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Limiting dynamics for stochastic wave equations

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Abstract

In this paper, relations between the asymptotic behavior for a stochastic wave equation and a heat equation are considered. By introducing almost surely $\mathcal{D}$–α-contracting property for random dynamical systems, we obtain a global random attractor of the stochastic wave equation $\nu u_{tt}^{\nu} + u_t^{\nu} - \Delta u^{\nu} + f(u^{\nu}) = \sqrt{\nu}\dot{W}$ endowed with Dirichlet boundary condition for any $0 < \nu \leqslant 1$. The upper semicontinuity of this global random attractor and the global attractor of the heat equation $z_t - \Delta z + f(z) = 0$ with Dirichlet boundary condition as ν goes to zero is investigated. Furthermore we show the stationary solutions of the stochastic wave equation converge in probability to some stationary solution of the heat equation.

Keywords: Random dynamical system; Random attractor; Singular perturbation; Stochastic wave equation; Stationary solution; Tightness

1. Introduction

The need for taking random effects into account in modeling, analyzing, simulating and predicting complex phenomena has been widely recognized in geophysical and climate dynamics, materials science, chemistry, biology and other areas [5,10,15,23]. Stochastic partial differential equations (SPDEs or stochastic PDEs) are appropriate mathematical models for complex systems under random influences [24].

Wave motion is one of the most commonly observed physical phenomena, which can be described by hyperbolic partial differential equations. Nonlinear wave or hyperbolic equations also

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have been studied a great deal recently in many modern problems such as sonic booms, bottleneck in traffic flows, nonlinear optics and quantum field theory, see [21,25]. However for many problems such as wave propagation through the atmosphere or the ocean, due to the presence of turbulence the media has random fluctuation, the more realistic models must take the random fluctuation into account, which leads to the introduction of stochastic wave equations, see [4,5].

Let $D \subset \mathbb{R}^3$ be a bounded open set with smooth boundary ∂D. In this paper we consider the following singular perturbed stochastic wave equation on D

$$\nu u_{tt}^{\nu} + u_t^{\nu} - \Delta u^{\nu} + f\big(u^{\nu}\big) = \sqrt{\nu}\dot{W} \tag{1.1}$$

with Dirichlet boundary condition and positive parameter ν. Δ is the Laplace operator on D, $W(t)$ is an infinite dimensional Wiener process and non-Lipschitzian nonlinear term f satisfies some increasing properties, see Section 2. If ν is small, a natural idea is using the following heat equation on D

$$z_t - \Delta z + f(z) = 0 \tag{1.2}$$

with Dirichlet boundary condition to approximate system (1.1) in some sense.

It is known that the study of asymptotic behavior as $t \to \infty$ is one of the most important parts in modern research of nonlinear evolutionary systems. So a natural question is what is the relation between the long time behavior of systems (1.1) and (1.2) as $\nu \to 0$. The present paper is devoted to determine the asymptotic behavior of systems (1.1) and (1.2) as $t \to \infty$ respectively and then find the relation between them.

First we apply the theories of random dynamical system and random attractor to study the long time behavior of system (1.1). It is proved that the dynamical behavior of system (1.1) can be described by a global random attractor $\mathcal{A}^{\nu}(\omega)$ in space $(H^2(D) \cap H_0^1(D)) \times H_0^1(D)$. Here $\omega \in (\Omega, \mathcal{F}, \mathbb{P})$. In our approach a stationary Ornstein–Uhlenbeck process which solves a linear stochastic wave equation, see Section 3, is introduced to transform the stochastic system (1.1) into a system with random coefficient. Then almost surely $\mathcal{D}$–α-contracting property for a continuous random dynamical system is introduced, see Definition 2.10, to obtain a global random attractor for the random dynamical system generated by the system with random coefficient. And some technique of energy estimates is used in our argument. A novelty of here is that we obtain the random attractor in space $(H^2(D) \cap H_0^1(D)) \times H_0^1(D)$. The random attractors, which are obtained in $H_0^1(D) \times L^2(D)$, for stochastic wave equations have been studied in [8,11,26] and others. And as far as the authors know almost surely $\mathcal{D}$–α-contracting property is the first time introduced for continuous random dynamical systems to study the existence of random attractors.

Next we pass the limit $\nu \to 0$ in system (1.1) and prove that $\mathcal{A}^{\nu}(\omega)$ is upper semicontinuous in space $H_0^1(D) \times L^2(D)$ at $\nu = 0$ in probability, that is

$$\lim_{\nu \to 0} \operatorname{dist}\big(\mathcal{A}^{\nu}(\omega), \mathcal{A}^0\big) = 0 \quad \text{with probability one.}$$

Here $\mathcal{A}^0$, which is a deterministic set, is the extension of a global attractor for system (1.2) in space $(H^2(D) \cap H_0^1(D)) \times H_0^1(D)$ and $\operatorname{dist}(\cdot,\cdot)$ is the Hausdorff semidistance

$$\operatorname{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y),$$

on a metric space (E, d). In [2], Caraballo et al. have studied the upper semicontinuity of random attractors with respect to noise for stochastic parabolic equations. Here we apply the result in [2] to a stochastic hyperbolic equation with singular perturbation.

Finally for any $T > 0$, we concern ourselves with the limit of stationary solutions $\bar{u}^\nu(\cdot)$ of system (1.1) in space $L^2(0, T; L^2(D))$ as $\nu \to 0$. It is proved that any limit point of a sequence of stationary solutions of system (1.1) in $L^2(0, T; L^2(D))$ as $\nu \to 0$ defines one stationary solution of system (1.2). In this approach a key step is the tightness of distributions of the sequence of stationary solutions in space $L^2(0, T; L^2(D))$. And the mean square convergence is proved on a new probability space $(\Omega_\varrho, \mathcal{F}_\varrho, \mathbb{P}_\varrho)$ for any $\varrho > 0$ which yields the convergence of stationary solutions in probability on $(\Omega, \mathcal{F}, \mathbb{P})$.

Here we point out that in [3] Cerrai and Freidlin have studied the following stochastic hyperbolic equation

$$\nu u_{tt}^\nu + u_t^\nu - \Delta u^\nu + f\big(u^\nu\big) = \dot{W} \tag{1.3}$$

with f is global Lipschitz continuous. And for any finite time $T > 0$ and $\kappa > 0$ the following approximation is obtained

$$\lim_{\nu \to 0} \mathbb{P}\big\{\big|u^\nu(t) - z(t)\big|_{C(0,T;L^2(D))} > \kappa\big\} = 0, \tag{1.4}$$

which is called Smoluchowski–Kramers approximation, with z solves the following stochastic heat equation

$$z_t - \Delta z + f(z) = \dot{W}. \tag{1.5}$$

And the stationary distribution of system (1.3) is coincide with that of system (1.5). In this paper we derive the approximation on long time interval for system (1.1) with non-Lipschitz nonlinear term f and rescaled white noise. Such approximation is not Smoluchowski–Kramers approximation. Here we just consider the case that the noise is of trace-class to which Itô formula can be applied. The study of global random attractors was initiated by Ruelle [18]. And the fundamental theory was developed in [7,8,19] and others.

The rest is organized as follows. In Section 2 some preliminaries are given. A stationary Ornstein–Uhlenbeck process with some basic properties is given in Section 3. And we transform system (1.1) into a random partial differential equation in Section 4, then a global random attractor for the random partial differential equation is obtained in Section 5. The main result about the upper semicontinuity of the random attractor is proved in Section 6. In last section we consider the convergence of stationary solutions for system (1.1).

2. Preliminary

Let $D \subset \mathbb{R}^n$, $n = 3$, be an open bounded set with smooth boundary. Define the following unbounded operator on $L^2(D)$

$$Au = -\Delta u, \quad D(A) = H^2(D) \cap H_0^1(D), \tag{2.1}$$

which is self-adjoint and strictly positive. For $s \in \mathbb{R}$ we can define $D(A^{s/2})$ endowed with the norm

$$\|\cdot\|_s = \|A^{s/2}\cdot\|_{L^2(D)}.$$

Consider the following singularly perturbed stochastic wave equation on D

$$\nu u_{tt} + u_t + Au + f(u) = \sqrt{\nu}\dot{W}, \tag{2.2}$$

$$u(0) = u_0, \qquad u_t(0) = u_1 \tag{2.3}$$

where $0 < \nu \leqslant 1$ and $\{W(t)\}_{t\in\mathbb{R}}$ is an $L^2(D)$-valued two-sided Q-Wiener process with

$$Qe_k = a_k e_k, \quad k = 1, 2, \ldots,$$

where $\{e_k\}$ is a complete orthonormal system in $L^2(D)$, a_k is a bounded sequence of nonnegative real numbers. Then $W(t)$ can be written as

$$W(t) = \sum_{k=1}^{\infty} \sqrt{a_k} e_k w_k(t)$$

where w_k are real mutually independent Brownian motions. Further we assume

$$\operatorname{Tr}(AQ) < \infty. \tag{2.4}$$

For more descriptions of Q-Wiener process we refer to [17].

We assume that the nonlinear term $f \in C^2$ on $\mathbb{R}$ and there exist positive constants C_1, C_2, C_3 and C_4 such that for any $s \in \mathbb{R}$

(1) $|f(s)| \leqslant C_1(1 + |s|^3)$;
(2) $|f'(s)| \leqslant C_2(1 + |s|^2)$;
(3) $F(s) \geqslant C_3(|s|^4 - 1)$;
(4) $sf(s) \geqslant C_4(F(s) - 1)$

where $F(s) = \int_0^s f(r)\,dr$.

Remark 2.1. Here we omit the superscript ν without causing confusion unless in Sections 6 and 7.

Remark 2.2. Note that the growth exponent of the nonlinearity is critical in $\mathbb{R}^3$. And to the best of our knowledge this is the first result for the random attractor for the stochastic wave equation with critical growth exponent.

Remark 2.3. Also note that the assumptions on f are satisfied for cubic nonlinearity $f(u) = u^3 - au$, $a \in \mathbb{R}$, and not satisfied for the sine-Gordon equation with $f(u) = \sin u$. However the calculations are simpler in that case since f and its derivative are bounded.

Let $\mathcal{H} = H_0^1(D) \times L^2(D)$ and $\mathcal{H}^1 = (H^2(D) \cap H_0^1(D)) \times H_0^1(D)$ with product norms written as $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}^1}$, respectively. We also consider the following heat equation

$$z_t + Az + f(z) = 0, \qquad z(0) = z_0. \tag{2.5}$$

Under the assumptions on f it is well known that system (2.5) possesses a global attractor $\tilde{\mathcal{A}} \subset H^2(D) \cap H_0^1(D)$, see [13,22] and others. Associated with the attractor $\tilde{\mathcal{A}}$ we define the set

$$\mathcal{A}^0 = \{(x, y)\colon x \in \tilde{\mathcal{A}},\ y = -Ax - f(x)\} \tag{2.6}$$

which is a natural embedding of the attractor $\tilde{\mathcal{A}}$ into $\mathcal{H}^1$.

For our purpose we introduce the following probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\Omega = \{\omega \in C(\mathbb{R}; L^2(D))\colon \omega(0) = 0\}$$

endowed with compact-open topology, $\mathbb{P}$ is the corresponding Wiener measure and $\mathcal{F}$ is the $\mathbb{P}$-completeness of Borel σ-algebra on Ω. Write $\mathbf{E}$ as the expectation with respect to $\mathbb{P}$. Let

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system with the filtration

$$\mathcal{F}_t := \bigvee_{s\leqslant t} \mathcal{F}_s^t, \quad t \in \mathbb{R},$$

where

$$\mathcal{F}_s^t = \sigma\{W(t_2) - W(t_1)\colon s \leqslant t_1 \leqslant t_2 \leqslant t\}$$

is the smallest σ-algebra generated by random variables $W(t_2) - W(t_1)$ for all $s \leqslant t_1 \leqslant t_2 \leqslant t$. The metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ defined above is ergodic. Since the above probability space is canonical we can define the Wiener process and its shift operator

$$W(t) = \omega(t), \qquad W(t, \theta_s\omega) = \omega(t + s) - \omega(s) = W(t + s) - W(s) \tag{2.7}$$

which is called the helix property.

Now we introduce the concept of the random dynamical system (RDS).

Definition 2.4. Let (E, d) be a Polish space and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Suppose the mapping

$$\phi : \mathbb{R}^+ \times \Omega \times E \to E$$

is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(E), \mathcal{B}(E))$-measurable and satisfies the following properties

- $\phi(0, \omega)x = x$;
- $\phi(t, \theta_\tau\omega) \circ \phi(\tau, \omega)x = \phi(t + \tau, \omega)x$ for all $t, \tau \in \mathbb{R}^+$, $x \in E$ and $\omega \in \Omega$ (cocycle property),

then ϕ is called an RDS with respect to $(\theta_t)_{t\in\mathbb{R}}$. If ϕ is continuous with respect to x for $t \geqslant 0$ and $\omega \in \Omega$, ϕ is called a continuous RDS.

For detail we refer to [1].

We intend to study the long time behavior of an RDS generated by system (2.2) by means of random attractors. Now we recall some knowledge of random attractors.

A set-valued map $B : \Omega \to 2^E$ is called a random closed set if $B(\omega)$ is closed, nonempty and $\omega \mapsto d(x, B(\omega))$ is measurable for all $x \in E$. A random set B is said to be tempered if for a.e. $\omega \in \Omega$ and all $\gamma > 0$

$$\lim_{t\to\infty} e^{-\gamma t}\operatorname{diam}\bigl(B(\theta_{-t}\omega)\bigr) = 0$$

where $\operatorname{diam}(B) = \sup_{x,y\in B} d(x, y)$. Let $\mathcal{D}$ be the collection of all tempered random sets in E. And for such a set we write as a $\mathcal{D}$-set. Then we give the following definition of the random attractor, see [7,8,19].

Definition 2.5. A random set $\mathcal{A}(\omega) \in \mathcal{D}$ is a random (pullback) attractor for RDS ϕ if

- $\mathcal{A}(\omega)$ is a random compact set, i.e., $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in E$ and $\mathcal{A}(\omega)$ is compact for a.e. $\omega \in \Omega$;
- $\mathcal{A}(\omega)$ is strictly invariant, i.e., $\phi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$ $\forall t \geqslant 0$ and for a.e. $\omega \in \Omega$;
- $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, i.e., for all $B \in \mathcal{D}$ and a.e. $\omega \in \Omega$ we have

$$\lim_{t\to\infty} \operatorname{dist}\bigl(\phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \mathcal{A}(\omega)\bigr) = 0.$$

The following theorem gives a sufficient condition for the existence of a random attractor, see [7,8,19].

Theorem 2.6. *Suppose ϕ is an RDS on a Polish space (E, d) and there exists a random compact set $B(\omega)$ absorbing every tempered random set $D \subset E$. Then, the set*

$$\mathcal{A}(\omega) = \bigcap_{\tau \geqslant t_B(\omega)} \overline{\bigcup_{t\geqslant\tau} \phi\bigl(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\bigr)}$$

is a global random attractor for RDS ϕ.

However this result is improper for the present situation since a wave equation generates a semigroup without compactness. So we give another method to obtain the global random attractor.

We review briefly the basic knowledge of Kuratowski measure of non-compactness, see [14,20] for detail.

Definition 2.7. For a bounded set A in a metric space E, the Kuratowski measure of non-compactness $\alpha(A)$ is defined as

$$\alpha(A) = \inf\bigl\{\varepsilon > 0\colon\ A \text{ has a finite open cover of sets } \{X_i\} \text{ with } \operatorname{diam}(X_i) \leqslant \varepsilon\bigr\}.$$

If A is an unbounded set in E define $\alpha(A) = \infty$. The following properties will be useful in our approach.

Lemma 2.8. *The Kuratowski measure of non-compactness $\alpha(A)$ on a complete metric space E satisfies the following properties*:

(1) $\alpha(A) = 0$ *if and only if A is precompact in E.*
(2) *If $B_1 \subset B_2 \subset E$, $\alpha(B_1) \leqslant \alpha(B_2)$.*
(3) $\alpha(A) = \alpha(\overline{A})$.
(4) *If $\{A_\tau\}_{\tau > \tau_0}$ is a family of nonempty, closed bounded sets such that $A_\tau \subset A_s$, $\tau \geqslant s$, and $\alpha(A_\tau) \to 0$ as $\tau \to \infty$, then $\bigcap_{\tau > \tau_0} A_\tau$ is a nonempty compact set in E.*
(5) *If (E, d) is a Banach space, then $\alpha(B_1 + B_2) \leqslant \alpha(B_1) + \alpha(B_2)$.*

Remark 2.9. For the set $A \in \mathcal{D}$, $\alpha(A)$ is a random variable.

For a random dynamical system $\phi(t, \omega)$ on a Polish space (E, d) we introduce the following definition, similar definition for cocycle, see [6].

Definition 2.10. A random dynamical system ϕ on a Polish space (E, d) is almost surely $\mathcal{D}$–α-contracting if for any set $A \in \mathcal{D}$

$$\alpha\big(\phi(t, \theta_{-t}\omega)A(\theta_{-t}\omega)\big) \to 0 \quad \text{a.s.}, \ t \to \infty.$$

The following theorem describes sufficient conditions for a random dynamical system to be almost surely $\mathcal{D}$–α-contracting.

Theorem 2.11. *For a random dynamical system $\phi(t, \omega)$ on a separable Banach space $(E, \|\cdot\|_E)$, if almost surely the following hold*

(1) $\phi(t, \omega) = \phi_1(t, \omega) + \phi_2(t, \omega)$;
(2) *for any tempered random variable $a \geqslant 0$, there exists $r = r(a)$, $0 \leqslant r < \infty$, a.s. such that for the closed ball B_a with radius a in E, $\phi_1(t, \theta_{-t}\omega)B_a(\theta_{-t}\omega)$ is precompact in E for $t > r(a)$;*
(3) $\|\phi_2(t, \theta_{-t}\omega)u\|_E \leqslant k(t, \theta_{-t}\omega, a)$, *$t > 0$, $u \in B_a(\theta_{-t}\omega)$ with $k(t, \omega, a)$ is a measurable function with respect to (t, ω, x) which satisfies $k(t, \theta_{-t}\omega, a) \to 0$ as $t \to \infty$,*

then $\phi(t, \omega)$ is almost surely $\mathcal{D}$–α-contracting.

Proof. Let $A \in \mathcal{D}$ and a tempered random variable $a \geqslant 0$ such that $A \subset B_a$. Then for any $\varepsilon > 0$ with $t > r(a)$

$$\begin{aligned}
\alpha\big(\phi(t, \theta_{-t}\omega)A(\theta_{-t}\omega)\big) &\leqslant \alpha\big(\phi(t, \theta_{-t}\omega)B_a(\theta_{-t}\omega)\big) \\
&= \alpha\big(\phi_1(t, \theta_{-t}\omega)B_a(\theta_{-t}\omega) + \phi_2(t, \theta_{-t}\omega)B_a(\theta_{-t}\omega)\big) \\
&\leqslant \alpha\big(\phi_2(t, \theta_{-t}\omega)B_a(\theta_{-t}\omega)\big)
\end{aligned}$$

$$\leqslant \operatorname{diam}\bigl(\phi_2(t,\theta_{-t}\omega)B_a(\theta_{-t}\omega)\bigr)+\varepsilon$$
$$\leqslant 2k(t,\theta_{-t}\omega,a)+\varepsilon.$$

Then the arbitrariness of ε yields the result. □

For an almost surely $\mathcal{D}$–α-contracting random dynamical system ϕ on a Polish space (E,d), we give the following result about a sufficient condition of existence of a global random attractor in E.

Theorem 2.12. *For a random dynamical system $\phi(t,\omega)$ on a Polish space (E,d), assume that*

(1) *$\phi(t,\omega)$ has an absorbing set $B(\omega)\in\mathcal{D}$;*
(2) *$\phi(t,\omega)$ is almost surely $\mathcal{D}$–α-contracting.*

Then $\phi(t,\omega)$ possesses a global random attractor in E.

Proof. The argument is just a small adaption of that in [14] or [20] for deterministic system.

Set

$$\mathcal{A}(\omega)=\bigcap_{\tau>0}\overline{\bigcup_{t\geqslant\tau}\phi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)}.$$

We claim that $\mathcal{A}(\omega)\in\mathcal{D}$ is a global random attractor of $\phi(t,\omega)$ in E. In fact it is known that, see [7,8,19], $\mathcal{A}(\omega)$ is nonempty, invariant and attracting all the sets in $\mathcal{D}$. So we just prove the compactness of $\mathcal{A}(\omega)$. Let

$$A_\tau(\omega)=\overline{\bigcup_{t\geqslant\tau}\phi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)}.$$

Then $A_\tau(\omega)\subset B_0(\omega)\in\mathcal{D}$ and $\alpha(A_\tau(\omega))\to 0$ a.s. as $\tau\to\infty$. By property (4) in Lemma 2.8, $\mathcal{A}(\omega)$ is compact. The proof is complete. □

3. Stationary Ornstein–Uhlenbeck process

In this section we construct a stationary Ornstein–Uhlenbeck (OU) process. First we consider the following deterministic linear system,

$$\varphi_t=\psi, \tag{3.1}$$
$$\nu\psi_t=-A\varphi-\psi, \tag{3.2}$$
$$\varphi(0)=\varphi_0,\qquad \psi(0)=\psi_0, \tag{3.3}$$

for any $(\varphi_0,\psi_0)\in\mathcal{H}$, and $\nu\in(0,1]$. Write (3.1)–(3.3) as the following abstract form

$$\frac{dZ(t)}{dt} = \mathbf{A}_\nu Z(t), \qquad Z(0) = (\varphi_0, \psi_0)^T \tag{3.4}$$

with $Z(t) = (\varphi(t), \psi(t))^T$ and

$$\mathbf{A}_\nu = \begin{pmatrix} 0 & 1 \\ \frac{-1}{\nu}A & \frac{-1}{\nu} \end{pmatrix}.$$

By the theory of semigroup, $\mathbf{A}_\nu$ is a generator of C_0 semigroup $S_\nu(t)$ on $\mathcal{H}$. Further there are $M_\nu > 0$, $\lambda_\nu > 0$ such that

$$\left|S_\nu(t)\right|_{\mathcal{L}} \leqslant M_\nu e^{-\lambda_\nu^* t}, \quad t \geqslant 0,$$

see [3]. Here $|\cdot|_{\mathcal{L}}$ denotes the operator norm. We claim that λ_ν^* can be taken as a positive real number independent of $\nu \in (0,1]$. In fact a simple calculation yields the eigenvalues of operator $\mathbf{A}_\nu$

$$\delta_k^{\pm} = -\frac{1}{2\nu} \pm \sqrt{\frac{1}{4\nu^2} - \frac{\lambda_k}{\nu}}$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of operator A satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. Then λ_ν^* can be taken as

$$\lambda_\nu^* = -\mathrm{Re}\,\delta_1^+ = \mathrm{Re}\left\{\frac{\lambda_1}{\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_1\nu}}\right\} \geqslant \min\left\{\lambda_1, \frac{1}{2}\right\}$$

where Re denotes the real part of a complex number.

Denote the adjoint semigroup of $S_\nu(t)$ by $S_\nu^*(t)$ which is defined by, see [3],

$$S_\nu^*(t)(u,v) = \left(\Pi_1 S_\nu(t)\left(u, \frac{-v}{\nu}\right), \Pi_2 S_\nu(t)(-\nu u, v)\right), \quad (u,v) \in \mathcal{H},$$

where Π_1 and Π_2 are the projections from $\mathcal{H}$ to $H_0^1(D)$ and $L^2(D)$, respectively.

Now we consider the following linear stochastic problem in $\mathcal{H}$

$$\nu\eta_{tt} + \eta_t + A\eta = \sqrt{\nu}\dot{W}, \tag{3.5}$$

$$\eta(0) = \eta_0, \qquad \eta_t(0) = \eta_1, \tag{3.6}$$

which is equivalent to

$$\frac{d\Xi(t)}{dt} = \mathbf{A}_\nu \Xi(t) + \frac{1}{\sqrt{\nu}}\dot{\tilde{W}}(t), \qquad \Xi(0) = (\eta_0, \eta_1)^T \tag{3.7}$$

with $\Xi(t,\omega) = (\eta(t,\omega), \eta_t(t,\omega))^T$ and $\tilde{W}(t) = (0, W(t))^T$.

Suppose condition (2.4) holds, by the classical analysis of SPDEs [17], system (3.7) has a solution $(\eta(t,\omega), \eta_t(t,\omega)) \in L^2(\Omega; C(0,T; H_0^1(D))) \times L^2(\Omega; L^2(0,T; L^2(D)))$ for any initial value $(\eta_0, \eta_1) \in \mathcal{H}$. In our approach a stationary solution of system (3.7) is useful. For such a solution $\Xi(t,\omega) = (\eta(t,\omega), \eta_t(t,\omega))^T$ we have the following result.

Lemma 3.1. *Assume* (2.4) *holds. Then*

(1) *system* (3.7) *has a unique stationary solution* (η, η_t) *which is a Gaussian process in* $\mathcal{H}$;
(2) $t \mapsto \|(\eta(\theta_t\omega), \eta_t(\theta_t\omega))\|_{\mathcal{H}}$ *grows sublinearly, that is*

$$\lim_{t\to\pm\infty} \frac{\|(\eta(\theta_t\omega), \eta_t(\theta_t\omega))\|_{\mathcal{H}}}{t} = 0;$$

(3)
$$\mathbf{E}\big\|\eta_t(\omega)\big\|_0^2 = \frac{1}{2}\operatorname{Tr} Q;$$

(4)
$$\frac{1}{t}\int_0^t \big\|\big(\eta(\theta_t\omega), \eta_t(\theta_t\omega)\big)\big\|_{\mathcal{H}}^m\, ds \to \mathbf{E}\big\|\big(\eta(\omega), \eta_t(\omega)\big)\big\|_{\mathcal{H}}^m, \quad t\to\infty,\ m\in\mathbb{Z}^+.$$

Proof. Consider system (3.7). Let

$$C_\nu = \int_0^\infty S_\nu(s) Q_\nu Q_\nu^* S_\nu^*(s)\, ds$$

where $Q_\nu : L^2(D) \to \mathcal{H}$ is defined as

$$Q_\nu h = \frac{1}{\sqrt{\nu}}\big(0, Q^{\frac{1}{2}} h\big), \quad h \in L^2(D),$$

and $Q_\nu^* : \mathcal{H} \to L^2(D)$ is the adjoint operator of Q_ν defined as

$$Q_\nu^*(u, v) = \frac{1}{\sqrt{\nu}} Q^{\frac{1}{2}} v, \quad (u, v) \in \mathcal{H}.$$

Following the proof of Proposition 5.1 of [3], we have

$$C_\nu(u, v) = \frac{1}{2}(\nu Q u, Q v), \quad (u, v) \in \mathcal{H},$$

with $\operatorname{Tr} C_\nu = \frac{\nu+1}{2}\sum_{k=1}^\infty a_k < \infty$. Thus by condition (2.4), operator C_ν is nonnegative, symmetric and of trace-class on $\mathcal{H}$. So by Theorem 11.7 of [17], system (3.7) has a unique invariant Gaussian measure $\mathcal{N}(0, C_\nu)$ on $\mathcal{H}$. Then

$$(\eta, \eta_t) = \int_{-\infty}^t S_\nu(t-s)\, d\tilde{W}(s)$$

which distributes as $\mathcal{N}(0, C_\nu)$ is the unique stationary solution of system (3.7). And by Birkhoff ergodic theorem, $\sup_{t\in[n,n+1]}\|(\eta(\theta_t\omega), \eta_t(\theta_t\omega))\|_{\mathcal{H}}/n$ tends to zero as n goes to infinity gives (2). Applying Itô formula to $\|\eta_t(t)\|_0^2$, see [4], we easily have

$$e\big(\eta(t)\big) = e\big(\eta(0)\big) + \int_0^t \operatorname{Tr} Q\,ds + 2\sqrt{\nu}\int_0^t \big\langle \eta_t(s), dW(s)\big\rangle - 2\int_0^t \big\|\eta_t(s)\big\|_0^2\,ds \tag{3.8}$$

where

$$e\big(\eta(t)\big) = \nu\big\|\eta_t(t)\big\|_0^2 + \big\|\eta(t)\big\|_1^2.$$

Taking expectation on both sides of (3.8), then the stationarity of (η, η_t) yields (3). And (4) also follows by Birkhoff ergodic theorem and the Gaussian property of (η, η_t). $\square$

Remark 3.2. Further by condition (2.4), (η, η_t) in fact takes value in $\mathcal{H}^1$. Formula (3.8) is an energy equation for system (3.7). And by the above result we see that $\mathbf{E}\|\eta_t(\omega)\|_0^2$ is independent of ν. Moreover $\eta(\theta_t\omega)$ coincides with the stationary solution of the following linear heat equation in distribution, see [3],

$$\eta_t + A\eta = \sqrt{\nu}\dot{W}. \tag{3.9}$$

Furthermore $\eta(\theta_t\omega)$ distributes as $\mathcal{N}(0, \frac{\nu}{2}Q)$ which means $\eta(\theta_t\omega)$ converges to zero when $\nu \to 0$ in $H^2(D) \cap H_0^1(D)$ a.s.

4. Conjugation between the stochastic PDE and random PDE

Consider the following random PDE for $\omega \in \Omega$ fixed,

$$\nu v_{tt} + v_t + Av + f(v+\eta) = 0, \tag{4.1}$$

$$v(0) = u_0 - \eta(\omega), \qquad v_t(0) = u_1 - \eta_t(\omega) \tag{4.2}$$

where (η, η_t) is the stationary solution of system (3.5)–(3.6). By the classical analysis in [22], for any $(u_0, u_1) \in \mathcal{H}$, system (4.1)–(4.2) has a unique solution $(v, v_t) \in C(0, T; H_0^1(D) \times L^2(D))$ for any $T > 0$ and a.s. $\omega \in \Omega$. (The global existence of solution follows the boundedness of solution in Section 5.1.) Then system (4.1)–(4.2) generates a continuous random dynamical system $\Phi^\nu(t, \omega)$ on $\mathcal{H}$ as

$$\Phi^\nu(t,\omega)\big(v(0), v_t(0)\big) = \big(v(t,\omega), v_t(t,\omega)\big). \tag{4.3}$$

Define an isomorphism on $\mathcal{H}$ as

$$\mathcal{T}(\omega, x) = x + \big(\eta(\omega), \eta_t(\omega)\big), \quad x \in \mathcal{H}, \tag{4.4}$$

then system (2.2)–(2.3) generates a random dynamical system $\Psi^\nu(t, \omega)$ on $\mathcal{H}$ as

$$\Psi^\nu(t,\omega)(u_0, u_1) = \mathcal{T} \circ \Phi^\nu \circ \mathcal{T}^{-1}(t,\omega)(u_0, u_1) = \big(u(t,\omega), u_t(t,\omega)\big).$$

For more about the dynamics between two conjugate RDSs, see [9].

We end the section by the following simple truth for system (4.1)–(4.2).

Lemma 4.1. *For any $(u_0, u_1) \in \mathcal{H}$ we have*

$$\int_0^t \|v_t(s)\|_0^2 \, ds = H(0) - H(t) - \int_0^t \langle f(u(s)), \eta_t(s) \rangle \, ds \tag{4.5}$$

where

$$H(t) = \frac{\nu}{2} \|v_t(t)\|_0^2 + \frac{1}{2} \|v(t)\|_1^2 + \langle F(u)(t), 1 \rangle.$$

We aim to prove the existence of random attractors for $\Psi^\nu(t, \omega)$ which is upper semicontinuous with respect to ν. By isomorphism $\mathcal{T}$, it is enough to prove the same result for $\Phi^\nu(t, \omega)$.

5. Random dynamics of stochastic wave equations

We always write C as some constant independent of ν, $Q_i(\omega)$, $i = 1, 2, 3, 4$, $Q_\mu(\omega)$, $\mu > 0$, are almost surely finite positive random variables in the form of some polynomials of $\|u_0\|_0^2$, $\|u_0\|_1^2$, $\|u_1\|_0^2$, $\|\eta\|_0^2$, $\|\eta\|_1^2$, $\|\eta_t\|_0^2$ and $\|\eta_t\|_1^2$ and are independent of ν. In this section we prove the existence of random attractors for $\Phi^\nu(t, \omega)$. Our approach is the classical energy method, see [7,8] for stochastic partial differential equations. We consider the system on a fixed ω-fiber.

5.1. Boundedness of solutions in $\mathcal{H}$

Let $\rho = v_t + \delta v$, where $\delta \in (0, \delta_0)$ with $\delta_0 = \min(\frac{1}{4}, \frac{\lambda_1}{2})$, then

$$\nu \rho_t + (1 - \delta\nu)\rho + (A - \delta(1 - \delta\nu))v + f(u) = 0. \tag{5.1}$$

Multiplying ρ on both sides of (5.1) and integrating on D, we have

$$\frac{1}{2} \frac{d}{dt} (\|v\|_1^2 + \nu\|\rho\|_0^2) + \delta\|v\|_1^2 + (1 - \delta\nu)\|\rho\|_0^2 - \delta(1 - \delta\nu)\langle v, \rho \rangle + \langle f(u), \rho \rangle = 0.$$

Notice the choice of δ and $\nu \in (0, 1]$, by a simple calculation we have

$$\delta\|v\|_1^2 + (1 - \delta\nu)\|\rho\|_0^2 - \delta(1 - \delta\nu)\langle v, \rho \rangle \geqslant \frac{\delta}{2}\|v\|_1^2 + \frac{1}{2}\|\rho\|_0^2.$$

Moreover by the assumptions of f we have

$$\langle f(u), u_t \rangle = \frac{d}{dt} \int_D F(u) \, dx,$$

$$\langle f(u), u \rangle = \int_D f(u)u \, dx \geqslant C_4 \int_D F(u) \, dx - C_4|D|,$$

and

$$\big\langle f(u), \eta_t + \delta\eta \big\rangle \leqslant \frac{\delta C_4}{2} \int_D F(u)\,dx + C\big(1 + \|\eta_t + \delta\eta\|_1^4\big)$$

for some positive constant C. Here we use the embedding of $H^1(D)$ into $L^6(D)$.

Combining the above estimates and by Hölder inequality we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|v\|_1^2 + \nu\|\rho\|_0^2 + 2\int_D F(u)\,dx + 2C_3|D|\right) \\
&\quad + \alpha\left(\|v\|_1^2 + \nu\|\rho\|_0^2 + 2\int_D F(u)\,dx + 2C_3|D|\right) \\
&\quad \leqslant g(\theta_t\omega)
\end{aligned} \tag{5.2}$$

where $\alpha = \min(\delta, \frac{\delta C_4}{2}, \frac{1}{\nu})$ and

$$g(\theta_t\omega) = 2\delta C_4|D| - 2\alpha C_3|D| + 2C\big(1 + \|\eta_t + \delta\eta\|_1^4\big).$$

Then Gronwall lemma yields

$$\begin{aligned}
&\|v\|_1^2 + \nu\|\rho\|_0^2 + 2\int_D F(u)\,dx + 2C_3|D| \\
&\quad \leqslant e^{-\alpha t}\left(\big\|v(0)\big\|_1^2 + \nu\big\|\rho(0)\big\|_0^2 + 2\int_D F(u_0)\,dx + 2C_3|D|\right) + \int_0^t e^{-\alpha(t-s)} g(\theta_s\omega)\,ds.
\end{aligned} \tag{5.3}$$

Since $\nu \in (0, 1]$, α can be chosen independent of ν and by the embedding theorem we have

$$\begin{aligned}
&\|v\|_1^2 + \|v_t\|_0^2 \\
&\quad \leqslant \big(1 + \delta{\lambda_1}^{-\frac{1}{2}}\big)\big(\|v\|_1^2 + \|v_t + \delta v\|_0^2\big) \\
&\quad \leqslant \big(1 + \delta{\lambda_1}^{-\frac{1}{2}}\big)\left(\|v\|_1^2 + \|\rho\|_0^2 + 2\int_D F(u)\,dx + 2C_3|D|\right) \\
&\quad \leqslant \big(1 + \delta{\lambda_1}^{-\frac{1}{2}}\big)e^{-\alpha t} Q_1(\theta_t\omega) + \big(1 + \delta{\lambda_1}^{-\frac{1}{2}}\big)\int_0^t e^{-\alpha(t-s)} g(\theta_s\omega)\,ds.
\end{aligned}$$

Let

$$r_1^2(\omega) = 1 + 2\big(1 + \delta{\lambda_1}^{-\frac{1}{2}}\big)\lim_{t\to\infty}\int_0^t e^{-\alpha(t-s)} g(\theta_{s-t}\omega)\,ds \tag{5.4}$$

which is a $\mathbb{P}$. a.s. finite random variable, since $g(\theta_t\omega)$ grows polynomially at most. And $Q_1(\omega)$ in fact is a $\mathbb{P}$. a.s. finite random variable, then for any $\mathcal{D}$-set B in $\mathcal{H}$, there is $T_0(B)>0$ such that

$$\big\|v(t,\theta_{-t}\omega)\big\|_1^2+\big\|v_t(t,\theta_{-t}\omega)\big\|_0^2\leqslant r_1^2(\omega),\quad \text{for } t\geqslant T_0(B)\ \mathbb{P}\text{. a.s.} \tag{5.5}$$

5.2. Boundedness of solutions in $\mathcal{H}^1$

To obtain the existence of random attractors we need the further estimates in $\mathcal{H}^1$ for solution (v,v_t) of system (4.1)–(4.2) with $(v(0),v_t(0))\in\mathcal{H}^1$.

For some $\beta>0$ small enough, multiplying (4.1) by $\Delta(v_t+\beta v)$ in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big[\nu\|\nabla v_t\|_0^2+\|\Delta v\|_0^2-2\beta\nu\langle v_t,\Delta v\rangle\big]+(1-\beta\nu)\|\nabla v_t\|_0^2+\beta\|\Delta v\|_0^2+\beta\langle\nabla v_t,\nabla v\rangle\\&\quad=\beta\big\langle f(u),\Delta v\big\rangle+\big\langle f(u),\Delta v_t\big\rangle.\end{aligned} \tag{5.6}$$

Notice that

$$\big\langle f(u),\Delta v_t\big\rangle=\frac{d}{dt}\big\langle f(u),\Delta v\big\rangle-\big\langle f'(u)u_t,\Delta v\big\rangle, \tag{5.7}$$

we focus on the last term. Then Hölder inequality yields

$$\big|\big\langle f'(u)u_t,\Delta v\big\rangle\big|\leqslant\big|f'(u)\big|_{L^6}|u_t|_{L^3}\|\Delta v\|_0.$$

By the assumptions of f and by classical interpolation results we have that for some positive constant C

$$\begin{gathered}\big|f'(u)\big|_{L^6}=\big|f'(v+\eta)\big|_{L^6}\leqslant C\big(1+|v|_{L^{12}}^2+\|\eta\|_2^2\big),\\ |v|_{L^{12}}^2\leqslant C\|v\|_1^{3/2}\|v\|_2^{1/2}\quad\text{and}\quad |u_t|_{L^3}\leqslant C\|u_t\|_0^{1/2}\|u_t\|_1^{1/2},\end{gathered}$$

then, by the similar calculation of [12], we have

$$\big|\big\langle f'(u)u_t,\Delta v\big\rangle\big|\leqslant Q_2(\theta_t\omega)\|u_t\|_0^{1/2}\|\Delta v\|_0^{3/2}\|\nabla u_t\|_0^{1/2}+Q_2(\theta_t\omega).$$

Write $u=v+\eta$, then applying Young inequality, for any $\mu>0$

$$\begin{aligned}&\big|\big\langle f'(u)u_t,\Delta v\big\rangle\big|\\&\quad\leqslant\mu\big(\|\nabla v_t\|_0^2+\|\Delta v\|_0^2\big)+Q_\mu(\theta_t\omega)\|v_t\|_0^2\|\Delta v\|_0^2+Q_\mu(\theta_t\omega)\|\eta_t\|_0^2\|\Delta v\|_0^2+Q_\mu(\theta_t\omega).\end{aligned}$$

Now fix $\mu>0$ small enough, by (5.6) and the above analysis we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big[\nu\|\nabla v_t\|_0^2+\|\Delta v\|_0^2-2\beta\nu\langle v_t,\Delta v\rangle-2\big\langle f(u),\Delta v\big\rangle\big]+\beta_1\big(\|\nabla v_t\|_0^2+\|\Delta v\|_0^2\big)\\&\quad\leqslant Q_\mu(\theta_t\omega)\big(\|v_t\|_0^2+\|\eta_t\|_0^2\big)\|\Delta v\|_0^2+C\big\|f(u)\big\|_0^2+Q_\mu(\theta_t\omega)\end{aligned}$$

for sufficiently small $\beta_1>0$ which is independent of ν.

Let

$$Y(t) = \nu\|\nabla v_t\|_0^2 + \|\Delta v\|_0^2 - 2\beta\nu\langle v_t, \Delta v\rangle - 2\langle f(u), \Delta v\rangle.$$

Then by Hölder inequality and the assumptions on f we have for some positive constants $\tilde{C}_1$ and $\tilde{C}_2$

$$\tilde{C}_1\big(\nu\|v_t\|_1^2 + \|v\|_2^2\big) - Q_0\big(\|u\|_1\big) \leqslant Y(t) \leqslant \tilde{C}_2\big(\nu\|v_t\|_1^2 + \|v\|_2^2\big) + Q_0\big(\|u\|_1\big) \tag{5.8}$$

with $Q_0(\|u\|_1)$ a polynomial of $\|u\|_1$. Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Y(t) + \big(\beta_1 - Q_\mu(\theta_t\omega)\big(\|v_t\|_0^2 + \|\eta_t\|_0^2\big)\big)Y(t) \\
&\quad \leqslant \big(\beta_1 + Q_\mu(\theta_t\omega)\big(\|v_t\|_0^2 + \|\eta_t\|_0^2\big)\big)\big[2\beta\nu\big|\langle v_t, \Delta v\rangle\big| + 2\big|\langle f(u), \Delta v\rangle\big|\big] \\
&\qquad + C\big\|f(u)\big\|_0^2 + Q_\mu(\theta_t\omega).
\end{aligned}$$

Applying Hölder inequality, using (5.8) and notice that $\|f(u)\|_0^2 \leqslant C(1 + \|u\|_1^6)$, we obtain

$$\frac{d}{dt}Y(t) + L(t)Y(t) \leqslant Q_3(\theta_t\omega)$$

with

$$L(t) = \beta_1 - 4Q_\mu(\theta_t\omega)\big(\|v_t\|_0^2 + \|\eta_t\|_0^2\big).$$

Then Gronwall lemma yields

$$Y(t) \leqslant e^{-\int_0^t L(s)\,ds}Y(0) + \int_0^t e^{-\int_s^t L(\tau)\,d\tau}Q_3(\theta_s\omega)\,ds. \tag{5.9}$$

By Hölder inequality

$$\frac{1}{t}\int_0^t Q_\mu(\theta_s\omega)\big\|\eta_t(s)\big\|_0^2\,ds \leqslant \left[\frac{1}{t}\int_0^t Q_\mu^2(\theta_s\omega)\big\|\eta_t(s)\big\|_0^2\,ds \times \frac{1}{t}\int_0^t \big\|\eta_t(s)\big\|_0^2\,ds\right]^{\frac{1}{2}}.$$

And by Lemma 4.1

$$\begin{aligned}
&\frac{1}{t}\int_0^t Q_\mu(\theta_s\omega)\big\|v_t(s)\big\|_0^2\,ds \\
&\quad \leqslant \left\{\frac{|H(t) - H(0)|}{t}Q_\mu^*(t,\omega) + Q_\mu^*(t,\omega)\left[\frac{1}{t}\int_0^t Q_4(\theta_s\omega)\,ds \times \frac{1}{t}\int_0^t \big\|\eta_t(s)\big\|_0^2\,ds\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}
\end{aligned}$$

where $Q_\mu^*(t,\omega)=\frac{1}{t}\int_0^t Q_\mu^2(\theta_s\omega)\|v_t(s)\|_0^2\,ds$. Write the right-hand side of (5.9) as $r(t,\omega)$. By Birkhoff ergodic theorem

$$\lim_{t\to\infty} Q_\mu^*(t,\omega)\leqslant \lim_{t\to\infty}\frac{1}{t}\int_0^t Q_\mu^2(\theta_s\omega)r_1^2(\theta_s\omega)\,ds=\mathbf{E}\big\{Q_\mu^2(\omega)r_1^2(\omega)\big\}<\infty,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Q_4(\theta_s\omega)\,ds=\mathbf{E}Q_4(\omega)<\infty$$

and notice that $\int_D F(u)\,dx\leqslant C(1+\|u\|_0+\|u\|_0^2+\|u\|_1^6)$,

$$\lim_{t\to\infty}\frac{|H(t,\theta_{-t}\omega)-H(0,\omega)|}{t}=0,\quad \text{a.s.}$$

Then by Lemma 3.1 and if $\operatorname{Tr} Q$ is sufficiently small we have

$$\lim_{t\to\infty} r(t,\theta_{-t}\omega)<\infty,\quad \mathbb{P}.\ \text{a.s.}$$

Then by (5.8) and (5.9) there is a $\mathbb{P}$. a.s. finite random variable $r_2(\omega)$ such that for any $\mathcal{D}$-set B in $\mathcal{H}^1$, there is $T_1(B)>0$ we have

$$\nu\big\|v_t(t,\theta_{-t}\omega)\big\|_1^2+\big\|v(t,\theta_{-t}\omega)\big\|_2^2\leqslant r_2^2(\omega),\quad t\geqslant T_1(B),\ \mathbb{P}.\ \text{a.s.} \tag{5.10}$$

5.3. Existence of the random attractor

Taking $\operatorname{Tr} Q$ sufficiently small as in the above subsection, it is shown that $\Phi^\nu(t,\theta_{-t}\omega)$ has an absorbing set in $\mathcal{H}^1$ which is independent of ν.

By the above analysis and Theorem 2.12, we just prove that $\Phi^\nu(t,\theta_{-t}\omega)$ is almost surely $\mathcal{D}$–α-contracting in $\mathcal{H}^1$. Write

$$v(t,\theta_{-t}\omega)=v_1(t,\theta_{-t}\omega)+v_2(t,\theta_{-t}\omega)$$

where v_1 and v_2 satisfy

$$\nu v_{1tt}+v_{1t}+Av_1=0,\qquad v_1(0)=v(0),\qquad v_{1t}(0)=v_t(0) \tag{5.11}$$

and

$$\nu v_{2tt}+v_{2t}+Av_2+f(v+\eta)=0,\qquad v_2(0)=0,\qquad v_{2t}(0)=0. \tag{5.12}$$

By a standard calculation for (5.11), we have for some positive constant $\gamma>0$ independent of ν

$$\big\|\tilde v_1(t)\big\|_0^2+\big\|v_1(t)\big\|_1^2\leqslant e^{-\gamma t}\big(\big\|v(0)\big\|_1^2+\big\|v_t(0)\big\|_0^2\big) \tag{5.13}$$

where $\tilde{v}_1 = v_{1t} + \delta v_1$. Then the condition (3) of Theorem 2.11 holds. Notice that $(v_2(0), v_{2t}(0)) = (0, 0) \in \mathcal{H}^1$, by the analysis of Section 5.2 we have

$$\left\| v_{2t}(t, \theta_{-t}\omega) \right\|_1^2 + \left\| v_2(t, \theta_{-t}\omega) \right\|_2^2 \leqslant r_2^2(\omega), \quad t > T_2, \ \mathbb{P}. \text{ a.s.} \tag{5.14}$$

for some $T_2 > 0$. For any tempered random ball $B_a \subset \mathcal{H}^1$, by the assumptions of f, (5.13), (5.14) and the embedding $H^2(D) \subset L^\infty(D)$, $\{f(v(s, \theta_{-s}\omega) + \eta(\omega))\colon 0 \leqslant s \leqslant t,\ (v(0), v_t(0)) \in B_a(\omega)\}$ is bounded in $H^2(D) \cap H_0^1(D)$ and therefore a compact set of $H_0^1(D)$ a.s. Then $\{(v_2(s, \theta_{-s}\omega), v_{2t}(s, \theta_{-s}\omega))\colon 0 \leqslant s \leqslant t,\ (v(0), v_t(0)) \in B_a(\omega)\}$ is precompact in $\mathcal{H}^1$ by the similar approach of [20] for deterministic nonlinear wave equations. So condition (2) of Theorem 2.11 holds, and $\Phi^\nu(t, \theta_{-t}\omega)$ is almost surely $\mathcal{D}$–α-contracting in $\mathcal{H}^1$. That means $\Phi^\nu(t, \theta_{-t}\omega)$ possesses a compact global random attractor $\mathcal{A}^\nu(\omega) \subset \mathcal{H}^1$. We summarize the main result of this section as the following theorem.

Theorem 5.1. *Suppose* (2.4) *and the assumptions on f hold, system* (2.2) *possesses a global random attractor $\mathcal{A}^\nu(\omega)$ in $\mathcal{H}^1$.*

6. Upper semicontinuity of $\mathcal{A}^\nu(\omega)$

In this section we prove that random attractor $\mathcal{A}^\nu(\omega)$ obtained in the last section is upper semicontinuous in probability with respect to ν. In fact we have our main result

Theorem 6.1.

$$\lim_{\nu \to 0} \operatorname{dist}\big(\mathcal{A}^\nu(\omega), \mathcal{A}^0\big) = 0 \quad \textit{with probability one.}$$

Before giving the proof of the above theorem we give some results about the upper semicontinuity of random attractors, for more see [2].

Suppose $\phi(t)$ is a dynamical system on a Polish space (E, d) and there exists a global attractor $\mathcal{A}^0$ in E. We perturb $\phi(t)$ by a random element depending on a parameter $\nu \in (0, 1]$, so that we obtain an RDS

$$\phi^\nu : \mathbb{R}^+ \times \Omega \times E \to E,$$

such that for $\mathbb{P}$. a.s. $\omega \in \Omega$ and all $t \in \mathbb{R}^+$

$$\text{(H1)} \qquad \phi^\nu(t, \theta_{-t}\omega)x \to \phi(t)x \quad \text{as } \nu \to 0,$$

uniformly on bounded sets of E. Then we can have the following result (see [2]).

Theorem 6.2. *Assume that for all $\nu \in (0, 1]$ there is a random attractor $\mathcal{A}^\nu(\omega)$ for RDS ϕ^ν which satisfies* (H1)*, and there exists a compact set K such that, $\mathbb{P}$. a.s.*

$$\text{(H2)} \qquad \lim_{\nu \to 0} \operatorname{dist}\big(\mathcal{A}^\nu(\omega), K\big) = 0.$$

Then

$$\lim_{\nu\to 0}\operatorname{dist}\big(\mathcal{A}^\nu(\omega),\mathcal{A}^0\big)=0\quad \textit{with probability one.}$$

In applications, (H2) will follow from a similar property for random absorbing sets which are in fact used to obtain random attractors (see [2]). So we only justify (H1). And since we consider the upper semicontinuity of attractors, we just verify condition (H1) for the orbits in attractors.

To apply the above result we need the following estimates.

Writing $y=v_t$ and by differentiation of (4.1), we have the following stochastic differential system

$$\nu y^\nu_{tt}+y^\nu_t+Ay^\nu+F^\nu(t,\theta_{-t}\omega)=0 \tag{6.1}$$

with

$$F^\nu(t,\omega)=f'\big(u^\nu\big)\big(v^\nu_t+\eta^\nu_t\big).$$

Applying Hölder inequality and noticing the assumption on f' with the embedding $H_1\subset L^6$, we have for some positive constant C

$$\big\|F^\nu(t,\omega)\big\|_0\leqslant C\big(1+\|u^\nu\|_1^2\big)\big\|v^\nu_t+\eta^\nu_t\big\|_0. \tag{6.2}$$

Notice that we restrict system (4.1) on $\mathcal{A}^\nu(\omega)$, by the same approach in Section 5.1 and (6.2), there is an a.s. finite positive random variable $r_3(\omega)$ such that

$$\nu\big\|y^\nu_t\big\|_0^2\leqslant r_3^2(\omega).$$

Then we have

$$\nu\big\|v^\nu_{tt}(t,\theta_{-t}\omega)\big\|_0^2\leqslant r_3^2(\omega). \tag{6.3}$$

Proof of Theorem 6.1. Let $\phi(t)$ be the dynamical system generated by (2.5) and $\phi^\nu(t,\omega)=\Phi^\nu(t,\omega)$. It is enough to verify condition (H1) for the orbits in random attractor $\mathcal{A}^\nu(\omega)$. That is for a fixed $\omega\in\Omega$, we prove for any $\nu_n\in(0,1]$, $\nu_n\to 0$ as $n\to\infty$, there is a subsequence, which we still write as ν_n, such that $(v^{\nu_n}(t,\theta_{-t}\omega),v^{\nu_n}_t(t,\theta_{-t}\omega))\to(v(t),v_t(t))$ with $(v^{\nu_n}(0),v^{\nu_n}_t(0))=(v(0),v_t(0))$, with (v,v_t) satisfying system (2.5).

Let $V^{\nu_n}=v^{\nu_n}-v$, then V^{ν_n} satisfies the following equation

$$V^{\nu_n}_t=\Delta V^{\nu_n}-f\big(v^{\nu_n}+\eta^{\nu_n}\big)+f(v)-\nu_n v^{\nu_n}_{tt} \tag{6.4}$$

with $V^{\nu_n}(0)=0$. Since we restrict the system on $\mathcal{A}^\nu(\omega)$, by the assumption of f', f is Lipschitz with Lipschitz constant $L(\omega)$ which is an a.s. finite positive random variable. Taking scalar product in $L^2(D)$ with $-\Delta V^{\nu_n}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|V^{\nu_n}\big\|_1^2&\leqslant\big\langle f\big(v^{\nu_n}+\eta^{\nu_n}\big)-f(v),\Delta V^{\nu_n}\big\rangle+\nu_n\big\langle v^{\nu_n}_{tt},\Delta V^{\nu_n}\big\rangle\\&\leqslant L(\omega)\big\|V^{\nu_n}\big\|_1^2+L(\omega)\big\|\Delta V^{\nu_n}\big\|_0\big\|\eta^{\nu_n}\big\|_0+\nu_n\big\|v^{\nu_n}_{tt}\big\|_0\big\|\Delta V^{\nu_n}\big\|_0.\end{aligned}$$

Then it is easy to verify that

$$\|V^{\nu_n}(t)\|_1 \to 0 \quad \text{a.s. as } n \to \infty, \text{ for } t \in \mathbb{R}, \tag{6.5}$$

by (6.3) and the fact that $\eta^{\nu_n} \to 0$ as $n \to \infty$ a.s.

Multiplying both sides of (6.4) with $V_t^{\nu_n}$ in $L^2(D)$, we have

$$\|V_t^{\nu_n}\|_0^2 \leqslant \|V^{\nu_n}\|_1 \|V_t^{\nu_n}\|_1 + \|f(v^{\nu_n} + \eta^{\nu_n}) - f(v)\|_0 \|V_t^{\nu_n}\|_0 + \nu \|v_{tt}^{\nu_n}\|_0 \|V_t^{\nu_n}\|_0.$$

Then also by $\eta^{\nu_n} \to 0$ as $n \to \infty$ a.s., (6.3) and (6.5),

$$\|V_t^{\nu_n}(t)\|_0 \to 0 \quad \text{a.s. as } n \to \infty, \text{ for } t \in \mathbb{R}. \tag{6.6}$$

Thus condition of (H1) in Theorem 6.2 is satisfied which ends the proof of our main result. □

7. The limit of stationary solutions

In this section we consider the stationary solutions of system (2.2). By the random dynamical system theory, the existence of a compact random attractor yields the existence of stationary solutions which are included in the random attractor, see [7]. In the following we consider the convergence of the stationary solutions of (2.2) as $\nu \to 0$ in the space $L^2(0, T; L^2(D))$ for any $T > 0$.

Let $(\bar{u}^\nu(t), \bar{u}_t^\nu(t))$ be a stationary solution of the stochastic wave equation (2.2) with $\nu > 0$. Then we have the following result.

Theorem 7.1. *For any $T > 0$, there is a positive constant M_T such that the stationary solution $(\bar{u}^\nu(t), \bar{u}_t^\nu(t))$ satisfies*

$$\mathbf{E}\int_0^T \|\bar{u}^\nu(s)\|_1^2\, ds \leqslant M_T, \qquad \mathbf{E}\int_0^T \|\bar{u}_t^\nu(s)\|_0^2\, ds \leqslant M_T.$$

Proof. Applying Itô formula to $\|u_t^\nu\|_0^2$, we have

$$\mathcal{E}(\bar{u}^\nu(t)) - \mathcal{E}(\bar{u}^\nu(0)) = -2\int_0^t \|\bar{u}_t^\nu(s)\|_0^2\, ds + \operatorname{Tr} Qt + 2\sqrt{\nu}\int_0^t \langle \bar{u}_t^\nu, dW(s)\rangle, \tag{7.1}$$

with

$$\mathcal{E}(\bar{u}^\nu) = \nu\|\bar{u}_t^\nu\|_0^2 + \|\bar{u}^\nu\|_1^2 + 2\langle F(\bar{u}^\nu), 1\rangle.$$

Then taking expectation on both sides of (7.1) and by the stationarity of $(\bar{u}^\nu, \bar{u}_t^\nu)$, we have

$$\mathbf{E}\int_0^T \|\bar{u}_t^\nu(s)\|_0^2\, ds = \frac{1}{2}\operatorname{Tr} QT.$$

By the definition of $\mathcal{E}(u^\nu)$, (7.1) and the assumptions on F we have

$$\left\|\bar{u}^\nu(t)\right\|_1^2 \leqslant \mathcal{E}\left(\bar{u}^\nu(0)\right) + \operatorname{Tr} Qt + 2C_3|D| + 2\sqrt{\nu}\int_0^t \left\langle \bar{u}_t^\nu, dW(s)\right\rangle.$$

Thus we have

$$\mathbf{E}\int_0^T \left\|\bar{u}^\nu(t)\right\|_1^2 dt \leqslant \mathbf{E}\mathcal{E}\left(\bar{u}^\nu(0)\right)T + \operatorname{Tr} QT^2 + 2C_3|D|T.$$

Define $M_T = \max\{\frac{1}{2}\operatorname{Tr} QT,\ \mathbf{E}\mathcal{E}(\bar{u}^\nu(0))T + \operatorname{Tr} QT^2 + 2C_3|D|T\}$. The proof is complete. □

For our purpose we introduce the following lemma about a compactness result, see [16]. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \subset (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}}) \subset (\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ be three reflective Banach spaces and $\mathcal{X} \subset \mathcal{Y}$ with compact and dense embedding. Define Banach space

$$\mathcal{G} = \left\{v\colon\ v \in L^2(0,T;\mathcal{X}),\ \frac{dv}{dt} \in L^2(0,T;\mathcal{Z})\right\}$$

with norm

$$\|v\|_{\mathcal{G}}^2 = \int_0^T \left\|v(s)\right\|_{\mathcal{X}}^2 ds + \int_0^T \left\|\frac{dv}{ds}(s)\right\|_{\mathcal{Z}}^2 ds, \quad v \in \mathcal{G}.$$

Lemma 7.2. *If K is bounded in $\mathcal{G}$, then K is precompact in $L^2(0,T;\mathcal{Y})$.*

Now let $\mathcal{X} = H_0^1(D)$, $\mathcal{Y} = \mathcal{Z} = L^2(D)$. Then by the estimates of Theorem 7.1 and Chebyshev inequality, we have, for any $\varrho > 0$ there is a bounded set $K_\varrho \subset \mathcal{G}$ such that

$$\mathbb{P}\left\{\bar{u}^\nu \in K_\varrho\right\} > 1 - \varrho.$$

Moreover by Lemma 7.2, K_ϱ is precompact in $L^2(0,T;L^2(D))$. Then the family of measures $\{\mathcal{L}(\bar{u}^\nu)\}_\nu$, the distributions of $\{\bar{u}^\nu(\cdot)\}_\nu$ in $L^2(0,T;L^2(D))$, is tight.

Now we introduce a new probability space $(\Omega_\varrho, \mathcal{F}_\varrho, \mathbb{P}_\varrho)$ as

$$\Omega_\varrho = \left\{\omega' \in \Omega\colon\ u^\nu(\omega') \in K_\varrho\right\},$$
$$\mathcal{F}_\varrho = \{S \cap \Omega_\varrho\colon\ S \in \mathcal{F}\}$$

and

$$\mathbb{P}_\varrho(S) = \frac{\mathbb{P}(S \cap \Omega_\varrho)}{\mathbb{P}(\Omega_\varrho)}, \quad \text{for } S \in \mathcal{F}_\varrho.$$

Denote by $\mathbf{E}_\varrho$ the expectation operator with respect to $\mathbb{P}_\varrho$.

We restrict the system on the new probability space $(\Omega_\varrho, \mathcal{F}_\varrho, \mathbb{P}_\varrho)$ and arbitrarily fix a sequence $\{\hat{\nu}_n\} \subset (0, 1]$, $\hat{\nu}_n \to 0$, $n \to \infty$. Consider the stationary process $\bar{u}^{\hat{\nu}_n}$. By the compactness of K_ϱ, there is a subsequence $\{\nu_n\} \subset \{\hat{\nu}_n\}$ such that for any $\omega' \in \Omega_\varrho$

$$\bar{u}^{\nu_n}(0) \to \bar{u}(0), \quad \text{strongly in } L^2(D),\ n \to \infty, \tag{7.2}$$

$$\bar{u}^{\nu_n} \to \bar{u}, \quad \text{strongly in } L^2\big(0, T; L^2(D)\big),\ n \to \infty, \tag{7.3}$$

$$\nabla\bar{u}^{\nu_n} \to \nabla\bar{u}, \quad \text{weakly in } L^2\big(0, T; \big(L^2(D)\big)^3\big),\ n \to \infty, \tag{7.4}$$

with $\bar{u} \in L^2(0, T; H_0^1(D))$. We aim to prove that $\bar{u}$ solves (2.5) in the following weak sense

$$\int_0^T \big\langle \bar{u}(t), \phi_t(t)\big\rangle\,dt = \big\langle \bar{u}(0), \phi(0)\big\rangle + \int_0^T \big\langle \nabla\bar{u}(t), \nabla\phi(t)\big\rangle\,dt + \int_0^T \big\langle f\big(\bar{u}(t)\big), \phi(t)\big\rangle\,dt \tag{7.5}$$

for any $\phi \in C^1(0, T; C_0^\infty(D))$ with $\phi(T) = 0$. In fact $\bar{u}^{\nu_n}$, the solution of Eq. (2.2), can be written in the following weak form

$$\begin{aligned}
&\int_0^T \big\langle \bar{u}^{\nu_n}(t), \phi_t(t)\big\rangle\,dt \\
&\quad = \big\langle \bar{u}^{\nu_n}(0), \phi(0)\big\rangle + \int_0^T \big\langle \nabla\bar{u}^{\nu_n}(t), \nabla\phi(t)\big\rangle\,dt + \int_0^T \big\langle f\big(\bar{u}^{\nu_n}(t)\big), \phi(t)\big\rangle\,dt \\
&\qquad - \nu_n \int_0^T \big\langle \bar{u}_t^{\nu_n}, \phi_t(t)\big\rangle\,dt + \nu_n\big\langle \bar{u}_t^{\nu_n}(0), \phi(0)\big\rangle - \sqrt{\nu_n}\int_0^T \big\langle \phi(t), dW(t)\big\rangle
\end{aligned} \tag{7.6}$$

for almost all $\omega' \in \Omega_\varrho$ and $\phi \in C^1(0, T; C_0^\infty(D))$ with $\phi(T) = 0$. By the property of stochastic integral

$$\lim_{n\to\infty} \mathbf{E}_\varrho\left[\sqrt{\nu_n}\int_0^T \big\langle \phi(t), dW(t)\big\rangle\right]^2 \leqslant \lim_{n\to\infty} \nu_n \int_0^T \big|\phi(t)\big|^2_{\mathcal{L}_2^Q}\,dt = 0 \tag{7.7}$$

where $|\cdot|_{\mathcal{L}_2^Q}$ is the Hilbert–Schmidt norm, see [17]. Then by (7.2)–(7.4), the assumptions on f and the estimates in Theorem 7.1, we can pass the limit $n \to \infty$ in (7.6) in the sense of $L^2(\Omega_\varrho)$ and reach (7.5). And by the stationarity of $\bar{u}^{\nu_n}$, $\bar{u}$ is also a stationary process with value in $L^2(D)$. Furthermore it is easy to see that the distribution of $\bar{u}$ is independent of $\varrho > 0$. Then by the arbitrariness of ϱ, $\bar{u}^{\nu_n}$ converges to $\bar{u}$ in probability as $n \to \infty$.

Remark 7.3. Here we should point out that $\bar{u}$ may be still a random process since the initial value $\bar{u}(0)$ may be an $L^2(D)$-valued random variable as the limit of random variable $\bar{u}^{\nu_n}(0)$. That is, $\bar{u}$ may be a stationary solution for the heat equation (2.5) with random initial value. However, if the heat equation (2.5) has just a unique equilibrium point $\bar{u}$, $\bar{u}^\nu$ converges in probability to the deterministic equilibrium point $\bar{u}$ in $L^2(0, T; L^2(D))$ as $\nu \to 0$.

Now we can formulate our main result of this section in the following theorem.

Theorem 7.4. *Assume the conditions in Theorem* 5.1 *hold, for any stationary solution $\bar{u}^\nu$ of system* (2.2)*, there is a subsequence $\{\nu_n\} \subset (0, 1]$, with $\nu_n \to 0$ as $n \to \infty$, and a stationary process $\bar{u}$ valued in $L^2(D)$ such that $\bar{u}^{\nu_n}$ converges to $\bar{u}$ in probability. Moreover $\bar{u}$ is a stationary solution of system* (2.5)*.*

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References

[1] L. Arnold, Random Dynamical Systems, Springer-Verlag, New York/Berlin, 1998.
[2] T. Caraballo, J.A. Langa, J.C. Robinson, Upper semicontinuity of attractors for small random perturbations of dynamical systems, Comm. Partial Differential Equations 23 (1998) 1557–1581.
[3] S. Cerrai, M. Freidlin, On the Smoluchowski–Kramers approximation for a system with an infinite number of degrees of freedom, Probab. Theory Related Fields 135 (3) (2006) 363–394.
[4] P. Chow, Stochastic wave equation with polynomial nonlinearity, Ann. Appl. Probab. 12 (1) (2002) 361–381.
[5] P. Chow, W. Kohler, G. Papanicolaou, Multiple Scattering and Waves in Random Media, North-Holland, Amsterdam, 1981.
[6] D. Cheban, Global Attractors of Nonautonomous Dissipative Dynamical Systems, Interdiscip. Math. Sci., vol. 1, World Scientific, 2004.
[7] H. Crauel, F. Flandoli, Attractor for random dynamical systems, Probab. Theory Related Fields 100 (1994) 365–393.
[8] H. Crauel, A. Debussche, F. Flandoli, Random attractors, J. Dynam. Differential Equations 9 (1997) 307–341.
[9] J. Duan, K. Lu, B. Schmalfuß, Invariant manifolds for stochastic partial differential equations, Ann. Probab. 31 (4) (2003) 2109–2135.
[10] W. E, X. Li, E. Vanden-Eijnden, Some recent progress in multiscale modeling, in: Multiscale Modelling and Simulation, in: Lect. Notes Comput. Sci. Eng., vol. 39, Springer-Verlag, Berlin, 2004, pp. 3–21.
[11] X.M. Fan, Random attractor for a damped sine-Gordon equation with white noise, Pacific J. Math. 216 (1) (2004) 63–76.
[12] P. Fabrie, C. Galusinski, A. Miranville, S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, Discrete Contin. Dyn. Syst. Ser. A 10 (1–2) (2004) 211–238.
[13] J.K. Hale, G. Rauge, Upper semicontinuity of the attractor for a singular perturbed hyperbolic equation, J. Differential Equations 73 (2) (1988) 197–215.
[14] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Amer. Math. Soc., Providence, RI, 1988.
[15] P. Imkeller, A. Monahan (Eds.), Stochastic climate dynamics, a Special Issue in the journal Stoch. Dyn. 2 (3) (2002).
[16] J.L. Lions, Quelques méthodes de résolution des problèmes non linéaires, Dunod, Paris, 1969.
[17] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, 1992.
[18] D. Ruelle, Characteristic exponents for a viscous fluid subjected to time dependent forces, Comm. Math. Phys. 93 (1984) 285–300.
[19] B. Schmalfuß, Backward cocycles and attractors of stochastic differential equations, in: V. Reitmann, T. Riedrich, N. Koksch (Eds.), International Seminar on Applied Mathematics–Nonlinear Dynamics: Attractor Approximation and Global Behaviour, 1992, pp. 185–192.
[20] G.R. Sell, Y. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.
[21] M. Reed, B. Simon, Methods of Modern Mathematical Physics II, Academic Press, New York, 1975.
[22] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, second ed., Springer-Verlag, New York, 1997.

[23] R. Temam, A. Miranville, Mathematical Modeling in Continuum Mechanics, second ed., Cambridge Univ. Press, Cambridge, 2005.
[24] E. Waymire, J. Duan (Eds.), Probability and Partial Differential Equations in Modern Applied Mathematics, IMA, vol. 140, Springer-Verlag, New York, 2005.
[25] G. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[26] S.F. Zhou, F.Q. Yin, Z.G. OuYang, Random attractor for damped nonlinear wave equations with white noise, SIAM J. Appl. Dyn. Syst. 4 (4) (2005) 883–903.