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Reverse Hölder inequalities for singular parabolic equations near the boundary

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Abstract

We show that weak solutions to a singular parabolic partial differential equation globally belong to a higher Sobolev space than assumed a priori. To this end, we prove that the gradients satisfy a reverse Hölder inequality near the boundary. The results extend to singular parabolic systems as well. Motivation for studying reverse Hölder inequalities comes partly from applications to regularity theory.

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1. Introduction

We study the global regularity properties of singular parabolic partial differential equations. Parabolic partial differential equations with the principal part in the divergence form are either degenerate or singular depending on the vanishing of the gradient. In particular, the parabolic p-Laplace equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

is singular when $1 < p < 2$ and degenerate when $p > 2$. In the degenerate case, the modulus of ellipticity, $|\nabla u|^{p-2}$, vanishes when $|\nabla u| = 0$, whereas in the singular case, it becomes un-

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bounded. The modulus of ellipticity describes the rate of diffusion, and therefore, the behavior of solutions is quite different between the two cases. For example, disturbances have a finite speed of propagation in the degenerate case, whereas solutions extinct in finite time in the singular case.

Weak solutions to degenerate equations belong to a slightly higher Sobolev space than assumed a priori. Moreover, this holds up to the boundary, as shown in [21]. In the singular case, there are several new phenomena and hence, it is not obvious that singular equations have the higher integrability property as well. In this paper, we show that weak solutions to singular parabolic partial differential equations also have the higher integrability property when $2n/(n+2) < p \leqslant 2$. Furthermore, the results extend to systems of the form

$$\frac{\partial u_i}{\partial t} = \operatorname{div} \mathcal{A}_i(x,t,\nabla u), \quad i = 1,2,\dots,N.$$

We assume that the complement of the domain satisfies a uniform capacity density condition, which is essentially sharp for our main results. In addition, the boundary values belong to an appropriate higher Sobolev space. Note, however, that the results of this paper are already nontrivial for regular domains and smooth boundary values. The proofs are based on Caccioppoli and Sobolev–Poincaré-type inequalities as well as on the careful analysis of level sets. We also apply intrinsic scaling and covering arguments. Intuitively, some properties of the heat equation can be restored in the intrinsic geometry that depends on the gradient itself. However, boundary effects and singularity cause extra difficulties: The covering now consists of three kind of intrinsic cylinders. Indeed, the cylinders may lie near the lateral boundary, near the initial boundary or inside the domain. Due to singularity, it is a delicate problem to cover the space–time domain in such a way that an appropriate reverse Hölder inequality holds. Moreover, the proof in the degenerate case utilizes the L^p-norm of the gradient, whereas in the singular case, we avoid the use of the L^2-norm of the gradient by applying a different scaling.

The first nonlinear parabolic higher integrability results apparently date back to a 1982 paper of Giaquinta and Struwe [11]. They studied the local higher integrability for systems of parabolic equations with quadratic growth conditions. However, for more general systems, the problem remained open for some time: In the year 2000 Kinnunen and Lewis settled the local higher integrability question in [16] when $p > 2n/(n+2)$. For recent results, see Acerbi and Mingione [1] and Parviainen [22]. See also Antontsev and Zhikov [3], Arkhipova [4], DiBenedetto [5], and Duzaar and Mingione [6] for further parabolic regularity results.

In the elliptic case, the same higher integrability proof applies to both degenerate and singular equations. Granlund showed in [12] that an elliptic minimizer has the global higher integrability property if the complement of the domain satisfies a measure density condition. Later, Kilpeläinen and Koskela generalized the elliptic results to a wider class of equations and to a uniform capacity density condition in [15]. The higher integrability estimates provide a useful tool in applications to partial regularity (see, for example, Giaquinta and Modica [10]) and stability, to mention a few. On the other hand, the regularity properties of solutions are often interesting in their own right.

2. Preliminaries

2.1. Parabolic setting

Let Ω be a bounded open set in $\mathbf{R}^n$, $n \geqslant 2$, and let $2n/(n+2) < p \leqslant 2$. We study the equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathcal{A}(x,t,\nabla u), \quad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

where $u : \Omega \times (0,T) \to \mathbf{R}$ and $\mathcal{A} : \Omega \times (0,T) \times \mathbf{R}^n \to \mathbf{R}^n$. We assume that $\mathcal{A}$ is a Carathéodory function, that is, $(x,t) \mapsto \mathcal{A}(x,t,\xi)$ is measurable for every ξ in $\mathbf{R}^n$ and $\xi \mapsto \mathcal{A}(x,t,\xi)$ is continuous for almost every $(x,t) \in \Omega \times (0,T)$. In addition, there exist constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\mathcal{A}(x,t,\xi) \cdot \xi \geqslant \alpha |\xi|^p \quad \text{and} \quad \bigl|\mathcal{A}(x,t,\xi)\bigr| \leqslant \beta |\xi|^{p-1}.$$

As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions in $L^p(\Omega)$ whose first distributional partial derivatives belong to $L^p(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The Sobolev space $W_0^{1,p}(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.

The parabolic space $L^p(0,T; W^{1,p}(\Omega))$ is a collection of measurable functions $u(x,t)$ such that for almost every $t \in (0,T)$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ and the norm

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} = \left(\int_0^T \|u\|^p_{W^{1,p}(\Omega)} \,\mathrm{d}t \right)^{1/p}$$

is finite. Analogously, the space $L^p(0,T; W_0^{1,p}(\Omega))$ is a collection of measurable functions $u(x,t)$ such that for almost every $t \in (0,T)$, the function $x \mapsto u(x,t)$ belongs to $W_0^{1,p}(\Omega)$ and

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} < \infty.$$

The parabolic Sobolev space $W^{1,2}(0,T; L^2(\Omega))$ consists of functions

$$\left\{ \varphi \in L^2\bigl(0,T; L^2(\Omega)\bigr) \colon \frac{\partial \varphi}{\partial t} \in L^2\bigl(0,T; L^2(\Omega)\bigr) \right\}$$

with the norm

$$\|\varphi\|_{W^{1,2}(0,T;L^2(\Omega))} = \|\varphi\|_{L^2(0,T;L^2(\Omega))} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))}.$$

Finally, the space $C([0,T]; L^2(\Omega))$ comprises all continuous functions $u : [0,T] \to L^2(\Omega)$ (that is, u is continuous with respect to t in the norm $\|\cdot\|_{L^2(\Omega)}$) such that

$$\|u\|_{C([0,T];L^2(\Omega))} = \max_{t \in [0,T]} \bigl\|u(\cdot,t)\bigr\|_{L^2(\Omega)} < \infty.$$

A function u belonging to the space $L^2_{\mathrm{loc}}(\Omega \times (0,T)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution to (2.1) if

$$-\int_0^T\int_\Omega u\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \mathcal{A}(x,t,\nabla u)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{2.2}$$

for every $\phi \in C_0^\infty(\Omega \times (0,T))$.

A Lebesgue-type initial condition and a Sobolev-type boundary condition turn out to be convenient for our purposes. To be more specific, we say that u is a global solution if $u \in L^2(\Omega \times (0,T)) \cap L^p(0,T;W^{1,p}(\Omega))$ satisfies (2.2) as well as the initial and boundary conditions:

$$\begin{aligned}&u(\cdot,t)-\varphi(\cdot,t)\in W_0^{1,p}(\Omega)\quad\text{for almost every } t\in(0,T)\quad\text{and}\\ &\frac{1}{h}\int_0^h\int_\Omega |u-\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\to 0\quad\text{as } h\to 0,\end{aligned} \tag{2.3}$$

for a given

$$\varphi\in W^{1,2}\big(0,T;L^2(\Omega)\big)\cap L^p\big(0,T;W^{1,p}(\Omega)\big)\cap C\big([0,T];L^2(\Omega)\big).$$

Observe that already smooth φ leads to a nontrivial theory.

There is a well-recognized difficulty in proving Caccioppoli-type estimates for weak solutions: We often use test function depending on u itself, but u may not be admissible. We treat this difficulty by using the standard convolution. We set

$$\phi_\varepsilon(x,t)=\int_{\mathbf{R}}\phi(x,t-s)\zeta_\varepsilon(s)\,\mathrm{d}s,$$

where $\phi \in C_0^\infty(\Omega \times (0,T))$ and $\zeta_\varepsilon(s)$ is a standard mollifier, whose support is contained in $(-\varepsilon,\varepsilon)$ with $\varepsilon < \operatorname{dist}(\operatorname{spt}(\phi), \Omega \times \{0,T\})$. We insert ϕ_ε into (2.2), change variables, and apply Fubini's theorem to obtain

$$-\int_0^T\int_\Omega u_\varepsilon\frac{\partial\phi}{\partial t}\,\mathrm{d}z+\int_0^T\int_\Omega \mathcal{A}(x,t,\nabla u)_\varepsilon\cdot\nabla\phi\,\mathrm{d}z=0. \tag{2.4}$$

Here u_ε and $\mathcal{A}(x,t,\nabla u)_\varepsilon$ denote the mollified functions in the time direction.

2.2. Notation

Let

$$\Omega_T=\Omega\times(0,T)$$

be a space–time cylinder. We denote the points of the cylinder by $z = (x,t)$ and employ a shorthand notation $\mathrm{d}z = \mathrm{d}x\,\mathrm{d}t$.

Let $z_0 = (x_0, t_0) \in \Omega_T$ and $\theta, \rho > 0$. Then we denote

$$B_\rho(x_0) = \{x \in \mathbf{R}^n \colon |x - x_0| < \rho\},$$
$$\overline{B}_\rho(x_0) = \{x \in \mathbf{R}^n \colon |x - x_0| \leqslant \rho\}$$

and

$$\Lambda_{\theta\rho^2}(t_0) = \left(t_0 - \frac{1}{2}\theta\rho^2, t_0 + \frac{1}{2}\theta\rho^2\right).$$

Further, a space–time cylinder in $\mathbf{R}^{n+1}$ is denoted by

$$Q_{\rho,\theta\rho^2}(z_0) = Q_{\rho,\theta\rho^2}(x_0, t_0) = B_\rho(x_0) \times \Lambda_{\theta\rho^2}(t_0).$$

When no confusion arises, we shall omit the reference points and simply write B_ρ, $\Lambda_{\theta\rho^2}$ and $Q_{\rho,\theta\rho^2}$. The integral average of u is denoted by

$$u_\rho(t) = \fint_{B_\rho} u(x,t)\,\mathrm{d}x = \frac{1}{|B_\rho|}\int_{B_\rho} u(x,t)\,\mathrm{d}x,$$

where $|B_\rho|$ denotes the Lebesgue measure of B_ρ. The power $2_* = 2n/(n+2)$ is used in the initial boundary term. Finally, ϕ' sometimes denotes the time derivative of ϕ instead of $\frac{\partial\phi}{\partial t}$.

2.3. Capacity

Let $1 < p < \infty$. The *variational p-capacity* of a compact set $C \subset \Omega$ is defined to be

$$\operatorname{cap}_p(C, \Omega) = \inf_g \int_\Omega |\nabla g|^p\,\mathrm{d}x,$$

where the infimum is taken over all the functions $g \in C_0^\infty(\Omega)$ such that $g = 1$ in C. To define the variational p-capacity of an open set $U \subset \Omega$, we take the supremum over the capacities of the compact sets belonging to U. The variational p-capacity of an arbitrary set $E \subset \Omega$ is defined by taking the infimum over the capacities of the open sets containing E. For the capacity of a ball, we obtain the simple formula

$$\operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) = c\rho^{n-p}, \tag{2.5}$$

where $c > 0$ depends only on n and p. For further details, see Chapter 4 of Evans and Gariepy [7], Chapter 2 of Heinonen, Kilpeläinen and Martio [14], or Chapter 2 of Malý and Ziemer [18].

In this paper, we assume that the complement of the domain satisfies a uniform capacity density condition. For the higher integrability results, this condition is essentially sharp as pointed out in Remark 3.3. of Kilpeläinen and Koskela [15] in the elliptic case.

Definition 2.6. A set $E \subset \mathbf{R}^n$ is *uniformly p-thick* if there exist constants $\mu, \rho_0 > 0$ such that

$$\operatorname{cap}_p\big(E \cap \overline{B}_\rho(x), B_{2\rho}(x)\big) \geqslant \mu \operatorname{cap}_p\big(\overline{B}_\rho(x), B_{2\rho}(x)\big),$$

for all $x \in E$ and for all $0 < \rho < \rho_0$.

If we replace the capacity with the Lebesgue measure in the definition above, we obtain a measure density condition. A set E, satisfying the measure density condition, is uniformly p-thick for all $p > 1$. Singularity does not play an essential role before Lemma 3.2, and, therefore, we mostly omit the proofs of first lemmas. For more details, we refer the reader to the degenerate proofs in [21]. Since Ω is bounded, the estimate in Definition 2.6 actually holds for every ρ. Moreover, the estimate is also valid inside a uniformly p-thick domain near the boundary as stated in the next lemma.

Lemma 2.7. *Let Ω be a bounded open set, and suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. Choose $y \in \Omega$ such that $B_{\frac{4}{3}\rho}(y) \setminus \Omega \neq \emptyset$. Then there exists a constant $\tilde{\mu} = \tilde{\mu}(\mu, \rho_0, n, p) > 0$ such that*

$$\operatorname{cap}_p\big(\overline{B}_{2\rho}(y) \setminus \Omega, B_{4\rho}(y)\big) \geqslant \tilde{\mu} \operatorname{cap}_p\big(\overline{B}_{2\rho}(y), B_{4\rho}(y)\big).$$

A uniformly p-thick domain has a deep self-improving property. This result was shown by Lewis in [17], see also Ancona [2]. For a good survey of the boundary regularity, see Section 8 of Mikkonen [20].

Theorem 2.8. *Let $1 < p \leqslant n$. If a set E is uniformly p-thick, then there exists a constant $q = q(n, p, \mu)$ such that $1 < q < p$ for which E is uniformly q-thick.*

We end this section by stating without a proof a capacitary version of a Sobolev-type inequality. A boundary version of Sobolev's inequality follows from this lemma coupled with the boundary regularity condition. For the proof, see Hedberg [13], Chapter 10 of Maz'ja's monograph [19] or Lemma 3.1 of Kilpeläinen and Koskela [15].

The lemma employs quasicontinuous representatives of the Sobolev functions. We call $u \in W^{1,p}(\Omega)$ *p-quasicontinuous* if for each $\varepsilon > 0$ there exists an open set U, $U \subset \Omega \subset B_{R'}$, such that $\operatorname{cap}_p(U, B_{2R'}) \leqslant \varepsilon$, and the restriction of u to the set $\Omega \setminus U$ is finite valued and continuous.

The p-quasicontinuous functions are closely related to the Sobolev space $W^{1,p}(\Omega)$: For example, if $u \in W^{1,p}(\Omega)$, then u has a p-quasicontinuous representative. In addition, the capacity can be written in terms of quasicontinuous representatives.

Lemma 2.9. *Suppose that $q \in (1, p)$ and that $u \in W^{1,q}(B_{2\rho})$ is q-quasicontinuous. Denote*

$$N_{B_\rho}(u) = \big\{x \in \overline{B}_\rho\colon u(x) = 0\big\}$$

and choose $\tilde{q} \in [q, q^*]$, where $q^* = qn/(n-q)$. Then there exists a constant $c = c(n, q) > 0$ such that

$$\left(\fint_{B_{2\rho}} |u|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}} \leqslant \left(\frac{c}{\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^q\,\mathrm{d}x\right)^{1/q}.$$

The above estimate also holds if the powers on both sides are replaced by p.

Lemma 2.10. *Suppose that $u \in W^{1,p}(B_{2\rho})$ is p-quasicontinuous and let $N_{B_\rho}(u)$ be as above. Then there exists a constant $c = c(n, p) > 0$ such that*

$$\left(\fint_{B_{2\rho}} |u|^p \, \mathrm{d}x \right)^{1/p} \leqslant \left(\frac{c}{\operatorname{cap}_p(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

3. Estimates near the boundary

In this section, we derive estimates near the lateral boundary $\partial\Omega \times (0, T)$. These estimates are applied in Section 4 in order to prove a reverse Hölder inequality. We start with a Caccioppoli-type inequality.

Lemma 3.1 *(Caccioppoli). Let u be a global solution with the boundary and initial conditions* (2.3). *Let $\theta > 0$, suppose that $0 < \theta\rho^2 < M$ for some $M > 0$, and let $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$. Then there exists a constant $c = c(n, p, M, \alpha, \beta) > 0$ such that*

$$\begin{aligned} &\int_{Q_{\rho,\theta\rho^2} \cap \Omega_T} |\nabla u|^p \, \mathrm{d}z + \operatorname*{ess\,sup}_{t \in \Lambda_{\theta\rho^2} \cap (0,T)} \int_{B_\rho \cap \Omega} |u - \varphi|^2 \, \mathrm{d}x \\ &\quad \leqslant \frac{c}{\theta\rho^2} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |u - \varphi|^2 \, \mathrm{d}z + \frac{c}{\rho^p} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |u - \varphi|^p \, \mathrm{d}z \\ &\qquad + c \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \big(|\varphi'|^2 + |\nabla \varphi|^p \big) \, \mathrm{d}z. \end{aligned}$$

Proof. The proof is virtually the same as in the degenerate case. Observe, however, that now the power 2 dominates over p. Formally, we choose in (2.4) the test function

$$\phi(x, t) = \eta^p(x, t)\big(u(x, t) - \varphi(x, t)\big)\chi^h_{0,t_1}(t),$$

where $\chi^h_{0,t_1}(t)$ is a piecewise linear approximation of a characteristic function approaching $\chi_{0,t_1}(t)$ as $h \to 0$. Furthermore, $\eta \in C_0^\infty(\mathbf{R}^{n+1})$ is a cut-off function such that $\operatorname{spt} \eta \subset Q_{4\rho,\theta(4\rho)^2}$, $\eta(x, t) = 1$ in $Q_{\rho,\theta\rho^2}$, $0 \leqslant \eta \leqslant 1$, and

$$\rho|\nabla\eta| + \theta\rho^2 \left| \frac{\partial\eta}{\partial t} \right| \leqslant c.$$

The assumption $\theta\rho^2 < M$ is utilized together with Young's inequality to estimate

$$\int_{\Omega \times (0,t_1)} |\varphi'| \eta^p |u - \varphi| \, \mathrm{d}z \leqslant \varepsilon \int_{\Omega \times (0,t_1)} |\varphi'|^2 \eta^p \, \mathrm{d}z + \frac{c}{\theta\rho^2} \int_{\Omega \times (0,t_1)} \eta^p |u - \varphi|^2 \, \mathrm{d}z$$

in the proof. Here c depends on M and ε. $\square$

In order to derive a reverse Hölder inequality, we estimate the right hand side of Caccioppoli's inequality in terms of the gradient. A natural idea is to use Sobolev's inequality, but there is a principal difficulty in the parabolic case: We assume little regularity for a weak solution in the time direction, and Sobolev's inequality is not applicable in space–time cylinders as such. Nevertheless, weak solutions satisfy the following parabolic Sobolev's inequality.

Lemma 3.2 *(Parabolic Sobolev). Let u be a global solution with the boundary and initial conditions* (2.3)*. Suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. Let $\theta > 0$, suppose that $0 < \theta\rho^2 < M$ for some $M > 0$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$ such that $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. Then there exists a positive constant $c = c(n, p, M, \mu, \rho_0, \alpha, \beta)$ such that*

$$\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int\limits_{B_\rho\cap\Omega} |u-\varphi|^2\,\mathrm{d}x \leqslant \frac{c}{\theta\rho^2} \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} |u-\varphi|^2\,\mathrm{d}z + c \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} \bigl|\nabla(u-\varphi)\bigr|^p\,\mathrm{d}z$$
$$+ c \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} \bigl(|\varphi'|^2 + |\nabla\varphi|^p\bigr)\,\mathrm{d}z.$$

Proof. The claim follows from Caccioppoli's inequality and Lemma 2.10 in a straightforward manner: We extend $u(\cdot,t) - \varphi(\cdot,t)$ by zero outside of Ω and use the same notation for the extension. For a given t, we denote

$$N_{B_{2\rho}}(u-\varphi) = \bigl\{x \in \overline{B}_{2\rho}\colon u(x,t) - \varphi(x,t) = 0\bigr\}.$$

We estimate the second term on the right side of Caccioppoli's inequality by using Hölder's inequality and Lemma 2.10. Consequently,

$$\frac{c}{\rho^p} \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} |u-\varphi|^p\,\mathrm{d}z \leqslant \frac{c\rho^n}{\rho^p} \int\limits_{\Lambda_{\theta(4\rho)^2}\cap(0,T)} \frac{1}{\operatorname{cap}_p(N_{B_{2\rho}}(u-\varphi), B_{4\rho})} \int\limits_{B_{4\rho}} \bigl|\nabla(u-\varphi)\bigr|^p\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick and $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$, we conclude by Lemma 2.7 and (2.5) that

$$\operatorname{cap}_p\bigl(N_{B_{2\rho}}(u-\varphi), B_{4\rho}(x_0)\bigr) \geqslant \tilde{\mu}\operatorname{cap}_p\bigl(\overline{B}_{2\rho}(x_0), B_{4\rho}(x_0)\bigr) = c\rho^{n-p}$$

for almost every $t \in [0, T]$. Notice that this estimate still holds true if we redefine $u(\cdot,t) - \varphi(\cdot,t)$ in a set of measure zero in Ω. $\square$

One of the difficulties in proving the first reverse Hölder inequality is the fact that both the powers 2 and p appear in the above inequalities. We combine the previous lemma with the following Sobolev-type inequality in order to estimate the terms on the right hand side of the Caccioppoli. Observe that the self-improving property of the capacity density condition plays an important role in the proof.

Lemma 3.3. *Let u be a global solution with the boundary and initial conditions* (2.3)*. Suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. Let $\theta > 0$, suppose that $0 < \theta\rho^2 < M$ for some $M > 0$, and* *choose* $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$ *such that* $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. *Then there exist constants* $\tilde{q} < p$ *and* $c = c(n, p, M, \mu, \rho_0) > 0$ *such that*

$$\frac{1}{|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |u-\varphi|^2 \,\mathrm{d}z \leqslant \frac{c\rho^{\tilde{q}}}{|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \left|\nabla(u-\varphi)\right|^{\tilde{q}} \,\mathrm{d}z$$
$$\times \left(\operatorname*{ess\,sup}_{t \in \Lambda_{\theta(4\rho)^2} \cap (0,T)} \frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho} \cap \Omega} |u-\varphi|^2 \,\mathrm{d}x \right)^{1-\tilde{q}/2}.$$

Proof. In order to prove the claim, we apply Hölder's and Sobolev's inequalities. First, divide the term on the left hand side of the claim as

$$\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho} \cap \Omega} |u-\varphi|^2 \,\mathrm{d}x = \left(\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho} \cap \Omega} |u-\varphi|^2 \,\mathrm{d}x \right)^{\tilde{q}/2}$$
$$\times \left(\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho} \cap \Omega} |u-\varphi|^2 \,\mathrm{d}x \right)^{1-\tilde{q}/2}, \tag{3.4}$$

where $\tilde{q} < p$ is fixed later. Next we extend $u(\cdot,t) - \varphi(\cdot,t)$ by zero outside of Ω, use the same notation for the extension, and set $\tilde{q}^* = \tilde{q}n/(n-\tilde{q})$. Furthermore, for a given t, denote

$$N_{B_{2\rho}}(u-\varphi) = \left\{ x \in \overline{B}_{2\rho}\colon\ u(\cdot,t) - \varphi(\cdot,t) = 0 \right\}.$$

According to Lemma 2.9, we have

$$\left(\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho}} |u-\varphi|^2 \,\mathrm{d}x \right)^{\tilde{q}/2} \leqslant \frac{c}{\operatorname{cap}_{\tilde{q}}(N_{B_{2\rho}}(u-\varphi), B_{4\rho})} \int\limits_{B_{4\rho}} \left|\nabla(u-\varphi)\right|^{\tilde{q}} \,\mathrm{d}x. \tag{3.5}$$

To continue, we would like to use the uniform capacity density condition, but this is not immediately possible since $\tilde{q} < p$ and since we only assumed that the complement of a domain is uniformly p-thick. Nevertheless, Theorem 2.8 asserts that the density condition satisfies the self-improving property. This, together with Lemma 2.7 and (2.5), implies

$$\operatorname{cap}_{\tilde{q}}\left(N_{B_{2\rho}}(u-\varphi), B_{4\rho}\right) \geqslant \tilde{\mu} \operatorname{cap}_{\tilde{q}}(\overline{B}_{2\rho}, B_{4\rho}) = c\rho^{n-\tilde{q}},$$

for almost every t and for large enough $\tilde{q} < p$. We combine this capacity estimate with (3.5) and (3.4), and end up with

$$\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho}} |u-\varphi|^2 \,\mathrm{d}x \leqslant \frac{c\rho^{\tilde{q}}}{|B_{4\rho}|} \int\limits_{B_{4\rho}} \left|\nabla(u-\varphi)\right|^{\tilde{q}} \,\mathrm{d}x \left(\frac{1}{|B_{4\rho}|} \int\limits_{B_{4\rho}} |u-\varphi|^2 \,\mathrm{d}x \right)^{1-\tilde{q}/2}.$$

The claim follows by integrating this estimate with respect to time. $\square$

4. Reverse Hölder inequalities

The proof of the main result, Theorem 6.1, consists of three cases: We consider cylinders near the lateral boundary, near the initial boundary and inside the domain. This section provides a reverse Hölder inequality near the lateral boundary for the gradient of a solution, and the next section deals with a reverse Hölder inequality near the initial boundary. Finally, Section 6 combines all the cases and shows that the reverse Hölder inequalities have a self-improving property.

We utilize the estimates from the previous section in scaled space–time cylinders. The scaling takes both singularity and boundary effects into account. In particular, the scaling allows us to absorb the additional terms into the left hand side in the next lemma. In addition, the right scaling helps in combining the initial and lateral boundary estimates in the proof of the main result. Due to singularity, the term with the power 2 is dominant contrary to the degenerate case.

Lemma 4.1 *(Reverse Hölder). Let u be a global solution with the boundary and initial conditions* (2.3). *Suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. Let $\lambda > 0$, set $\theta = \lambda^{2-p}$, suppose that $0 < \theta\rho^2 < M$ for some $M > 0$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$ such that $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. Further, denote*

$$\mathcal{B}_\rho = \frac{1}{|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2} \cap \Omega_T} |\varphi'|^2 \, dz + \frac{1}{\theta|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2} \cap \Omega_T} |\nabla\varphi|^2 \, dz \tag{4.2}$$

for short. Suppose then that there exists a constant $c_1 \geqslant 1$ for which

$$\begin{aligned} c_1^{-1}\lambda^p &\leqslant \frac{1}{|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2} \cap \Omega_T} \left(\frac{|u-\varphi|^2}{\theta\rho^2} + |\nabla u|^p \right) dz + \mathcal{B}_\rho \\ &\leqslant \frac{c_1}{|Q_{20\rho,\theta(20\rho)^2}|} \int\limits_{Q_{20\rho,\theta(20\rho)^2} \cap \Omega_T} \left(\frac{|u-\varphi|^2}{\theta\rho^2} + |\nabla u|^p \right) dz + c_1\mathcal{B}_{20\rho} \leqslant c_1^2\lambda^p. \end{aligned} \tag{4.3}$$

Then there exist constants $c = c(n, p, M, c_1, \mu, \rho_0, \alpha, \beta) > 0$ and $\tilde{q} = \tilde{q}(n, p, \mu) < p$ such that

$$\frac{1}{|Q_{20\rho,\theta(20\rho)^2}|} \int\limits_{Q_{20\rho,\theta(20\rho)^2} \cap \Omega_T} |\nabla u|^p \, dz \leqslant \left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |\nabla u|^{\tilde{q}} \, dz \right)^{p/\tilde{q}} + c\mathcal{B}_{4\rho}.$$

Proof. To prove the claim, we estimate the terms on the right hand side of Caccioppoli's inequality with the gradient by using the parabolic version of Sobolev's inequality. Observe first that Lemma 3.1 provides the estimate

$$\begin{aligned} &\frac{1}{|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2} \cap \Omega_T} \left(|\nabla u|^p + \frac{|u-\varphi|^2}{\theta\rho^2} \right) dz + \mathcal{B}_\rho \\ &\quad\leqslant \frac{c}{\theta\rho^2|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |u-\varphi|^2 \, dz + \frac{c}{|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |\nabla\varphi|^p \, dz \end{aligned}$$

$$+\frac{c}{\rho^p|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^p\,\mathrm{d}z+c\mathcal{B}_{4\rho}. \tag{4.4}$$

Notice that we inserted some extra terms to the above inequality. This will help us at the end of the proof to absorb terms into the left.

Since $p \leqslant 2$ and $\theta=\lambda^{2-p}$, we may estimate the third term on the right in terms of the first by using Hölder's and Young's inequalities. We conclude that

$$\begin{aligned}
&\frac{c}{\rho^p|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^p\,\mathrm{d}z\\
&\quad\leqslant \theta^{p/2}\biggl(\frac{c}{\theta\rho^2|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^2\,\mathrm{d}z\biggr)^{p/2}\\
&\quad\leqslant \lambda^p\varepsilon+\frac{c}{\theta\rho^2|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^2\,\mathrm{d}z,
\end{aligned}\tag{4.5}$$

and hence it is enough to estimate the first term on the right hand side of (4.4).

In view of Lemma 3.3, there exists a constant $\tilde q<p$ such that

$$\begin{aligned}
&\frac{1}{|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^2\,\mathrm{d}z\\
&\quad\leqslant \frac{c\rho^{\tilde q}}{|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}\bigl|\nabla(u-\varphi)\bigr|^{\tilde q}\,\mathrm{d}z\\
&\qquad\times\biggl(\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(4\rho)^2}\cap(0,T)}\frac{1}{|B_{4\rho}|}\int\limits_{B_{4\rho}\cap\Omega}|u-\varphi|^2\,\mathrm{d}x\biggr)^{1-\tilde q/2}.
\end{aligned}\tag{4.6}$$

The first integral on the right hand side is of the correct form, but the second integral should be estimated from above by the gradient. To accomplish this, we apply Lemma 3.2, Hölder's inequality, and assumption (4.3). First, according to Hölder's inequality and (4.3), we have

$$\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|\nabla\varphi|^p\,\mathrm{d}z\leqslant\biggl(\frac{1}{\theta|Q_{4\rho,\theta(4\rho)^2}|}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|\nabla\varphi|^2\,\mathrm{d}z\biggr)^{p/2}\theta^{p/2}|Q_{4\rho,\theta(4\rho)^2}|\leqslant\rho^{n+2}\lambda^2,$$

since $\theta=\lambda^{2-p}$. This leads to

$$\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int\limits_{B_\rho\cap\Omega}|u-\varphi|^2\,\mathrm{d}x\leqslant\frac{c}{\theta\rho^2}\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}|u-\varphi|^2\,\mathrm{d}z+c\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T}\bigl|\nabla(u-\varphi)\bigr|^p\,\mathrm{d}z$$

$$+ c \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \big(|\varphi'|^2 + |\nabla \varphi|^p\big)\,\mathrm{d}z \leqslant c\rho^{n+2}\lambda^2. \tag{4.7}$$

To continue, we merge estimates (4.6) and (4.7), apply Young's inequality, and conclude that

$$\begin{aligned}
&\frac{1}{\theta\rho^2 |Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |u-\varphi|^2\,\mathrm{d}z \\
&\quad \leqslant \frac{\rho^{\tilde q} c}{\theta\rho^2 |Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \big|\nabla(u-\varphi)\big|^{\tilde q}\,\mathrm{d}z \big(\rho^2\lambda^2\big)^{1-\tilde q/2} \\
&\quad \leqslant \left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \big|\nabla(u-\varphi)\big|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q} + \varepsilon\lambda^p,
\end{aligned}$$

since $(\theta\rho^2)^{-1}\rho^{\tilde q}(\rho^2\lambda^2)^{1-\tilde q/2} = \lambda^{p-\tilde q}$.

We combine the previous estimate with (4.4) and (4.5). Furthermore, we deduce by Hölder's and Young's inequalities that the second term on the right hand side of (4.4) can be estimated as

$$\begin{aligned}
\frac{1}{|Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |\nabla\varphi|^p\,\mathrm{d}z &\leqslant \theta^{p/2}\left(\frac{1}{\theta |Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} |\nabla\varphi|^2\,\mathrm{d}z\right)^{p/2} \\
&\leqslant \varepsilon\lambda^p + c\mathcal{B}_{4\rho}.
\end{aligned}$$

Combining the facts, we end up with

$$\begin{aligned}
&\frac{1}{|Q_{\rho,\theta\rho^2}|} \int_{Q_{\rho,\theta\rho^2} \cap \Omega_T} \left(|\nabla u|^p + \frac{|u-\varphi|^2}{\theta\rho^2}\right)\mathrm{d}z + \mathcal{B}_\rho \\
&\quad \leqslant 3\varepsilon\lambda^p + \left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|} \int_{Q_{4\rho,\theta(4\rho)^2} \cap \Omega_T} \big|\nabla(u-\varphi)\big|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q} + c\mathcal{B}_{4\rho}.
\end{aligned} \tag{4.8}$$

Next we absorb $3\varepsilon\lambda^p$ into the left. To accomplish this, we employ scaling of the time direction and choose $\varepsilon > 0$ small enough. Finally, since (4.3) implies

$$\frac{1}{|Q_{20\rho,\theta(20\rho)^2}|} \int_{Q_{20\rho,\theta(20\rho)^2} \cap \Omega_T} |\nabla u|^p\,\mathrm{d}z \leqslant \frac{c}{|Q_{\rho,\theta\rho^2}|} \int_{Q_{\rho,\theta\rho^2} \cap \Omega_T} \left(|\nabla u|^p + \frac{|u-\varphi|^2}{\theta\rho^2}\right)\mathrm{d}z + c\mathcal{B}_\rho,$$

we have proven the claim. □

5. Estimates near the initial boundary

This section provides estimates near the initial boundary $\Omega \times \{0\}$. Here we compare the solution with its average instead of the boundary function, and the estimates become somewhat different.

The proof uses the weighted mean

$$u^{\eta}_{2\rho}(t) = \frac{\int_{B_{2\rho}} \eta^p(x,t)u(x,t)\,\mathrm{d}x}{\int_{B_{2\rho}} \eta^p(x,t)\,\mathrm{d}x},$$

instead of the standard mean

$$u_{2\rho}(t) = \fint_{B_{2\rho}} u(x,t)\,\mathrm{d}x.$$

The weighted mean should be close to the standard mean, and therefore the weight $\eta \in C_0^\infty(\mathbf{R}^{n+1})$ is defined to be a cut-off function such that

$$\operatorname{spt}\eta \subset Q_{2\rho,\theta(2\rho)^2}(x_0,t_0), \quad 0 \leqslant \eta \leqslant 1, \quad \text{and} \quad \eta = 1 \quad \text{in } Q_{\rho,\theta\rho^2}(x_0,t_0),$$

where $\theta > 0$. In addition,

$$\sup_{x\in B_{2\rho}} \eta(x,t) \leqslant \tilde{c} \fint_{B_{2\rho}} \eta(x,t)\,\mathrm{d}x, \quad t \in \Lambda_{\theta(2\rho)^2}(t_0), \tag{5.1}$$

where

$$\Lambda_{\theta(2\rho)^2}(t_0) = \left(t_0 - \frac{1}{2}\theta(2\rho)^2, t_0 + \frac{1}{2}\theta(2\rho)^2\right).$$

The following lemma gives a useful connection between the standard mean and the weighted mean.

Lemma 5.2. *Suppose that $B_{2\rho} \Subset \Omega$, let $u(\cdot,t) \in L^p_{\mathrm{loc}}(\Omega)$, where $p > 1$, and let η, $u^{\eta}_{2\rho}(t)$, $u_{2\rho}(t)$ be as above. Then there exists a constant $c = c(p,\tilde{c}) > 0$ such that*

$$\int_{B_{2\rho}} \left|u - u_{2\rho}(t)\right|^p \mathrm{d}x \leqslant c \int_{B_{2\rho}} \left|u - u^{\eta}_{2\rho}(t)\right|^p \mathrm{d}x \leqslant c^2 \int_{B_{2\rho}} \left|u - u_{2\rho}(t)\right|^p \mathrm{d}x.$$

Here $\tilde{c}$ is the constant in (5.1).

Proof. Let us begin with the first inequality. We add and subtract $u^{\eta}_{2\rho}(t)$, which leads to

$$\int_{B_{2\rho}} \left|u - u^{\eta}_{2\rho}(t) + u^{\eta}_{2\rho}(t) - u_{2\rho}(t)\right|^p \mathrm{d}x \leqslant c \int_{B_{2\rho}} \left|u - u^{\eta}_{2\rho}(t)\right|^p \mathrm{d}x + c|B_{2\rho}|\left|u^{\eta}_{2\rho}(t) - u_{2\rho}(t)\right|^p$$

since $p > 1$. This implies the desired estimate since

$$|B_{2\rho}|\left|u^{\eta}_{2\rho}(t) - u_{2\rho}(t)\right|^p \leqslant \int_{B_{2\rho}} \left|u^{\eta}_{2\rho}(t) - u\right|^p \,\mathrm{d}x$$

due to Hölder's inequality.

To obtain the second inequality of the claim, we add and subtract $u_{2\rho}(t)$. It follows that

$$\fint_{B_{2\rho}} \left|u - u^{\eta}_{2\rho}(t)\right|^p \,\mathrm{d}x \leqslant c \fint_{B_{2\rho}} \left|u - u_{2\rho}(t)\right|^p \,\mathrm{d}x + c\left|u_{2\rho}(t) - u^{\eta}_{2\rho}(t)\right|^p.$$

Then we estimate the last terms on the right hand side by using the definition of $u^{\eta}_{2\rho}(t)$, Hölder's inequality, and assumption (5.1). We conclude that

$$\begin{aligned}
\left|u^{\eta}_{2\rho}(t) - u_{2\rho}(t)\right| &\leqslant \frac{\fint_{B_{2\rho}} |u - u_{2\rho}(t)|\eta^p \,\mathrm{d}x}{\fint_{B_{2\rho}} \eta^p \,\mathrm{d}x} \leqslant \left(\frac{\sup_{x\in B_{2\rho}} \eta}{\fint_{B_{2\rho}} \eta \,\mathrm{d}x}\right)^p \fint_{B_{2\rho}} \left|u - u_{2\rho}(t)\right| \mathrm{d}x \\
&\leqslant \tilde{c}^p \left(\fint_{B_{2\rho}} \left|u - u_{2\rho}(t)\right|^p \,\mathrm{d}x\right)^{1/p},
\end{aligned}$$

which completes the proof. $\square$

We suppress the explicit dependence on $\tilde{c}$ in the notation, since this constant is fixed as soon as the weight is fixed. From now on, we assume that the cut-off function η, defined at the beginning of the section, also satisfies

$$\rho|\nabla\eta| + \theta\rho^2\left|\frac{\partial\eta}{\partial t}\right| \leqslant c. \tag{5.3}$$

The next lemma provides a Caccioppoli-type inequality near the initial boundary. We assume that $\varphi(\cdot, 0) \in W^{1,2_*+\delta}(\Omega)$ and, thus, the boundary term in the next lemma is well defined.

Lemma 5.4 *(Caccioppoli). Let u be a global solution with the boundary and initial conditions* (2.3)*. Let $\theta > 0$ and let $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$ be such that $B_{4\rho}(x_0) \subset \Omega$ and $0 \in \Lambda_{\theta(2\rho)^2}(t_0)$. Then there exists a constant $c = c(n, p, \alpha, \beta) > 0$ such that*

$$\begin{aligned}
&\int_{Q_{\rho,\theta\rho^2}\cap\Omega_T} |\nabla u|^p \,\mathrm{d}z + \operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int_{B_\rho} \left|u - u^{\eta}_{2\rho}(t)\right|^2 \mathrm{d}x \\
&\quad\leqslant \frac{c}{\theta\rho^2} \int_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u - u_{2\rho}(t)\right|^2 \mathrm{d}z + \frac{c}{\rho^p} \int_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u - u_{2\rho}(t)\right|^p \mathrm{d}z \\
&\qquad + c\left(\int_{B_{2\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\right)^{2/2_*},
\end{aligned}$$

where $2_* = 2n/(n+2)$.

Proof. Formally, we choose a test function

$$\phi(x,t)=\eta^p(x,t)\big(u(x,t)-u^\eta_{2\rho}(t)\big)\chi^h_{0,t_1}(t),\qquad t_1\in\Lambda_{\theta\rho^2}\cap(0,T),$$

where $u^\eta_{2\rho}(t)$ is the weighted mean and otherwise the notation is the same as in Lemma 3.1.

The weighted mean is utilized in the estimation of the first term of (2.4). We add and subtract $u^\eta_{2\rho}(t)\phi'$ to obtain

$$-\int_{\Omega_T}u\phi'\,\mathrm{d}z=-\int_{\Omega_T}\big(u-u^\eta_{2\rho}(t)\big)\phi'\,\mathrm{d}z-\int_{\Omega_T}u^\eta_{2\rho}(t)\phi'\,\mathrm{d}z.$$

The last term in the above expression vanishes. To see this, we integrate by parts, use the definition of $u^\eta_{2\rho}(t)$, and have

$$-\int_{\Omega_T}u^\eta_{2\rho}(t)\phi'\,\mathrm{d}z=\int_0^{t_1}\chi^h_{0,t_1}(t)\bigg(\int_{B_{2\rho}}u\eta^p\,\mathrm{d}x-\frac{\int_{B_{2\rho}}\eta^p\,\mathrm{d}x\int_{B_{2\rho}}\eta^p u\,\mathrm{d}x}{\int_{B_{2\rho}}\eta^p\,\mathrm{d}x}\bigg)\big(u^\eta_{2\rho}(t)\big)'\,\mathrm{d}t=0.$$

The rest of the proof is almost similar to the degenerate case and we omit it. □

The following lemma asserts that a parabolic Poincaré-type inequality is also valid near the initial boundary.

Lemma 5.5 *(Parabolic Poincaré). With the assumptions of the previous lemma, there exists a constant $c=c(n,p,\alpha,\beta)>0$ such that*

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho}\big|u-u^\eta_{2\rho}(t)\big|^2\,\mathrm{d}x\leqslant{}&\frac{c}{\theta\rho^2}\int_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T}\big|u-u_{2\rho}(t)\big|^2\,\mathrm{d}z\\&+c\int_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T}|\nabla u|^p\,\mathrm{d}z+c\bigg(\int_{B_{2\rho}}\big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}x\bigg)^{2/2_*}.\end{aligned}$$

Proof. This is an immediate consequence of Lemma 5.4 since Lemma 5.2 and Poincaré's inequality implies

$$\frac{c}{\rho^p}\int_{Q_{2\rho,\theta(2\rho^2)}}\big|u-u^\eta_{2\rho}(t)\big|^p\,\mathrm{d}z\leqslant c\int_{Q_{2\rho,\theta(2\rho^2)}}|\nabla u|^p\,\mathrm{d}z.\qquad\square$$

The following lemma helps us to combine Caccioppoli's inequality with parabolic Poincaré's inequality. The proof is a straightforward application of Hölder's and Poincaré's inequalities.

Lemma 5.6. *Let $u \in L^{2_*}(0,T; W^{1,2_*}_{\mathrm{loc}}(\Omega))$, let $\theta > 0$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbf{R}^{n+1}$ such that $B_{4\rho}(x_0) \subset \Omega$ and $0 \in \Lambda_{\theta(2\rho)^2}(t_0)$. Then there exists a constant $c = c(n) > 0$ such that*

$$\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T} \left|u-u_\rho(t)\right|^2 \mathrm{d}z \leqslant c \int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T} |\nabla u|^{2_*}\,\mathrm{d}z \Bigg(\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int\limits_{B_\rho} \left|u-u_{2\rho}(t)\right|^2 \mathrm{d}x\Bigg)^{2_*/n}.$$

Proof. First, we divide the left hand side into two parts as

$$\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T} \left|u-u_\rho(t)\right|^2 \mathrm{d}z = \int\limits_{\Lambda_{\theta\rho^2}\cap(0,T)} \Bigg(\int\limits_{B_\rho}\left|u-u_\rho(t)\right|^2 \mathrm{d}x\Bigg)^{1-\frac{2_*}{2}} \Bigg(\int\limits_{B_\rho}\left|u-u_\rho(t)\right|^2 \mathrm{d}x\Bigg)^{\frac{2_*}{2}} \mathrm{d}t.$$

Then we apply Poincaré's inequality to the second part, replace $u_\rho(t)$ by $u_{2\rho}(t)$ in the first the part, and take the essential supremum. □

The following lemma provides a counterpart for Lemma 4.1 near the initial boundary. Here we can ignore the lateral boundary terms in the scaling.

Lemma 5.7 *(Reverse Hölder). Let u be a global solution with the boundary and initial conditions* (2.3). *Let $\lambda > 0$, set $\theta = \lambda^{2-p}$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbf{R}^{n+1}$ such that $B_{40\rho}(x_0) \subset \Omega$ and $0 \in \Lambda_{\theta(4\rho)^2}(t_0)$. Suppose that there exists $c_1 > 1$ such that*

$$\begin{aligned} c_1^{-1}\lambda^p &\leqslant \frac{1}{|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T} \left(\frac{|u-u_\rho(t)|^2}{\theta\rho^2} + |\nabla u|^p\right)\mathrm{d}z \\ &\leqslant \frac{c_1}{|Q_{20\rho,\theta(20\rho)^2}|} \int\limits_{Q_{20\rho,\theta(20\rho)^2}\cap\Omega_T} \left(\frac{|u-u_{20\rho}|^2}{\theta\rho^2} + |\nabla u|^p\right)\mathrm{d}z \leqslant c_1^2\lambda^p. \end{aligned} \tag{5.8}$$

Then there exists a positive constant $c = c(n,p,c_1,\alpha,\beta)$ such that

$$\begin{aligned} &\frac{1}{|Q_{20\rho,\theta(20\rho)^2}|} \int\limits_{Q_{20\rho,\theta(20\rho)^2}\cap\Omega_T} |\nabla u|^p\,\mathrm{d}z \\ &\quad\leqslant \Bigg(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|} \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} |\nabla u|^{2_*}\,\mathrm{d}z\Bigg)^{p/2_*} + \frac{c}{\theta}\Bigg(\fint\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*}\mathrm{d}x\Bigg)^{2/2_*}. \end{aligned}$$

Proof. In view of Lemma 5.4, we have

$$\frac{1}{|Q_{\rho,\theta\rho^2}|} \int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T} \left(|\nabla u|^p + \frac{|u-u_\rho(t)|^2}{\theta\rho^2}\right)\mathrm{d}z$$

$$
\begin{aligned}
&\leqslant \frac{c}{\theta\rho^2|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u-u_{2\rho}(t)\right|^2 \mathrm{d}z \\
&\quad + \frac{c}{\rho^p|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u-u_{2\rho}(t)\right|^p \mathrm{d}z \\
&\quad + \frac{c}{\theta}\Bigg(\fint\limits_{B_{2\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\Bigg)^{2/2_*}. \qquad (5.9)
\end{aligned}
$$

Since $p \leqslant 2$ and $\theta = \lambda^{2-p}$, we can estimate the second term on the right hand side in terms of the first in the same way as in (4.5). Thus, we can concentrate on the first term on the right of (5.9).

Recalling Lemma 5.6, we have

$$
\begin{aligned}
&\frac{1}{\theta\rho^2|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u-u_{2\rho}(t)\right|^2 \mathrm{d}z \\
&\quad\leqslant \frac{c}{\theta\rho^2|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} |\nabla u|^{2_*} \mathrm{d}z \operatorname*{ess\,sup}_{t\in\Lambda_{\theta(2\rho)^2}\cap(0,T)} \Bigg(\int\limits_{B_{2\rho}} \left|u-u^\eta_{4\rho}(t)\right|^2 \mathrm{d}x\Bigg)^{2_*/n}.
\end{aligned}
$$

We also applied Lemma 5.2 to manipulate the last integral. Lemma 5.5 implies

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(2\rho)^2}\cap(0,T)} \int\limits_{B_{2\rho}} \left|u-u^\eta_{4\rho}(t)\right|^2 \mathrm{d}x &\leqslant \frac{c}{\theta\rho^2} \int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} \left|u-u_{4\rho}(t)\right|^2 \mathrm{d}z \\
&\quad + c\int\limits_{Q_{4\rho,\theta(4\rho)^2}\cap\Omega_T} |\nabla u|^p \,\mathrm{d}z + c\Bigg(\int\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\Bigg)^{2/2_*} \\
&\leqslant c\rho^{n+2}\lambda^2 + c\Bigg(\int\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\Bigg)^{2/2_*} \qquad (5.10)
\end{aligned}
$$

since $\theta = \lambda^{2-p}$ and $|Q_{4\rho,\theta(4\rho)^2}| = c\,\theta\rho^{n+2}$.

Collecting the facts, we end up with

$$
\begin{aligned}
&\frac{1}{\theta\rho^2|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} \left|u-u^\eta_{2\rho}(t)\right|^2 \mathrm{d}z \\
&\quad\leqslant \frac{c}{\theta\rho^2|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2}\cap\Omega_T} |\nabla u|^{2_*} \mathrm{d}z \Bigg(\rho^{n+2}\lambda^2 + \Bigg(\int\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\Bigg)^{2/2_*}\Bigg)^{2_*/n}.
\end{aligned}
$$

Observe that $\rho^{-2} = \rho^{-(n+2)2_*/n}$ and, on the other hand, $\rho^{-2} = (\rho^{-n})^{2/n}$. Young's inequality now leads to

$$\begin{aligned}
&\frac{1}{\rho^p |Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2} \cap \Omega_T} \left|u - u_{2\rho}(t)\right|^p \mathrm{d}z \\
&\quad\leqslant \left(\frac{c}{|Q_{2\rho,\theta(2\rho)^2}|} \int\limits_{Q_{2\rho,\theta(2\rho)^2} \cap \Omega_T} |\nabla u|^{2_*} \mathrm{d}z\right)^{p/2_*} \\
&\qquad + c\left(\;\fint\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\right)^{p/2_*} + \varepsilon\lambda^p.
\end{aligned} \tag{5.11}$$

Furthermore, since the power 2 dominates over p, we estimate

$$\begin{aligned}
\left(\;\fint\limits_{B_{2\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\right)^{p/2_*} &\leqslant \frac{1}{\theta^{p/2}}\left(\;\fint\limits_{B_{2\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\right)^{p/2_*} \theta^{p/2} \\
&\leqslant \varepsilon\lambda^p + \frac{c}{\theta}\left(\;\fint\limits_{B_{4\rho}} \left|\nabla\varphi(x,0)\right|^{2_*} \mathrm{d}x\right)^{2/2_*}.
\end{aligned} \tag{5.12}$$

Next we combine (5.9), (5.11), and (5.12), as well as recall the remark after (5.9). Finally, we absorb the terms containing λ^p into the left by choosing $\varepsilon > 0$ small enough. This is possible due to assumption (5.8). $\square$

6. The main result

This section provides an improved version of a reverse Hölder inequality. The proof employs covering arguments and the reverse Hölder inequalities from the previous sections. In the case $p = 2$, we could use the well-known Giaquinta–Modica lemma, which can be found from Giaquinta and Modica [10] or from Giaquinta [9]. See also Gehring [8], Stredulinsky [23] and Giaquinta and Struwe [11]. Due to singularity, we follow a different strategy.

We define that $\tilde{V}^2_\delta(0, T; \Omega)$ comprises functions in

$$W^{1,2+\delta}\big(0, T; L^{2+\delta}(\Omega)\big) \cap L^{2+\delta}\big(0, T; W^{1,2+\delta}(\Omega)\big) \cap C\big([0, T]; L^2(\Omega)\big)$$

with $\delta > 0$, and, furthermore, we assume that if $\varphi \in \tilde{V}^2_\delta(0, T; \Omega)$ then $\varphi(\cdot, 0) \in W^{1,2_*+\delta}(\Omega)$.

Theorem 6.1. *Let u be a global solution to* (2.2) *satisfying the boundary and initial conditions* (2.3) *for a boundary function $\varphi \in \tilde{V}^2_\delta(0, T; \Omega)$, where $\delta > 0$. Suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick and choose $Q_{R,R^2} = Q_{R,R^2}(x_0, t_0) \subset \mathbf{R}^{n+1}$ such that $Q_{4R,(4R)^2}$ intersects the lateral and initial boundaries. Then there exist constants $\varepsilon_0 = \varepsilon_0(n, p, \delta, \rho_0, \mu, \alpha, \beta) > 0$ and $c > 0$ with the same dependencies such that for all $0 \leqslant \varepsilon < \varepsilon_0$, we have*

$$\frac{1}{|Q_{R,R^2}|}\int_{Q_{R,R^2}\cap\Omega_T}|\nabla u|^{p+\varepsilon}\,\mathrm{d}z \leqslant \left(\frac{c}{|B_{4R}|}\int_{B_{4R}\cap\Omega}\left|\nabla\varphi(x,0)\right|^{2_*+\varepsilon}\,\mathrm{d}x\right)^{(2+\varepsilon)/(2_*+\varepsilon)}$$
$$+\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap\Omega_T}\left(|\nabla u|^p+g^{p+\varepsilon}\right)\mathrm{d}z$$
$$+\left(\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap\Omega_T}\left(f+g^p\right)\mathrm{d}z\right)^{\nu},$$

where

$$f=\frac{|u-\varphi|^2}{R^2}+\frac{|u-\tilde{u}_{4R}(t)|^2}{R^2}+|\nabla u|^p,$$
$$\tilde{u}_{4R}(t)=\frac{1}{|B_{4R}|}\int_{B_{4R}\cap\Omega}u\,\mathrm{d}x,$$
$$g=\left(|\nabla\varphi|^2+|\varphi'|^2\right)^{1/p},$$

and $\nu=(\varepsilon+\beta)/\beta$, $\beta=((n+2)p-2n)/2>0$.

Proof. The proof consists of several steps:

(1) The general idea is to divide the space–time cylinder into a good and a bad set. In the good set, the function $|\nabla u|^p$ is in control by definition, and in the bad set, we can estimate the average of the gradient by using the reverse Hölder inequality. The Calderón–Zygmund decomposition is usually applied for this, but here we use a different strategy which seems to work better in the nonlinear parabolic setting, in particular, in the global case. In the local setting, Kinnunen and Lewis developed this strategy in [16].
(2) To estimate the gradient in the bad set, we cover the space–time cylinder with intrinsic cylinders in such a way that we can apply reverse Hölder inequalities and control the dependence on a location of a cylinder. The main difference from the degenerate case is in the local geometry.
(3) We consider three possibilities: An intrinsic cylinder either lies near the lateral boundary or it does not. If it does not, then it may lie near the initial boundary or inside a domain. In addition, the intrinsic scaling should correspond to a right reverse Hölder inequality.
(4) Finally, we obtain the higher integrability by using Fubini's theorem.

Let us then carry out these steps.
Step (1): We denote $Q_0=Q_{4R,(4R)^2}(z_0)=Q_{4R,(4R)^2}(x_0,t_0)$. First, we choose the scaling $\lambda>0$ so that condition (4.3) or (5.8) holds in the cylinders having a center point in the bad set, where the size of the gradient is large. To this end, set

$$\beta=\frac{(n+2)p-2n}{2},$$

and

$$\lambda_0' = \left(\frac{1}{|Q_0|} \int_{Q_0 \cap \Omega_T} \left(f + g^p\right) \mathrm{d}z\right)^{1/\beta},$$

and choose λ such that

$$\lambda > \max\left(\lambda_0', 1\right) = \lambda_0.$$

Furthermore, set

$$\sigma = \frac{2n+8}{(n+2)((n+2)p-2n)}.$$

Step (2): Next we divide Q_0 into the Whitney-type cylinders

$$Q_i = Q_{r_i, r_i^2}(y_i, \tau_i), \quad i = 1, 2, \ldots,$$

where r_i is comparable to the parabolic distance of Q_i to the ∂Q_0. *Parabolic distance* is defined to be

$$\operatorname{dist}_p(E, F) = \inf\left\{|x - \bar{x}| + |t - \bar{t}|^{1/2}\colon (x, t) \in E,\ (\bar{x}, \bar{t}) \in F\right\}.$$

In addition, cylinders Q_i are of bounded overlap, meaning that every z belongs at most to a fixed finite number of cylinders, and

$$Q_{5r_i, (5r_i)^2} \subset Q_0.$$

For $(x, t) \in Q_0 \cap \Omega_T$, we define

$$h(x, t) = \frac{1}{c_2 |Q_0|^\sigma} \min\left\{|Q_i|^\sigma\colon (x, t) \in Q_i\right\} \left|\nabla u(x, t)\right|,$$

where $c_2 \geqslant 1$ is fixed later. Further, choose $(\tilde{x}, \tilde{t}) \in \Omega_T$ such that

$$h(\tilde{x}, \tilde{t}) > \lambda$$

and fix Q_i for which $(\tilde{x}, \tilde{t}) \in Q_i \cap \Omega_T$. We define

$$\alpha = \alpha(\tilde{x}, \tilde{t}) = \left(|Q_0|/|Q_i|\right)^\sigma,$$

and

$$\theta = (\lambda\alpha)^{2-p}.$$

Next we show that the second inequality in condition (4.3) is valid due to the definition of λ. To accomplish this, set $r = (\lambda\alpha)^{p/2-1} r_i'$ with $r_i/20 \leqslant r_i' \leqslant r_i$. Thus, $Q_{r,\theta r^2} \subset Q_0$, and for $Q_{r,\theta r^2} = Q_{r,\theta r^2}(\tilde{x}, \tilde{t})$, $r_i/20 \leqslant {r_i}' \leqslant r_i$, we obtain

$$
\begin{aligned}
&\frac{1}{|Q_{r,\theta r^2}|}\int\limits_{Q_{r,\theta r^2}\cap\Omega_T}\frac{1}{\theta r^2}|u-\varphi|^2\,\mathrm{d}z\\
&\quad\leqslant c|Q_0|r_i^{-(n+4)}(\lambda\alpha)^{(1-p/2)n}\frac{1}{|Q_0|}\int\limits_{Q_0\cap\Omega_T}|u-\varphi|^2\,\mathrm{d}z\\
&\quad\leqslant c\left(\frac{|Q_0|}{|Q_i|}\right)^{(n+4)/(n+2)}(\lambda\alpha)^{(1-p/2)n}\frac{1}{|Q_0|}\int\limits_{Q_0\cap\Omega_T}\frac{|u-\varphi|^2}{R^2}\,\mathrm{d}z\\
&\quad\leqslant c_2{}^p\alpha^p\lambda^p.
\end{aligned}
\tag{6.2}
$$

We also have

$$
\begin{aligned}
\frac{1}{|Q_{r,\theta r^2}|}\int\limits_{Q_{r,\theta r^2}\cap\Omega_T}\left(|\nabla u|^p+|\varphi'|^2\right)\mathrm{d}z&\leqslant c\frac{|Q_0|}{|Q_i|}(\alpha\lambda)^{-n(p-2)/2}\lambda^\beta\leqslant c_2{}^p\alpha^{(2(n+p))/(n+4)}\lambda^p\\
&\leqslant c_2{}^p\alpha^p\lambda^p,
\end{aligned}
$$

since $\alpha>1$ and $p\geqslant\frac{2(n+p)}{n+4}$ for $2n/(n+2)<p\leqslant 2$. Furthermore, we can estimate

$$
\begin{aligned}
\frac{1}{\theta|Q_{r,\theta r^2}|}\int\limits_{Q_{r,\theta r^2}\cap\Omega_T}|\nabla\varphi|^2\,\mathrm{d}z&\leqslant c\frac{|Q_0|}{|Q_i|}(\alpha\lambda)^{(1-n/2)(p-2)}\lambda^\beta\\
&\leqslant c_2^p\alpha^{((6+n)p-8)/(4+n)}\lambda^{(1-n/2)(p-2)+\beta}\\
&\leqslant c_2^p\alpha^p\lambda^p,
\end{aligned}
$$

since $\alpha,\lambda>1$ and $p\geqslant\frac{(6+n)p-8}{n+4}$ as well as $p\geqslant(1-\frac{2}{n})(p-2)+\beta$. We combine the estimates and obtain

$$
\frac{1}{|Q_{r,\theta r^2}|}\int\limits_{Q_{r,\theta r^2}\cap\Omega_T}\left(\frac{1}{\theta r^2}|u-\varphi|^2+|\nabla u|^p\right)\mathrm{d}z+\mathcal{B}_r\leqslant c_2{}^p\alpha^p\lambda^p,
$$

where c_2 is chosen to be large enough and $\mathcal{B}_r$ was defined in (4.2). The first inequality in (4.3) will be valid for small cylinders due to Lebesgue's differentiation theorem, and, thus

$$
\lim_{r'\to 0}\frac{1}{|Q_{r',\theta r'^2}|}\int\limits_{Q_{r',\theta r'^2}(\tilde{x},\tilde{t})}\left(\frac{1}{\theta r'^2}|u-\varphi|^2+|\nabla u|^p\right)\mathrm{d}z+\mathcal{B}_{r'}>c_2{}^p\alpha^p\lambda^p,
\tag{6.3}
$$

which holds for almost every $(\tilde{x},\tilde{t})\in Q_i\cap\Omega_T$ such that $h(\tilde{x},\tilde{t})>\lambda$.

Observe that the integral above is continuous with respect to r. Furthermore, the integral is less than or equal to $c_2{}^p\alpha^p\lambda^p$ for all r, $r_i/20\leqslant(\alpha\lambda)^{1-p/2}r\leqslant r_i$, and greater than $c_2{}^p\alpha^p\lambda^p$ for r small enough. Thus, there exists ρ_1, $0<\rho_1\leqslant(\alpha\lambda)^{p/2-1}r_i/20$, such that the integral equals $c_2{}^p\alpha^p\lambda^p$. Moreover, for all larger values of radius, the integral is less than or equal to $c_2{}^p\alpha^p\lambda^p$. We arrive at

$$
\begin{aligned}
c^{-1}\alpha^p\lambda^p &\leqslant \frac{1}{|Q_{\rho_1,\theta\rho_1^2}|}\int\limits_{Q_{\rho_1,\theta\rho_1^2}\cap\Omega_T}\left(\frac{1}{\theta\rho_1}|u-\varphi|^2+|\nabla u|^p\right)\mathrm{d}z+\mathcal{B}_{\rho_1}\\
&\leqslant \frac{c}{|Q_{20\rho_1,\theta(20\rho_1)^2}|}\int\limits_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap\Omega_T}\left(\frac{1}{\theta\rho_1}|u-\varphi|^2+|\nabla u|^p\right)\mathrm{d}z+c\mathcal{B}_{20\rho_1}\\
&\leqslant c^2\alpha^p\lambda^p. \qquad (6.4)
\end{aligned}
$$

At this point, we remark that $\alpha,\lambda>1$, and, hence, by construction, $Q_{20\rho_1,\theta(20\rho_1)^2}\subset Q_0$ as well as $\theta r^2<R^2<M$ for some $M>0$ as required.

Step (3): We shall also consider cylinders near the initial boundary. We suppose that $B_\rho\subset\Omega$, add and subtract $\tilde{u}_{4R}(t)$, and estimate

$$
\frac{1}{|Q_{\rho,\theta\rho^2}|}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}\left|u-\tilde{u}_\rho(t)\right|^2\mathrm{d}z\leqslant\frac{c}{|Q_{\rho,\theta\rho^2}|}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}\left|u-\tilde{u}_{4R}(t)\right|^2\mathrm{d}z.
$$

Thus, we can essentially repeat calculation (6.2). We can also repeat calculation (6.3), and, consequently, there exists ρ_2 such that

$$
\begin{aligned}
c^{-1}\alpha^p\lambda^p &\leqslant \frac{1}{|Q_{\rho_2,\theta\rho_2^2}|}\int\limits_{Q_{\rho_2,\theta\rho_2^2}\cap\Omega_T}\left(\frac{1}{\theta\rho_2}\left|u-\tilde{u}_{\rho_2}(t)\right|^2+|\nabla u|^p\right)\mathrm{d}z\\
&\leqslant \frac{c}{|Q_{20\rho_2,\theta(20\rho_2)^2}|}\int\limits_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap\Omega_T}\left(\frac{1}{\theta\rho_2}\left|u-\tilde{u}_{20\rho_2}(t)\right|^2+|\nabla u|^p\right)\mathrm{d}z\\
&\leqslant c^2\alpha^p\lambda^p. \qquad (6.5)
\end{aligned}
$$

We now have two alternatives: Either $B_{\frac{4}{3}\rho_1}(\tilde{x})\setminus\Omega\neq\emptyset$ and scaling (6.4) holds, or $B_{\frac{4}{3}\rho_2}(\tilde{x})\setminus\Omega=\emptyset$ and scaling (6.5) holds. Indeed, suppose that $B_\rho\subset\Omega$ and estimate

$$
\begin{aligned}
&\frac{1}{\theta\rho^2}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}\left|u-u_\rho(t)\right|^2\mathrm{d}z\\
&\quad\leqslant\frac{c}{\theta\rho^2}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}|u-\varphi|^2+\left|\varphi-\varphi_\rho(t)\right|^2+\left|\varphi_\rho(t)-u_\rho(t)\right|^2\mathrm{d}z. \qquad (6.6)
\end{aligned}
$$

Furthermore,

$$
\frac{1}{\theta\rho^2}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}\left|\varphi_\rho(t)-u_\rho(t)\right|^2\mathrm{d}z\leqslant\frac{1}{\theta\rho^2}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}|\varphi-u|^2\,\mathrm{d}z.
$$

Finally, Poincaré's inequality implies

$$\frac{1}{\theta\rho^2}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}\left|\varphi-\varphi_\rho(t)\right|^2\,\mathrm{d}z\leqslant\frac{c}{\theta}\int\limits_{Q_{\rho,\theta\rho^2}\cap\Omega_T}|\nabla\varphi|^2\,\mathrm{d}z.$$

Consequently, by multiplying the integrals in (6.4) by a constant $c=c(n,p)$ if necessary, we can make sure that they are larger than the integrals in (6.5). Hence, $\rho_2\leqslant\rho_1$ whenever $B_{\rho_2}\subset\Omega$, which shows that one of the two alternatives always holds.

Let us assume that the first alternative holds. If λ is replaced by $\alpha\lambda$, then (6.4) shows that condition (4.3) in Lemma 4.1 holds with ρ_1 whenever $h(\tilde{x},\tilde{t})>\lambda$. Thus, Lemma 4.1 implies

$$\frac{1}{|Q_{\rho_1,\theta\rho_1^2}|}\int\limits_{Q_{\rho_1,\theta\rho_1^2}\cap\Omega_T}|\nabla u|^p\,\mathrm{d}z\leqslant\left(\frac{c}{|Q_{4\rho_1,\theta(4\rho_1)^2}|}\int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\Omega_T}|\nabla u|^{\tilde{q}}\,\mathrm{d}z\right)^{p/\tilde{q}}+c\mathcal{B}_{4\rho_1},\tag{6.7}$$

for some $\tilde{q}<p$.

Assume then that the second alternative holds. If $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ does not intersect the initial boundary, then we obtain a local result

$$\begin{aligned}\frac{1}{|Q_{\rho_2,\theta\rho_2^2}|}\int\limits_{Q_{\rho_2,\theta\rho_2^2}\cap\Omega_T}|\nabla u|^p\,\mathrm{d}z&\leqslant c\left(|Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}|^{-1}\int\limits_{Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}\cap\Omega_T}|\nabla u|^{2_*}\,\mathrm{d}z\right)^{p/2_*}\\&\leqslant\left(\frac{c}{|Q_{4\rho_2,\theta(4\rho_2)^2}|}\int\limits_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap\Omega_T}|\nabla u|^{\tilde{q}}\,\mathrm{d}z\right)^{p/\tilde{q}},\end{aligned}\tag{6.8}$$

by essentially repeating the proof of Lemma 5.7 without the initial boundary terms. If the second alternative holds and if $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ intersects the initial boundary, then Lemma 5.7 can be adjusted to the current setting. Thus, by Hölder's inequality, we have

$$\begin{aligned}\frac{1}{|Q_{\rho_2,\theta\rho_2^2}|}\int\limits_{Q_{\rho_2,\theta\rho_2^2}\cap\Omega_T}|\nabla u|^p\,\mathrm{d}z&\leqslant\left(\frac{c}{|Q_{4\rho_2,\theta(4\rho_2)^2}|}\int\limits_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap\Omega_T}|\nabla u|^{\tilde{q}}\,\mathrm{d}z\right)^{p/\tilde{q}}\\&\quad+\frac{c}{\theta}\left(\frac{1}{|B_{4\rho_2}|}\int\limits_{B_{4\rho_2}\cap\Omega}\left|\nabla\varphi(x,0)\right|^{2_*}\,\mathrm{d}x\right)^{2/2_*}.\end{aligned}\tag{6.9}$$

For convenience, we only used integer multiples of radii in Lemma 5.7, but the proof holds verbatim for noninteger multiples as well.

Let us now return to the first alternative. By (6.4), we obtain

$$\begin{aligned} c^{-1}\lambda^p &\leqslant \frac{1}{|Q_{\rho_1,\theta\rho_1^2}|} \int\limits_{Q_{\rho_1,\theta\rho_1^2}\cap\Omega_T} \left(h^p + \frac{\alpha^{-p}}{\theta\rho_1^2}|u-\varphi|^2\right)\mathrm{d}z + \alpha^{-p}\mathcal{B}_{\rho_1} \\ &\leqslant \frac{1}{|Q_{20\rho_1,\theta(20\rho_1)^2}|} \int\limits_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap\Omega_T} \left(h^p + \frac{\alpha^{-p}}{\theta\rho_1^2}|u-\varphi|^2\right)\mathrm{d}z + \alpha^{-p}\mathcal{B}_{20\rho_1} \\ &\leqslant c^2\lambda^p, \end{aligned} \tag{6.10}$$

since the volumes of all the Whitney cylinders intersecting $Q_{20\rho_1,\theta(20\rho_1)^2}$ are comparable. In view of (6.7) and (6.10), we have

$$\begin{aligned} &\frac{1}{|Q_{20\rho_1,\theta(20\rho_1)^2}|} \int\limits_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap\Omega_T} \left(h^p + \frac{\alpha^{-p}}{\theta\rho_1^2}|u-\varphi|^2\right)\mathrm{d}z + \alpha^{-p}\mathcal{B}_{20\rho_1} \\ &\quad\leqslant \left(\frac{c}{|Q_{4\rho_1,\theta(4\rho_1)^2}|} \int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\Omega_T} h^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q} + c\alpha^{-p}\mathcal{B}_{4\rho_1}. \end{aligned} \tag{6.11}$$

Observe that the term $\int \frac{\alpha^{-p}}{\theta\rho_1^2}|u-\varphi|^2\,\mathrm{d}z$ is not needed on the right hand side. Indeed, this term can be estimated by the right hand side due to the Sobolev-type inequality as done in the proof of Lemma 4.1. See, in particular, (4.8).

Next we decompose Q_0 into level sets in the spirit of Step (1). We define

$$G(\lambda) = \{(x,t) \in Q_0 \cap \Omega_T\colon\ h(x,t) > \lambda\}$$

and

$$\tilde G(\lambda) = \{(x,t) \in Q_0 \cap \Omega_T\colon\ g(x,t) > \lambda\}.$$

Since $h(x,t) > \lambda$ in $G(\lambda)$, we can later use the previous estimates in $G(\lambda)$. Observe that

$$h(x,t) \leqslant \eta\lambda \quad \text{whenever } (x,t) \in (Q_{4\rho_1,\theta(4\rho_1)^2} \cap \Omega_T) \setminus G(\eta\lambda),$$

and

$$g(x,t) \leqslant \eta\lambda \quad \text{whenever } (x,t) \in (Q_{4\rho_1,\theta(4\rho_1)^2} \cap \Omega_T) \setminus \tilde G(\eta\lambda).$$

Furthermore, since $\alpha \geqslant 1$ and $\alpha^{-p}/\theta \leqslant 1$, we obtain by (6.11) and the previous estimates that

$$\begin{aligned} &\frac{1}{|Q_{20\rho_1,\theta(20\rho_1)^2}|} \int\limits_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap\Omega_T} \left(h^p + \frac{\alpha^{-p}}{\theta\rho_1^2}|u-\varphi|^2\right)\mathrm{d}z + \alpha^{-p}\mathcal{B}_{20\rho_1} \\ &\quad\leqslant c\eta^p\lambda^p + \left(\frac{c}{|Q_{4\rho_1,\theta(4\rho_1)^2}|} \int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q} \end{aligned}$$

$$+\frac{c}{|Q_{4\rho_1,\theta(4\rho_1)^2}|}\int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\tilde{G}(\eta\lambda)} g^p\,\mathrm{d}z. \tag{6.12}$$

By Hölder's inequality and (6.10), there exists a constant $c \geqslant 1$ such that

$$\left(\frac{1}{|Q_{4\rho_1,\theta(4\rho_1)^2}|}\int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\Omega_T} h^{\tilde{q}}\,\mathrm{d}z\right)^{(p-\tilde{q})/\tilde{q}} \leqslant c\lambda^{p-\tilde{q}}. \tag{6.13}$$

To continue, we choose $\eta > 0$ small enough to absorb the first two terms on the right hand side of (6.12) into the left. This is possible due to (6.10). We combine the result with (6.13), multiply by $|Q_{20\rho_1,\theta(20\rho_1)^2}|$ and get

$$\int\limits_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap\Omega_T} h^p\,\mathrm{d}z \leqslant c\lambda^{p-\tilde{q}}\int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap G(\eta\lambda)} h^{\tilde{q}}\,\mathrm{d}z + c\int\limits_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\tilde{G}(\eta\lambda)} g^p\,\mathrm{d}z. \tag{6.14}$$

If the second alternative holds and if $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ does not intersect the initial boundary, then we obtain a local version of the above estimate by using (6.5) and (6.8). Consequently,

$$\begin{aligned}\int\limits_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap\Omega_T} h^p\,\mathrm{d}z &\leqslant c\lambda^{p-\tilde{q}}\int\limits_{Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde{q}}\,\mathrm{d}z\\ &\leqslant c\lambda^{p-\tilde{q}}\int\limits_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde{q}}\,\mathrm{d}z.\end{aligned} \tag{6.15}$$

Finally, if the second alternative holds and if $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ intersects the initial boundary, then we obtain an initial boundary version by using (6.5) and (6.9). Indeed, observe first that

$$\frac{\alpha^{-p}}{\theta}\left(\frac{1}{|B_{4\rho_2}|}\int\limits_{(B_{4\rho_2}\cap\Omega)\setminus\overline{G}(\eta\lambda)} \left|\nabla\varphi(x,0)\right|^{2_*}\mathrm{d}x\right)^{2/2_*} \leqslant \frac{\alpha^{-p}}{(\alpha\lambda)^{2-p}}(\eta\lambda)^2 \leqslant \frac{\eta^2\lambda^p}{\alpha^2},$$

where

$$\overline{G}(\eta\lambda) = \left\{x \in B_{4R}(x_0)\cap\Omega\colon \left|\nabla\varphi(x,0)\right| > \eta\lambda\right\}.$$

Thus, by repeating the above reasoning and observing that $\rho_2^{-n2/2_*} = \rho_2^{-(n+2)}$, we deduce

$$\begin{aligned}\int\limits_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap\Omega_T} h^p\,\mathrm{d}z &\leqslant c\lambda^{p-\tilde{q}}\int\limits_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde{q}}\,\mathrm{d}z\\ &\quad + c\left(\int\limits_{B_{4\rho_2}\cap\overline{G}(\eta\lambda)} \left|\nabla\varphi(x,0)\right|^{2_*}\mathrm{d}x\right)^{2/2_*}.\end{aligned} \tag{6.16}$$

As a next step, we use a covering argument to extend the estimates to the whole of $G(\lambda)$. By Vitali's covering theorem, we have a disjoint set of cylinders

$$\big\{Q_{4\rho_i',\theta(4\rho_i')}(\tilde z_i)\big\}_{i=1}^{\infty}, \quad \tilde z_i \in G(\lambda),\ \tilde z_i = (\tilde x_i, \tilde t_i) \tag{6.17}$$

such that almost everywhere

$$G(\lambda) \subset \bigcup_{i=1}^{\infty} Q_{20\rho_i',\theta(20\rho_i')^2}(\tilde z_i) \subset Q_0,$$

and either (6.14), (6.15), or (6.16) holds in each of the cylinders. This is possible, since one of the two alternatives always holds. Then we sum over i and obtain

$$\begin{aligned}\int\limits_{G(\lambda)} h^p\,\mathrm{d}z &\leqslant \sum_{i=1}^{\infty} \int\limits_{Q_{20\rho_i',\theta(20\rho_i')^2}(\tilde z_i)\cap\Omega_T} h^p\,\mathrm{d}z \leqslant c\sum_{i=1}^{\infty}\Bigg(\lambda^{p-\tilde q} \int\limits_{Q_{4\rho_i',\theta(4\rho_i')^2}(\tilde z_i)\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z + b_i\Bigg)\\ &\leqslant c\lambda^{p-\tilde q}\int\limits_{G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z + c\int\limits_{\tilde G(\eta\lambda)} g^p\,\mathrm{d}z + c\Bigg(\int\limits_{\overline G(\eta\lambda)}\big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}x\Bigg)^{2/2_*},\end{aligned} \tag{6.18}$$

where b_i is either the lateral boundary term, initial boundary term, or zero depending on the corresponding estimate. When summing over the initial boundary terms, we used the fact $2/2_* > 1$. Step (4): The higher integrability result is now a consequence of (6.18) and Fubini's theorem. To see this, we integrate over $G(\lambda_0)$ and use (6.18) together with Fubini's theorem. Thus,

$$\begin{aligned}\int\limits_{G(\lambda_0)} h^{p+\varepsilon}\,\mathrm{d}z &= \int\limits_{G(\lambda_0)}\Bigg(\int\limits_{\lambda_0}^{h}\varepsilon\lambda^{\varepsilon-1}\,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\Bigg)h^p\,\mathrm{d}z = \varepsilon\int\limits_{\lambda_0}^{\infty}\lambda^{\varepsilon-1}\int\limits_{G(\lambda)} h^p\,\mathrm{d}z\,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\int\limits_{G(\lambda_0)} h^p\,\mathrm{d}z\\ &\leqslant c\int\limits_{\lambda_0}^{\infty}\Bigg(\varepsilon\lambda^{\varepsilon-1+p-\tilde q}\int\limits_{G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z + \varepsilon\lambda^{\varepsilon-1}\int\limits_{\tilde G(\eta\lambda)} g^p\,\mathrm{d}z\\ &\quad + \varepsilon\lambda^{\varepsilon-1}\Bigg(\int\limits_{\overline G(\eta\lambda)}\big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}x\Bigg)^{2/2_*}\Bigg)\,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\int\limits_{G(\lambda_0)} h^p\,\mathrm{d}z.\end{aligned} \tag{6.19}$$

We estimate the right hand side in three parts. Similarly as in the degenerate case, we first apply Fubini's theorem and end up with

$$\begin{aligned}&\varepsilon\int\limits_{\lambda_0}^{\infty}\lambda^{\varepsilon-1+p-\tilde q}\int\limits_{G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z\,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\int\limits_{G(\lambda_0)} h^p\,\mathrm{d}z\\ &\quad = c\varepsilon\int\limits_{G(\eta\lambda_0)}\int\limits_{\lambda_0}^{h/\eta}\lambda^{\varepsilon-1+p-\tilde q}h^{\tilde q}\,\mathrm{d}\lambda\,\mathrm{d}z + (\lambda_0)^{\varepsilon}\int\limits_{G(\lambda_0)} h^p\,\mathrm{d}z\end{aligned}$$

$$\leqslant \frac{c\varepsilon}{\varepsilon+p-\tilde{q}} \int_{G(\lambda_0)} h^{\varepsilon+p}\eta^{\tilde{q}-p-\varepsilon}\,\mathrm{d}z + c(\lambda_0)^{\varepsilon} \int_{G(\eta\lambda_0)} h^p\,\mathrm{d}z, \tag{6.20}$$

where we also dropped a negative term on the right hand side and used the fact that $\lambda_0 \geqslant h$ in $G(\eta\lambda_0) \setminus G(\lambda_0)$.

Let us now estimate the lateral boundary term in (6.19). We utilize Fubini's theorem and obtain

$$\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \int_{\tilde{G}(\eta\lambda)} g^p\,\mathrm{d}z\,\mathrm{d}\lambda = \int_{\tilde{G}(\eta\lambda_0)} \big((g/\eta)^{\varepsilon} - (\lambda_0)^{\varepsilon}\big) g^p\,\mathrm{d}z \leqslant c \int_{\tilde{G}(\eta\lambda_0)} g^{p+\varepsilon}\,\mathrm{d}z. \tag{6.21}$$

To estimate the initial boundary term in (6.19), we divide the term into two parts and apply Fubini's theorem as well as Hölder's inequality. It follows that

$$\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \Big(\int_{\overline{G}(\eta\lambda)} \big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}x \Big)^{2/2_*} \mathrm{d}\lambda \\
&\quad \leqslant \Big(\int_{\overline{G}(\eta\lambda_0)} \big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}x \Big)^{2/2_*-1} \int_{\overline{G}(\eta\lambda_0)} \int_{\lambda_0}^{|\nabla\varphi(x,0)|/\eta} \varepsilon\lambda^{\varepsilon-1} \big|\nabla\varphi(x,0)\big|^{2_*}\,\mathrm{d}\lambda\,\mathrm{d}x \\
&\quad \leqslant cR^{2\varepsilon/(2_*+\varepsilon)} \Big(\int_{\overline{G}(\eta\lambda_0)} \big|\nabla\varphi(x,0)\big|^{2_*+\varepsilon}\,\mathrm{d}x \Big)^{(2+\varepsilon)/(2_*+\varepsilon)}.
\end{aligned} \tag{6.22}$$

Now we are ready to collect the estimates. We combine (6.20), (6.21), and (6.22) with (6.19). Then we choose $\varepsilon > 0$ small enough to absorb the term containing $h^{p+\varepsilon}$ into the left hand side and get

$$\begin{aligned}
\int_{G(\lambda_0)} h^{p+\varepsilon}\,\mathrm{d}z &\leqslant c\,(\lambda_0)^{\varepsilon} \int_{G(\eta\lambda_0)} h^p\,\mathrm{d}z + c \int_{\tilde{G}(\eta\lambda_0)} g^{p+\varepsilon}\,\mathrm{d}z \\
&\quad + cR^{2\varepsilon/(2_*+\varepsilon)} \Big(\int_{\overline{G}(\eta\lambda_0)} \big|\nabla\varphi(x,0)\big|^{2_*+\varepsilon}\,\mathrm{d}x \Big)^{(2+\varepsilon)/(2_*+\varepsilon)}.
\end{aligned} \tag{6.23}$$

Notice that if the term we would like to absorb is infinite, then we can replace h by $h_k = \min\{h,k\}$, $k > \lambda_0$ similarly as in the degenerate case.

Since $h \leqslant \lambda_0$ in $(Q_0 \cap \Omega_T) \setminus G(\lambda_0)$, estimate (6.23) extends to the whole of $Q_{R,R^2} \cap \Omega_T$. Indeed,

$$\int_{Q_{R,R^2}\cap\Omega_T} h^{p+\varepsilon}\,\mathrm{d}z \leqslant (\lambda_0)^{\varepsilon} \int_{(Q_0\cap\Omega_T)\setminus G(\lambda_0)} h^p\,\mathrm{d}z + \int_{G(\lambda_0)} h^{p+\varepsilon}\,\mathrm{d}z$$

$$\leqslant c(\lambda_0)^{\varepsilon} \int\limits_{Q_0 \cap \Omega_T} h^p \, dz + c \int\limits_{Q_0 \cap \Omega_T} g^{p+\varepsilon} \, dz + cR^{2\varepsilon/(2_*+\varepsilon)} \left(\int\limits_{B_0 \cap \Omega} \left| \nabla \varphi(x,0) \right|^{2_*+\varepsilon} dx \right)^{(2+\varepsilon)/(2_*+\varepsilon)}.$$

Next we divide the estimate by $|Q_0|$ and apply the definition of $h(z)$. Since Q_{R,R^2} lies far away from the boundary of $Q_0 = Q_{4R,(4R)^2}$, there exists $c > 0$, independent of R, such that

$$\frac{1}{|Q_0|} \int\limits_{Q_{R,R^2} \cap \Omega_T} |\nabla u|^{p+\varepsilon} \, dz \leqslant \frac{c(\lambda_0)^{\varepsilon}}{|Q_0|} \int\limits_{Q_0 \cap \Omega_T} |\nabla u|^p \, dz + \frac{c}{|Q_0|} \int\limits_{Q_0 \cap \Omega_T} g^{p+\varepsilon} \, dz + \left(\frac{c}{|B_0|} \int\limits_{B_0 \cap \Omega} \left| \nabla \varphi(x,0) \right|^{2_*+\varepsilon} dx \right)^{(2+\varepsilon)/(2_*+\varepsilon)}.$$

Next we take the cut-off level into account. Remember that either

$$\lambda_0 = 1 \quad \text{or} \quad \lambda_0 = \lambda_0'.$$

The first case is clear. Moreover, if $\lambda_0 = \lambda_0'$, then Young's inequality and the definition of λ_0' leads to

$$\frac{1}{|Q_{R,R^2}|} \int\limits_{Q_{R,R^2} \cap \Omega_T} |\nabla u|^{p+\varepsilon} \, dz \leqslant \left(\frac{c}{|Q_0|} \int\limits_{Q_0 \cap \Omega_T} \left(f + g^p \right) dz \right)^{(\varepsilon+\beta)/\beta} + \frac{c}{|Q_0|} \int\limits_{Q_0 \cap \Omega_T} g^{p+\varepsilon} \, dz + \left(\frac{c}{|B_0|} \int\limits_{B_0 \cap \Omega} \left| \nabla \varphi(x,0) \right|^{2_*+\varepsilon} dx \right)^{(2+\varepsilon)/(2_*+\varepsilon)}.$$

This finishes the proof. □

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