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The Regularized Boussinesq equation: Instability of periodic traveling waves

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In this work we study the linear instability of periodic traveling waves associated with a generalization of the Regularized Boussinesq equation. By using analytic and asymptotic perturbation theory, we establish sufficient conditions for the existence of exponentially growing solutions to the linearized problem and so the linear instability of periodic profiles is obtained. With respect to applications of this approach, we prove the linear/nonlinear instability of cnoidal wave solutions for the modified Regularized Boussinesq equation and for a system of two coupled Boussinesq equations.

1. Introduction

This paper is concerned with the linear instability of periodic traveling waves for the following type of Regularized Boussinesq equation (RBou-type henceforth)

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$$\partial_t^2(1+\mathcal{M})u - \partial_x^2 u - \partial_x^2 f(u) = 0, \tag{1.1}$$

where $u = u(x,t)$, $x, t \in \mathbb{R}$, is a real-valued function, $\mathcal{M}$ is a differential or pseudo-differential operator in the framework of periodic functions, which is defined as a Fourier multiplier operator by

$$\widehat{\mathcal{M}g}(n) = \alpha(n)\widehat{g}(n), \quad n \in \mathbb{Z},$$

and f is assumed to be a smooth nonlinear function. The symbol α of $\mathcal{M}$ (representing the dispersive effects) is assumed to be a measurable, locally bounded, even function on $\mathbb{R}$, satisfying the conditions

$$a_1|n|^{m_1} \leqslant \alpha(n) \leqslant a_2\bigl(1+|n|\bigr)^{m_2},$$

for $1 \leqslant m_1 \leqslant m_2$, $|n| \geqslant k_0$, $\alpha(n) > b$, for all $n \in \mathbb{Z}$, $a_i > 0$, $i = 1, 2$. Furthermore, in the applications we shall consider $f(u) = u^{p+1}$, with integer $p \geqslant 1$.

For $\mathcal{M} = -\partial_x^2$ and $f(u) = u^2$ Eq. (1.1) arises in the modelling of acoustic waves on elastic rods with circular cross-section, when transverse motion and nonlinear effects are considered. In particular, it is used to describe the wave propagation at right angles to the magnetic field and also to approach the "bad" Boussinesq equation (see Makhankov [31]) or to study ion-sound waves (see Bogolubsky [15]).

The RBou equation has been studied by many authors and from many points of view. For instance, Yang and Wang in [39] studied the local well-posedness and blow up of solutions on the spatial interval $(0, 1)$ by the Galerkin method (see also Zhijian [40], Guowang and Shubin [23]). On the real line Liu [30] proved the existence of local and global solutions for the Pochhammer–Chree equation,

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0, \quad x, t \in \mathbb{R},$$

where $f(u)$ satisfies $f \in C^m(\mathbb{R})$, with m a positive integer, and $f(0) = 0$ (note that we obtain the Regularized Boussinesq equation if $f(u) = u + u^2$). The local well-posedness in $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ is obtained for $s \geqslant 1$ if $m \geqslant s$. Moreover, the global well-posedness on the same space is attained if $F(u) := \int_0^u f(x)\,dx \geqslant 0$ and there exists $q > 1/2$ such that $|f(x)| \lesssim F(x)^{1/q} + |x|$, for all $x \in \mathbb{R}$. Liu also obtained a nonlinear scattering result for the Pochhammer–Chree equation, which was later improved by Cho and Ozawa in [20]. The RBou equation has also been studied in $\mathbb{R}^n$, where explicit solitary and periodic solutions have been established (see for example [1,18,36,37]).

We want to call the attention to the generalized Regularized Boussinesq equation (gRBou), that is, when $\mathcal{M} = -\partial_x^2$ and $f(u) = \frac{1}{p+1}u^{p+1}$ in (1.1). For this equation, the existence of solitary waves is well known, with profile given by

$$u_c(x) = \alpha\,\operatorname{sech}^{2/p}(\gamma x), \tag{1.2}$$

where

$$c^2 > 1, \quad \alpha = \left[\frac{1}{2}\bigl(c^2-1\bigr)(p+2)(p+1)\right]^{\frac{1}{p}} \quad \text{and} \quad \gamma = \frac{1}{2}p\bigl[\bigl(c^2-1\bigr)/c^2\bigr]^{\frac{1}{2}}.$$

Pego and Weinstein [33] proved that if $1 < c^2 < \frac{3p}{4+2p}$ (and consequently $p > 4$) the family of solitary waves $c \mapsto u_c$ is linearly exponentially unstable by the flow of the gRBou. The nonlinear stability or instability for the solitary waves (1.2) with $1 \leqslant p \leqslant 4$ remains an open problem (the situation being the same in the periodic case). In the present work, and in the periodic case, we present a partial answer to this problem, for $p = 2$ (see Section 5 below). Specifically, we show that the cnoidal solution's profile remains nonlinearly unstable by the flow of the modified RBou.

We are interested in writing equation (1.1) as an equivalent system,

$$\begin{cases} v_t + (\mathcal{M}v)_t - \big(f(u)\big)_x - u_x = 0, \\ u_t - v_x = 0. \end{cases} \tag{1.3}$$

For (1.3) we now wish to find solutions of the form

$$\big(u_c(x,t), v_c(x,t)\big) = \big(\phi_c(x-ct), \psi_c(x-ct)\big), \tag{1.4}$$

with profiles $\phi_c, \psi_c : \mathbb{R} \to \mathbb{R}$ being L-periodic smooth functions and $c \in \mathbb{R}$ the wave-speed. If we substitute (1.4) in (1.3), the following pseudo-differential system is obtained after an integration

$$\begin{cases} c\mathcal{M}\psi_c + c\psi_c + f(\phi_c) + \phi_c = A_{\phi_c,\psi_c}, \\ c\phi_c + \psi_c = B_{\phi_c,\psi_c}, \end{cases} \tag{1.5}$$

where A_{ϕ_c,ψ_c} and B_{ϕ_c,ψ_c} are constants, which are considered zero in our theory.

Some symmetry considerations deserve to be mentioned, before talking about instability. Since Eq. (1.3) is invariant under *translations* (if $(u(x,t), v(x,t))$ is a solution for (1.3), then $(u(x+y,t), v(x+y,t))$ is also a solution for every $y \in \mathbb{R}$) we obtain that the one-parameter group of unitary operators $\{T(y)\}_{y\in\mathbb{R}}$, defined by $T(y)f(\cdot) = f(\cdot + y)$, determines the (ϕ_c, ψ_c)-orbit

$$\Omega_{(\phi_c,\psi_c)} := \big\{\big(T(y)\phi_c, T(y)\psi_c\big);\ y \in \mathbb{R}\big\}.$$

Then, we say that $\Omega_{(\phi_c,\psi_c)}$ is stable in the periodic space $X = H_{per}^{\frac{m_2}{2}}([0,L]) \times L_{per}^2([0,L])$ by the flow of Eq. (1.3), if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\|(u_0, v_0) - (\phi_c, \psi_c)\|_X < \delta$ and $(u(t), v(t))$ is the global solution of (1.3) with initial data $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$, then

$$\inf_{y\in\mathbb{R}} \big\|\big(u(t), v(t)\big) - \big(T(y)\phi_c, T(y)\psi_c\big)\big\|_X < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Otherwise, the (ϕ_c, ψ_c)-orbit is said to be orbitally unstable in X.

The instability behavior would happen if the solutions ceased to exist after a finite time (blow-up case), for a class of initial data close to (ϕ_c, ψ_c). This kind of behavior for models of dispersive type is in general a very difficult task to be addressed.

Consider $w(x,t) = u(x+ct,t) - \phi_c(x)$ and $z(x,t) = v(x+ct,t) - \psi_c(x)$ in (1.3). Then, we obtain via the Taylor Theorem the next system

$$\begin{cases} (\partial_t - c\partial_x)(z + \mathcal{M}z) - \partial_x\big(w + f'(\phi_c)w\big) + O\big(\big\|(w,z)\big\|^2\big) = 0, \\ (\partial_t - c\partial_x)w - z_x + O\big(\big\|(w,z)\big\|\big) = 0. \end{cases}$$

Therefore, the following system,

$$\begin{cases} (\partial_t - c\partial_x)(z + \mathcal{M}z) - \partial_x\big(w + f'(\phi_c)w\big) = 0, \\ (\partial_t - c\partial_x)w - z_x = 0, \end{cases} \tag{1.6}$$

represents the linearization of (1.3) around (ϕ_c, ψ_c). Our objective will be to provide sufficient conditions which imply that the solution $(w,z) \equiv (0,0)$ is unstable by the linear flow of (1.6). More exactly, we are interested in finding a *growing mode solution* for (1.6) of the form $(e^{\lambda t}u(x), e^{\lambda t}v(x))$ with $\operatorname{Re}\lambda > 0$. Hence, for this to hold (u, v) has to satisfy the non-local differential equation

$$\begin{cases} v + \mathcal{M}v - \dfrac{\partial_x}{\lambda - c\partial_x}\big(u + f'(\phi_c)u\big) = 0, \\ u = \dfrac{\partial_x}{\lambda - c\partial_x} v, \end{cases} \tag{1.7}$$

where the expression $\frac{\partial_x}{\lambda - c\partial_x}$, with $\operatorname{Re}\lambda > 0$, is a notation for the linear operator $\partial_x(\lambda - c\partial_x)^{-1}$ defined in $L^2_{per}([0,L])$. From Eq. (1.7) it is clear that u and v have zero mean. Using the second equation of (1.7) to replace u in the first one, we arrive at

$$v + \mathcal{M}v - \left(\frac{\partial_x}{\lambda - c\partial_x}\right)^2 \big(1 + f'(\phi_c)\big)v = 0. \tag{1.8}$$

Next, we consider the space $\mathbb{V}$ of zero mean functions, more precisely

$$\mathbb{V} := \left\{ f \in L^2_{per}\big([0,L]\big) \colon \langle f \rangle = \frac{1}{L}\int_0^L f(x)\,dx = 0 \right\}$$

and the orthogonal projection on $\mathbb{V}$, $Q : L^2_{per}([0,L]) \to \mathbb{V}$, given by $Qu = u - \langle u \rangle$. Define $X^0_{m_2} = H^{m_2}_{per}([0,L]) \cap \mathbb{V}$. Then, based on Eq. (1.8) (see [29]), we consider the family of closed linear operators, for $\operatorname{Re}\lambda > 0$, $\mathcal{A}^\lambda : X^0_{m_2} \to \mathbb{V}$ given by

$$\mathcal{A}^\lambda w := (\mathcal{M} + 1)w - \left(\frac{\partial_x}{\lambda - c\partial_x}\right)^2 Q\big(w + f'(\phi_c)w\big). \tag{1.9}$$

We note that $\partial_x^2(\lambda - c\partial_x)^{-2} : \mathbb{V} \to \mathbb{V}$ and that $\mathcal{A}^\lambda$ is also well defined in $H^{m_2}_{per}([0,L])$. From the analyticity of the resolvent associated to the operator ∂_x, $\lambda \in \mathcal{S} \to (\lambda - c\partial_x)^{-1}$, for $\mathcal{S} = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$, we obtain that $\lambda \in \mathcal{S} \to \mathcal{A}^\lambda$ represents an analytical family of operators of type-A, namely,

(1) $D(\mathcal{A}^\lambda) = H^{m_2}_{per}([0,L])$ for all $\lambda \in \mathcal{S}$,
(2) for $u \in H^{m_2}_{per}([0,L])$, $\lambda \in \mathcal{S} \mapsto \mathcal{A}^\lambda u$ is analytic in the topology of $L^2_{per}([0,L])$.

Therefore, we obtain that all discrete eigenvalues of $\mathcal{A}^\lambda$ ($\operatorname{Re}\lambda > 0$) are stable (see Kato [28]).

In order to deduce the existence of a growing mode solution for (1.8), it is sufficient to find $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ such that the operator $\mathcal{A}^\lambda$ possesses a nontrivial kernel. Indeed, for $u \in H^{m_2}_{per}([0,L]) \cap \mathbb{V}$, $u \neq 0$, such that $\mathcal{A}^\lambda u = 0$ we obtain

$$\begin{aligned} 0 &= (\lambda - c\partial_x)^2(\mathcal{M} + 1)u + \partial_x^2\big[u + f'(\phi_c)u - \big\langle f'(\phi_c)u \big\rangle\big] \\ &= (\lambda - c\partial_x)^2(\mathcal{M} + 1)u + \partial_x^2\big[u + f'(\phi_c)u\big]. \end{aligned}$$

In our approach we find a growing mode solution for $\lambda > 0$ via asymptotic analytic perturbation theory. Indeed, since for

$$\mathcal{L}_0 = (\mathcal{M} + 1) - \frac{1}{c^2}\big(1 + f'(\phi_c)\big), \tag{1.10}$$

we have that

$$\mathcal{A}^\lambda \to Q\mathcal{L}_0 \quad \text{as } \lambda \to 0^+,$$

strongly in $H_{per}^{m_2}([0,L]) \cap \mathbb{V}$, the usual perturbation theories do not apply and so we extend the asymptotic perturbation arguments due to Vock and Hunziker [35] and Lin [29] (see also Hislop and Sigal [25]) to the periodic case in order to deduce the existence of a purely growing mode. In our analysis we need to count the number of eigenvalues of $\mathcal{A}^\lambda$ (λ small) in the left-half plane. Since the kernel of $\mathcal{L}_0$ is nontrivial we need to know how the zero eigenvalue of $Q\mathcal{L}_0$ is perturbed, so we deduce a moving kernel formula that allows us to decide when zero is moving to the right or left (see Lemma 4.1 below). We also establish the stability of all discrete eigenvalues of $Q\mathcal{L}_0$ when they are analytically perturbed by the operator $\mathcal{A}^\lambda$, for $\lambda > 0$ and small enough (see Definition 3.1 and Lemma 3.4 below).

The linearized instability result for the RBou equation (1.3) is the following:

Theorem 1.1 *(Instability criterion for the RBou-type equation). Define the space $X_{m_2}^0 = H_{per}^{m_2}([0,L]) \cap \mathbb{V}$ and let $c \mapsto (\phi_c, \psi_c) \in X_{m_2}^0 \times X_{m_2}^0$ be a smooth curve of periodic solutions for Eq. (1.5) with $c^2 > 1$. Assume that*

$$\ker(Q\mathcal{L}_0) = \left[\frac{d}{dx}\phi_c\right].$$

Denote by $n^-(Q\mathcal{L}_0)$ the number (counting multiplicity) of negative eigenvalues of the operator $Q\mathcal{L}_0$ defined in $X_{m_2}^0$. Then, there is a purely growing mode $(e^{\lambda t}u(x), e^{\lambda t}v(x))$ to the linearized equation (1.6), with $\lambda > 0$, $u, v \in X_{m_2}^0 - \{(0,0)\}$, if one of the following conditions is true:

(i) *$n^-(Q\mathcal{L}_0)$ is even and $I(c) < 0$,*
(ii) *$n^-(Q\mathcal{L}_0)$ is odd and $I(c) > 0$.*

Here,

$$I(c) := -\frac{1}{\|\phi_c'\|_{L^2_{per}}^2}\frac{1}{c^2}\frac{dP}{dc}$$

with P given by

$$P(c) = c\big\langle(\mathcal{M}+1)\phi_c, \phi_c\big\rangle_{L^2_{per}}.$$

Theorem 1.1 enables us to establish a novel proof of the linear instability of *cnoidal* wave's profiles associated with the modified RBou equation (mRBou henceforth)

$$u_{tt} - u_{xxtt} - u_{xx} - \left(3u^2u_x\right)_x = 0,$$

i.e., $f(u) = u^3$ and $\mathcal{M} = -\partial_x^2$ in Eq. (1.1), provided $c \in (c^*, +\infty)$, $c^* > 1$ (see Theorem 5.2 below). In Section 6 we adapt the results due to Henry, Perez and Wreszinski [24] (see Theorem 6.1 below) to the case of dispersive equations and we obtain that the linear instability result implies nonlinear instability. The proof that *linear instability* implies *nonlinear instability* is obtained because the data-solution mapping associated with the mRBou equation is at least of class C^2.

In the last part of the paper, the theory of instability established for the RBou-type equation is also used to study the linear instability of periodic traveling waves for two coupled Boussinesq equations

$$\begin{cases} v_{tt} - v_{xxtt} - \left(v - \beta_0 v^p + w^p\right)_{xx} = 0, \\ w_{tt} - w_{xxtt} - \left(w + pvw^{p-1}\right)_{xx} = 0 \end{cases} \tag{1.11}$$

where $v = v(x,t)$, $w = w(x,t)$, $p \in \mathbb{N}$, $p \geqslant 2$, and $\beta_0 \in \mathbb{R} - \{0\}$. The case $p = 2$ in (1.11) models weakly nonlinear vibrations in a cubic lattice (see Christiansen, Lomdahl and Muto [19], Pego, Smereka and Weinstein [34]), where $v = v(x,t)$ is the longitudinal strain and $w = w(x,t)$ is the transverse strain. In the case $p = 3$ we adapt the theory of linear instability established for the RBou equation to obtain the nonlinear instability of periodic traveling wave solutions of cnoidal type for system (1.11).

Our paper is organized as follows. In Section 2 we present notation and preliminaries. Section 3 presents the main properties of the operator $\mathcal{A}^\lambda$ and sets that all the eigenvalues of $Q\mathcal{L}_0$ are stable by the perturbations $\mathcal{A}^\lambda$, for λ small enough. Section 4 establishes the moving kernel formula and the instability proof. In Sections 5 and 6, we provide the theories of instability of two families of traveling wave solutions, with cnoidal profile, for the mRBou and a coupled system of Regularized Boussinesq equations, respectively.

2. Notation and preliminaries

The L^2-based Sobolev spaces of periodic functions are defined as follows (for further details see Iorio and Iorio [27]). Let $\mathcal{P} = C^\infty_{per}$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ which are C^∞ and periodic with period $L > 0$. The collection $\mathcal{P}'$ of all continuous linear functionals from $\mathcal{P}$ into $\mathbb{C}$ is the set of *periodic distributions*. If $\Psi \in \mathcal{P}'$ and $\varphi \in \mathcal{P}$, we denote the value of Ψ at φ by $\langle \Psi, \varphi\rangle$. Define the functions $\Theta_k(x) = \exp(2\pi ikx/L)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$. The Fourier transform of Ψ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by $\widehat{\Psi}(k) = \frac{1}{L}\langle \Psi, \Theta_k\rangle$, for all $k \in \mathbb{Z}$. So, if Ψ is a periodic function with period L, we have $\widehat{\Psi}(k) = \frac{1}{L}\int_0^L \Psi(x) e^{-\frac{2k\pi ix}{L}}\,dx$. For $s \in \mathbb{R}$, the Sobolev space of order s, denoted by $H^s_{per}([0,L])$, is the set of all $f \in \mathcal{P}'$ such that $(1+|k|^2)^{\frac{s}{2}}\widehat{f}(k) \in l^2(\mathbb{Z})$, with norm $\|f\|^2_{H^s_{per}} = L\sum_{k=-\infty}^{\infty}(1+|k|^2)^s|\widehat{f}(k)|^2$. We also note that H^s_{per} is a Hilbert space with respect to the inner product $(f|g)_s = L\sum_{n=-\infty}^{\infty}(1+|k|^2)^s\widehat{f}(k)\overline{\widehat{g}(k)}$. The space H^0_{per} will be denoted by L^2_{per} and its norm will be $\|\cdot\|_{L^2_{per}}$. Of course $H^s_{per} \subseteq L^2_{per}$, for any $s \geqslant 0$.

The normal elliptic integral of first type (see Byrd and Friedman [17]) is defined by

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\phi,k)$$

where $y = \sin\phi$ and $k \in (0,1)$. k is called the modulus and ϕ the argument. When $y = 1$, we denote $F(\pi/2,k)$ by $K = K(k)$. The Jacobian elliptic functions are denoted by $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$ (called snoidal, cnoidal and dnoidal, respectively), and are defined via the previous elliptic integral. More precisely, let $u(y;k) := u = F(\phi,k)$, then $y = \sin\phi := \mathrm{sn}(u;k)$, $\mathrm{cn}(u;k) = \sqrt{1-\mathrm{sn}^2(u;k)}$ and $\mathrm{dn}(u;k) = \sqrt{1-k^2\,\mathrm{sn}^2(u;k)}$. We have the following asymptotic formulas: $\mathrm{sn}(x;1) = \tanh(x)$, $\mathrm{cn}(x;1) = \mathrm{sech}(x)$ and $\mathrm{dn}(x;1) = \mathrm{sech}(x)$.

3. Stability of the eigenvalues of $Q\mathcal{L}_0$ by $\mathcal{A}^\lambda$

In this section we show that all the discrete eigenvalues of $Q\mathcal{L}_0$ are stable by the family of linear operators $\mathcal{A}^\lambda$ in (1.9), for λ positive and small enough. We begin by establishing some basic structure of the family $\mathcal{A}^\lambda$.

3.1. Properties of $\mathcal{A}^\lambda$

Let us define the differential operators $\mathcal{D} = c\partial_x$ and $\mathcal{E}^\lambda = \frac{\lambda}{\lambda-\mathcal{D}}$. Then we can rewrite the operator $\mathcal{A}^\lambda$ in (1.9) as

$$\mathcal{A}^\lambda u = (\mathcal{M}+1)u - \frac{1}{c^2}\left(\mathcal{E}^\lambda - 1\right)^2 Q\left(u + f'(\phi_c)u\right).$$

Proposition 3.1. *For $\lambda > 0$, the operator $\mathcal{A}^\lambda$ converges to $\mathcal{A}^0 := Q\mathcal{L}_0$ strongly in $\mathbb{V}$ when $\lambda \to 0^+$, and converges to $\mathcal{M}+1$ strongly in L^2_{per} when $\lambda \to +\infty$.*

Proof. Consider $\varphi \in \mathbb{V}$. Then for $H\varphi := (1+f'(\phi_c))\varphi$ we have

$$\left\|\left(\mathcal{A}^\lambda - Q\mathcal{L}_0\right)\varphi\right\|_{L^2_{per}} = \frac{1}{c^2}\left\|\left[\left(\mathcal{E}^\lambda - 1\right)^2 - 1\right]QH\varphi\right\|_{L^2_{per}},$$

and

$$\left\|\left(\mathcal{A}^\lambda - (\mathcal{M}+1)\right)\varphi\right\|_{L^2_{per}} = \frac{1}{c^2}\left\|\left(\mathcal{E}^\lambda - 1\right)^2 QH\varphi\right\|_{L^2_{per}}.$$

Thus, since for $\lambda > 0$ the operator $\mathcal{E}^\lambda$ is continuous in $L^2_{per}([0,L])$ with respect to λ and satisfies the properties $\|\mathcal{E}^\lambda\|_{B(L^2_{per})} \leqslant 1$, $\|\mathcal{E}^\lambda - 1\|_{B(L^2_{per})} \leqslant 1$ and $\mathcal{E}^\lambda$ converges to 0 strongly (uniformly) in $\mathbb{V}$ as $\lambda \to 0^+$ and $\mathcal{E}^\lambda$ converges to I strongly in $L^2_{per}([0,L])$ as $\lambda \to +\infty$, we immediately obtain the proposition. □

Now, since the spectrum of $T = \mathcal{M}+1$ is discrete and $\mathcal{A}^\lambda - T$ is T-compact, we have for $\sigma_{ess}(A)$ denoting the essential spectrum of the operator A, that $\sigma_{ess}(\mathcal{A}^\lambda) = \sigma_{ess}(T) = \emptyset$. Therefore, the spectrum of $\mathcal{A}^\lambda$ with domain $H^2_{per} \cap \mathbb{V}$ is also discrete. So, for $\sigma_p(A)$ denoting the discrete spectrum of the operator A, we obtain the following proposition.

Proposition 3.2. *For any $\lambda > 0$, we have $\sigma(\mathcal{A}^\lambda) = \sigma_p(\mathcal{A}^\lambda)$.*

The next result is established for $\mathcal{A}^\lambda$ with domain $H^{m_2}_{per}$, and in particular it gives a localization of the spectrum of $\mathcal{A}^\lambda$ for every $\lambda > 0$.

Lemma 3.1. *Let $c^2 > 1$. There exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, $\mathcal{A}^\lambda$ does not have eigenvalues $z \in \mathbb{C}$ satisfying $\operatorname{Re} z \leqslant 0$.*

Proof. We follow the ideas established by Lin in [29]. Suppose by contradiction that there exists a sequence $\lambda_n \to +\infty$ and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, $\{u_n\}_{n\in\mathbb{N}} \subset H^{m_2}_{per}$, such that $\operatorname{Re} b_n \leqslant 0$ and $(\mathcal{A}^{\lambda_n} - b_n)u_n = 0$. Now, the inequality

$$\left\|\mathcal{A}^\lambda u - (\mathcal{M}+1)u\right\|_{L^2_{per}} \leqslant c^{-2}\left\|\left(\mathcal{E}^\lambda - 1\right)^2 QHu\right\|_{L^2_{per}} \leqslant C\|u\|_{L^2_{per}}$$

where $C > 0$ does not depend on $\lambda > 0$, implies for $z \in \sigma_p(\mathcal{A}^\lambda)$ and $\mathcal{A}^\lambda\psi = z\psi$ ($\|\psi\| = 1$) that

$$\left(\operatorname{Re} z - \left\langle\psi, (\mathcal{M}+1)\psi\right\rangle\right)^2 + (\operatorname{Im} z)^2 + \left\|(\mathcal{M}+1)\psi\right\|^2_{L^2_{per}} - \left\langle\psi, (\mathcal{M}+1)\psi\right\rangle^2 \leqslant C^2.$$

Since $\mathcal{M}+1$ is a self-adjoint positive operator, we obtain from Cauchy–Schwarz inequality that all eigenvalues of $\mathcal{A}^\lambda$ must lie in the closed subset

$$D_C := \left\{z \in \mathbb{C}\colon \operatorname{Re} z \geqslant -C \text{ and } |\operatorname{Im} z| \leqslant C\right\},$$

$\sigma_p(\mathcal{A}^\lambda) \subset D_C$. Then, there exists $b_\infty \in D_C$ such that $b_n \to b_\infty$, as $n \to +\infty$ and $\operatorname{Re} b_\infty \leqslant 0$. Denote $e(x) = [f'(\phi_c(x))]^2$, thus we can normalize u_n such that $\|u_n\|_{L^2_{per,e}} := \int u_n^2(x)e(x)\,dx = 1$. Now, since the equation $(\mathcal{A}^{\lambda_n} - b_n)u_n = 0$ implies (see Lemma 3.3 below)

$$\|u_n\|_{H^{\frac{m_1}{2}}_{per}} \leqslant M,$$

where M does not depend on n, there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, that we still denote by $\{u_n\}_{n\in\mathbb{N}}$, such that $u_n \rightharpoonup u_\infty$ in $H^{\frac{m_1}{2}}_{per}$, as $n \to +\infty$. On the other hand, since the embedding $H^{\frac{m_1}{2}}_{per} \hookrightarrow L^2_{per}$ is compact we deduce $u_n \to u_\infty$ in L^2_{per} and $u_n \to u_\infty$ in $L^2_{per,e}$, as $n \to +\infty$. Hence $\|u_\infty\|_{L^2_{per,e}} = 1$. Next, by using Proposition 3.1 we have $\mathcal{A}^{\lambda_n} u_n - (\mathcal{M}+1)u_n \to 0$ as $n \to +\infty$. So, since $\mathcal{A}^{\lambda_n} u_n \to b_\infty u_\infty$ we obtain that $(\mathcal{M}+1)u_n \to b_\infty u_\infty$. Therefore, as $\mathcal{M}$ is a closed operator, we have $u_\infty \in H^{m_2}_{per}$ and $(\mathcal{M}+1)u_\infty = b_\infty u_\infty$. However, since $\operatorname{Re} b_\infty \leqslant 0$ we obtain a contradiction because $\mathcal{M}+1$ is a positive operator. $\square$

Remark 3.1. Since $\lambda \in \mathcal{S} := \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\} \mapsto \mathcal{A}^\lambda$ is an analytic family of type-A, we obtain that every $\eta \in \sigma_p(\mathcal{A}^\lambda)$ $(\eta \in D_C)$ is stable, in other words, there is $\delta > 0$ such that for $\lambda_0 \in B(\eta;\delta)$ we obtain that $\mathcal{A}^{\lambda_0}$ has $\eta_i(\lambda_0)$ eigenvalues close to η with total algebraic multiplicity equal to that of η.

Using Lemma 3.1 we obtain

Lemma 3.2. *Let $c^2 > 1$. There exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, $\mathcal{A}^\lambda$ with domain $H^{m_2}_{per}([0,L]) \cap \mathbb{V}$ does not have eigenvalues $z \in \mathbb{C}$ satisfying* $\operatorname{Re} z \leqslant 0$.

3.2. The stability of eigenvalues for λ small enough

In this subsection we study the spectra of the family of linear operators $\mathcal{A}^\lambda$ in $X^0_{m_2}$, for $0 < \lambda \ll 1$, that is, $\lambda > 0$ is small enough. In order to obtain the results we extend the arguments of asymptotic perturbation theory in Hislop and Sigal [25, Chapter 19] and Kato [28, Chapter VIII] to the periodic context. We start with the following definition. Consider the self-adjoint linear operator $Q\mathcal{L}_0 : X^0_{m_2} \to \mathbb{V}$, given by

$$Q\mathcal{L}_0 g = \mathcal{L}_0 g + \frac{1}{c^2 L}\langle g, f'(\phi_c)\rangle.$$

Definition 3.1. An eigenvalue $\mu_0 \in \sigma(Q\mathcal{L}_0) = \sigma_p(Q\mathcal{L}_0)$ is stable with respect to the family $\mathcal{A}^\lambda$ defined in (1.9) if the following two conditions hold:

(i) There is $\delta > 0$ such that the punctured region $\mathcal{Q}_\delta := \{z \in \mathbb{C};\ 0 < |z-\mu_0| < \delta\}$ satisfies

$$\mathcal{Q}_\delta \subset \rho(Q\mathcal{L}_0) \cap \Delta_b,$$

where $\rho(Q\mathcal{L}_0)$ is the resolvent set of $Q\mathcal{L}_0$ and Δ_b is the region of boundedness for the family $\mathcal{A}^\lambda$, defined by

$$\Delta_b := \big\{z \in \mathbb{C};\ \big\|R_\lambda(z)\big\|_{B(\mathbb{V})} \leqslant M,\ \forall 0 < \lambda \ll 1\big\}.$$

Here $M = M(z) > 0$ does not depend on λ and $R_\lambda(z) = (\mathcal{A}^\lambda - z)^{-1} : \mathbb{V} \to X^0_{m_2}$.

(ii) Let Γ be a simple closed curve about μ_0 such that $\Gamma \subset \mathcal{Q}_\delta \subset \rho(Q\mathcal{L}_0) \cap \rho(\mathcal{A}^\lambda)$, for all λ small, and define the associated Riesz projector for $\mathcal{A}^\lambda$

$$P_\lambda = -\frac{1}{2\pi i}\int_\Gamma R_\lambda(z)\,dz.$$

Then,

$$\lim_{\lambda\to 0^+} \|P_\lambda - P_{\mu_0}\|_{B(\mathbb{V})} = 0,$$

where P_{μ_0} is the Riesz projector for $Q\mathcal{L}_0$ and μ_0.

Remark 3.2. It follows from Definition 3.1 that, for all $0 < \lambda \ll 1$, the operators $\mathcal{A}^\lambda$ have discrete spectra inside the domain determined by Γ with total algebraic multiplicity equal to that of μ_0, because $\dim(\operatorname{Im} P_\lambda) = \dim(\operatorname{Im} P_{\mu_0})$ for λ small. In order to simplify the notation, we write $\dim(P_\lambda)$ to refer $\dim(\operatorname{Im} P_\lambda)$.

The next lemma is a periodic version of Lemma 2.8 in Lin [29] (see also [9]) and because of this, we omit its proof.

Lemma 3.3. *Let $c^2 > 1$. For all $\lambda > 0$ small enough, consider $u \in H^{m_2}_{per}([0, L])$ satisfying the equation $(\mathcal{A}^\lambda - z)u = v$, where $z \in \mathbb{C}$ with $\operatorname{Re} z \leqslant \frac{1}{2}(1 - \frac{1}{c^2})$ and $v \in L^2_{per}([0, L])$. Then, we have the estimative*

$$\|u\|_{H^{\frac{m_1}{2}}_{per}} \leqslant M\big(\|u\|_{L^2_{per,e}} + \|v\|_{L^2_{per}}\big),$$

for some constant $M > 0$ which does not depend on $\lambda > 0$.

The following result gives us sufficient conditions for determining when a complex number belongs to the region of boundedness for the family $\{\mathcal{A}^\lambda\}$.

Lemma 3.4. *Let $c^2 > 1$. For $z \in \mathbb{C}$ with $\operatorname{Re} z \leqslant \frac{1}{2}(1 - \frac{1}{c^2})$, we have $z \in \Delta_b$ if and only if $z \in \rho(Q\mathcal{L}_0)$.*

Proof. Let $z \in \Delta_b$. Then for all $u \in C^\infty_{per}([0, L]) \cap \mathbb{V}$ we have

$$\big\|\big(\mathcal{A}^\lambda - z\big)u\big\|_{L^2_{per}} \geqslant \varepsilon\|u\|_{L^2_{per}} > 0, \tag{3.1}$$

for all $0 < \lambda \ll 1$ and $\varepsilon > 0$ does not depend on λ. From Proposition 3.1 and (3.1) we obtain, making $\lambda \to 0^+$, that

$$\big\|(Q\mathcal{L}_0 - z)u\big\|_{L^2_{per}} \geqslant \varepsilon\|u\|_{L^2_{per}}.$$

Since $Q\mathcal{L}_0$ is self-adjoint it follows that $z \in \rho(Q\mathcal{L}_0)$.

Next, we suppose that $z \in \rho(Q\mathcal{L}_0)$ but $z \notin \Delta_b$. Then, we guarantee the existence of a sequence $\{u_\lambda\} \subset C^\infty_{per}([0, L]) \cap \mathbb{V}$, with $\|u_\lambda\|_{L^2_{per}} = 1$ such that

$$\big\|\big(\mathcal{A}^\lambda - z\big)u_\lambda\big\|_{L^2_{per}} \to 0, \quad \text{as } \lambda \to 0^+.$$

Denote by $v_\lambda = (\mathcal{A}^\lambda - z)u_\lambda$. From Lemma 3.3 we thus have, for λ small,

$$\|u_\lambda\|_{H^{\frac{m_1}{2}}_{per}} \leqslant M\big(\|u_\lambda\|_{L^2_{per,e}} + \|v_\lambda\|_{L^2_{per}}\big) \leqslant K.$$

Hence, from the compact embedding $H_{per}^{\frac{m_1}{2}} \hookrightarrow L^2_{per}$, we have (modulo a subsequence) that $u_\lambda \rightharpoonup u$ in $H_{per}^{\frac{m_1}{2}}$ and $u_\lambda \to u$ in $\mathbb{V}$ as $\lambda \to 0^+$. Then $\|u\|_{L^2_{per}} = 1$. Next, for any $v \in D((\mathcal{A}^\lambda)^*) = D(Q\mathcal{L}_0)$ we conclude

$$0 = \lim_{\lambda \to 0^+} \langle v, (\mathcal{A}^\lambda - z)u_\lambda \rangle_{L^2_{per}} = \lim_{\lambda \to 0^+} \langle ((\mathcal{A}^\lambda)^* - \bar{z})v, u_\lambda \rangle_{L^2_{per}} = \langle (Q\mathcal{L}_0 - \bar{z})v, u \rangle_{L^2_{per}}.$$

Therefore, $u \in D(Q\mathcal{L}_0)$ and $(Q\mathcal{L}_0 - z)u = 0$. Since $z \in \rho(Q\mathcal{L}_0)$, we conclude that $u = 0$. This last fact yields a contradiction because $\|u\|_{L^2_{per}} = 1$. The proof of the theorem is now complete. $\square$

Using Lemma 3.4 we obtain the following main result.

Theorem 3.1. *Let $\mathcal{A}^\lambda$ be the linear operator defined in (1.9). Suppose that $\mu_0 \in \sigma(Q\mathcal{L}_0)$ (therefore μ_0 is a discrete eigenvalue). Then μ_0 is stable in the sense of Definition 3.1.*

Proof. We only give a sketch of the proof (see [9]). Let $\mu_0 \in \sigma(Q\mathcal{L}_0)$ and $\delta > 0$ such that $\Omega_\delta = \{z \in \mathbb{C}\colon\ 0 < |z - \mu_0| < \delta\} \subset \rho(Q\mathcal{L}_0)$. From Lemma 3.4, we see that $\Omega_\delta \subset \Delta_b$. Then for $z \in \Omega_\delta$

$$\|R_\lambda(z)\|_{B(\mathbb{V})} \leqslant M, \quad \text{for } 0 < \lambda \ll 1. \tag{3.2}$$

Therefore since $\mathcal{A}^\lambda u \to Q\mathcal{L}_0 u$ when $\lambda \to 0^+$ and $\rho(Q\mathcal{L}_0) \cap \Delta_b \neq \emptyset$, from Kato [28] we have that for all $z \in \Omega_\delta$ and $u \in C^\infty_{per}([0, L]) \cap \mathbb{V}$, $\lim_{\lambda \to 0^+} R_\lambda(z)u = R_0(z)u$. Then, the strong resolvent convergence $R_\lambda(z) \to R_0(z)$ is uniform on the circle $\Gamma = \{z\colon\ |z - \mu_0| = r < \delta\}$. Hence, since $\lim_{\lambda \to 0^+} P_\lambda u = P_{\mu_0} u$ and $\lim_{\lambda \to 0^+} P^*_\lambda u = P_{\mu_0} u$, for $u \in C^\infty_{per}([0, L]) \cap \mathbb{V}$, we have $\dim(P_\lambda) \geqslant \dim(P_{\mu_0})$ (see Lemma 1.23 in Kato [28, p. 438]). Next, from Lemma 1.24 in Kato [28] the condition

$$\dim(P_\lambda) \leqslant \dim(P_{\mu_0}), \quad 0 < \lambda \ll 1, \tag{3.3}$$

is sufficient to establish the condition (ii) in Definition 3.1. Thus, let us suppose that (3.3) does not occur. Then, since P_{μ_0} is an orthogonal projection, we can find a sequence $u_\lambda \in \mathbb{V}$, $\|u_\lambda\|_{L^2_{per}} = 1$ such that $P_\lambda u_\lambda = u_\lambda$ and $P_{\mu_0} u_\lambda = 0$. Hence, there is a subsequence, still denoted by $\{u_\lambda\}$, such that $u_\lambda \rightharpoonup u_0$ in L^2_{per}. Therefore, $u_0 = 0$.

On the other hand, for $z \in \Omega_\delta - \Gamma$ we have from the first resolvent identity that

$$(\mathcal{A}^\lambda - z)P_\lambda u_\lambda = -\frac{1}{2\pi i} \int_\Gamma [u_\lambda - (z - \eta)R_\lambda(\eta)u_\lambda]\, d\eta.$$

Therefore, from (3.2) and the compactness of Γ, we obtain for $0 < \lambda \ll 1$ that,

$$\|(\mathcal{A}^\lambda - z)P_\lambda u_\lambda\|_{L^2_{per}} \leqslant M_0 \Big[1 + \sup_{\eta \in \Gamma} |z - \eta|\Big].$$

Hence,

$$\|\mathcal{A}^\lambda u_\lambda\|_{L^2_{per}} \leqslant \|(\mathcal{A}^\lambda - z)P_\lambda u_\lambda\|_{L^2_{per}} + \|zP_\lambda u_\lambda\|_{L^2_{per}} \leqslant M, \tag{3.4}$$

where $M > 0$ does not depend on $\lambda > 0$. Inequality (3.4) implies that u_λ is bounded in $H^{m_2}_{per}$. So, we obtain (modulo a subsequence) that there is $u \in L^2_{per}$ such that $u_\lambda \to u$ in L^2_{per}, as $\lambda \to 0^+$, with $\|u\|_{L^2_{per}} = 1$. Since u_λ converges weakly to zero in L^2_{per} we obtain a contradiction from the uniqueness of the weak limit. This finishes the proof of the theorem. $\square$

4. The moving kernel formula and the instability proof

In this section we study the perturbation of the eigenvalue $\mu = 0$ associated with the linear operator $Q\mathcal{L}_0$ with respect to the operator $\mathcal{A}^\lambda$ for small $\lambda > 0$. For this purpose we derive a moving kernel formula in order to determine an instability criterion. Let us suppose that $\ker(Q\mathcal{L}_0) = \ker(\mathcal{L}_0) = [\frac{d}{dx}\phi_c]$. Then, since $\dim P_0 = 1$ and from Theorem 3.1 one has $\dim P_\lambda = 1$ for all $0 < \lambda \ll 1$. We note that since the eigenvalues of $\mathcal{A}^\lambda$ appear in conjugate pairs, we have that there is only one real eigenvalue b_λ of $\mathcal{A}^\lambda$ inside $B(0;\delta)$. The idea of the following result is to determine the sign of b_λ, for λ small, in other words, we want to know when the zero eigenvalue is moving to the left or to the right of the real axis.

Lemma 4.1. *Let $c^2 > 1$ and assume that $\ker(Q\mathcal{L}_0) = [\frac{d}{dx}\phi_c]$. For $\lambda > 0$ small enough, let $b_\lambda \in \mathbb{R}$ be the only eigenvalue of $\mathcal{A}^\lambda$ near the origin. Then,*

$$\lim_{\lambda\to 0^+} \frac{b_\lambda}{\lambda^2} = -\frac{1}{c^2}\frac{1}{\|\phi_c'\|^2_{L^2_{per}}}\frac{dP}{dc} \tag{4.1}$$

with $P(c) = c\langle(\mathcal{M}+1)\phi_c, \phi_c\rangle_{L^2_{per}}$.

Proof. From Theorem 3.1 there exists $u_\lambda \in H^{m_2}_{per}([0,L]) \cap \mathbb{V}$ such that $(\mathcal{A}^\lambda - b_\lambda)u_\lambda = 0$, $b_\lambda \in \mathbb{R}$ and $\lim_{\lambda\to 0^+} b_\lambda = 0$. We set $\|u_\lambda\|_{L^2_{per,e}} = 1$. So, from Lemma 3.3 we have that $\|u_\lambda\|_{H^{\frac{m_1}{2}}_{per}} \leqslant C$, for some constant $C > 0$ which does not depend on $\lambda > 0$. Then, modulo a subsequence, we have that $u_\lambda \rightharpoonup u_0$ in $H^{\frac{m_1}{2}}_{per}([0,L])$, as $\lambda \to 0^+$, and,

$$u_\lambda \to u_0 \quad \text{in } \mathbb{V}, \text{ as } \lambda \to 0^+. \tag{4.2}$$

Hence, since $\mathcal{A}^0 u_0 = Q\mathcal{L}_0 u_0$ and $\ker(Q\mathcal{L}_0) = [\phi_c']$, we have $u_0 = c_0\phi_c'$ with $c_0 \neq 0$. We can assume $c_0 = 1$ by normalizing the sequence. Moreover, from the equality $(\mathcal{A}^\lambda - b_\lambda)(u_\lambda - u_0) = b_\lambda u_0 + (\mathcal{A}^0 - \mathcal{A}^\lambda)u_0$, we obtain from Lemma 3.3, Proposition 3.1 and (4.2) that $u_\lambda \to u_0$ in $H^{\frac{m_1}{2}}_{per}([0,L])$ as $\lambda \to 0^+$.

Next, we show that $\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda} = 0$. Indeed, since $(\mathcal{A}^\lambda - b_\lambda)u_\lambda = 0$ we obtain

$$\frac{b_\lambda}{\lambda}u_\lambda = \frac{1}{\lambda}\mathcal{A}^\lambda u_\lambda = \frac{1}{\lambda}\mathcal{A}^0 u_\lambda + \frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}u_\lambda.$$

Then, $\mathcal{A}^0\phi_c' = 0$ implies that $\frac{b_\lambda}{\lambda}\langle u_\lambda, \phi_c'\rangle = \langle\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}u_\lambda, \phi_c'\rangle$. Using the arguments in Proposition 3.1 and from the formula

$$\left\langle\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}u_\lambda, \phi_c'\right\rangle = \frac{1}{c^2}\left\langle\frac{\partial_x(2c\partial_x - \lambda)}{(\lambda - c\partial_x)^2}QHu_\lambda, \phi_c\right\rangle,$$

we conclude that

$$\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda} = \frac{2}{c^3\|\phi'\|^2}\langle Q\left[1 + f'(\phi_c)\right]\phi_c', \phi_c\rangle = \frac{2}{c^3\|\phi'\|^2}\langle\left[1 + f'(\phi_c)\right]\phi_c', \phi_c\rangle = 0. \tag{4.3}$$

The following step is to compute the $\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda^2}$. We write $u_\lambda = c_\lambda\phi_c' + \lambda v_\lambda$, with $c_\lambda = \frac{\langle u_\lambda, \phi_c'\rangle}{\|\phi_c'\|^2}$. Then $\langle v_\lambda, \phi_c'\rangle = 0$ and $c_\lambda \to 1$ as $\lambda \to 0^+$. Next, we obtain the bound,

$$\|v_\lambda\|_{H_{per}^{\frac{m_2}{2}}} \leqslant C, \tag{4.4}$$

where $C > 0$ does not depend on $\lambda > 0$. Indeed, first note that

$$\mathcal{A}^\lambda v_\lambda = \frac{b_\lambda}{\lambda} u_\lambda - c_\lambda \frac{\mathcal{A}^\lambda \phi_c'}{\lambda} = \frac{b_\lambda}{\lambda} u_\lambda - c_\lambda \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} \right) \phi_c'. \tag{4.5}$$

So, by denoting $w_\lambda := (\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda})\phi_c'$, we get that

$$w_\lambda = -\frac{1}{c^2} \frac{(2c\partial_x - \lambda)\partial_x}{(\lambda - c\partial_x)^2} Q\left[\phi_c + f(\phi_c)\right] \tag{4.6}$$

and therefore $\|w_\lambda\|_{L^2_{per}} \leqslant C$, for some $C > 0$ which does not depend on $\lambda > 0$. Since $\langle \phi_c \rangle = 0$ Eq. (1.5) implies that $\langle \phi_c + f(\phi_c) \rangle = 0$ and so from (4.6) we have for $\lambda \to 0^+$,

$$\omega_\lambda \to -\frac{1}{c^2} \frac{2}{c} \big(\phi_c + f(\phi_c)\big) = -\frac{2}{c} (\mathcal{M} + 1)\phi_c. \tag{4.7}$$

Therefore, if we combine (4.3), (4.7) and Lemma 3.3 we get the desired result in (4.4). Thus $v_\lambda \rightharpoonup v_0$ in $H_{per}^{\frac{m_2}{2}}([0, L])$ and $v_\lambda \to v_0$ in $\mathbb{V}$, as $\lambda \to 0^+$. From (4.5) and (4.7) we obtain the equality $Q\mathcal{L}_0 v_0 = \frac{2}{c}(\mathcal{M} + 1)\phi_c$. Now, from (1.5) we have

$$\mathcal{L}_0\left(\frac{d}{dc}\phi_c \right) = -\frac{2}{c}(\mathcal{M} + 1)\phi_c,$$

that is, $Q\mathcal{L}_0(v_0 + \frac{d}{dc}\phi_c) = 0$. From $\ker(Q\mathcal{L}_0) = [\phi_c']$, there is $\theta \in \mathbb{R}$ such that $v_0 + \frac{d}{dc}\phi_c = \theta\phi_c'$. Next, defining $\overline{c_\lambda} := c_\lambda + \lambda\theta$ and $\overline{v_\lambda} := v_\lambda - \theta\phi_c'$ we obtain

$$u_\lambda = \overline{c_\lambda}\phi_c' + \lambda\overline{v_\lambda}.$$

From the limit $v_\lambda \to v_0$ in $L^2_{per,e}([0, L])$ and the fact that

$$\big(\mathcal{A}^\lambda - b_\lambda\big)(v_\lambda - v_0) = \frac{b_\lambda}{\lambda} u_\lambda - c_\lambda \omega_\lambda - \mathcal{A}^\lambda v_0 - b_\lambda v_\lambda + b_\lambda v_0 \to 0,$$

in $L^2_{per}([0, L])$, as $\lambda \to 0^+$, we obtain from Lemma 3.3 that $\|v_\lambda - v_0\|_{H_{per}^{\frac{m_1}{2}}} \to 0$, as $\lambda \to 0^+$. Then, $\overline{v_\lambda} \to v_0 - \theta\phi_c' = -\frac{d}{dc}\phi_c$, as $\lambda \to 0^+$.

Now, for $\Theta_\lambda = \lambda^{-1}(\mathcal{A}^\lambda - \mathcal{A}^0)$ we have that $\frac{b_\lambda}{\lambda^2} u_\lambda = \frac{1}{\lambda}\Theta_\lambda u_\lambda + \frac{1}{\lambda^2} Q\mathcal{L}_0 u_\lambda = \overline{c_\lambda}\frac{1}{\lambda}\Theta_\lambda\phi_c' + \Theta_\lambda\overline{v_\lambda} + \frac{1}{\lambda^2} Q\mathcal{L}_0 u_\lambda$. Hence, we obtain

$$\mathcal{J}(\lambda) := \lambda^{-2}\langle b_\lambda u_\lambda, \phi_c'\rangle_{L^2_{per}} = \lambda^{-2}\langle Q\mathcal{L}_0 u_\lambda, \phi_c'\rangle_{L^2_{per}} + \lambda^{-1}\overline{c_\lambda}\langle \Theta_\lambda\phi_c', \phi_c'\rangle_{L^2_{per}} + \langle \Theta_\lambda\overline{v_\lambda}, \phi_c'\rangle_{L^2_{per}}. \tag{4.8}$$

We now need to handle with the last two terms in (4.8) for $0 < \lambda \ll 1$ small enough. In fact, from Proposition 3.1 we obtain, when $\lambda \to 0^+$,

$$\langle \Theta_\lambda \overline{v_\lambda}, \varphi_c' \rangle_{L^2_{per}} = c^{-2} \left\langle \frac{2c\partial_x^2 - \lambda\partial_x}{(\lambda - c\partial_x)^2} Q\left(1 + f'(\phi_c)\right)\overline{v_\lambda}, \phi_c \right\rangle_{L^2_{per}}$$

$$\to \frac{2}{c^3}\left\langle Q\left[1 + f'(\phi_c)\right]\left(-\frac{d}{dc}\phi_c\right), \phi_c \right\rangle_{L^2_{per}} = -\frac{2}{c^3}\left\langle \left[1 + f'(\phi_c)\right]\frac{d}{dc}\phi_c, \phi_c \right\rangle_{L^2_{per}},$$

and from (1.5),

$$\begin{aligned}
\lambda^{-1}\langle \Theta_\lambda \phi_c', \phi_c' \rangle_{L^2_{per}} &= -c^{-2}\left\langle \left[\frac{2c^2\partial_x^2}{\lambda(\lambda - c\partial_x)} - \frac{c^2\partial_x^2}{(\lambda - c\partial_x)^2}\right](\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}} \\
&= -\frac{2}{c^2}\left\langle \left(\mathcal{E}^\lambda - 1\right)(\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}} + \frac{2}{c^2\lambda}\left\langle c\partial_x(\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}} \\
&\quad + \frac{1}{c^2}\left\langle \left(\mathcal{E}^\lambda - 1\right)^2(\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}} \\
&\to \frac{3}{c^2}\left\langle (\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}},
\end{aligned}$$

where we have used that $\mathcal{E}^\lambda \to 0$ and $\langle c\partial_x(\mathcal{M}+1)\phi_c, \phi_c \rangle_{L^2_{per}} = 0$. Then, from the equality

$$\left(1 + f'(\phi_c)\right)\frac{d}{dc}\phi_c = c^2(\mathcal{M}+1)\frac{d}{dc}\phi_c + 2c(\mathcal{M}+1)\phi_c$$

we obtain from (4.8),

$$\lim_{\lambda \to 0^+} \mathcal{J}(\lambda) = -\frac{2}{c}\left\langle (\mathcal{M}+1)\frac{d}{dc}\phi_c, \phi_c \right\rangle_{L^2_{per}} - \frac{1}{c^2}\left\langle (\mathcal{M}+1)\phi_c, \phi_c \right\rangle_{L^2_{per}}.$$

Thus we finally conclude that

$$\lim_{\lambda \to 0^+} \frac{b_\lambda}{\lambda^2} = \lim_{\lambda \to 0^+} \frac{\mathcal{J}(\lambda)}{\langle u_\lambda, \phi_c' \rangle_{L^2_{per}}} = -\frac{1}{c^2}\frac{1}{\|\phi_c'\|^2_{L^2_{per}}}\frac{dP}{dc}. \qquad \square$$

Next we give a sufficient condition to have the relation $\ker(Q\mathcal{L}_0) = \ker(\mathcal{L}_0) = [\frac{d}{dx}\phi_c]$.

Lemma 4.2. *Consider the operator $\mathcal{L}_0 : H^{m_2}_{per} \to L^2_{per}$ defined in (1.10) and suppose $\ker(\mathcal{L}_0) = [\frac{d}{dx}\phi_c]$. Then the operator $Q\mathcal{L}_0 : X^0_{m_2} \to \mathbb{V}$ satisfies $\ker(Q\mathcal{L}_0) = \ker(\mathcal{L}_0)$ provided that for all $g \in \mathcal{L}_0^{-1}(1)$ we have $\langle g \rangle \neq 0$.*

Proof. Let $\psi \in X^0_{m_2}$ such that $Q\mathcal{L}_0\psi = 0$. Then $\mathcal{L}_0\psi = r$, $r \in \mathbb{R}$. Suppose $r \neq 0$, then $\frac{1}{r}\psi \in \mathcal{L}_0^{-1}(1)$ but $\langle \frac{1}{r}\psi \rangle = 0$. Therefore $r = 0$ and so $\psi = \theta\frac{d}{dx}\phi_c$. $\square$

The proof of Theorem 1.1 follows the ideas of Lin [29]. We present it here just for the sake of completeness.

Instability criterion for the RBou-type equation. We sketch the proof assuming (ii) since the arguments can be mimicked if we suppose (i). Assume that $n^-(Q\mathcal{L}_0)$ is odd and $I(c) > 0$. Consider $k_1^-, k_2^-, \dots, k_l^-$, with $l \leqslant n^-(Q\mathcal{L}_0)$, all the distinct negative eigenvalues of $Q\mathcal{L}_0$. Since the eigenvalues k_i^-, $i = 1, 2, \dots, l$ are isolated we can choose a $\delta_1 > 0$ such that the l open disks $B_{\delta_1}(k_i^-)$

are disjoint and still lie in the left-half plane ($\operatorname{Re} z < 0$). Using Theorem 3.1, there exists $\lambda_1 > 0$ and $\delta > 0$ small enough, $\delta < \delta_1$, such that for $0 < \lambda < \lambda_1$, $\mathcal{A}^\lambda$ has $n^-(Q\mathcal{L}_0)$ eigenvalues (counting multiplicity) in $\bigcup_{i=1}^{l} B_\delta(k_i^-)$. Since $I(c) > 0$, we see from Lemma 4.1 that the zero eigenvalue of $\mathcal{A}^0 = Q\mathcal{L}_0$ is perturbed to a positive eigenvalue $0 < b_\lambda < \delta$ of $\mathcal{A}^\lambda$ for small λ_0. Let us consider the region $D_C^0 := \{z \in \mathbb{C}\colon -2C < \operatorname{Re} z < 0 \text{ and } |\operatorname{Im} z| < 2C\}$, where $C > 0$ is the same constant which appears in Lemma 3.1. If we repeat the same arguments as in Lemma 3.1, we get that $\mathcal{A}^\lambda$ has exactly $n^-(Q\mathcal{L}_0) + 1$ eigenvalues (counting multiplicity) in $D_C^0 := \{z \in \mathbb{C}\colon -2C < \operatorname{Re} z < 2\delta \text{ and } |\operatorname{Im} z| < 2C\}$. Then, all eigenvalues of $\mathcal{A}^\lambda$ with real part no greater than 2δ lie in $\bigcup_{i=1}^{l} B_\delta(k_i^-) \cup B_\delta(0)$. Thus for small λ, $\mathcal{A}^\lambda$ has exactly $n^-(Q\mathcal{L}_0)$ eigenvalues in D_C^0.

Now, we assume that the conclusion of the theorem does not occur. Then, $\ker(\mathcal{A}^\lambda) = \{0\}$ for any $\lambda > 0$. Let $n_{D_C^0}(\lambda)$ be the number of eigenvalues (with multiplicity) of $\mathcal{A}^\lambda$ in D_C^0. Since the spectrum of $\mathcal{A}^\lambda$ is discrete and $\overline{D_C^0}$ is compact, we conclude that $n_{D_C^0}(\lambda)$ is always a finite integer for every $\lambda > 0$. Moreover, for $\lambda > 0$ small enough $n_{D_C^0}(\lambda) = n^-(Q\mathcal{L}_0)$ is odd and there is $\Lambda > 0$ such that for $\lambda > \Lambda$ we conclude $n_{D_C^0}(\lambda) = 0$. Define now the two non-empty sets $S_{odd} = \{\lambda > 0;\ n_{D_C^0}(\lambda) \text{ is odd}\}$, and $S_{even} = \{\lambda > 0;\ n_{D_C^0}(\lambda) \text{ is even}\}$. Because the complex eigenvalues of $\mathcal{A}^\lambda$ appear in conjugate pairs, the number of pure complex eigenvalues is even (since $\ker(\mathcal{A}^\lambda) = \{0\}$) and due to the analyticity of the mapping $\lambda \in (0, +\infty) \to \mathcal{A}^\lambda$ (see p. 4 above) we can conclude that both S_{odd} and S_{even} are open disjoint subsets such that $(0, +\infty) = S_{odd} \cup S_{even}$, which is a contradiction.

So, there exists $\lambda > 0$ and $0 \neq u \in X_{m_2}^0$ such that $\mathcal{A}^\lambda u = 0$ and therefore $e^{\lambda t} u(x)$ is a purely growing mode solution to (1.8). With this solution at hand, it is easy to obtain a solution for (1.7) of the form $(e^{\lambda t} u(x), e^{\lambda t} v(x))$. This finishes the proof of Theorem 1.1. $\square$

5. Linear instability for the mRBou equation

In this section we study the instability of a family of periodic traveling wave solutions for the mRBou equation associated to the equation

$$u_{tt} - u_{xxtt} - u_{xx} - \left(3u^2 u_x\right)_x = 0.$$

For studying the linear instability we apply the results established in the last sections.

The mBou equation is equivalent to the system

$$\begin{cases} v_t - v_{xxt} - u_x - 3u^2 u_x = 0, \\ u_t - v_x = 0. \end{cases} \tag{5.1}$$

So, suppose that (u, v), with $u(x,t) = \phi_c(x - ct)$ and $v(x,t) = \psi(x - ct)$, is a solution of (5.1). Then (ϕ_c, ψ_c) satisfy

$$\begin{cases} c\psi_c - c\psi_c'' + \phi_c + \phi_c^3 = 0, \\ c\phi_c + \psi_c = 0, \end{cases} \tag{5.2}$$

where we considered all the constants of integration equal to zero. Using the last system we have that the profile ϕ_c satisfies

$$c^2 \phi_c'' - \left(c^2 - 1\right)\phi_c + \phi_c^3 = 0. \tag{5.3}$$

In order to obtain the existence results of periodic waves solutions for (5.3), we will apply the phase portrait analytical technique. Eq. (5.3) is equivalent to the Hamiltonian system

$$\frac{d\phi}{dx} = y, \qquad \frac{dy}{dx} = \frac{c^2-1}{c^2}\phi - \frac{1}{c^2}\phi^3, \tag{5.4}$$

with the Hamiltonian $H(\phi, y) = \frac{y^2}{2} - \frac{c^2-1}{2c^2}\phi^2 + \frac{1}{4c^2}\phi^4 = h$, where h is an integral constant. For $c^2 - 1 > 0$, the system (5.4) has three equilibrium points. Denote them by $P_0(0,0)$, $P_1(\sqrt{c^2-1}, 0)$ and $P_2(-\sqrt{c^2-1}, 0)$, respectively. By the qualitative theory of ordinary differential equations, we have that $P_0(0,0)$ is a saddle point and $P_1(\sqrt{c^2-1}, 0)$ and $P_2(-\sqrt{c^2-1}, 0)$ are two centers. For a fixed $h \in \mathbb{R}$, the curve

$$\Gamma_h = \{(\phi, y) \in \mathbb{R} \times \mathbb{R}\colon\ H(\phi, y) = h\}$$

is called a level curve with the energy level h. Each orbit of Eq. (5.4) is a branch of certain energy curve. Next we investigate the relation between the bounded orbit of (5.4) and the energy level h. For $U_h(\phi) = 2c^2h + (c^2-1)\phi^2 - \frac{1}{2}\phi^4$, we have that the three extreme points are: $\phi_0 = 0$ and $\phi_\pm = \pm\sqrt{c^2-1}$. Let $h_c = H(\sqrt{c^2-1}, 0) = -\frac{(c^2-1)^2}{4c^2}$. Then for each given h we have:

(1) If $h_c < h < 0$, then $U_h(\phi) = 0$ has four different real roots. Denote them by $\pm\phi_1, \pm\phi_2$ with $0 < \phi_2 < \phi_1$;
(2) If $h = 0$, then $U_h(\phi) = 0$ has a zero root and two different nonzero real roots which are denoted by $\pm\phi_3$ with $\phi_3 > 0$;
(3) If $h > 0$, then $U_h(\phi) = 0$ has two different real roots which are denoted by $\pm\phi_4$ with $\phi_4 > 0$.

Now, since the energy curves Γ_h are equivalent to the curves defined by $y^2 = \frac{1}{c^2}U_h(\phi)$ (the quadrature form associated to (5.3)), from the above arguments we obtain the following results:

(1) System (5.4) does not have any *bounded orbit* with energy level h satisfying $h \leqq h_c$;
(2) System (5.4) has two families of periodic orbits $\Gamma_h = \{(\phi, y) \in \mathbb{R} \times \mathbb{R}\colon\ H(\phi, y) = h,\ h_c < h < 0\}$ which lie on the inside of the different bounded regions determined by two homoclinic orbits. The periodic orbits on the right-hand side of the plane (ϕ, y) correspond to the so-called *dnoidal waves* and they are positive solutions of (5.3) (see Angulo [2]);
(3) System (5.4) has two homoclinic orbits with energy level 0. The homoclinic orbit on the right-hand side of the plane (ϕ, y) corresponds to the positive solitary wave solutions of (5.3) (see (5.6) below);
(4) System (5.4) has a family of periodic orbits $\Gamma_h = \{(\phi, y) \in \mathbb{R} \times \mathbb{R}\colon\ H(\phi, y) = h,\ h > 0\}$ which lie on the outside of the bounded region determined by two homoclinic orbits. These periodic orbits in the plane (ϕ, y) correspond to the so-called *cnoidal waves* and they are periodic sign-changing solutions for (5.3).

The two Jacobian elliptic profile solutions of (5.3), the dnoidal and cnoidal solutions are of important interest in applications. In this work we only establish a result of instability for the cnoidal solutions. The stability or instability for the dnoidal waves remains an open problem. Next we obtain the specific profiles of the cnoidal waves.

5.1. Linear instability of cnoidal waves for the mRBou equation

Here we apply the results established in Section 4 to conclude the linear instability of cnoidal waves for the mRBou equation. From the above arguments, we have from (5.3) the differential equation in quadrature form,

$$[\phi_c']^2 = \frac{1}{2c^2}\left[-\phi_c^4 + 2(c^2-1)\phi_c^2 + 4h_{\phi_c}\right],$$

where h_{ϕ_c} is a nonzero integration constant. For $c^2-1>0$ and $h_{\phi_c}>0$, the polynomial $Q(t)=-t^4+2(c^2-1)t^2+4h_{\phi_c}$ has two symmetric real roots, $-b<0<b$, and a pure imaginary root ia. The periodic solutions arising in this case have the cnoidal's profile (see Angulo [2])

$$\phi_c(x)=b\,\mathrm{cn}\left(\frac{\beta}{c}x;k\right), \tag{5.5}$$

which is a periodic sign-changing solution for (5.3). Here, we have

$$k^2=\frac{b^2}{a^2+b^2},\qquad b^2-a^2=2\big(c^2-1\big)\quad\text{and}\quad \beta=\sqrt{\frac{a^2+b^2}{2}}.$$

We note that $4h_{\phi_c}=b^2a^2$ and for $c^2>1$ we get that $b^2>2(c^2-1)$. Since $k^2=\frac{b^2}{j(b,c)}$ for $j(b,c)=2b^2-2(c^2-1)$, we must have $k^2\in(\frac{1}{2},1)$. Next, since the cnoidal has real fundamental period equal to $4K$ we obtain that the fundamental period for ϕ_c can be regarded as a function of b,

$$T_{\phi_c}(b)=\frac{4\sqrt{2}c}{\sqrt{j(b,c)}}K\big(k(b)\big).$$

If $a\to 0$ then $h_{\phi_c}\to 0$ and so from the above arguments we get the corresponding homoclinic orbit in the (ϕ,y)-phase plane. In fact, in this case $k\to 1^-$ and $T_{\phi_c}\to+\infty$ and so the cnoidal wave loses its periodicity and we obtain the *solitary wave* solution for (5.3) in the form

$$\phi_s(x)=\sqrt{2\big(c^2-1\big)}\,\mathrm{sech}\left(\frac{\sqrt{c^2-1}}{c}x\right). \tag{5.6}$$

Next, applying a similar argument as in Theorem 2.3 in Angulo [2] we can deduce, from the Implicit Function Theorem, the following result.

Theorem 5.1. *Let $L>0$ be fixed and $k^2\in(\frac{1}{2},1)$, satisfying $L^2>16K^2(k)(2k^2-1)$. Then,*

(i) *For every $c>1$ there is a unique $b=b(c)\in(\sqrt{2(c^2-1)},+\infty)$ such that the map $c\in(1,+\infty)\mapsto b(c)$ is a strictly increasing smooth function and $L=\frac{4\sqrt{2}c}{\sqrt{j(b(c),c)}}K(k)$. The modulus $k=k(c)$ is given by $k^2(c)=\frac{b^2(c)}{j(b(c),c)}$ and $\frac{dk}{dc}>0$.*

(ii) *For every $c>1$ and $h(c)\equiv\frac{1}{2c^2}j(b(c),c)$, the wave*

$$\big(\phi_c(x),\psi_c(x)\big)=\Big(b\,\mathrm{cn}\big(\sqrt{h(c)}\cdot x;k\big),-cb\,\mathrm{cn}\big(\sqrt{h(c)}\cdot x;k\big)\Big),$$

has fundamental period L and satisfies Eq. (5.2). Moreover, the mapping $c\in(1,+\infty)\mapsto(\phi_c,\psi_c)\in H^n_{per}([0,L])\times H^n_{per}([0,L])$ is a smooth function for all $n\in\mathbb{N}$.

Next, for $\mathcal{L}_0$ defined in (1.10) with $f(u)=u^3$ we consider the eigenvalue problem in $H^2_{per}([0,L])$,

$$\begin{cases}\mathcal{L}_0\psi=\eta\psi,\\ \psi(0)=\psi(L),\qquad \psi'(0)=\psi'(L),\end{cases} \tag{5.7}$$

which immediately implies the existence of a countable set of eigenvalues $\{\eta_i\}_{i\geqq 0}$. From (5.5) and the transformation $\Phi(x)=\psi(\sqrt{c}x/\beta)$ we obtain the eigenvalue problem

$$\mathcal{L}\Phi := -\Phi'' + 6k^2\operatorname{sn}^2(x;k)\Phi = \theta\Phi, \qquad \Phi(0)=\Phi\bigl(4K(k)\bigr), \qquad \Phi'(0)=\Phi'\bigl(4K(k)\bigr). \tag{5.8}$$

Here θ will be an eigenvalue satisfying

$$\theta = \frac{1}{\beta^2}\bigl[3b^2 - \bigl(c^2-1\bigr) + c^2\eta\bigr]. \tag{5.9}$$

Now, it is known that for (5.8) the set of eigenvalues $\{\theta_i\}_{i\geqq 0}$ has the distribution

$$\theta_0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 \leqq \theta_6 < \cdots,$$

this means that the first five eigenvalues are simple and that all other eigenvalues have multiplicity two (see Magnus and Winkler [32]). Since the exact value of these five eigenvalues, as well as their associated eigenfunctions, will be useful for all subsequent calculations, we have the following (see Ince [26]).

$$\begin{aligned}
&\theta_0 = 2\bigl[1+k^2-r(k)\bigr]; &&\Phi_0(x) = 1-\bigl(1+k^2-r(k)\bigr)\operatorname{sn}^2(x),\\
&\theta_1 = 1+k^2; &&\Phi_1(x) = \partial_x\operatorname{sn}(x) = \operatorname{cn}(x)\operatorname{dn}(x),\\
&\theta_2 = 1+4k^2; &&\Phi_2(x) = \partial_x\operatorname{cn}(x) = -\operatorname{sn}(x)\operatorname{dn}(x),\\
&\theta_3 = 4+k^2; &&\Phi_3(x) = \partial_x\operatorname{dn}(x) = -k^2\operatorname{sn}(x)\operatorname{cn}(x),\\
&\theta_4 = 2\bigl[1+k^2+r(k)\bigr]; &&\Phi_4(x) = 1-\bigl(1+k^2+r(k)\bigr)\operatorname{sn}^2(x),
\end{aligned} \tag{5.10}$$

where $r(k)=\sqrt{1-k^2+k^4}$. Moreover, the following basic computation shows that for $j\neq 0$ and $j\neq 4$ we have that the associated eigenfunction Φ_j has zero mean. Indeed, $L\Phi_j=\theta_j\Phi_j$ implies

$$\theta_j\langle\Phi_j,1\rangle = 6\bigl\langle k^2\operatorname{sn}^2,\Phi_j\bigr\rangle = 6\langle\Phi_j,\Phi_4\rangle + 2\bigl[1+k^2-r(k)\bigr]\langle\Phi_j,1\rangle.$$

Since $\theta_j > \theta_0$ we obtain $\langle\Phi_j,1\rangle = 0$.

Now, using the same steps as in the proof of Theorem 2.3 in Angulo [2] we obtain that $n^-(\mathcal{L}_0)=2$ and $\ker(\mathcal{L}_0)=[\frac{d}{dx}\phi_c]$ (from (5.9) the eigenvalues θ_0, θ_1 and θ_2 determine η_0, η_1 and $\eta_2=0$, respectively. Moreover, $\eta_0<\eta_1<\eta_2=0$). Furthermore, since $\langle\Phi_1,1\rangle=\int_0^{4K(k)}\partial_x\operatorname{sn}(x;k)\,dx=0$ we conclude that the eigenvalue η_1 belongs to the negative spectrum of $Q\mathcal{L}_0$, so we have that $1\leqslant n^-(Q\mathcal{L}_0)\leqslant 2$. Next, we obtain the sign of $I(c)$ given in (4.1). From Theorem 5.1 we get

$$\frac{dP(c)}{dc} = \int_0^L\bigl(\phi_c'^2+\phi_c^2\bigr)\,dx + 4c\frac{d}{dc}\left\{\frac{b^2}{\alpha}\int_0^K\bigl[\operatorname{cn}^2(x;k)+\alpha^2\operatorname{sn}^2(x;k)\operatorname{dn}^2(x;k)\bigr]\,dx\right\}, \tag{5.11}$$

with $\alpha=\frac{4K}{L}$. Moreover, for all $L>0$ we can write $b^2=\frac{32c^2k^2K^2(k)}{L^2}$ and the wave speed $c>1$ by the expression

$$c^2 = \frac{L^2}{L^2-16K^2(k)(2k^2-1)}, \quad k^2\in(1/2,1).$$

Now, from Byrd and Friedman [17], we have

$$\int_0^K \mathrm{cn}^2(x;k)\,dx = \frac{1}{k^2}\big[E(k) - \big(1-k^2\big)K(k)\big] := f_1(k),$$

with $E = E(k)$ the complete elliptic integral of second kind, and

$$\alpha^2 \int_0^K \mathrm{sn}^2(x;k)\,\mathrm{dn}^2(x;k)\,dx = \frac{16K(k)^2}{3k^2L^2}\big[\big(2k^2-1\big)E(k) + \big(1-k^2\big)K(k)\big] := f_2(k).$$

Since $\frac{dk}{dc} > 0$ implies that $c \mapsto c^2k^2K(k)$ is a strictly increasing function and $k \mapsto f_i(k)$ are positive strictly increasing functions in $(\frac{\sqrt{2}}{2}, 1)$, we have from (5.11) that $\frac{dP(c)}{dc} > 0$ and thus $I(c) < 0$.

Next, we show that $n^-(Q\mathcal{L}_0) = 2$ for a specific range of the elliptic modulus k. Let $\{\psi_i\}_{i\geqslant 0}$ be the complete orthonormal system of eigenfunctions associated with the periodic problem (5.7). Then, by considering $\psi \in D(Q\mathcal{L}_0)$ such that $Q\mathcal{L}_0\psi = \lambda\psi$ we obtain the following three relations:

$$\psi = \sum_{i=0}^{\infty} \langle \psi, \psi_i\rangle \psi_i, \qquad Q\mathcal{L}_0\psi = \sum_{i=0}^{\infty} \eta_i \langle \psi, \psi_i\rangle \psi_i + \frac{3}{c}\big\langle \phi_c^2 \psi\big\rangle = \lambda\psi,$$

and

$$(\lambda - \eta_i)\langle \psi, \psi_i\rangle = \frac{3}{c}\big\langle \phi_c^2\psi\big\rangle \int_0^L \psi_i\,dx. \tag{5.12}$$

Now, we obtain immediately from (5.12) that if $\langle \psi_i\rangle = 0$ then $\eta_i \in \sigma(Q\mathcal{L}_0)$. So, if $\lambda \neq \eta_i$ we obtain from (5.12)

$$\psi = \frac{3L}{c}\big\langle \phi_c^2\psi\big\rangle \sum_{i=0}^{\infty} \frac{\langle \psi_i\rangle}{\lambda - \eta_i}\psi_i, \tag{5.13}$$

where the sum in (5.13) is taken over those eigenfunctions $\psi_i \notin \mathbb{V}$. From the analysis above we know that for $i = 0$ and $i = 4$,

$$\psi_i(x) = \sqrt{\frac{4K}{L}}\,\frac{1}{\|\Phi_i\|_{L^2_{per}([0,4K])}}\,\Phi_i\big(\beta x/\sqrt{c}\,\big)$$

satisfies $\langle \psi_i\rangle \neq 0$. Therefore, since $\frac{3}{c}\langle \phi_c^2\psi\rangle \neq 0$ and $\langle \psi\rangle = 0$ we obtain from (5.13) that λ must be a zero of the meromorphic function

$$J(\lambda) = \frac{\langle \psi_0\rangle^2}{\lambda - \eta_0} + \frac{\langle \psi_4\rangle^2}{\lambda - \eta_4}. \tag{5.14}$$

We note that J is a two variable function depending on λ and $k \in (0,1)$. Moreover, since $Q\mathcal{L}_0$ is self-adjoint we are interested in the zeros of the function $J(\lambda)$. So, since $J'(\lambda) < 0$ for $\lambda \notin \{\eta_0, \eta_4\}$, we see that every (real) zero of J must be simple and we guarantee the existence of a unique zero

$\lambda^* \in (\eta_0, \eta_4)$ (this last fact comes from (5.9) and (5.10) since in this case we have $\eta_0 < 0 < \eta_4$). Next, we show that there is a unique k^* such that $J(0) < 0$ for $k \in (k^*, 1)$ and $J(0) \geqslant 0$ for $k \in (0, k^*]$. Indeed, from (5.9) and (5.10) the exactly values of η_0 and η_4 are

$$\frac{L^2}{16K^2}\eta_0 = 1 - 2k^2 - 2r(k), \quad \text{and} \quad \frac{L^2}{16K^2}\eta_4 = 1 - 2k^2 + 2r(k). \tag{5.15}$$

Using the identities for $k \in (0, 1)$,

$$\begin{aligned} \int_0^{2K(k)} k^2 \operatorname{sn}^2(x; k)\, dx &= 2\big(K(k) - E(k)\big), \\ \int_0^{2K(k)} k^4 \operatorname{sn}^4(x; k)\, dx &= \frac{2}{3}\big(\big(2 + k^2\big)K(k) - 2\big(1 + k^2\big)E(k)\big) \end{aligned} \tag{5.16}$$

we obtain for $j = 0$ and $j = 4$ the following,

$$\begin{aligned} \int_0^{4K(k)} \Phi_j(x)\, dx &= \frac{1}{k^2}\big(4k^2 K(k) - 4a_j\big(K(k) - E(k)\big)\big) \equiv \frac{1}{k^2} J_j(k), \\ \int_0^{4K(k)} \Phi_j^2(x)\, dx &= 4K(k) - \frac{8a_j}{k^2}\big(K(k) - E(k)\big) + \frac{4a_j^2}{3k^4}\big(\big(2 + k^2\big)K(k) - 2\big(1 + k^2\big)E(k)\big), \end{aligned} \tag{5.17}$$

where $a_j = 1 + k^2 \pm r(k)$, with sign "$-$" for $j = 0$ and sign "$+$" for $j = 4$. Therefore, we conclude that $J(0) < 0$ if and only if the function

$$F(k) = \frac{1}{1 - 2k^2 - 2r(k)} \frac{J_0^2(k)}{\|\Phi_0\|^2_{L^2_{per}([0,4K])}} + \frac{1}{1 - 2k^2 + 2r(k)} \frac{J_4^2(k)}{\|\Phi_4\|^2_{L^2_{per}([0,4K])}},$$

is positive. Doing the necessary calculations we found the value $k^* \approx 0.909$. Thus, if $k \in (k^*, 1)$ we see that $F(k) > 0$, and for $k \in (0, k^*]$ we deduce $F(k) \leqslant 0$. Therefore, for $k \in (k^*, 1)$ we obtain $n^-(Q\mathcal{L}_0) = 2$, and for $k \in (\frac{\sqrt{2}}{2}, k^*)$, $n^-(Q\mathcal{L}_0) = 1$.

Next we see that $\ker(Q\mathcal{L}_0) = [\frac{d}{dx}\phi_c]$. By Lemma 4.2 it is sufficient to show that if $\mathcal{L}_0 g = 1$ then $\langle g\rangle \neq 0$. Suppose that $\langle g\rangle = 0$. Initially, from (1.10) we obtain $\frac{3}{c}\langle\phi_c^2 g\rangle = -1$. So, from (5.12) we have $\eta_i\langle g, \psi_i\rangle = -\frac{3}{c}\langle\phi_c^2 g\rangle \int_0^L \psi_i\, dx = L\langle\psi_i\rangle$, for $i = 0, 4$, and therefore

$$0 = \langle g, \psi_0\rangle\langle\psi_0\rangle + \langle g, \psi_4\rangle\langle\psi_4\rangle = -LJ(0). \tag{5.18}$$

This is a contradiction because $J(0) < 0$.

From Theorem 1.1 the following result is obtained.

Theorem 5.2 *(Linear instability of cnoidal waves for the mRBou equation). The solution* (ϕ_c, ψ_c) *where* $\psi_c = -c\phi_c$ *and* ϕ_c *is given in* (5.5) *is linearly unstable for the mRBou equation* (5.1)*, provided that the wave speed* $c \in (c^*, +\infty)$*, where*

$$c^{*2} = \frac{L^2}{L^2 - 16K^2(k^*)(2k^{*2} - 1)}. \tag{5.19}$$

Remark 5.1. We can numerically determine the value of $c^{*2} \approx \frac{L^2}{L^2 - 56{,}277}$ and so our minimal period L must satisfy *a priori* the lower bound $L^2 > 56{,}277$.

6. Nonlinear instability of cnoidal waves for the mRBou equation

In this section we establish the nonlinear instability of the linearly unstable cnoidal waves for the mRBou equation determined in Theorem 5.2.

6.1. Linking nonlinear instability and linear instability

The following theorem is the link to achieving nonlinear instability from a linear instability result.

Theorem 6.1. *Let* Y *be a Banach space and* $\Omega \subset Y$ *an open set containing* 0*. Suppose* $T : \Omega \to Y$ *satisfies* $T(0) = 0$*, and for some* $p > 1$ *and continuous linear operator* $\mathcal{L}$*, with spectral radius* $r(\mathcal{L}) > 1$*, we have that* $\|T(x) - \mathcal{L}x\|_Y = O(\|x\|_Y^p)$ *as* $x \to 0$*. Then* 0 *is unstable as a fixed point of* T*.*

Proof. See Theorem 2 in Henry, Perez and Wreszinski [24]. □

Remark 6.1. In Theorem 6.1, 0 is unstable as a fixed point of T if there is $\varepsilon_0 > 0$ such that for all $B(0; \eta)$ and arbitrarily large $N_0 \in \mathbb{N}$, there is an $N > N_0$ and $x \in B(0; \eta)$ such that $\|T^N(x)\|_Y \geqslant \varepsilon_0$.

By using Taylor's Theorem, Theorem 6.1 implies immediately the following result.

Corollary 6.1. *Let* $S : \Omega \subset Y \to Y$ *be a* C^2 *map defined in an open neighborhood of a fixed point* φ *of* S*. If there is an element* $\mu \in \sigma(S'(\varphi))$ *with* $|\mu| > 1$ *then* φ *is an unstable fixed point of* S*.*

6.2. Local and global well-posedness for the gRBou equation

In this subsection we study the specific initial value problem

$$\begin{cases} v_t - v_{xxt} - u_x - \left(u^{p+1}\right)_x = 0, \\ u_t - v_x = 0, \\ \big(u(0), v(0)\big) = (u_0, v_0), \end{cases} \tag{6.1}$$

for $p \geqslant 1$, $p \in \mathbb{N}$, in the periodic setting and it which will be necessary to apply Theorem 6.1 above. Write (6.1) in the form

$$\begin{cases} iu_t = -\psi(D_x)v, \\ iv_t = -\varphi(D_x)\big(u + u^{p+1}\big), \\ \big(u(0), v(0)\big) = (u_0, v_0) \end{cases} \tag{6.2}$$

where $\widehat{\varphi(D_x)u}(\xi) = \frac{\xi}{1+|\xi|^2}\widehat{u}(\xi) = \varphi(\xi)\widehat{u}(\xi)$ and $\widehat{\psi(D_x)u}(\xi) = \xi\widehat{u}(\xi) = \psi(\xi)\widehat{u}(\xi)$. Solving the linear problem

$$\begin{cases} iu_t = -\psi(D_x)v, \\ iv_t = -\varphi(D_x)u, \\ \big(u(0), v(0)\big) = (u_0, v_0) \end{cases}$$

we get the solution $(u(t), v(t)) = S(t)(u_0, v_0)$, where

$$\begin{pmatrix} \widehat{u}(t,\xi) \\ \widehat{v}(t,\xi) \end{pmatrix} = \begin{pmatrix} \cos(\alpha(\xi)t) & i\sqrt{1+|\xi|^2}\sin(\alpha(\xi)t) \\ \frac{i}{\sqrt{1+|\xi|^2}}\sin(\alpha(\xi)t) & \cos(\alpha(\xi)t) \end{pmatrix} \begin{pmatrix} \widehat{u}_0(\xi) \\ \widehat{v}_0(\xi) \end{pmatrix}$$

with $\alpha(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$. Then, (6.2) may be rewritten as the integral equation

$$\big(u(x,t), v(x,t)\big) = S(t)(u_0, v_0) + i\int_0^t S(t-\tau)G\big[u(x,\tau)\big]\,d\tau$$

where G is given by $G(u) = [0, \varphi(D_x)(u^{p+1})]$. This latter equation can be solved locally in time, using the fact that H^s_{per}, for $s > 1/2$, is a Banach algebra and performing a Picard iteration in the space X^s_T of continuous functions defined on $[-T, T]$ with values in $X^s := H^s_{per} \times H^{s+1}_{per}$, equipped with the usual norm

$$\big\|(u,v)\big\|_{X^s_T} = \sup_{t\in[0,T]}\big\|(u,v)(\cdot,t)\big\|_{X^s}.$$

More precisely, argue as follows. For any $t \geqslant 0$ and $s \in \mathbb{R}$ it easy to see that $\|S(t)(v_0, v_0)\|_{X^s} \lesssim \|(u_0, v_0)\|_{X^s}$ and consequently, for any $T > 0$,

$$\big\|S(\cdot)(v_0, v_0)\big\|_{X^s_T} \lesssim \big\|(u_0, v_0)\big\|_{X^s}.$$

Using the properties of $S(t)$, we obtain for $s > 1/2$ that

$$\begin{aligned}\left\|\int_0^t S(t-\tau)G\big[u(\cdot,\tau)\big]\,d\tau\right\|_{X^s} &\leqslant \int_0^t \big\|\varphi(D_x)\big(u^{p+1}\big)\big(t'\big)\big\|_{H^{s+1}}\,dt' \leqslant \int_0^t \big\|u^{p+1}\big(t'\big)\big\|_{H^s}\,dt' \\ &\leqslant \int_0^t \big\|u\big(t'\big)\big\|^{p+1}_{H^s}\,dt \leqslant C_s T\big\|(u,v)\big\|^{p+1}_{X^s_T},\end{aligned}$$

where C_s is a constant depending only on s. Therefore

$$\left\|\int_0^t S(t-\tau)G\big[u(\cdot,\tau)\big]\,d\tau\right\|_{X^s_T} \lesssim T\big\|(u,v)\big\|^{p+1}_{X^s_T}.$$

Similarly, we arrive at

$$\left\|\int_0^t S(t-\tau)\big[G\big(u(\tau)\big) - G\big(u(\tau)\big)\big]\,d\tau\right\|_{X^s_T} \lesssim T\big\|(u,v)-(w,z)\big\|_{X^s_T}\big[\big\|(u,v)\big\|^p_{X^T_s} + \big\|(w,z)\big\|^p_{X^s_T}\big].$$

Using the last two inequalities we get the next result.

Theorem 6.2. *The Cauchy problem associated with the gRBou equation (6.1) is locally well-posed in $X^s = H^s_{per} \times H^{s+1}_{per}$, for $s > 1/2$, that is, if $(u_0, v_0) \in X^s$ there is $T > 0$ and a unique mild solution $(u, v) \in C([0,T]; X^s)$ of (6.1). Moreover, the data-solution mapping associated to the gRBou equation,*

$$\Upsilon : X^s \to C\big([0,T]; X^s\big),$$
$$(u_0, v_0) \mapsto \Upsilon(u_0, v_0) = (u_{u_0}, v_{v_0}),$$

is smooth and we have that the following quantities,

$$E(u,v) = \frac{1}{2}\int\left(u^2 + v^2 + v_x^2 + \frac{2}{p+2}u^{p+2}\right)dx, \qquad F(u,v) = \int (uv + u_x v_x)\,dx,$$
$$I_1(u,v) = \int u\,dx \quad \text{and} \quad I_2(u,v) = \int v\,dx, \tag{6.3}$$

are conserved by the flow of the gRBou equation.

If $\int u^{p+2}\,dx \geqslant 0$, the Cauchy problem (6.1) is globally well-posed in X^s, for $s \geqslant 1$.

Proof. The proof of the global well-posedness is obtained in the same way as Theorem 2.4 in Liu [30]. The proof that the data-solution mapping is smooth is based on the Implicit Function Theorem as in Angulo and Natali [9]. □

6.3. Nonlinear instability for cnoidal waves for the mRBou

In this subsection we achieve the main result of this section.

Theorem 6.3. *The cnoidal profile solution (ϕ_c, ψ_c) where $\psi_c = -c\phi_c$ and ϕ_c is given in (5.5) is nonlinearly unstable in X^s, with $s > 1/2$, for the mRBou equation (5.1), provided that the wave speed $c \in (c^*, +\infty)$, with c^* defined in (5.19).*

Proof. In system (5.1) we replace $(u(x,t), v(x,t))$ by $(u(x+ct,t), v(x+ct,t))$ and then we obtain

$$\begin{cases} v_t - cv_x + cv_{xxx} - v_{xxt} - u_x - 3u^2u_x = 0, \\ u_t - cu_x - v_x = 0. \end{cases} \tag{6.4}$$

Then (ϕ_c, ψ_c), is an equilibrium solution for Eq. (6.4). Defining $G(u,v) = E(u,v) + cF(u,v)$, where E and F are defined in (6.3), we have that equation in (6.4) can be written as

$$\big((1-\partial_x^2)u, v\big)_t = JG'(u,v), \tag{6.5}$$

where $J = \left[\begin{smallmatrix} 0 & \partial_x(1-\partial_x^2)^{-1} \\ \partial_x & 0 \end{smallmatrix}\right]$. Moreover, from (6.5) we see that the linearized equation, at the equilibrium point (ϕ_c, ψ_c), is $((1-\partial_x^2)w, z)_t = JH_0(w,z)$ (see (1.6)), where H_0 is the linear self-adjoint operator defined by

$$H_0 = \begin{bmatrix} 1 + 3\phi_c^2 & c(1-\partial_x^2) \\ c(1-\partial_x^2) & 1-\partial_x^2 \end{bmatrix}.$$

Let us define $S : X^s \to X^s$ as $S(u_0, v_0) = (u_{u_0}(1), v_{v_0}(1))$, where $(u_{u_0}(t), v_{v_0}(t))$ is the solution of (6.4) with initial data $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$. If one considers $\Upsilon_c : X^s \to C([0,T]; X^s)$ the data-solution mapping related to Eq. (6.4), we see from Theorem 6.2 that Υ_c is smooth. Furthermore $S(\phi_c, \psi_c) = (\phi_c, \psi_c)$ and S is a C^2 map defined on a neighborhood of (ϕ_c, ψ_c) (this last fact follows since translation in x is a linear continuous map in X^s). Moreover, for $(g, h) \in X^s$ we have $S'(\phi_c, \psi_c)(g, h) = (w_g(1), z_h(1))$, where $(w_g(1), z_h(1))$ is the solution of the linear initial value problem

$$\begin{cases} \left((1-\partial_x^2)w, z\right)_t = JH_0(w, z), \\ (w, z)(0) = (g, h), \end{cases}$$

evaluated at $t = 1$. Then, from arguments established in Section 4 and Section 5.1, we deduce that there is $\lambda > 0$ and $(w_0, z_0) \in X^1 - \{0\}$ such that $JH_0(w_0, z_0) = \lambda((1-\partial_x^2)w_0, z_0)$. Hence for $(w_{w_0}(t), z_{z_0}(t)) = e^{\lambda t}(w_0, z_0)$ and $\mu := e^{\lambda}$ we obtain $S'(\phi_c, \psi_c)(w_0, z_0) = \mu(w_0, z_0)$. Therefore $\mu \in \sigma(S'(\phi_c, \psi_c))$ and from Corollary 6.1 we obtain the nonlinear instability in X^s of the cnoidal solution (ϕ_c, ψ_c), provided $c \in (c^*, +\infty)$. $\square$

7. Instability for coupled Boussinesq equations in lattice vibrations

In this section we study the instability of periodic traveling waves for two coupled Boussinesq equations, namely

$$\begin{cases} v_{tt} - v_{xxtt} - \left(v - \beta_0 v^p + w^p\right)_{xx} = 0, \\ w_{tt} - w_{xxtt} - \left(w + p v w^{p-1}\right)_{xx} = 0 \end{cases} \tag{7.1}$$

where $v = v(x, t)$, $w = w(x, t)$, $p \in \mathbb{N}$, $p \geqslant 2$, and $\beta_0 \in \mathbb{R} - \{0\}$.

By supposing $v(x, t) = \phi_c(x - ct)$ and $w(x, t) = \alpha\phi_c(x - ct)$ we obtain from (7.1) the system

$$\begin{cases} \left(c^2 - 1\right)\phi_c - c^2\phi_c'' + \left(\beta_0 - \alpha^p\right)\phi_c^p = 0, \\ \left(c^2 - 1\right)\phi_c - c^2\phi_c'' - p\alpha^{p-2}\phi_c^p = 0, \end{cases} \tag{7.2}$$

which implies the condition $\beta_0 + p\alpha^{p-2} = \alpha^p$. In the case $p = 2$ we obtain the following profile of solitary wave solution, satisfying the second equation in (7.2), for $c^2 > 1$:

$$\phi_c(\xi) = \frac{3}{4}\left(c^2 - 1\right)\operatorname{sech}^2(\gamma\xi), \qquad \alpha = \sqrt{2 + \beta_0}, \qquad \gamma = \frac{1}{2}\sqrt{\frac{c^2 - 1}{c^2}}. \tag{7.3}$$

Moreover, in the case where ϕ_c is a periodic profile, from the ideas in Angulo [4] (see also Angulo, Bona and Scialom [8]) we can derive the cnoidal type solution, for $c^2 > 1$:

$$\phi_c(\xi) = \frac{1}{4}\beta_2 + \frac{1}{4}(\beta_3 - \beta_2)\operatorname{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12c^2}}\xi; k\right), \tag{7.4}$$

where

$$\beta_1 < \beta_2 < \beta_3, \qquad \beta_1 + \beta_2 + \beta_3 = 3\left(c^2 - 1\right), \qquad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}. \tag{7.5}$$

It follows immediately from (7.4) and (7.5) that ϕ_c must take values in the range $\beta_2 \leqslant \phi_c \leqslant \beta_3$, with $\beta_2 > 0$. Therefore, ϕ_c does not have zero mean (we note that the same can be deduced by integrating the second equation in (7.2)). We now consider some degenerate cases. Suppose, say, that c and β_1

are fixed. If $\beta_2 \to \beta_1^+$ then $k^2 \to 1$, and so the elliptic function cn converges, uniformly on compacts sets, to the hyperbolic function sech and (7.4) becomes, in this limit,

$$\phi(\xi) = \phi_\infty + \theta \operatorname{sech}^2\left(\sqrt{\frac{\theta}{3c^2}}\xi\right) \tag{7.6}$$

with $\phi_\infty = \frac{1}{4}\beta_1$ and $\theta = \frac{1}{4}(\beta_3 - \beta_1)$. If $\beta_1 = 0$, we obtain the solitary wave in (7.3).

The stability of the traveling wave profiles in (7.3) and (7.4) might be studied via Lyapunov's direct method (see [3–14,16,21,22,38]). In this approach (for both solitary and periodic waves), the traveling wave profile is characterized as a minimum of a modified Hamiltonian, subject to the constraint that the momentum is fixed. In the case of the solitary wave profile it is well known that this method fails for (7.1), since the second variation of the appropriate conserved functional turns out to have an infinite-dimensional indefiniteness, namely, there are two infinite-dimensional subspaces of variations, such that the second variation is negative definite on one subspace, and positive definite on the other (see Pego, Smereka and Weinstein [34]). To date, no variational method of proving stability has been designed to handle the strong indefiniteness. A similar phenomenon is well known to occur for the RBou equation (1.1), with $f(u) = \frac{1}{p+1}u^{p+1}$ and the solitary wave solutions

$$\phi(\xi) = \left[\frac{1}{2}\left(c^2 - 1\right)(p+1)(p+2)\right]^{1/p} \operatorname{sech}^{2/p}(p\gamma\xi),$$

with γ given by (7.3). If one tries to deduce the stability of the cnoidal profile (7.4) by the Lyapunov's method, the approach fails for the same reasons as it does for the system (7.1).

7.1. Linear instability of cnoidal waves for coupled Boussinesq equations

We are now interested in the linear instability of the cnoidal profiles for system (7.1), with $p = 3$. In fact, from (5.3), (5.5) and Theorem 5.1 we have a smooth family $c \to (\phi_{c,\alpha}, \alpha\phi_{c,\alpha})$ ($c^2 > 1$ and $\alpha > 0$) of periodic solutions for (7.1), where

$$\phi_{c,\alpha}(\xi) = \frac{1}{\sqrt{3\alpha}} b \operatorname{cn}\left(\frac{\beta}{c}\xi; k\right) \tag{7.7}$$

and $b = b(c)$, $\beta = \beta(c)$ and $k = k(c)$ are smooth functions determined by Theorem 5.1.

We proceed by describing a Hamiltonian formulation of system (7.1). We define $J = (1 - \partial_x^2)^{1/2}$ and the bounded operator $\mathcal{B} = \partial_x J^{-1}$, introducing the new variables u, z such that

$$\begin{cases} v_t = \mathcal{B}u, & u_t = \mathcal{B}\left(v - \beta_0 v^p + w^p\right), \\ w_t = \mathcal{B}z, & z_t = \mathcal{B}\left(w + p v w^{p-1}\right). \end{cases} \tag{7.8}$$

The Hamiltonian defined by

$$H(v, w, u, z) = \int \frac{1}{2}u^2 + \frac{1}{2}z^2 + \frac{1}{2}v^2 + \frac{1}{2}w^2 - \frac{\beta_0}{p+1}v^{p+1} + v w^p \, dx, \tag{7.9}$$

is a conserved quantity for (7.8), which allows (7.8) to be written as

$$\partial_t \mathcal{Y}(t) = \mathcal{J} H'\left(\mathcal{Y}(t)\right) \tag{7.10}$$

where $\mathcal{Y} = (v, w, u, z)$ and $\mathcal{J}$ is the skew-symmetric operator

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & \mathcal{B} & 0 \\ 0 & 0 & 0 & \mathcal{B} \\ \mathcal{B} & 0 & 0 & 0 \\ 0 & \mathcal{B} & 0 & 0 \end{pmatrix}. \tag{7.11}$$

In addition to the functional H, there is the conserved quantity, called momentum, arising from the translation invariance of the system (7.1),

$$M(v, w, u, z) = \int uJv + zJw\,dx. \tag{7.12}$$

Then, if we consider the functional $\mathcal{F} = H + cM$, a traveling wave solution of the system (7.8) is a critical point of this functional. Therefore, (7.10) has periodic traveling wave solutions of the form

$$\mathcal{Y}_c = (\phi_c, \alpha\phi_c, -cJ\phi_c, -\alpha cJ\phi_c),$$

with α satisfying $\beta_0 + p\alpha^{p-2} = \alpha^p$ and ϕ_c defined by (7.7).

To study linear instability we look for solutions of (7.10) of the form

$$\mathcal{W}(x,t) = \mathcal{Y}(x+ct, t) - \mathcal{Y}_c$$

and neglect terms which are $O(\|\mathcal{W}\|^2)$, thus leading to the linear evolution equation

$$\partial_t \mathcal{W}(t) = \mathcal{J}\mathcal{S}\big(\mathcal{W}(t)\big) \tag{7.13}$$

where $\mathcal{S} = \mathcal{F}''(\mathcal{Y}_c)$ is given by

$$\mathcal{S} = \begin{pmatrix} I - p\beta_0\phi_c^{p-1} & p\alpha^{p-1}\phi_c^{p-1} & cJ & 0 \\ p\alpha^{p-1}\phi_c^{p-1} & I + p(p-1)\alpha^{p-2}\phi_c^{p-1} & 0 & cJ \\ cJ & 0 & I & 0 \\ 0 & cJ & 0 & I \end{pmatrix}. \tag{7.14}$$

At this point, we search for solutions of (7.13) of the form $e^{\lambda t}Y(x)$ with $Y(x) = (v(x), w(x), u(x), z(x))^t$ and $\operatorname{Re}\lambda > 0$. Eliminating u and z from the equations

$$(\lambda - c\partial_x)v = \mathcal{B}u, \qquad (\lambda - c\partial_x)w = \mathcal{B}z,$$

we find that v and w satisfy

$$(\lambda - c\partial_x)^2 J^2 \begin{pmatrix} v \\ w \end{pmatrix} = \partial_x^2 \left[I + p\phi_c^{p-1} \begin{pmatrix} -\beta_0 & \alpha^{p-1} \\ \alpha^{p-1} & (p-1)\alpha^{p-2} \end{pmatrix} \right] \begin{pmatrix} v \\ w \end{pmatrix}. \tag{7.15}$$

Changing variables via the transformation

$$\begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{1+\alpha^2} \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{7.16}$$

the eigenvalue problem (7.15) becomes

$$\begin{cases} (\lambda - c\partial_x)^2 J^2 f = \partial_x^2\big(1 + p^2\alpha^{p-2}\phi_c^{p-1}\big) f, \\ (\lambda - c\partial_x)^2 J^2 g = \partial_x^2\big(1 - p\phi_c^{p-1}(\beta_0 + \alpha^{p-2})\big) g. \end{cases} \tag{7.17}$$

Thus from the theory developed in Sections 4 and 5 above, we have for the linear operator associated to the second equation in (7.2)

$$\mathcal{L}_1 = \big(1 - \partial_x^2\big) - \frac{1}{c^2}\big(1 + p^2\alpha^{p-2}\phi_c^{p-1}\big) \tag{7.18}$$

the following instability criterion for the Boussinesq type system (7.8).

Theorem 7.1. *Define the space $X_1^0 = [H^1_{per}([0,L])]^4 \cap \mathbb{V}$ and let $c \mapsto (\phi_c, \alpha\phi_c, -cJ\phi_c, -\alpha cJ\phi_c) \in X_1^0$ be a smooth curve of periodic solutions such that $(\phi_c, \alpha\phi_c)$ satisfies the system (7.2) with $c^2 > 1$. Assume that*

$$\ker(Q\mathcal{L}_1) = \left[\frac{d}{dx}\phi_c\right]. \tag{7.19}$$

Denote by $n^-(Q\mathcal{L}_1)$ the number (counting multiplicity) of negative eigenvalues of the operator $Q\mathcal{L}_1$ defined in $H^1_{per}([0,L]) \cap \mathbb{V}$. Then there is a purely growing mode $e^{\lambda t}(v(x), w(x), u(x), z(x))^t$ with $\lambda > 0$, $(v, w, u, z) \in X_1^0 - \{(0,0,0,0)\}$, to the linearized equation (7.13) if one of the following conditions holds:

(i) *$n^-(Q\mathcal{L}_1)$ is even and $V(c) < 0$,*
(ii) *$n^-(Q\mathcal{L}_1)$ is odd and $V(c) > 0$.*

Here,

$$V(c) := -\frac{1}{\|\phi_c'\|^2_{L^2_{per}}}\frac{1}{c^2}\frac{dP}{dc}$$

with P given by

$$P(c) = c\big\langle\big(1 - \partial_x^2\big)\phi_c, \phi_c\big\rangle_{L^2_{per}}.$$

Proof. The first eigenvalue problem in (7.17) induced the following family of linear operators for $\operatorname{Re}\lambda > 0$, $\mathcal{F}^\lambda : X_1^0 \to \mathbb{V}$ given by

$$\mathcal{F}^\lambda f := \big(1 - \partial_x^2\big) f - \frac{\partial_x^2}{(\lambda - c\partial_x)^2} Q\big(1 + p^2\alpha^{p-2}\phi_c^{p-1}\big) f,$$

which belong to the class of operators defined in (1.9) with $\mathcal{M} = -\partial_x^2$ and $f(\phi_c) = p^2\alpha^{p-2}\phi_c^p$. Moreover, we have the convergence

$$\mathcal{F}^\lambda \to Q\mathcal{L}_1 \quad \text{as } \lambda \to 0^+$$

strongly in $H^1_{per}([0,L]) \cap \mathbb{V}$. Since the kernel of $Q\mathcal{L}_1$ satisfies (7.19) we obtain, from the proof of Theorem 1.1, that the conditions (i) and (ii) above imply the existence of a coupled $(f_0, \lambda_0) \in (\operatorname{Ker}(\mathcal{F}^\lambda) - \{0\}) \times \mathbb{R}^+$ which satisfies the first equation in (7.17). Next, if we consider $g \equiv 0$ for the second equation in (7.17) then we obtain from (7.16) that

$$\begin{pmatrix} v \\ w \end{pmatrix} \equiv \begin{pmatrix} f_0 \\ \alpha f_0 \end{pmatrix},$$

satisfies the eigenvalue problem (7.15). Therefore,

$$Y_0(x) = e^{\lambda_0 t}\big(f_0, \alpha f_0, \partial_x^{-1} J(\lambda_0 - c\partial_x) f_0, \alpha \partial_x^{-1} J(\lambda_0 - c\partial_x) f_0\big)$$

is a purely growing mode to the linearized equation (7.13). This finishes the proof of the theorem. □

We can immediately provide an application of Theorem 7.1 for the case of the cnoidal profiles in (7.7), namely, $p = 3$, $c^2 - 1 > 0$ and α being a positive root of $\beta_0 + 3\alpha = \alpha^3$ in (7.2). From (7.7), (5.5) and (7.18) we obtain that

$$\mathcal{L}_1 = \mathcal{L}_0 = \big(1 - \partial_x^2\big) - \frac{1}{c^2}\big(1 + 3\phi_c^2\big)$$

with $c \to \phi_c$ being the smooth curve of cnoidal profiles determined by Theorem 5.1. Therefore from the analysis in Section 5.1 we have $\ker(Q\mathcal{L}_1) = [\frac{d}{dx}\phi_{c,\alpha}]$, $n^-(Q\mathcal{L}_1) = 1$ for $k \in (\frac{\sqrt{2}}{2}, k^*)$ and $n^-(Q\mathcal{L}_1) = 2$ for $k \in (k^*, 1)$, where $k^* \sim 0.909$. Moreover, we obtain from (5.11) that

$$P_\alpha(c) = c\big\langle\big(1 - \partial_x^2\big)\phi_{c,\alpha}, \phi_{c,\alpha}\big\rangle = \frac{c}{3\alpha}\big\langle\big(1 - \partial_x^2\big)\phi_c, \phi_c\big\rangle,$$

hence $\frac{d}{dc}P_\alpha(c) > 0$. Therefore, from Theorem 7.1 we obtain the linear instability of the cnoidal profiles in (7.7) for the Boussinesq system (7.8) provided the elliptic modulus k satisfies $k \in (k^*, 1)$.

Before to establish the result of nonlinear instability for the cnoidal waves, we need a result of local or global well-posedness, we make this on the next subsection.

7.2. Local well-posedness for coupled Boussinesq equations

In this subsection we study the initial value problem

$$\begin{cases} v_t = \mathcal{B}u, & u_t = \mathcal{B}\big(v - \beta_0 v^p + w^p\big), \\ w_t = \mathcal{B}z, & z_t = \mathcal{B}\big(w + p v w^{p-1}\big), \\ \big(v(0), u(0), w(0), z(0)\big) = (u_0, v_0, w_0, z_0). \end{cases} \tag{7.20}$$

Solving the linear problem associated to (7.20) we get the solution $(v(t), w(t), u(t), z(t)) = S(t)(v_0, w_0, u_0, z_0)$, where

$$\begin{pmatrix} \widehat{v}(t,\xi) \\ \widehat{w}(t,\xi) \\ \widehat{u}(t,\xi) \\ \widehat{z}(t,\xi) \end{pmatrix} = \begin{pmatrix} \cos(\theta(\xi)t) & 0 & i\sin(\theta(\xi)t) & 0 \\ 0 & \cos(\theta(\xi)t) & 0 & i\sin(\alpha(\xi)t) \\ i\sin(\theta(\xi)t) & 0 & \cos(\theta(\xi)t) & 0 \\ 0 & i\sin(\theta(\xi)t) & 0 & \cos(\theta(\xi)t) \end{pmatrix} \begin{pmatrix} \widehat{v_0}(\xi) \\ \widehat{w_0}(\xi) \\ \widehat{u_0}(\xi) \\ \widehat{z_0}(\xi) \end{pmatrix}$$

with $\theta(\xi) = \frac{\xi}{\langle\xi\rangle}$ and $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$. Then, (7.20) may be rewritten as the integral equation

$$\big[v(x,t), w(x,t), u(x,t), z(x,t)\big] = S(t)[v_0, w_0, u_0, z_0] + i\int_0^t S(t-\tau)G\big[(v,w)(x,\tau)\big]\,d\tau$$

where G is given by $G(v,w) = [0, 0, \mathcal{B}(w^p - \beta_0 v^p), \mathcal{B}(p v w^{p-1})]$.

Similarly to Theorem 6.2, we obtain the following result.

Theorem 7.2. *The Cauchy problem associated to the coupled Boussinesq equations* (7.20) *is locally well-posed in $X^s = (H^s_{per})^4$, for $s > 1/2$, that is, if $(v_0, u_0, w_0, z_0) \in X^s$ there is $T > 0$ and a unique mild solution $(v, u, w, z) \in C([0, T]; X^s)$ of* (7.20)*. Moreover, the data-solution mapping associated to the RBou system,*

$$\Upsilon : X^s \to C\big([0, T]; X^s\big),$$
$$(v_0, w_0, u_0, z_0) \mapsto \Upsilon(v_0, w_0, u_0, z_0) = (v_{v_0}, w_{w_0}, u_{u_0}, z_{z_0}),$$

is smooth and we have that the quantities (7.9), (7.12),

$$I_1(v, w, u, z) = \int v\,dx, \qquad I_2(v, w, u, z) = \int u\,dx,$$
$$I_2(v, w, u, z) = \int w\,dx \quad \textit{and} \quad I_2(v, w, u, z) = \int z\,dx,$$

are conserved by the flow of the coupled Boussinesq equations.

7.3. Nonlinear instability for cnoidal waves for coupled Boussinesq equations

Finally, in this section we obtain the nonlinear instability of cnoidal waves solutions for system (7.1). Next, we present the main result of this section.

Theorem 7.3. *The cnoidal profile solution $\Phi_c := (\phi_c, \alpha\phi_c, -cJ\phi_c, -c\alpha J\phi_c)$, where ϕ_c is given in* (7.7)*, is nonlinearly unstable in X^s, with $s > 1/2$, for the coupled equations* (7.8)*, provided that the wave speed $c \in (c^*, +\infty)$ with c^* defined in* (5.19)*.*

Proof. In system (7.8) we replace $(v(x,t), w(x,t), u(x,t), z(x,t))$ by $(v(x+ct,t), w(x+ct,t), v(x+ct,t), z(x+ct,t))$ yielding

$$\begin{cases} v_t - cv_x = \mathcal{B}u, & u_t - cu_x = \mathcal{B}\big(v - \beta_0 v^p + w^p\big), \\ w_t - cw_x = \mathcal{B}z, & z_t - cz_x = \mathcal{B}\big(w + pvw^{p-1}\big). \end{cases} \tag{7.21}$$

Then, Φ_c is an equilibrium solution for Eq. (7.21). Defining $\mathcal{F} = H + cM$, where H and M are given in (7.9) and (7.12), we have that system (7.21) can be written as

$$(v, w, u, z)_t = \mathcal{J}\mathcal{F}'(v, w, u, z), \tag{7.22}$$

where $\mathcal{J}$ is given by (7.11). Moreover, from (7.22) we see that the linearized equation at the equilibrium point Φ_c is $(v, w, u, z)_t = \mathcal{J}\mathcal{S}(v, w, u, z)$, where $\mathcal{S}$ is the linear self-adjoint operator defined by (7.14).

Let us define $S : X^s \to X^s$ as $S(v_0, w_0, u_0, z_0) = (v_{v_0}(1), w_{w_0}(1), u_{u_0}(1), z_{z_0}(1))$, where

$$\big(v_{v_0}(t), w_{w_0}(t), v_{v_0}(t), w_{w_0}(t)\big)$$

is the solution of (7.20) with initial data $(v(x,0), w(x,0), u(x,0), z(x,0)) = (v_0(x), w_0(x), u_0(x), z_0(x))$. If one considers $\Upsilon_c : X^s \to C([0, T]; X^s)$ the data-solution mapping related to the system (7.20), we see from Theorem 7.2 that Υ_c is smooth. Furthermore, $S(\Phi_c) = \Phi_c$ and S is a C^2 map defined on a neighborhood of Φ_c (this last fact is a consequence of translation in x being a linear continuous map in X^s). For $(f,g,h,i) \in X^s$ we have $S'(\Phi_c)(f,g,h,i) = (a_f(1), b_g(1), c_h(1), d_i(1))$, where $(a_f(1), b_g(1), c_h(1), d_i(1))$, is the solution of the linear initial value problem

$$\begin{cases} (a,b,c,d)_t = \mathcal{J}\mathcal{S}(a,b,c,d), \\ (a,b,c,d,)(0) = (f,g,h,i), \end{cases}$$

evaluated at $t = 1$. Then, from arguments established in Section 4 and Section 5.1, we deduce that there is $\lambda > 0$ and $(a_0, b_0, c_0, d_0) \in X^s - \{0\}$ such that $\mathcal{J}\mathcal{S}(a_0, b_0, c_0, d_0) = \lambda(a_0, b_0, c_0, d_0)$. Hence, for $(a_{a_0}(t), b_{b_0}(t), c_{c_0}(t), d_{d_0}(t)) = e^{\lambda t}(a_0, b_0, c_0, d_0)$ and $\mu := e^{\lambda}$ we obtain $S'(\Phi_c)(a_0, b_0, c_0, d_0) = \mu(a_0, b_0, c_0, d_0)$. Therefore $\mu \in \sigma(S'(\Phi_c))$ and from Corollary 6.1 we obtain the nonlinear instability in X^s of the solution Φ_c, provided $c \in (c^*, +\infty)$. $\square$

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