phi(\xi)} \hat{F}(\xi, \tau) d\tau \right)^{\vee} \right\|_{H^s} \\
 &\leq C\delta^{(1-2)/2} \left( \int (1 + |\xi|)^s \left( \int_{|\tau + \phi(\xi)| \geq 1/2} \frac{|\hat{F}(\xi, \tau)|}{1 + |\tau + \phi(\xi)|} d\tau \right)^2 d\xi \right)^{1/2} \\
 &\leq C\delta^{(1-2)/2} \|F\|_{X_{s,b-1}}.
 \end{aligned} \tag{2.6}$$

The second term can be estimated using Lemma 2.1, thus

$$\begin{aligned}
 \|I_2\|_{X_{s,b}} &\leq c\delta^{(1-2b)/2} \left\| \iint e^{ix\xi + it\tau} \frac{1 - \psi(\tau + \phi(\xi))}{i(\tau + \phi(\xi))} \hat{F}(\xi, \tau) d\xi d\tau \right\|_{X_{s,b}} \\
 &\leq c\delta^{(1-2b)/2} \left( \iint \frac{|\hat{F}(\xi, \tau)|^2}{|\tau + \phi(\xi)|^2} (1 + |\tau + \phi(\xi)|)^{2b} (1 + |\xi|)^{2s} d\xi d\tau \right)^{1/2} \\
 &\leq C\delta^{(1-2)/2} \|F\|_{X_{s,b-1}}.
 \end{aligned} \tag{2.7}$$

Finally, by a Taylor expansion we write  $I_1$  as

$$I_1 = \sum_{k=1}^{\infty} \frac{i^k}{k!} t^k \psi_{\delta}(t) \int e^{ix\xi - it\phi(\xi)} \left( \int \hat{F}(\xi, \tau) (\tau + \phi(\xi))^{k-1} \psi(\tau + \phi(\xi)) d\tau \right) d\xi.$$

So using estimate (2.3) and the hypotheses on  $b$  we obtain the next chain of inequalities

$$\begin{aligned}
 \|I_1\|_{X_{s,b}} &\leq \sum_{k=1}^{\infty} \frac{\|t^k \psi_\delta\|_{\infty}}{k!} \left\| \left( \int \hat{F}(\xi, \tau) (\tau + \phi(\xi))^{k-1} \psi(\tau + \phi(\xi)) d\tau \right)^\vee \right\|_{H^s} \\
 &\leq C \delta^{(1-2b)} \sum_{k=1}^{\infty} \frac{\delta^k (k+1)}{k!} \left( \int (1 + |\xi|)^{2s} \right. \\
 &\quad \times \left. \left( \int_{|\tau + \phi(\xi)| \leq 1/2} \hat{F}(\xi, \tau) d\tau \right)^2 d\xi \right)^{1/2} \\
 &\leq C \delta^{(1-2b)} \left( \int (1 + |\xi|)^{2s} \left( \int \frac{\hat{F}(\xi, \tau)}{(1 + |\tau + \phi(\xi)|)^{1-b}} \right. \right. \\
 &\quad \times \left. \left. \frac{1}{(1 + |\tau + \phi(\xi)|)^b} d\tau \right)^2 d\xi \right)^{1/2} \\
 &\leq C \delta^{(1-2b)} \|F\|_{X_{s,b-1}}
 \end{aligned} \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) the estimate (2.4) follows. ■

As a consequence of this lemma we have the next regularity result.

**COROLLARY 2.3.** *Let  $b \in (1/2, 1)$  and  $\delta(0, 1)$ . Then*

$$\left\| \psi_\delta \int_0^t V(t-t') F(t') dt' \right\|_{L^\infty((0, T); H_x^s)} \leq C \delta^{(1-2)/2} \|F\|_{X_{s,b-1}}.$$

The next inequalities will be used to estimate the nonlinear terms. They are given in [KPV1] (Lemma 2.3).

**LEMMA 2.4.** *If  $b > 1/2$ , there exists  $C > 0$  such that*

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + |x - \alpha|)^{2b} (1 + |x - \beta|)^{2b}} \leq \frac{C}{(1 + |\alpha - \beta|)^{2b}}, \tag{2.9}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + |x|)^{2b} |\sqrt{a-x}|} \leq \frac{C}{(1 + |a|)^{1/2}}. \tag{2.10}$$



LEMMA 2.5. *If  $1/2 < b \leq 11/12$  and  $b' \in (1/2, b]$ , then*

$$\begin{aligned} & \frac{|\xi|}{(1 + |\tau + \phi(\xi)|)^{1-b}} \\ & \times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\xi_1 d\tau_1}{(1 + |\tau_1 + \phi(\xi_1)|)^{2b'} (1 + |(\tau - \tau_1) + \phi(\xi - \xi_1)|)^{2b'}} \right)^{1/2} \leq C. \end{aligned} \quad (2.11)$$

*Proof.* We take  $l=1$  to make an easier exposition of the argument of proof. The general case works similarly. So we begin applying (2.9) in Lemma 2.4 to obtain

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\xi_1 d\tau_1}{(1 + |\tau_1 + \phi(\xi_1)|)^{2b'} (1 + |(\tau - \tau_1) + \phi(\xi - \xi_1)|)^{2b'}} \right)^{1/2} \\ & \leq \left( \int_{-\infty}^{\infty} \frac{d\xi_1}{(1 + |\tau + \phi(\xi_1) + \phi(\xi - \xi_1)|)^{2b'}} \right)^{1/2} \\ & = \left( \int_{-\infty}^{\infty} \frac{d\xi_1}{(1 + |\tau - |\xi - \xi_1| (\xi - \xi_1) - |\xi_1| \xi_1 + \xi^3 - 3\xi^2 \xi_1 + 3\xi \xi_1^2)|)^{2b'}} \right)^{1/2} \\ & \equiv E(\xi, \tau). \end{aligned} \quad (2.12)$$

To estimate (2.12) we should distinguish the following cases

$$(i) \quad \xi \geq \xi_1, \quad \xi_1 \geq 0$$

$$(ii) \quad \xi \geq \xi_1, \quad \xi_1 < 0$$

$$(iii) \quad \xi < \xi_1, \quad \xi_1 \geq 0$$

$$(iv) \quad \xi < \xi_1, \quad \xi_1 < 0.$$

For case (i) we make the following change of variables in (2.12):

$$s = \tau - \xi^2 + \xi^3 + (2\xi - 3\xi^2) \xi_1 + (3\xi - 2) \xi_1^2,$$

$$ds = (2\xi_1 - \xi)(3\xi - 2) d\xi_1,$$

and

$$\xi_1 = \frac{1}{2} \left\{ \xi \pm \frac{\sqrt{-4\tau + 2\xi^2 - \xi^3 + 4s}}{\sqrt{|3\xi - 2|}} \right\}.$$

Then using inequality (10) in the previous lemma gives

$$\begin{aligned} E(\xi, \tau) &= \frac{1}{|3\xi - 2|^{1/4}} \left( \int_{-\infty}^{\infty} \frac{ds}{(1 + |s|)^{2b'} |\sqrt{-(-\xi^3 + 2\xi^2 + 4\tau) + 4s}|^{2b'}} \right)^{1/2} \\ &\leq \frac{C}{|3\xi - 2|^{1/4} (1 + |\tau - \xi^3/4 + \xi^2/2 + \tau|)^{1/4}}. \end{aligned}$$

Therefore, if  $b \leq 11/12$  we get

$$\begin{aligned} &\frac{|\xi|}{(1 + |\tau + \phi(\xi)|)^{1-b}} E(\xi, \tau) \\ &\leq \frac{|\xi|}{(1 + |\tau - \xi^2 + \xi^3|)^{1-b}} \frac{C}{|3\xi - 2|^{1/4} (1 + |\tau - \frac{1}{4}\xi^3 + \frac{1}{2}\xi^2|)^{1/4}} \\ &\leq \frac{C |\xi|^{3/4}}{(1 + |\tau - \xi^2 + \xi^3|)^{1-b} (1 + |\tau - \frac{1}{4}\xi^3 + \frac{1}{2}\xi^2|)^{1/4}} \leq C. \end{aligned}$$

*Case (ii).* The integral  $E(\xi, \tau)$  in (2.12) has the following expression

$$\left( \int_{-\infty}^{\infty} \frac{d\xi_1}{(1 + |\tau - \xi^2 + \xi^3 + 2\xi\xi_2 - 3\xi^2\xi_1 + 3\xi\xi_1^2|)^{2b'}} \right)^{1/2}. \quad (2.13)$$

Following the argument in the previous case we make the change of variables as

$$\begin{aligned} s &= \tau - \xi^2 + \xi^3 + (2\xi - 3\xi^2) \xi_1 + 3\xi\xi_1^2, \\ ds &= \xi(2 - 3\xi + 6\xi_1) d\xi_1, \end{aligned}$$

and use inequality (2.10) in Lemma 2.4 to obtain

$$E(\xi, \tau) \leq \frac{C}{|\xi|^{1/4} (1 + |\tau - \frac{5}{6}\xi^2 + \frac{3}{4}\xi^3|)^{1/4}}.$$

Thus if  $b \leq 11/12$  it follows that

$$\begin{aligned} &\frac{|\xi|}{(1 + |\tau + \phi(\xi)|)^{1-b}} E(\xi, \tau) \\ &\leq \frac{C |\xi|^{3/4}}{(1 + |\tau - \xi^2 + \xi^3|)^{1-b} (1 + |\tau - \frac{5}{6}\xi^2 + \frac{3}{4}\xi^3|)^{1/4}} \leq C. \end{aligned}$$

The cases (iii) and (iv) are treated similarly. Thus inequality (2.11) is proved. ■

The next lemma takes care of the nonlinear term in the Eq. (1).

LEMMA 2.6. *If  $u \in X_{s,b}$ , then there exists  $b > 1/2$  such that  $\partial_x(u^2) \in X_{s,b-1}$  and*

$$\|\partial_x(u^2)\|_{X_{s,b-1}} \leq C \|u\|_{X_{s,b}}^2. \quad (2.14)$$

*Proof.* We will prove the case  $s=0$ , the proof for the case  $s>0$  follows from this case therefore it will be omitted. Let  $f(\tau, \xi) = (1 + |\tau + \phi(\xi)|)^b \hat{u}(\tau, \xi)$ . Then

$$\begin{aligned} \|\partial_x(u^2)\|_{X_{0,b'-1}} &= \|(1 + |\tau + \phi(\xi)|)^{b'-1} \partial_x(u^2)\|_{L_\tau^2 L_\xi^2} \\ &= \left\| \frac{i\xi}{(1 + |\tau + \phi(\xi)|)^{1-b'}} \hat{u} * \hat{u} \right\|_{L_\tau^2 L_\xi^2} \\ &\leq \left\| \frac{|\xi|}{(1 + |\tau + \phi(\xi)|)^{1-b'}} \right\| \left\| \frac{f}{(1 + |\tau + \phi(\xi)|)^b} \right. \\ &\quad \left. * \frac{f}{(1 + |\tau + \phi(\xi)|)^b} \right\|_{L_\tau^2 L_\xi^2} \\ &\leq \frac{|\xi|}{(1 + |\tau + \phi(\xi)|)^{1-b'}} \\ &\quad \times \left( \iint \frac{d\tau_1 d\xi_1}{(1 + |\tau + \phi(\xi)|)^{2b} (1 + |\tau - \tau_1 - \phi(\xi + \xi_1)|)^{2b}} \right)^{1/2} \\ &\quad \times \|(|f|^2 * |f|^2)^{1/2}\|_{L_\tau^2 L_\xi^2} \\ &\leq C \|f\|_{L_\tau^2 L_\xi^2}^2. \end{aligned}$$

In the last inequality we used Lemma 2.5. The estimate (2.14) then follows. ■

LEMMA 2.7. Let  $u, v \in X_{s,b}$  with  $s \geq 0$ ,  $1/2 < b < 11/12$ . Then there exists a constant  $C > 0$  such that for  $b' < -1/6$

$$\|\partial_x(u^2) - \partial_x(v^2)\|_{X_{s,b'}} \leq C \|u + v\|_{X_{s,b}} \|u - v\|_{X_{s,b}}. \quad (2.15)$$

### 3. PROOF OF THEOREM 1.1

To prove Theorem 1.1 we will use the contraction mapping principle. Thus for  $u_0 \in H^s(\mathbb{R})$ ,  $s \geq 0$  we define the operator

$$\Phi_{u_0}(v) = \Phi(v) = \psi_1(t) V(t) u_0 - \psi_1(t) \int_0^t V(t-t') \psi_s(t') \partial_x(v^2(t')) dt'$$

and the set

$$\mathcal{B}_a = \{v \in X_{s,b} : \|v\|_{X_{s,b}} \leq a\},$$

where  $a = 2C \|u_0\|_{H^s}$  and  $\psi_\delta(t)$  is the cut-off function defined in Section 2.

To show that  $\Phi$  is a contraction on  $\mathcal{B}_a$ , we first prove that

$$\Phi(\mathcal{B}_a) \subset \mathcal{B}_a. \tag{3.1}$$

Using Lemmas (2.2), (2.1), and (2.6), we have for  $b < b' < 5/6$  and  $\theta = (b' - b)/4b'$  the next chain of inequalities

$$\begin{aligned} \|\Phi(v)\|_{X_{s,b}} &\leq c \|u_0\|_{H^s} + C \|\psi_\delta(t) \partial_x v^2(\cdot, t)\|_{X_{s,b-1}} \\ &\leq c \|u_0\|_{H^s} + C \delta^\theta \|\partial_x v^2(\cdot, t)\|_{X_{b'-1}} \\ &\leq c \|u_0\|_{H^s} + C \delta^\theta \|v\|_{X_{s,b}}^2. \end{aligned}$$

Thus we have

$$\|\Phi(v)\|_{X_{s,b}} \leq \frac{a}{2} + C \delta^\theta a^2.$$

Letting  $\delta^\theta \leq 1/2Ca$ , (3.1) follows.

On the other hand, if  $u, v \in \mathcal{B}_a$ , from inequality (2.15) in Lemma 2.7 it follows that

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{X_{s,b}} &\leq C \delta^\theta \|u + v\|_{X_{s,b}} \|u - v\|_{X_{s,b}} \\ &\leq 2Ca \delta^\theta \|u - v\|_{X_{s,b}} \\ &\leq \|u - v\|_{X_{s,b}}, \end{aligned}$$

where in the last inequality we use  $\delta^\theta \leq 1/2Ca$ . This shows that the map  $\Phi$  is a contraction on  $\mathcal{B}_a$ . Therefore we obtain a unique fixed point which solves the equation for  $T < \delta$ . ■

4. PERIODIC CASE

In this section we will give the needed estimates to establish Theorem 1.4. Most of the proofs of the results here follow the arguments given either by Bourgain in [Bo2] or by Kenig, Ponce, and Vega in [KPV2], so in several cases we will only refer to those works or simply give an sketch of the proof.

First we shall prove the estimate, corresponding to the bilinear estimate in [KPV2].

PROPOSITION 4.1.

$$\|\partial_x u^2\|_{Y_{0,-1/2}} \leq c \|u\|_{Y_{0,1/2}}^2. \quad (4.1)$$

If  $u \in Y_{0,b}$  then  $f(n, \tau) = (1 + |\tau - |n||n + n^3|)^b \hat{u}(n, \tau) \in L_\tau^2(\mathbb{R}; l_n^2(\mathbb{T}))$ . Thus the inequality (4.1) is equivalent to

$$\begin{aligned} & \left( \sum_{n \neq 0} \int_{-\infty}^{\infty} \frac{|n|^2}{(1 + |\tau - |n||n + n^3|)^{2(1-b)}} \right. \\ & \quad \times \left| \sum_{\substack{n_1 \neq 0 \\ n_1 \neq n}} \int \frac{f(n - n_1, \tau - \tau_1) f(n_1, \tau_1)}{\left[ \frac{(1 + |\tau_1 - |n_1||n_1 + n_1^3|)}{(1 + |\tau - \tau_1 - |n - n_1||n - n_1| + (n - n_1)^3|)} \right]} d\tau_1 \right|^2 \Big)^{1/2} \\ & \leq C \|f\|_{L_\tau^2(\mathbb{R}; l_n^2(\mathbb{T}))}^2. \end{aligned} \quad (4.2)$$

To prove inequality (4.2) we will make use of the next three lemmas. We first define

$$\begin{aligned} E(n, n_1) & \equiv (\tau - n|n| + n^3) - (\tau_1 - n_1|n_1| + n_1^3) \\ & \quad - (\tau - \tau_1 - (n - n_1)|n - n_1| + (n - n_1)^3) \\ & = -n|n| + n_1|n_1| + (n - n_1)|n - n_1| + 3nn_1(n - n_1). \end{aligned}$$

LEMMA 4.2. (i) If  $n \neq 0$ , then

$$|E(n, n_1)| \geq \frac{n^2}{4}. \quad (4.3)$$

(ii) There exists  $C > 0$  such that for any  $n \neq 0$  and  $\lambda \in \mathbb{R}$

$$\sum_{n \neq 0, n_1 \neq n} \frac{\log(2 + |\lambda - E(n, n_1)|)}{1 + |\lambda - E(n, n_1)|} \leq C. \quad (4.4)$$

*Proof.* The proof of (4.3) is a straight forward calculation. The proof of inequality (4.4) can be done using the same argument employed to prove inequality (5.2) in [KPV2]. ■

LEMMA 4.3

$$\frac{|n|}{(1 + |\tau - n| |n| + n^3)^{1/2}} \left( \sum_{n_1 \in A} \int_A \frac{d\tau_1}{\left[ (1 + |\tau_1 - n_1| |n_1| + n_1^3) (1 + |\tau - \tau_1 - (n - n_1)| |n - n_1| + (n - n_1)^3) \right]} \right)^{1/2} < C.$$

$$A = A(n, \tau) = \{ (n_1, \tau_1) : n_1 \neq 0, n_1 \neq n, \\ |\tau - \tau_1 - (n - n_1)| |n - n_1| + (n - n_1)^3 | \\ \leq |\tau_1 - n_1| |n_1| + n_1^3 \leq |\tau - n| |n| + n^3 \}.$$

*Proof.* It follows by combining (4.3) and (4.4) in Lemma 4.2. ■

LEMMA 4.4.

$$\frac{1}{(1 + |\tau_1 - n_1| |n_1| + n_1^3)^{1/2}} \times \left( \sum_{n \in B} \int_B \frac{|n|^2}{\left[ (1 + |\tau - n| |n| + n^3) (1 + |\tau - \tau_1 - (n - n_1)| |n - n_1| + (n - n_1)^3) \right]} d\tau \right)^{1/2} < C.$$

$$B = B(n_1, \tau_1) = \{ (n, \tau) : n \neq 0, n \neq n_1, |\tau - n| |n| + n^3 \leq |\tau_1 - n_1| |n_1| + n_1^3 \\ \text{and } |\tau - \tau_1 - (n - n_1)| |n - n_1| + (n - n_1)^3 \leq |\tau_1 - n_1| |n_1| + n_1^3 \}.$$

*Proof.* A similar argument as in Lemma 4.3 gives the result. ■

*Sketch of the Proof of Proposition 4.1.* The left hand side of inequality (4.2) is bounded by

$$\left( \sum_{n \neq 0} \int_{-\infty}^{\infty} \frac{n^2}{(1 + |\tau + \phi(n)|)} \right. \\ \times \left( \sum_{\substack{n_1 \neq 0 \\ n_1 \neq n}} \int_{-\infty}^{\infty} \frac{d\tau_1}{(1 + |\tau_1 + \phi(n_1)|)(1 + |\tau - \tau_1 + \phi(n - n_1)|)} \right) \\ \times \left( \sum_{\substack{n_1 \neq 0 \\ n_1 \neq n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(n_1, \tau_1) f(n - n_1, \tau - \tau_1)|^2 d\tau \right)^{1/2} \\ \leq \|G\|_{L_{\tau}^{\infty} l_n^{\infty}} \|f\|_{L_{\tau}^2 l_n^2}^2, \quad (4.5)$$

where

$$G(\tau, n) = \frac{n}{(1 + |\tau + \phi(n)|)^{1/2}} \times \left( \sum_{\substack{n_1 \neq 0 \\ n_1 \neq n}} \int_{-\infty}^{\infty} \frac{d\tau_1}{(1 + |\tau_1 + \phi(n_1)|)(1 + |\tau - \tau_1 + \phi(n - n_1)|)} \right)^{1/2}.$$

Next we apply Lemmas 4.3 and 4.4 to bound the  $L_\tau^\infty l_n^\infty$ -norm of the function  $G$ . Inequality (4.2) then follows. ■

As we saw above the bilinear estimate holds for  $b = 1/2$ , so we cannot use the same argument to obtain the regularity statement as we did for the the IVP (1.1). We need to estimate the  $\|\cdot\|_{L_t^\infty L_x^2}$  norm. To do so we use the version of Proposition 7.15 and Lemma 7.42 in Bourgain [Bo2] for our case.

LEMMA 4.5. *For functions on  $\mathbb{T}^2$  the estimate*

$$\|f\|_{L^4(\mathbb{T}^2)} \leq C \left( \sum_{m, n \in \mathbb{Z}} (1 + |n - m| m + m^3)^{2/3} |\hat{f}(m, n)|^2 \right)^{1/2} \quad (4.6)$$

holds.

*Proof.* The proof follows the same argument as in [Bo2] so it will be omitted. ■

LEMMA 4.6.

$$\sum_{n \neq 0} \left( \int \frac{|\widehat{\partial_x u^2}(n, \tau)|}{1 + |\tau - |n|| n + n^3|} d\tau \right)^2 \leq C \|u\|_{Y_{s, 1/2}}^2. \quad (4.7)$$

*Proof.* Once estimates (4.1) and (4.6) have been established the proof follows the same argument used by Bourgain in [Bo2, Lemma 7.42]. ■

Finally we have

LEMMA 4.7

$$\left\| \psi(t) \int_0^t V(t - t') \partial_x u^2(t') dt' \right\|_{L^2} \leq C \|u\|_{Y_{0, 1/2}}^2. \quad (4.8)$$

*Proof.* We write the Fourier transform of the expression on the left hand side of (4.8) as

$$\begin{aligned} &\psi(t) \sum_{n \neq 0} e^{inx} \int_{-\infty}^{\infty} \frac{e^{i\tau t} - e^{-it\phi(n)}}{\tau + \phi(n)} \widehat{\partial_x u^2}(n, \tau) \, d\tau. \\ &= \psi(t) \sum_{n \neq 0} e^{inx} \int_{-\infty}^{\infty} e^{i\tau t} \frac{1 - e^{-it(\tau + \phi(n))}}{\tau + \phi(n)} \psi(\tau + \phi(n)) \widehat{\partial_x u^2}(n, \tau) \, d\tau \\ &\quad + \psi(t) \sum_{n \neq 0} e^{inx} \int_{-\infty}^{\infty} \frac{e^{i\tau} - e^{-it\phi(n)}}{\tau + \phi(n)} (1 - \psi(\tau + \phi(n))) \widehat{\partial_x u^2}(n, \tau) \, d\tau \\ &= \sum_{j=1}^{\infty} \frac{(i)^j}{j!} t^j \psi(t) \left\{ \sum_{n \neq 0} e^{inx} \int_{-\infty}^{\infty} \psi(\tau + \phi(n)) (\tau + \phi(n))^{j-1} \widehat{\partial_x u^2}(n, \tau) \, d\tau \right\} \\ &\quad + \psi(t) \sum_{n \neq 0} e^{inx} \int_{-\infty}^{\infty} \frac{e^{i\tau} - e^{-it\phi(n)}}{\tau + \phi(n)} (1 - \psi(\tau + \phi(n))) \widehat{\partial_x u^2}(n, \tau) \, d\tau \\ &= S_1 + S_2. \end{aligned}$$

The same argument as in Lemma 2.3 (2.8) gives

$$\begin{aligned} \|S_1\|_{L^2(\mathbb{T})} &\leq C \left( \sum_{n \neq 0} \left( \int_{|\tau + \phi(n)| \leq 1} |\widehat{\partial_x u^2}(n, \tau)| \, d\tau \right)^2 \right)^{1/2} \\ &\leq C \sum_{n \neq 0} \left( \int \frac{|\widehat{\partial_x u^2}(n, \tau)|}{1 + |\tau + \phi(n)|} \, d\tau \right)^2 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \|S_2\|_{L^2(\mathbb{T})} &\leq C \left( \sum_{n \neq 0} \left( \int_{|\tau + \phi(n)| \geq 1/2} \frac{|\widehat{\partial_x u^2}(n, \tau)|}{|\tau + \phi(n)|} \, d\tau \right)^2 \right)^{1/2} \\ &\leq C \left( \sum_{n \neq 0} \left( \int \frac{|\widehat{\partial_x u^2}(n, \tau)|}{1 + |\tau + \phi(n)|} \, d\tau \right)^2 \right)^{1/2}. \end{aligned} \tag{4.10}$$

Applying Lemma 4.6 in (4.9) and (4.10) the lemma follows. ■

The proof of Theorem 1.4 now follows the same argument used in the proof of Theorem 1.1.

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