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Summability of divergent solutions of the n-dimensional heat equation

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Abstract

The Cauchy problem for n-dimensional complex heat equation is considered. The Borel summability of formal solutions is characterized in terms of analytic continuation with an appropriate growth condition of the spherical mean of the Cauchy data.

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1. Introduction

For the last several years the theory of the summability or the multisummability of the formal series has been developed. The application of this theory to ordinary differential equations gives very interesting results. In particular, it was proved that formal solutions of meromorphic differential equations are multisummable (see Braaksma [6]). It means that for every formal solution of such equation, one can derive an analytic solution in some sector, having the formal one as its asymptotic expansion.

For partial differential equation this problem is more complicated. The first result on summability of formal solutions was obtained for the heat equation by Lutz, Miyake and Schäfke [11] and generalized by Balser [1]. In the subsequent papers, analogous questions for more general

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classes of linear partial differential equations were studied by Balser [3], Balser and Malek [4], Balser and Miyake [5], Hibino [8], Ichinobe [9,10], Miyake [13], Ōuchi [14] and others.

In the paper we consider the Cauchy problem for the n-dimensional complex heat equation

$$\partial_\tau u(\tau, z) = \Delta_z u(\tau, z), \quad u(0, z) = \varphi(z), \tag{1}$$

where $\tau \in \mathbb{C}$, $z \in \mathbb{C}^n$, $\Delta_z := \sum_{i=1}^n \partial_{z_i}^2$ and $\varphi(z)$ is analytic in a complex neighbourhood $G \subseteq \mathbb{C}^n$ of the origin.

The Cauchy problem (1) has the unique formal solution

$$\hat{u}(\tau, z) = \sum_{k=0}^{\infty} u_k(z)\tau^k \quad \text{with } u_k(z) := \frac{\Delta^k \varphi(z)}{k!}, \tag{2}$$

which diverges for a general initial condition $\varphi(z)$. Precisely, the formal solution (2) convergences if and only if the Cauchy data $\varphi(z)$ is an entire function of exponential order at most 2 (see Introduction in [11]).

The aim of this paper is answer to the question: under what conditions on the Cauchy data is the formal solution (2) Borel summable? In the one-dimensional case this problem was solved by Lutz, Miyake and Schäfke [11]. They showed that for $n = 1$ the formal solution (2) is 1-summable in a direction θ if and only if $\varphi(z)$ can be analytically continued to infinity in some sectors in directions $\theta/2$ and $\pi + \theta/2$ and the continuation is of exponential size at most 2 when $z \to \infty$ in these directions. Similar result for more general initial data was given by W. Balser [1]. The multidimensional heat equation was investigated by Balser and Malek [4], but their result is not stated immediately in the term of the initial data.

In our paper we generalize the results of Lutz, Miyake and Schäfke [11] to the higher spatial dimensions as follows:

The formal solution (2) *of the Cauchy problem* (1) *is* 1*-summable in a direction* θ *if and only if the function*

$$\Phi_n(\tau, z) = \begin{cases} \int_{\partial B^n(1)} \varphi(z + \tau x)\, dS(x) & \text{if } n \text{ is odd}, \\ \int_{B^n(1)} \frac{\varphi(z+\tau x)\, dx}{\sqrt{1-|x|^2}} & \text{if } n \text{ is even}, \end{cases}$$

is analytically continued to infinity in some sectors in directions $\theta/2$ *and* $\pi + \theta/2$ (*with respect to* τ) *and to some ball with a center at origin* (*with respect to* z) *and this continuation is of exponential growth of order at most two as* $\tau \to \infty$.

For the precise formulation see Corollary 1.

Following Balser (see [1–5]) we shall use the modified Borel transform of $\hat{u}(\tau, z)$ instead of the Borel transform. This modified transform is more suitable for a study of formal solutions of PDE. The crucial fact is that after appropriate change of variables this transform satisfies the wave equation.

2. Notation

The complex (respectively real) ball with a center at $z_0 \in \mathbb{C}^n$ (respectively $x_0 \in \mathbb{R}^n$) and a radius $r > 0$ is denoted by $D^n(z_0, r) := \{z \in \mathbb{C}^n\colon\ |z - z_0| < r\}$ (respectively $B^n(x_0, r) := \{x \in \mathbb{R}^n\colon\ |x - x_0| < r\}$), where $|\cdot|$ is the Euclidean norm $|z| := \sqrt{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n}$ in $\mathbb{C}^n$ (respectively $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$ in $\mathbb{R}^n$). To simplify notation we write $D(z_0, r)$ (respectively $B(x_0, r)$) for $n = 1$, $D^n(r)$ and $D(r)$ (respectively $B^n(r)$ and $B(r)$) for $z_0 = 0$ (respectively $x_0 = 0$).

The mean values of a function f over a ball $B^n(r)$ and over a sphere $\partial B^n(r)$ are denoted by

$$\fint_{B^n(r)} f(x)\,dx := \frac{1}{\alpha(n)r^n} \int_{B^n(r)} f(x)\,dx$$

and

$$\fint_{\partial B^n(r)} f(y)\,dS(y) := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B^n(r)} f(y)\,dS(y),$$

where $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of the unit ball $B^n(1)$ and $n\alpha(n)$ is the surface of the unit sphere $S^{n-1} := \partial B^n(1)$.

For $\theta \in \mathbb{R}$, $\varepsilon > 0$ and $\delta > 0$ we set $E_+(\theta, \varepsilon) := \{s \in \mathbb{C}\colon\ \operatorname{dist}(s, e^{i\theta}\mathbb{R}_+) < \varepsilon\}$ and $\Omega(\theta/2, \delta) := \{z \in \mathbb{C}\colon\ \operatorname{dist}(z, e^{i\theta/2}\mathbb{R}) < \delta\}$.

A sector in the universal covering space of $\mathbb{C} \setminus \{0\}$ is denoted by

$$S(\theta, \alpha, T) := \bigl\{z \in \mathbb{C}\colon\ z = re^{i\varphi},\ \theta - \alpha/2 < \varphi < \theta + \alpha/2,\ 0 < r < T\bigr\}$$

for $\theta \in \mathbb{R}$, $\alpha > 0$ and $0 < T \leqslant +\infty$. In case $T = +\infty$ we denote it briefly by $S(\theta, \alpha)$. A sector S' is called a *proper subsector* of $S(\theta, \alpha, T)$ if its closure in $\mathbb{C}$ is contained in $S(\theta, \alpha, T) \cup \{0\}$.

Sometimes we will denote a point $x \in \mathbb{R}^{n+1}$ by $(x', x_{n+1}) = (x_1, \ldots, x_n, x_{n+1})$ and $z \in \mathbb{C}^{n+1}$ by $(z', z_{n+1}) = (z_1, \ldots, z_n, z_{n+1})$.

By $\mathcal{O}(D)$ we denote the space of analytic functions on a domain $D \subseteq \mathbb{C}^n$. The Banach space of analytic functions on $D^n(r)$, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z|\leqslant r} |\varphi(z)|$ is denoted by $\mathbb{E}_n(r)$.

3. Borel summability

We recall some fundamental facts about the Borel summability following [2,11].

Definition 1. We say that a function $u(t, z) \in \mathcal{O}(S(\theta, \varepsilon) \times D^n(r))$ (respectively $u(t, z) \in \mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$, $u(t, z) \in \mathcal{O}(\Omega(\theta, \varepsilon) \times D^n(r))$) is of *exponential growth of order at most k as $t \to \infty$ in $S(\theta, \varepsilon)$* (respectively *in $E_+(\theta, \varepsilon)$*, *in $\Omega(\theta, \varepsilon)$*) if and only if for any $r_1 \in (0, r)$ and any $\varepsilon_1 \in (0, \varepsilon)$ there exist positive constants C and B such that

$$\max_{|z|\leqslant r_1}\big|u(t,z)\big| < Ce^{B|t|^k} \quad \text{for every } t\in S(\theta,\varepsilon_1) \quad \big(\text{respectively } t\in E_+(\theta,\varepsilon_1),\ t\in\Omega(\theta,\varepsilon_1)\big).$$

If $k=1$, we say for short that $u(t,z)$ is of *exponential growth as* $t\to\infty$ *in* $S(\theta,\varepsilon)$ (respectively *in* $E_+(\theta,\varepsilon)$, *in* $\Omega(\theta,\varepsilon)$).

Analogously, we say that a function $\varphi(z)\in\mathcal{O}(S(\theta,\varepsilon))$ (respectively $\varphi(z)\in\mathcal{O}(\Omega(\theta,\varepsilon))$) is of *exponential growth of order at most k as* $z\to\infty$ *in* $S(\theta,\varepsilon)$ (respectively *in* $\Omega(\theta,\varepsilon)$) if and only if for any $\varepsilon_1\in(0,\varepsilon)$ there exist positive constants C and B such that

$$\big|\varphi(z)\big| < Ce^{B|z|^k} \quad \text{for every } z\in S(\theta,\varepsilon_1) \quad \big(\text{respectively } z\in\Omega(\theta,\varepsilon_1)\big).$$

If $k=1$, we say for short that $\varphi(z)$ is of *exponential growth as* $z\to\infty$ *in* $S(\theta,\varepsilon)$ (respectively in $\Omega(\theta,\varepsilon)$).

Definition 2. Fix any $r>0$. We say that

$$\hat{u}(\tau,z) := \sum_{k=0}^{\infty} u_k(z)\tau^k \quad \text{with } u_k(z)\in\mathbb{E}_n(r),$$

is 1-*Gevrey formal power series* if its coefficients satisfy

$$\max_{|z|\leqslant r}\big|u_k(z)\big| \leqslant AB^k k! \quad \text{for } k=0,1,\ldots,$$

with some positive constants A and B.

The set of 1-Gevrey formal power series in τ over $\mathbb{E}_n(r)$ is denoted by $\mathbb{E}_n(r)[\![\tau]\!]_1$. We also set $\mathbb{E}_n[\![\tau]\!]_1 := \bigcup_{r>0}\mathbb{E}_n(r)[\![\tau]\!]_1$.

Definition 3. Let $\theta\in\mathbb{R}$, $\alpha>0$ and $u(\tau,z)\in\mathcal{O}(S(\theta,\alpha,T)\times D^n(r))$ with some $r>0$ and $T>0$. Then $\hat{u}(\tau,z)\in\mathbb{E}_n[\![\tau]\!]_1$ is called the *Gevrey asymptotic expansion of* $u(\tau,z)$ *in* $S(\theta,\alpha)$ if for any proper subsector $S'\subset S(\theta,\alpha,T)$ there exist positive constants A, B and $r_1\in(0,r)$ such that $\hat{u}(\tau,z)\in\mathbb{E}_n(r_1)[\![\tau]\!]_1$ and

$$\max_{|z|\leqslant r_1}\left|u(\tau,z)-\sum_{k=1}^{K-1}u_k(z)\tau^k\right| \leqslant AB^K K!|\tau|^K \quad \text{for } \tau\in S',\ K=1,2,\ldots. \tag{3}$$

If we take $\alpha\leqslant\pi$ then, according to Ritt's theorem for Gevrey asymptotics (see [2, Proposition 10]), for every $\hat{u}(\tau,z)\in\mathbb{E}_n[\![\tau]\!]_1$ there exists an analytic function $u(\tau,z)$ such that $\hat{u}(\tau,z)$ is the Gevrey asymptotic expansion of $u(\tau,z)$. However, this $u(\tau,z)$ is not unique.

On the other hand, if $\alpha>\pi$ and if $\hat{u}(\tau,z)\in\mathbb{E}_n[\![\tau]\!]_1$ is the Gevrey asymptotic expansion of some $u(\tau,z)$ then, by Watson's lemma (see [2, Proposition 11]), this $u(\tau,z)$ is unique. This observation motivates the following definition.

Definition 4. We say that $\hat{u}(\tau,z)\in\mathbb{E}_n[\![\tau]\!]_1$ is 1-*summable in* $S(\theta,\alpha)$ if $\alpha>\pi$ and if there exist $r>0$, $T>0$ and $u(\tau,z)\in\mathcal{O}(S(\theta,\alpha,T)\times D^n(r))$ such that $\hat{u}(\tau,z)$ is the Gevrey asymptotic expansion of $u(\tau,z)$ in $S(\theta,\alpha)$. This $u(\tau,z)$ is called the 1-*sum of* $\hat{u}(\tau,z)$ *in* $S(\theta,\alpha)$.

We say that $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is 1-*summable in a direction* θ if $\hat{u}(\tau, z)$ is 1-summable in $S(\theta, \alpha)$ for some $\alpha > \pi$.

The 1-summability can be characterized as follows:

Proposition 1. (See [2, Theorem 33].) *Let $\theta \in \mathbb{R}$, $\alpha > \pi$. A formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is* 1-*summable in $S(\theta, \alpha)$ if and only if its Borel transform*

$$\hat{v}(s, z) := \sum_{k=0}^{\infty} u_k(z) \frac{s^k}{k!}$$

is analytic in $S(\theta, \alpha - \pi) \times D^n(r)$ (for some $r > 0$) and is of exponential growth as $s \to \infty$ in $S(\theta, \alpha - \pi)$. The 1-*sum of $\hat{u}(\tau, z)$ in $S(\theta, \alpha)$ is represented by the Laplace transform of $\hat{v}(s, z)$,*

$$u^{\varphi}(\tau, z) := \frac{1}{\tau} \int_0^{e^{i\varphi}\infty} e^{-s/\tau} \hat{v}(s, z)\, ds,$$

where the integration is taken over the ray $e^{i\varphi}\mathbb{R}_+ := \{re^{i\varphi}\colon r \geqslant 0\}$ for $\varphi \in (\theta - \alpha + \pi, \theta + \alpha - \pi)$.

In the case of $\alpha = \pi$, a function $u(\tau, z)$ satisfying a Gevrey asymptotic expansion (3) of $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is not unique. One can remove this inconvenience replacing subsectors by open balls.

Definition 5. We say that $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is *fine* 1-*summable in a direction* θ if there exist $T > 0$, $r > 0$ and $u(\tau, z) \in \mathcal{O}(D(Te^{i\theta}, T) \times D^n(r))$ such that for some $T' \in (0, T)$ and for every $r_1 \in (0, r)$,

$$\max_{|z| \leqslant r_1} \left| u(\tau, z) - \sum_{k=1}^{K-1} u_k(z) \tau^k \right| \leqslant AB^K K! |\tau|^K \quad \text{for } \tau \in D\big(T'e^{i\theta}, T'\big),\ K = 1, 2, \ldots,$$

with some positive constants A and B. This $u(\tau, z)$ is called the *fine* 1-*sum of $\hat{u}(\tau, z)$ in a direction θ*.

As in the 1-summable case, one can prove that if $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is fine 1-summable in a direction θ then its fine 1-sum is unique (see [12, Section 1.4.2]).

The fine 1-summability is also characterized as follows:

Proposition 2. (See [12, Theorem 1.4.2.1].) *A formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is fine* 1-*summable in a direction θ if and only if its Borel transform $\hat{v}(s, z)$ belongs to $\mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth as $s \to \infty$ in $E_+(\theta, \varepsilon)$. The fine* 1-*sum of $\hat{u}(\tau, z)$ in $S(\theta, \alpha)$ is represented by the Laplace transform of $\hat{v}(s, z)$ in a direction θ.*

Since $1/4^k \leqslant (k!)^2/(2k)! \leqslant 1$ for every $k \in \mathbb{N}$, according to the general theory of moment summability (see [2, Section 6.5]) a formal series $\hat{u}(\tau, z) = \sum u_k(z)\tau^k$ is 1-summable in $S(\theta, \alpha)$ (respectively 1-fine summable in a direction θ) if and only if the same holds for the series

$$\sum u_k(z)\frac{(k!)^2}{(2k)!}\tau^k.$$

Consequently, we obtain analogous characterization of 1-summability and of fine 1-summability as in Propositions 1 and 2 if we replace the Borel transform by the *modified Borel transform*

$$v(s, z) := \sum_{k=0}^{\infty} u_k(z)\frac{k!s^k}{(2k)!} \tag{4}$$

and the Laplace transform by the *Ecalle acceleration operator*

$$u^{\varphi}(\tau, z) = \frac{1}{\sqrt{\tau}} \int_0^{e^{i\varphi}\infty} v(s, z)C_2(\sqrt{s/\tau}\,)\,d\sqrt{s}. \tag{5}$$

The integration in the last formula is taken over the ray $e^{i\varphi}\mathbb{R}_+$ for $\varphi \in (\theta - \alpha + \pi, \theta + \alpha - \pi)$ and $C_2(\zeta)$ is defined by

$$C_2(\zeta) := \frac{1}{2\pi i}\int_{\gamma} \frac{e^{u-\zeta\sqrt{u}}}{\sqrt{u}}\,du$$

with a path of integration γ as in the Hankel integral for the inverse Gamma function: from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray.

Hence the modified version of Proposition 2 we can formulate as follows:

Proposition 3. *A formal series $\hat{u}(\tau, z) \in \mathbb{E}_n[\![\tau]\!]_1$ is fine 1-summable in a direction θ if and only if its modified Borel transform $v(s, z)$ belongs to $\mathcal{O}(E_+(\theta, \varepsilon) \times D^n(r))$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth as $s \to \infty$ in $E_+(\theta, \varepsilon)$. The fine 1-sum of $\hat{u}(\tau, z)$ in $S(\theta, \alpha)$ is represented by the Ecalle acceleration operator (5) of $v(s, z)$ in a direction θ.*

4. The main result

In the proof of the main theorem we shall use the following auxiliary lemmas.

Lemma 1. *Let $\varphi(z)$ be analytic in a complex neighbourhood of the origin. Then the formal solution (2) of the Cauchy problem (1) is 1-Gevrey formal power series. Moreover, if the Cauchy data $\varphi(z) \in \mathcal{O}(D^n(\tilde{r}))$ then for any $r \in (0, \tilde{r})$ the formal solution $\hat{u}(\tau, z) \in \mathbb{E}_n(r)[\![\tau]\!]_1$.*

Proof. Take $\tilde{r} > 0$ such that $\varphi(z) \in \mathcal{O}(D^n(\tilde{r}))$. We need to show that for any $r \in (0, \tilde{r})$ the formal solution $\hat{u}(\tau, z) \in \mathbb{E}_n(r)[\![\tau]\!]_1$. To this end take any $r, r_1 \in (0, \tilde{r})$, $r < r_1$, and put $\epsilon := \frac{r_1 - r}{\sqrt{n}+1}$. Observe that for any $z \in D^n(r)$ the set $\{\zeta \in \mathbb{C}^n\colon\ |\zeta_i - z_i| = \epsilon \text{ for } i = 1, \ldots, n\}$ is contained in $D^n(r_1)$. Hence, by the Cauchy integral formula, the coefficients $(u_k(z))_{k=0}^{\infty}$ of the formal solution (2) satisfy

$$\begin{aligned}
\max_{|z|\leqslant r}\left|u_k(z)\right| &= \max_{|z|\leqslant r}\left|\frac{(\partial_{z_1}^2+\cdots+\partial_{z_n}^2)^k\varphi(z)}{k!}\right| \\
&\leqslant \max_{|z|\leqslant r}\sum_{\substack{i_1+\cdots+i_n=k\\ i_1,\ldots,i_n\geqslant 0}}\frac{1}{i_1!\cdots i_n!}\left|\partial_{z_1}^{2i_1}\cdots\partial_{z_n}^{2i_n}\varphi(z)\right| \\
&\leqslant \max_{|z|\leqslant r}\sum_{\substack{i_1+\cdots+i_n=k\\ i_1,\ldots,i_n\geqslant 0}}\frac{(2i_1)!\cdots(2i_n)!}{i_1!\cdots i_n!(2\pi)^n} \\
&\quad\times\left|\int\limits_{|\zeta_1-z_1|=\epsilon}\cdots\int\limits_{|\zeta_n-z_n|=\epsilon}\frac{\varphi(\zeta)}{(\zeta_1-z_1)^{2i_1+1}\cdots(\zeta_n-z_n)^{2i_n+1}}\,d\zeta\right| \\
&\leqslant \sum_{\substack{i_1+\cdots+i_n=k\\ i_1,\ldots,i_n\geqslant 0}}\frac{4^k i_1!\cdots i_n!}{\epsilon^{2k}}\max_{|z|\leqslant r_1}\left|\varphi(z)\right| \leqslant \max_{|z|\leqslant r_1}\left|\varphi(z)\right|\left(\frac{4n}{\epsilon^2}\right)^k k! = AB^k k!
\end{aligned}$$

for $k=0,1,\ldots,$ with positive constants $A=\max_{|z|\leqslant r_1}|\varphi(z)|$ and $B=4n/\epsilon^2$. □

Lemma 2. *Let $v(s,z)$ be the modified Borel transform* (4) *of the formal solution* (2) *of the Cauchy problem* (1). *Then $w(\tau,z):=v(\tau^2,z)$ is a solution of the Cauchy problem for the complex n-dimensional wave equation*

$$\partial_\tau^2 w(\tau,z)=\Delta_z w(\tau,z),\quad w(0,z)=\varphi(z),\ \partial_\tau w(0,z)=0. \tag{6}$$

Moreover, if for some $z\in\mathbb{C}^n$ the function $\tau\mapsto w(\tau,z)$ is analytic on $D(r)$ then the function $s\mapsto v(s,z)$ is analytic on $D(r^2)$.

Proof. Notice that by (2) and (4)

$$v(s,z)=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}s^k,$$

hence

$$w(\tau,z)=v\big(\tau^2,z\big)=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}\tau^{2k}.$$

A trivial verification shows that $w(\tau,z)$ is a solution of the Cauchy problem for the wave equation (6).

Take $z\in\mathbb{C}^n$ such that $\tau\mapsto w(\tau,z)$ is analytic on $D(r)$. Hence, by the Cauchy–Hadamard formula,

$$\frac{1}{\limsup_{k\to\infty}\sqrt[2k]{\frac{|\Delta^k\varphi(z)|}{(2k)!}}}\geqslant r,$$

and consequently

$$\frac{1}{\limsup_{k\to\infty}\sqrt[k]{\frac{|\Delta^k\varphi(z)|}{(2k)!}}}\geqslant r^2,$$

which implies that the function $s\mapsto v(s,z)$ is analytic on $D(r^2)$. $\square$

Now, we are ready to formulate the main result

Theorem 1 *(The main theorem). The formal solution $\hat{u}(\tau,z)$ given by* (2) *is fine* 1*-summable in a direction θ if and only if there exists $\delta>0$ such that*:

- *For $n=1$, the Cauchy data $\varphi(z)$ is analytically continued to $\Omega(\theta/2,\delta)$ and is of exponential growth of order at most* 2 *as $z\to\infty$ in $\Omega(\theta/2,\delta)$.*
- *For $n>1$ and any $\tilde{\delta}\in(0,\delta)$, the function*

$$\Phi_n(\tau,z)=\begin{cases}\int_{\partial B^n(1)}\varphi(z+\tau x)\,dS(x) & \text{for } n \text{ odd},\\ \int_{B^n(1)}\frac{\varphi(z+\tau x)\,dx}{\sqrt{1-|x|^2}} & \text{for } n \text{ even},\end{cases}\tag{7}$$

is analytically continued to $\Omega(\theta/2,\delta-\tilde{\delta})\times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 *as $\tau\to\infty$ in $\Omega(\theta/2,\delta-\tilde{\delta})$.*

Proof. The one-dimensional part of this theorem was proved by Lutz, Miyake and Schäfke [11]. For the completeness we give a new proof of their result.

($\Leftarrow$) For $n=1$. By Lemma 2 and the d'Alembert formula we have

$$v\big(\tau^2,z\big)=w(\tau,z)=\frac{1}{2}\big[\varphi(z+\tau)+\varphi(z-\tau)\big].$$

Fix $r\in(0,\delta)$. The Cauchy data $\varphi(z)$ is analytic on $\Omega(\theta/2,\delta)$, hence for any $z\in D(r)$ the function $\tau\mapsto w(\tau,z)$ is analytic on $\Omega(\theta/2,\delta-r)$. Moreover, for any $r_1\in(0,r)$ and any $\varepsilon_1\in(0,\delta-r)$,

$$\begin{aligned}\max_{|z|\leqslant r_1}\big|w(\tau,z)\big|&=\max_{|z|\leqslant r_1}\frac{1}{2}\big|\varphi(z+\tau)+\varphi(z-\tau)\big|\\&\leqslant\max_{|z|\leqslant r_1}\frac{1}{2}\big|Ce^{\widetilde{B}|z+\tau|^2}+Ce^{\widetilde{B}|z-\tau|^2}\big|\leqslant Ce^{B|\tau|^2},\quad \tau\in\Omega(\theta/2,\varepsilon_1),\end{aligned}$$

with some positive constants B, $\widetilde{B}$ and C. Since $v(s,z)=w(\sqrt{s},z)=w(-\sqrt{s},z)$, it follows that $v(s,z)$ is analytic on

$$\big(\big\{s\in\mathbb{C}\colon \operatorname{dist}\big(\sqrt{s},e^{i\theta/2}\mathbb{R}\big)<\delta-r\big\}\cap S(\theta,\pi/2)\big)\times D(r).$$

On the other hand, by Lemma 2 and observation that $w(\tau,z)\in\mathcal{O}(D(\delta-r)\times D(r))$, for any $z\in D(r)$ the function $s\mapsto v(s,z)$ is analytic on $D((\delta-r)^2)$. Consequently, $v(s,z)$ is analytic on $E_+(\theta,\varepsilon)\times D(r)$ with $\varepsilon:=(\delta-r)^2$ and for any $r_1\in(0,r)$ and any $\varepsilon_1\in(0,\varepsilon)$,

$$\max_{|z|\leqslant r_1}\left|v(s,z)\right|=\max_{|z|\leqslant r_1}\left|w(\pm\sqrt{s},z)\right|\leqslant Ce^{B|s|},\quad s\in E_+(\theta,\varepsilon_1),$$

with some positive constants B and C. The assertion follows by Proposition 3.

For $n>1$. By Lemma 2 and the generalization of the Kirchhoff and Poisson formula (see Evans [7]), we have:

(a) for $n=2k+1$, $k=1,2,\dots,$

$$\begin{aligned}v\left(\tau^2,z\right)=w(\tau,z)&=\frac{1}{(n-2)!!}\partial_\tau\left(\tau^{-1}\partial_\tau\right)^{\frac{n-3}{2}}\left(\tau^{n-2}\fint_{\partial B^n(1)}\varphi(z+\tau x)\,dS(x)\right)\\&=\frac{1}{\alpha(n)n!!}\partial_\tau\left(\tau^{-1}\partial_\tau\right)^{\frac{n-3}{2}}\tau^{n-2}\Phi_{2k+1}(\tau,z);\end{aligned}$$

(b) for $n=2k$, $k=1,2,\dots,$

$$\begin{aligned}v\left(\tau^2,z\right)=w(\tau,z)&=\frac{1}{n!!}\partial_\tau\left(\tau^{-1}\partial_\tau\right)^{\frac{n-2}{2}}\left(\tau^{n-1}\fint_{B^n(1)}\frac{\varphi(z+\tau x)\,dx}{\sqrt{1-|x|^2}}\right)\\&=\frac{1}{\alpha(n)n!!}\partial_\tau\left(\tau^{-1}\partial_\tau\right)^{\frac{n-2}{2}}\tau^{n-1}\Phi_{2k}(\tau,z).\end{aligned}$$

Fix $r,\tilde r\in(0,\delta)$, $r<\tilde r$. By assumption, $\Phi_n(\tau,z)$ given by (7) is analytic on $D(\delta-\tilde r)\times D^n(\tilde r)$. In particular, the function

$$z\mapsto\Phi_n(0,z)=\begin{cases}\int_{\partial B^n(1)}\varphi(z)\,dS(x)=n\alpha(n)\varphi(z) & \text{for } n \text{ odd},\\ \int_{B^n(1)}\frac{\varphi(z)\,dx}{\sqrt{1-|x|^2}}=\frac{n+1}{2}\alpha(n+1)\varphi(z) & \text{for } n \text{ even},\end{cases}$$

is analytic on $D^n(\tilde r)$. Hence $\varphi(z)\in\mathcal{O}(D^n(\tilde r))$. Therefore, by Lemma 1, the formal solution $\hat u(\tau,z)$ given by (2) belongs to $\mathbb{E}_n(r)[\![\tau]\!]_1$.

Since $\Phi_n(\tau,z)$ is analytic on $\Omega(\theta/2,\delta-r)\times D^n(r)$, we see that $w(\tau,z)\in\mathcal{O}(\Omega(\theta/2,\delta-r)\times D^n(r))$. We show that $w(\tau,z)$ is of exponential growth of order at most 2 as $\tau\to\infty$ in $\Omega(\theta/2,\delta-r)$. To this end take any $m\in\mathbb{N}$. By assumption on the growth of $\Phi_n(\tau,z)$ and by the Cauchy inequalities for holomorphic functions, we have for any $r_1\in(0,r)$ and any $\varepsilon_1\in(0,\delta-r)$,

$$\begin{aligned}\max_{|z|\leqslant r_1}\left|\partial_\tau^m\Phi_n(\tau,z)\right|&\leqslant\max_{|z|\leqslant r_1}\max_{|\tau-\tilde\tau|=\frac{\delta-r-\varepsilon_1}{2}}\frac{|\Phi_n(\tilde\tau,z)|}{\left(\frac{\delta-r-\varepsilon_1}{2}\right)^m}\\&\leqslant\frac{2^m\widetilde{C}e^{\widetilde{B}(|\tau|+\frac{\delta-r-\varepsilon_1}{2})^2}}{(\delta-r-\varepsilon_1)^m}\leqslant Ce^{B|\tau|^2},\quad\tau\in\Omega(\theta/2,\varepsilon_1),\end{aligned}$$

with some positive constants B, $\widetilde{B}$, C and $\widetilde{C}$. Analogously

$$\max_{|z|\leqslant r_1}\left|\tau^m \Phi_n(\tau,z)\right| \leqslant \tau^m C e^{\widetilde{B}|\tau|^2} \leqslant C e^{B|\tau|^2}, \quad \tau\in\Omega(\theta/2,\varepsilon_1).$$

Therefore $w(\tau,z)$ is of exponential growth of order at most 2 as $\tau\to\infty$ in $\Omega(\theta/2,\delta-r)$.

Since $v(s,z)=w(\pm\sqrt{s},z)$, taking $\varepsilon=(\delta-r)^2$, we conclude (analogously as in 1-dimensional case), that $v(s,z)$ is analytic on $E_+(\theta,\varepsilon)\times D^n(r)$. Moreover, for any $r_1\in(0,r)$ and any $\varepsilon_1\in(0,\varepsilon)$,

$$\max_{|z|\leqslant r_1}\left|v(s,z)\right| = \max_{|z|\leqslant r_1}\left|w(\pm\sqrt{s},z)\right| \leqslant C e^{B|s|}, \quad s\in E_+(\theta,\varepsilon_1),$$

with some positive constants B and C. By Proposition 3, this yields the assertion.

($\Rightarrow$) For $n=1$. Fix $\delta\in(0,r)$. By Proposition 3 and Lemma 2, we can regard the function $w(\tau,z)=v(\tau^2,z)$ as the solution of the Cauchy problem to the wave equation

$$\partial_\tau^2 w(\tau,z)=\partial_z^2 w(\tau,z), \quad w(\tau,z_0)=\psi_0(\tau),\ w_z(\tau,z_0)=\psi_1(\tau),$$

for any fixed $z_0\in D(\delta)$. By assumption on $v(\tau^2,z)$, the functions $\psi_0(\tau)$ and $\psi_1(\tau)$ are analytic on $\{\tau\in\mathbb{C}\colon \operatorname{dist}(\tau^2,e^{i\theta}\mathbb{R}_+)<\varepsilon\}$ (for some $\varepsilon>0$) and are of exponential growth of order at most 2 as $\tau\to\infty$.

By the d'Alembert formula we obtain

$$w(\tau,z)=\frac{1}{2}\left[\psi_0(\tau+z-z_0)+\psi_0(\tau-z+z_0)+\int_{\tau-z+z_0}^{\tau+z-z_0}\psi_1(y)\,dy\right].$$

Hence the function

$$\varphi(z)=w(0,z)=\frac{1}{2}\left[\psi_0(z-z_0)+\psi_0(-z+z_0)+\int_{-z+z_0}^{z-z_0}\psi_1(y)\,dy\right]$$

is analytic on $z_0+\{z\in\mathbb{C}\colon \operatorname{dist}(z^2,e^{i\theta}\mathbb{R}_+)<\varepsilon\}$. Changing $z_0\in D(\delta)$, we see that $\varphi(z)$ is analytically continued to the domain

$$\bigcup_{|z_0|<\delta}\left(z_0+\left\{z\in\mathbb{C}\colon \operatorname{dist}\left(z^2,e^{i\theta}\mathbb{R}_+\right)<\varepsilon\right\}\right),$$

which contains $\{z\in\mathbb{C}\colon \operatorname{dist}(z,e^{i\theta/2}\mathbb{R})<\delta\}=\Omega(\theta/2,\delta)$.

Moreover, $\varphi(z)$ is of exponential growth of order at most 2, because for any $\varepsilon_1\in(0,\delta)$,

$$\begin{aligned}\left|\varphi(z)\right|=\left|w(0,z)\right| &\leqslant \frac{1}{2}\left|\psi_0(z-z_0)\right|+\frac{1}{2}\left|\psi_0(-z+z_0)\right| \\ &\quad + 2|z-z_0|\max_{y\in[-z+z_0,z-z_0]}\left|\psi_1(y)\right| \leqslant A e^{B|z|^2}, \quad z\in\Omega(\theta/2,\varepsilon_1),\end{aligned}$$

with some positive constants A and B.

For $n = 2k+1$, $k \geqslant 1$. By Proposition 3, $w(\tau, z) = v(\tau^2, z)$ is analytic on $\{\tau \in \mathbb{C}\colon \operatorname{dist}(\tau^2, e^{i\theta}\mathbb{R}_+) < \varepsilon\} \times D^n(r)$ (for some $\varepsilon > 0$ and $r > 0$) and, by Lemma 2, satisfies the n-dimensional wave equation. Using spherical means we can reduce this equation to 1-dimensional case (see Evans [7]). To this end let us define the function

$$\tilde{w}(\tau, \varrho, z) := \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^n(1)} w(\tau, z + \varrho x)\, dS(x),$$

which is analytic on $\{\tau \in \mathbb{C}\colon \operatorname{dist}(\tau^2, e^{i\theta}\mathbb{R}_+) < \varepsilon\} \times D(\delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ with $\delta = r$ and any fixed $\tilde{\delta} \in (0, r)$. Moreover, for any $\varrho_0 \in D(\delta - \tilde{\delta})$ the function $(\tau, z) \mapsto \tilde{w}(\tau, \varrho_0, z)$ is of exponential growth of order at most 2 as $\tau \to \infty$. We can regard $\tilde{w}(\tau, \varrho, z)$ as the solution of the Cauchy problem to the wave equation

$$\partial_\tau^2 \tilde{w}(\tau, \varrho, z) = \partial_\varrho^2 \tilde{w}(\tau, \varrho, z), \quad \tilde{w}(\tau, \varrho_0, z) = \tilde{\psi}_0(\tau, z),\ \partial_\varrho \tilde{w}(\tau, \varrho_0, z) = \tilde{\psi}_1(\tau, z),$$

for any fixed $\varrho_0 \in D(\delta - \tilde{\delta})$. By assumption on $v(\tau^2, z)$, $\tilde{\psi}_0(\tau, z)$ and $\tilde{\psi}_1(\tau, z)$ are analytic on $\{\tau \in \mathbb{C}\colon \operatorname{dist}(\tau^2, e^{i\theta}\mathbb{R}_+) < \varepsilon\} \times D^n(\tilde{\delta})$ and are of exponential growth of order at most 2 as $\tau \to \infty$. By the d'Alembert formula

$$\tilde{w}(\tau, \varrho, z) = \frac{1}{2}\left[\tilde{\psi}_0(\tau + \varrho - \varrho_0, z) + \tilde{\psi}_0(\tau - \varrho + \varrho_0, z) + \int_{\tau - \varrho + \varrho_0}^{\tau + \varrho - \varrho_0} \tilde{\psi}_1(y, z)\, dy\right].$$

Observe that for any $\varrho_0 \in D(\delta - \tilde{\delta})$ the function

$$\tilde{w}(0, \varrho, z) = \frac{1}{2}\left[\tilde{\psi}_0(\varrho - \varrho_0, z) + \tilde{\psi}_0(-\varrho + \varrho_0, z) + \int_{-\varrho + \varrho_0}^{\varrho - \varrho_0} \tilde{\psi}_1(y, z)\, dy\right]$$

is analytic on $(\varrho_0 + \{\varrho \in \mathbb{C}\colon \operatorname{dist}(\varrho^2, e^{i\theta}\mathbb{R}_+) < \varepsilon\}) \times D^n(\tilde{\delta})$. Changing $\varrho_0 \in D(\delta - \tilde{\delta})$ we conclude that $\tilde{w}(0, \varrho, z)$ is analytically continued to

$$\bigcup_{\varrho_0 \in D(\delta - \tilde{\delta})} \left(\varrho_0 + \left\{\varrho \in \mathbb{C}\colon \operatorname{dist}\left(\varrho^2, e^{i\theta}\mathbb{R}_+\right) < \varepsilon\right\}\right) \times D^n(\tilde{\delta}),$$

so also to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta}) = \{\varrho \in \mathbb{C}\colon \operatorname{dist}(\varrho, e^{i\theta/2}\mathbb{R}) < \delta - \tilde{\delta}\} \times D^n(\tilde{\delta})$.

On the other hand,

$$\begin{aligned}
\tilde{w}(0, \varrho, z) &= \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^n(1)} w(0, z + \varrho x)\, dS(x) \\
&= \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^n(1)} \varphi(z + \varrho x)\, dS(x).
\end{aligned}$$

Notice that for $l = 0, 1, \ldots, k-2$ the function

$$\varrho \mapsto \left(\varrho^{-1}\partial_\varrho\right)^l \varrho^{2k-1} \fint_{\partial B^n(1)} \varphi(z + \varrho x)\, dS(x)$$

is equal to 0 for $\varrho = 0$. Therefore,

$$\frac{1}{\varrho^{2k-1}} \int_0^{\varrho} \varrho_{k-1} \int_0^{\varrho_{k-1}} \varrho_{k-2} \cdots \int_0^{\varrho_2} \varrho_1 \tilde{w}(0, \varrho_1, z)\, d\varrho_1\, d\varrho_2 \cdots d\varrho_{k-1} = \fint_{\partial B^n(1)} \varphi(z + \varrho x)\, dS(x).$$

For any fixed $z \in D^n(\tilde{\delta})$ the left-hand side is analytic for $\varrho \in \Omega(\theta/2, \delta - \tilde{\delta}) \setminus \{0\}$ and the right-hand side is analytic for $|\varrho| < \delta - \tilde{\delta}$, hence the function

$$\Phi_{2k+1}(\varrho, z) = \int_{\partial B^n(1)} \varphi(z + \varrho x)\, dS(x)$$

is analytically continued to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 as $\varrho \to \infty$.

For $n = 2k$, $k \geqslant 1$. Let $z = (z', z_{n+1}) = (z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}$ and define the function $\overline{w}(\tau, z', z_{n+1}) := w(\tau, z') = v(\tau^2, z')$. By Proposition 3, this function is analytic on $\{\tau \in \mathbb{C}\colon \operatorname{dist}(\tau^2, e^{i\theta}\mathbb{R}_+) < \varepsilon\} \times D^n(r) \times \mathbb{C}$ (for some $\varepsilon > 0$ and $r > 0$) and, by Lemma 2, satisfies the $(n+1)$-dimensional wave equation.

Following previous case we have

$$\tilde{w}(\tau, \varrho, z) := \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^{n+1}(1)} \overline{w}(\tau, z + \varrho x)\, dS(x)$$

and $\tilde{w}(0, \varrho, z)$ is analytically continued to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta}) \times \mathbb{C}$ for $\delta = r$ and for any fixed $\tilde{\delta} \in (0, r)$. On the other hand,

$$\begin{aligned}\tilde{w}(0, \varrho, z) &= \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^{n+1}(1)} w(0, z' + \varrho x')\, dS(x) \\ &= \left(\varrho^{-1}\partial_\varrho\right)^{k-1} \varrho^{2k-1} \fint_{\partial B^{n+1}(1)} \varphi(z' + \varrho x')\, dS(x)\end{aligned}$$

with $x = (x', x_{n+1}) = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. Hence the function

$$\Phi_{2k}(\varrho, z') = \frac{1}{2} \int_{\partial B^{n+1}(1)} \varphi(z' + \varrho x')\, dS(x) = \int_{B^n(1)} \frac{\varphi(z' + \varrho x')}{\sqrt{1 - |x'|^2}}\, dx'$$

is analytically continued to $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 as $\varrho \to \infty$. $\square$

As a corollary to Theorem 1 we can characterize 1-summability of the formal solution (2) (see [11, Proof of Theorem 3.2])

Corollary 1. *The formal solution $\hat{u}(\tau, z)$ given by* (2) *is* 1*-summable in a sector $S(\theta, \alpha)$ if and only if the Cauchy data $\varphi(z)$ satisfies*:

(1) *For $n = 1$, $\varphi(z)$ is analytically continued to a double sector*

$$\widetilde{S}(\theta, \alpha) := S\big(\theta/2, (\alpha - \pi)/2\big) \cup S\big(\pi + \theta/2, (\alpha - \pi)/2\big)$$

and is of exponential growth of order at most 2 *as $z \to \infty$ in $\widetilde{S}(\theta, \alpha)$.*

(2) *For $n > 1$, the function*

$$\Phi_n(\tau, z) = \begin{cases} \int_{\partial B^n(1)} \varphi(z + \tau x)\, dS(x) & \text{for } n \text{ odd}, \\ \int_{B^n(1)} \frac{\varphi(z+\tau x)\, dx}{\sqrt{1-|x|^2}} & \text{for } n \text{ even}, \end{cases}$$

is analytically continued to $\widetilde{S}(\theta, \alpha) \times D^n(\delta)$ for some $\delta > 0$ and is of exponential growth of order at most 2 *as $\tau \to \infty$ in $\widetilde{S}(\theta, \alpha)$.*

If $\varphi(z)$ is analytic on $\{z \in \mathbb{C}^n\colon \operatorname{dist}(z, e^{i\theta/2}\mathbb{R}^n) < \delta\}$ for some $\delta > 0$ and is of exponential growth of order at most 2 as $z \to \infty$ then the function $\Phi_n(\tau, z)$ is analytic on $\Omega(\theta/2, \delta - \tilde{\delta}) \times D^n(\tilde{\delta})$ and is of exponential growth of order at most 2 as $\tau \to \infty$ in $\Omega(\theta/2, \delta - \tilde{\delta})$. Hence we have

Corollary 2. *If the Cauchy data $\varphi(z)$ is analytically continued to $\Omega^n(\theta/2, \delta) := \{z \in \mathbb{C}^n\colon \operatorname{dist}(z, e^{i\theta/2}\mathbb{R}^n) < \delta\}$ for some $\delta > 0$ and is of exponential growth of order at most* 2 *as $z \to \infty$ in $\Omega^n(\theta/2, \delta)$ (i.e., for every $\varepsilon_1 \in (0, \delta)$ there exists positive constants B and C such that $|\varphi(z)| \leqslant Ce^{B|z|^2}$ for $z \in \Omega^n(\theta/2, \varepsilon_1)$) then the formal solution $\hat{u}(\tau, z)$ given by* (2) *is fine* 1*-summable in the direction θ.*

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