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# Energy decay rates for solutions of the wave equations with nonlinear damping in exterior domain

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**Abstract**

In this paper we study the behaviors of the energy of solutions of the wave equations with localized nonlinear damping in exterior domains.


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*Keywords:* Wave equation; Nonlinear damping; Decay rate; Exterior domain

## 1. Introduction and statement of the results

Let $O$ be a compact domain of $\mathbb{R}^d$ $(d \geq 1)$ with $C^\infty$ boundary $\Gamma$ and $\Omega = \mathbb{R}^d \backslash O$. Consider the following wave equation with localized nonlinear damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x) \left|\partial_t u\right|^{r-1} \partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(0,x) = u_0 \quad \text{and} \quad \partial_t u(0,x) = u_1, \end{cases} \tag{1.1}$$

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here $\Delta$ denotes the Laplace operator in the space variables. $a(x)$ is a nonnegative function in $L^\infty(\Omega)$. Throughout this paper we assume that $1 < r \le 1 + \frac{2}{d}$. Below $r_0 > 0$ is a fixed constant such that $O \subset B_{r_o} = \{x \in \mathbb{R}^d; |x| < r_0\}$.

The existence and uniqueness of global solutions to the problem (1.1) is standard (see [16]). If $(u_0, u_1)$ is in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, then the system (1.1), admits a unique solution $u$ in the class

$$u \in C^0\Big(\mathbb{R}_+, H_0^1(\Omega)\Big) \cap C^1\Big(\mathbb{R}_+, L^2(\Omega)\Big) \text{ and } \partial_t u \in L^\infty\Big(\mathbb{R}_+, H_0^1(\Omega)\Big) \cap W^{1,\infty}\Big(\mathbb{R}_+, L^2(\Omega)\Big).$$

Let us consider the energy at instant $t$ defined by

$$E_u(t) = \frac{1}{2} \int_\Omega \Big(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\Big) dx.$$

The energy functional satisfies the following identity

$$E_u(T) + \int_0^T \int_\Omega a(x) |\partial_t u|^{r+1} dx dt = E_u(0), \tag{1.2}$$

for every $T \ge 0$. Moreover, we have

$$\begin{aligned} &\|\nabla \partial_t u\|^2_{L^\infty(\mathbb{R}_+, L^2(\Omega))} + \left\|\partial_t^2 u\right\|^2_{L^\infty(\mathbb{R}_+, L^2(\Omega))} \\ &\le 2(1 + \|a\|_{L^\infty}) \Big(\|u_0\|^2_{H^2} + \|u_1\|^2_{H^1} + \|u_1\|^{2r}_{H^1}\Big). \end{aligned} \tag{1.3}$$

The study of the behaviors of the energy decay of solutions of the damped wave equation has a very long history. First we give a summary of results on the asymptotic behavior of the energy of solutions of the nonlinear system (1.1) in the free space $\mathbb{R}^d$ and for a globally distributed damping. For the Klein–Gordon equation with localized nonlinear damping, under the Lion's condition a polynomial decay rate is derived by Nakao [19] for compactly supported initial data and he show in this case that

$$E_u(t) \le C(1+t)^{-\gamma}, \text{ if } 1 < r < 1 + \frac{2}{d}, \tag{1.4}$$

where $\gamma = \frac{2+d-dr}{r-1}$ and

$$E_u(t) \le C(\ln(2+t))^{-d}, \text{ if } r = 1 + \frac{2}{d}. \tag{1.5}$$

Mochizuki and Motai [17] give a decay rate estimate for weighted initial data. More precisely, they show that if $1 < r < 1 + \frac{2}{d}$, the energy decays according to

$$E_u(t) \le C(1+t)^{-\gamma}, \text{ where } 0 < \gamma < \frac{2+d-dr}{r-1} \text{ and } \gamma \le 1. \tag{1.6}$$

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If $r > 1 + \frac{2}{d}$, Mochizuki and Motai [17] establish a complementary non-decay result for a dense set of initial data in $H^1\left(\mathbb{R}^d\right) \times L^2\left(\mathbb{R}^d\right)$.

For the wave equation we first quote the result of Ono [22], in which the author considers the wave equation with a damping term equal to $\partial_t u + g\left(\partial_t u\right)$ where $g$ is superlinear and has a polynomial growth. He showed the polynomial decay of the energy. We note that in this case the $L^2$ norm of the time derivative on $\mathbb{R}_+ \times \mathbb{R}^d$ of the solution is bounded by the energy of the initial data. Mochizuki and Motai in [17] obtained a logarithmic decay rate when $1 < r \leq 1 + \frac{2}{d}$ and for a kind of weighted initial data. More precisely, they show that

$$E_u\left(t\right) \leq C\left(\ln\left(2+t\right)\right)^{-\gamma}, \text{ with } 0 < \gamma < \frac{2}{r-1}. \tag{1.7}$$

The corresponding non-decay result in [17] requires $r > 1 + \frac{2}{d-1}$. Todorova and Yordanov in [25] showed that for compactly supported initial data there exists a positive constant $\tau$ such that $E_u\left(t\right) \leq C\left(1+t\right)^{-\tau}$, when $1 < r \leq 1 + \frac{2}{d+1}$ and $d \geq 3$. The main idea in this paper is to use the "parabolic" effects coming from the presence of the damping term. Recently, Wakasa and Yordanov in [26] studied the energy decay for dissipative nonlinear wave equations in one space dimension with global distributed damping term. They established polynomial decay estimates for the energy for compactly supported initial data. More explicitly they show that $E_u\left(t\right) \leq C\left(1+t\right)^{-\tau}$, when $1 < r < 3$ with $\tau < \min\left(\frac{1}{2}, \frac{3-r}{r-1}\right)$.

In the case of exterior domain we mention the result of Nakao and Jung [21] who consider a dissipation which is allowed to be nonlinear only in a ball, but outside that ball the dissipation must be linear. For the generalized Klein–Gordon equation we quote the result of Nakao [20].

In the case of linear damping the literature is more furnished and the problem has been intensively studied. Before going any further, let us mention some results. First we note that if we assume that $a\left(x\right) \geq \epsilon_0 > 0$ in all of $\Omega$, then we know that

$$E_u\left(t\right) \leq C_0\left(1+t\right)^{-1} \text{ and } \left\|u\left(t\right)\right\|_{L^2} \leq C_0, \text{ for all } t \geq 0, \tag{1.8}$$

for weak solution $u$ to the system (1.1) with initial data in $H_0^1\left(\Omega\right) \times L^2\left(\Omega\right)$ and $r = 1$. Nakao in [18] obtained the same estimates in (1.8) for a damper $a$ which is positive near some part of the boundary (Lions's condition) and near infinity. The same result has been proved by Daoulatli in [8] under the (GCC) condition.

Furthermore, Ikehata and Matsuyama in [10] obtained a more precise decay estimate for the total energy of solutions of the problem (1.1) with $a\left(x\right) = 1$, $r = 1$ and for weighted initial data

$$E_u\left(t\right) \leq C_2\left(1+t\right)^{-2} \text{ and } \left\|u\left(t\right)\right\|_{L^2}^2 \leq C_2\left(1+t\right)^{-1} \text{ for all } t \geq 0. \tag{1.9}$$

Ikehata in [11] derived a fast decay rate like (1.9) for solutions of the system (1.1) with weighted initial data and assuming that $a\left(x\right) \geq \epsilon_0 > 0$ at infinity and $O = \mathbb{R}^d \backslash \Omega$ is star shaped with respect to the origin. This result has been obtained by Daoulatli in [8] under the (GCC) condition.

Recently, Aloui et al. [1] showed that under the (GCC) condition and for compactly supported initial data

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$$E_u(t) \leq C_2 (1+t)^{-\gamma} \text{ and } \|u(t)\|_{L^2}^2 \leq C_2 (1+t)^{-\frac{d}{2}} \text{ for all } t \geq 0, \tag{1.10}$$

where $\gamma = \min\left(1 + \frac{d}{2}, \frac{3d}{4}\right)$.

Hence establishing decay estimates for the nonlinear problem is more delicate and leads to weaker decay rates. For another type of total energy decay property we refer the reader to [13,14,24,2,23,12] and references therein. Finally, we note that to our knowledge no results seem to be known for the problem of the wave equation with localized nonlinear damping in the free space or in exterior domain.

Before introducing our results we shall state several assumptions:

**Hyp A:** There exists $L > r_0$ such that

$$a(x) \geq \epsilon_0 > 0 \text{ for } |x| \geq L.$$

**Definition 1.** Let $\omega$ be an open set of $\Omega$.

(1) $(\omega, T)$ geometrically controls $\Omega$, i.e. every generalized geodesic travelling with speed 1 and issued at $t = 0$, enters the set $\omega$ in a time $t < T$.
(2) We say that $\omega$ satisfies GCC if there exists $T > 0$ such that $(\omega, T)$ geometrically controls $\Omega$.

This condition is called Geometric Control Condition (see e.g. [3]). We shall relate the open subset $\omega$ with the damper $a$ by

$$\omega \subset \{x \in \Omega; a(x) > \epsilon_0 > 0\}.$$

We note that according to [3] and [4] the Geometric Control Condition of Bardos et al. is a necessary and sufficient condition for the exponential decay of solutions of the wave equation in bounded domain.

In this paper, we deal with real solutions, the general case can be treated in the same way. Throughout this paper we use the following notations

$$q(x) = \left(1 + |x|^2\right)^{\frac{1}{2}}, \text{ for } x \in \Omega,$$

and

$$p = \begin{cases} 2(r+1) & \text{if } d \leq 2 \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

Now we state the results of this paper.

**Theorem 1.** *We assume that Hyp A holds and $\omega$ satisfies GCC. Let*

$$\begin{aligned} &\gamma > 0 && \text{if } 1 < r < 1 + \tfrac{2}{d} \\ &0 < \gamma < d && \text{if } r = 1 + \tfrac{2}{d}. \end{aligned}$$

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*Then there exists $C_0 > 0$ such that the following estimate*

$$E_u(t) \leq C_0 (\ln(2+t))^{-\gamma} I_0, \text{ for all } t \geq 0,$$

*holds for every solution u of (1.1) with initial data $(u_0, u_1)$ in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, such that*

$$\left\| (\ln(1+q))^{\frac{\gamma}{2}} \nabla u_0 \right\|_{L^2}^2 + \left\| (\ln(1+q))^{\frac{\gamma}{2}} u_1 \right\|_{L^2}^2 < +\infty,$$

*where*

$$I_0 = \|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r} + \|u_0\|_{L^{r+1}}^{r+1} + \left\| (\ln(1+q))^{\frac{\gamma}{2}} \nabla u_0 \right\|_{L^2}^2 + \left\| (\ln(1+q))^{\frac{\gamma}{2}} u_1 \right\|_{L^2}^2 + \left( \|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r} \right)^{\frac{p}{2}} + 1.$$

**Remark 1.** When $1 < r < 1 + \frac{2}{d}$, we obtain the result of Mochizuki and Motai in [17], for the problem in the free space with global distributed damping.

In the result above we see that when $1 < r < 1 + \frac{2}{d}$, we can take any $\gamma > 0$, so we expect that we can obtain a rate of decay of the energy for a weight with a polynomial growth.

**Theorem 2.** *We assume that Hyp A holds and $\omega$ satisfies GCC. We suppose that $1 < r < 1 + \frac{2}{d}$. We take*

$$0 < \gamma < \min\left( \frac{r}{r+2}, \frac{d+2-dr}{r-1} \right).$$

*Then there exists $C_1 > 0$ such that the following estimate*

$$E_u(t) \leq C_1 (1+t)^{-\gamma} I_1, \text{ for all } t \geq 0,$$

*holds for every solution u of (1.1) with initial data $(u_0, u_1)$ in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, such that*

$$\left\| (1+q)^{\frac{\gamma}{2}} \nabla u_0 \right\|_{L^2}^2 + \left\| (1+q)^{\frac{\gamma}{2}} u_1 \right\|_{L^2}^2 < +\infty,$$

*where*

$$I_1 = \|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r} + \|u_0\|_{L^{r+1}}^{r+1} + \left\| (1+q)^{\frac{\gamma}{2}} \nabla u_0 \right\|_{L^2}^2 + \left\| (1+q)^{\frac{\gamma}{2}} u_1 \right\|_{L^2}^2 + \left( \|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r} \right)^{\frac{p}{2}} + 1.$$

**Remark 2.** We note that, using the energy estimates (1.2) and (2.3), we can show that

$$\int_0^\infty \int_\Omega a(x) (1+t)^\gamma |\partial_t u|^{r+1} \, dx dt \leq C I_1.$$

So we deduce that

$$\int_0^\infty \int_\Omega a(x)(1+t)^{\gamma-r-1}|u(t,x)|^{r+1}\,dxdt \le CI_1,$$

and this allows us to obtain a bound of $\|u(t)\|_{L^{r+1}}$. More precisely, we obtain

$$\|u(t)\|_{L^{r+1}} \le C(1+t)^{\frac{r-\gamma}{r+1}} I_1.$$

The case of initial data with compact support.

**Theorem 3.** *We assume that Hyp A holds and $\omega$ satisfies GCC. We suppose that $1 < r < 1+\frac{2}{d}$. We take*

$$0 < \gamma < \min\left(\frac{2r}{r+3}, \frac{d+2-dr}{r-1}\right).$$

*Then there exists $C_1 > 0$ such that the following estimate*

$$E_u(t) \le C_1 (R+t)^{-\gamma} I_2, \text{ for all } t \ge 0,$$

*holds for every solution $u$ of (1.1) with initial data $(u_0, u_1)$ in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ such that the support of the initial data is contained in $B_R$, where*

$$\begin{aligned} I_2 &= \|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r} + \|u_0\|_{L^{r+1}}^{r+1} \\ &+ \left(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r}\right)^{\frac{p}{2}} + 1. \end{aligned}$$

**Remark 3.**

(1) Our results are also valid for the case $\Omega = \mathbb{R}^d$, $d \ge 3$, where the boundary condition is dropped.
(2) As in the case of weighted initial data, we can show that

$$\|u(t)\|_{L^{r+1}} \le C(1+t)^{\frac{r-\gamma}{r+1}} I_2.$$

(3) When $d = 1$, we obtain that

$$\begin{aligned} 1/2 < \gamma < \tfrac{2r}{r+3} \quad &\text{if } 1 < r \le \tfrac{2}{3}\sqrt{7} + \tfrac{1}{3} \\ 0 < \gamma < \tfrac{3-r}{r-1} \quad &\text{if } \tfrac{2}{3}\sqrt{7} + \tfrac{1}{3} < r < 3. \end{aligned}$$

Our decay rate is better than or equal to the one obtained by Wakasa and Yordanov in [26].
(4) We remark that there is a gap between the rate of decay when $r = 1$ and $r = 1 + \epsilon$ for $\epsilon$ positive and close to zero. In addition, when $r$ is close to 1 the values of $\gamma$ increase as $r$ increase. This fact is unusual in the literature and we are unable to prove the optimality of our results.

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(5) In Theorem 3 we assume that the initial data is with compact support so we don't need to take a weight which depends on the space variable $x$. Therefore in the weighted energy estimate (2.3) we will take $\eta = 0$ and this fact give us a better decay estimate compared to the case of weighted initial data.

The main difficulty in establishing such results is the lack of control of the $L^2$ norm of the solution. This is an essential difference with the equation in a bounded domain or the Klein–Gordon equation or in the case of unbounded domain with finite measure [5]. The other difficulty is that the $L^2$ norm of the time derivative on $\mathbb{R}_+ \times \Omega$ is not controlled by the initial energy.

To prove our results it is sufficient to show the integrability of $\varphi' E_u$ over $(0, \infty)$. For this purpose we show an estimate on a Lyapunov functional $X(t)$ (Lemmas 2, 5 and 7) which control the weighted energy functional (see, for example, [8] and [9] for similar idea). Also we prove a weighted observability estimate for the local energy of solutions the wave equation with external force (Proposition 2). Combining these results and making some computations we end up with the last problem which is how to control these two quantities

$$\int \int_{\Omega} a(x) \varphi'(q(x) + s) |\partial_t u|^2 \, dx ds$$

and

$$\int \int_{\Omega} a(x) \varphi'(q(x) + s) |\partial_t u|^{2r} \, dx ds,$$

by

$$\epsilon \int \int_{\Omega} a(x) \varphi(q(x) + s) |\partial_t u|^{r+1} \, dx ds + C(\epsilon) \int \int_{\Omega} g(s, x) \, dx ds, \text{ for all } \epsilon > 0,$$

such that $\int_0^\infty \int_\Omega g(s, x) \, dx ds < \infty$.

The rest of the paper is organized as follows. In section 2 we present some results on the weighted energy and we give a weighted observability estimate for the local energy. Section 3 is devoted to the proof of Theorem 1 and in section 4 we give the proof of Theorem 2. In the last section we give the needed results to show Theorem 3.

## 2. Weighted observability estimate

The next result concerns the weighted energy estimate for solutions of (1.1) with initial data with finite weighted energy.

**Proposition 1.** *Let $\varphi$ be a positive function in $C^2(\mathbb{R}_+)$ such that $\varphi' \in L^\infty(\mathbb{R}_+)$ and $\varphi'' \in L^\infty(\mathbb{R}_+)$. Let $u$ be a solution of (1.1) with initial data $(u_0, u_1)$ in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$. We set*

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$$E_{\varphi}(u)(t)=\frac{1}{2}\int_{\Omega}\varphi(\eta q(x)+\alpha t)\left(|\nabla u|^2+|\partial_t u|^2\right)dx. \tag{2.1}$$

*If $E_{\varphi}(u)(0)<\infty$, then*

$$\sqrt{\varphi}\nabla u\in L^{\infty}_{loc}\left(\mathbb{R}_+,\left(L^2(\Omega)\right)^d\right) \text{ and } \sqrt{\varphi}\partial_t u\in L^{\infty}_{loc}\left(\mathbb{R}_+,L^2(\Omega)\right). \tag{2.2}$$

*Moreover, we have*

$$\begin{aligned} &E_{\varphi}(u)(t+T)+\int_t^{t+T}\int_{\Omega}a(x)\varphi(s,x)|\partial_t u|^{r+1}dxds \\ &\leq E_{\varphi}(u)(t)+\tfrac{\alpha+\eta}{2}\int_t^{t+T}\int_{\Omega}\left|\varphi'(s,x)\right|\left(|\nabla u(s)|^2+|\partial_t u(s)|^2\right)dxds, \end{aligned} \tag{2.3}$$

*for every $t\geq 0$ and $T\geq 0$, where $\varphi^{(j)}(t,x)=\varphi^{(j)}(\eta q(x)+\alpha t)$, for $j=0,1,2$ and $\alpha,\eta\geq 0$.*

**Proof.** We remind that

$$u\in L^{\infty}_{loc}\left(\mathbb{R}_+,H^1_0(\Omega)\cap H^2(\Omega)\right)\cap W^{1,\infty}\left(\mathbb{R}_+,H^1_0(\Omega)\right)\cap W^{2,\infty}\left(\mathbb{R}_+,L^2(\Omega)\right).$$

To prove (2.2) we use the Yosida approximation of the nonlinearity to obtain some energy estimates. Then using the fact that $\varphi'$ is in $L^{\infty}(\mathbb{R}_+)$ and $\varphi''(1+\varphi)^{-1/2}\in L^{\infty}(\mathbb{R}_+)$, we obtain (2.2) for the approximated solution. Then using classical method (see, for example, [16]) we show that (2.2) holds for the solution $u$.

Now we will prove the energy estimate (2.3). Let $R\gg 1$ and set $S(R)=\partial B_R$. It is easy to see that

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\int_{\Omega\cap B_R}\varphi\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx+\int_{\Omega\cap B_R}a(x)\varphi|\partial_t u(t)|^{r+1}dx \\ &=\tfrac{\alpha}{2}\int_{\Omega\cap B_R}\varphi'\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx+\int_{\Omega\cap B_R}\varphi\nabla u(t)\cdot\nabla\partial_t u(t)+\varphi\partial_t u(t)\,\partial_t^2 u(t)\,dx \\ &+\int_{\Omega\cap B_R}a(x)\varphi|\partial_t u(t)|^{r+1}dx \\ &=\tfrac{\alpha}{2}\int_{\Omega\cap B_R}\varphi'\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx+\int_{\Omega\cap B_R}\nabla u(t)\cdot\nabla(\varphi\partial_t u(t))+\varphi\partial_t u(t)\,\partial_t^2 u(t)\,dx \\ &+\int_{\Omega\cap B_R}a(x)\varphi|\partial_t u(t)|^{r+1}dx-\eta\int_{\Omega\cap B_R}\varphi'\tfrac{x\cdot\nabla u(t)}{q(x)}\partial_t u(t)\,dx. \end{aligned}$$

Green's formula along with the fact that $u$ is a solution of (1.1),

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_R}\varphi\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx+\int_{\Omega\cap B_R}a(x)\varphi|\partial_t u(t)|^{r+1}dx$$
$$=\frac{\alpha}{2}\int_{\Omega\cap B_R}\varphi'\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx-\eta\int_{\Omega\cap B_R}\varphi'\frac{x\cdot\nabla u(t)}{q(x)}\partial_t u(t)\,dx$$
$$+\int_{S(R)}\varphi\frac{x\cdot\nabla u(t)}{R}\partial_t u(t)\,dS.$$

Integrating the estimate above between $t$ and $t+T$, we obtain

$$\int_{\Omega\cap B_R}\varphi\left(|\nabla u(t+T)|^2+|\partial_t u(t+T)|^2\right)dx+\int_t^{t+T}\int_{\Omega\cap B_R}a(x)\varphi|\partial_t u|^{r+1}dxds$$
$$\leq E_\varphi(u)(t)+\frac{\alpha}{2}\int_t^{t+T}\int_\Omega|\varphi'|\left(|\nabla u(s)|^2+|\partial_t u(s)|^2\right)dxds \tag{2.4}$$
$$+\eta\int_t^{t+T}\int_\Omega\left|\varphi'\frac{x\cdot\nabla u(s)}{q(x)}\partial_t u(s)\right|dxds+\int_t^{t+T}\int_{S(R)}\varphi\left|\frac{x\cdot\nabla u(s)}{R}\partial_t u(s)\right|dSds.$$

Using Young's inequality

$$\int_t^{t+T}\int_{S(R)}\varphi\left|\frac{x\cdot\nabla u}{R}\partial_t u\right|dSd\tau\leq\frac{1}{2}\int_t^{t+T}\int_{S(R)}\left(|\partial_r u|^2+|\partial_t u|^2\right)\varphi dSd\tau.$$

Moreover, using (2.2), we infer that

$$\liminf_{R\longrightarrow+\infty}\int_t^{t+T}\int_{S(R)}\varphi\left|\frac{x\cdot\nabla u}{R}\partial_t u\right|dSd\tau=0.$$

Passing to the limit in (2.4), we get

$$E_\varphi(u)(t+T)+\int_t^{t+T}\int_\Omega a(x)\varphi|\partial_t u|^{r+1}dxds\leq E_\varphi(u)(t)$$
$$+\frac{\alpha}{2}\int_t^{t+T}\int_\Omega|\varphi'|\left(|\nabla u(s)|^2+|\partial_t u(s)|^2\right)dxds+\eta\int_t^{t+T}\int_\Omega\left|\varphi'\frac{x\cdot\nabla u}{q(x)}\partial_t u(s)\right|dxds.$$

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Young's inequality gives

$$E_\varphi(u)(t+T) + \int_t^{t+T} \int_\Omega a(x)\,\varphi\,|\partial_t u|^{r+1}\,dxds$$
$$\leq E_\varphi(u)(t) + \frac{\alpha+\eta}{2} \int_t^{t+T} \int_\Omega |\varphi'| \left( |\nabla u(s)|^2 + |\partial_t u(s)|^2 \right) dxds. \qquad \square$$

**Remark 4.** We note that the result above remains valid for

$$\begin{aligned} r \geq 1 \quad & \text{if } d = 1, 2 \\ 1 \leq r \leq \tfrac{d}{d-2} \quad & \text{if } d \geq 3. \end{aligned}$$

The proofs of our results need a weighted observability estimate for the local energy and to show such result we need to prove a unique continuation result for the wave equation.

**Lemma 1.** *We assume that Hyp A holds and $(\omega, T)$ geometrically controls $\Omega$. Then the only solution of the system*

$$\begin{cases} \partial_t^2 z - \Delta z = 0 & in\ (0,T) \times \Omega, \\ z = 0 & on\ (0,T) \times \Gamma, \\ a(x)\,\partial_t z = 0 & on\ (0,T) \times \Omega, \end{cases} \tag{2.5}$$

*in the class*

$$C^0([0,T]; H_D(\Omega)) \cap C^1\left([0,T]; L^2(\Omega)\right),$$

*is the null one, where $H_D(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm*

$$\|\varphi\|_{H_D}^2 = \int_\Omega |\nabla \varphi(x)|^2\,dx.$$

**Proof.** Let $\chi \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $\chi = 1$ on $\{|x| \leq L\}$ and the support of $\chi$ is contained in $\{|x| \leq 2L\}$. First we note that $H_D(\Omega) \subset H_{loc}^1(\Omega)$. Let $z$ be a solution of the system (2.5). We set $w = \chi z$, we observe that

$$\begin{cases} \partial_t^2 w - \Delta w = -2\nabla\chi\nabla z - z\Delta\chi & \text{in } (0,T) \times \Omega \cap B_{2L}, \\ w = 0 & \text{on } (0,T) \times \Gamma \cup \{|x| = 2L\}, \\ (w_0, w_1) \in H_0^1(\Omega \cap B_{2L}) \times L^2(\Omega \cap B_{2L}) & \\ a(x)\,\partial_t w = 0 & \text{on } (0,T) \times \Omega. \end{cases}$$

From linear semi-group theory, we infer that

$$w \in C^0\left([0,T]; H_0^1(\Omega \cap B_{2L})\right) \cap C^1\left([0,T]; L^2(\Omega \cap B_{2L})\right).$$

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We set

$$v_n(t,x) = n\left(w\left(t+\frac{1}{n},x\right) - w(t,x)\right).$$

Since

$$a(x) \geq \epsilon_0 > 0 \text{ for } |x| \geq L,$$

and $\chi = 1$ on $\{|x| \leq L\}$, therefore, $v_n$ is a solution of

$$\begin{cases} \partial_t^2 v_n - \Delta v_n = 0 & \text{in } (0,T)\times\Omega\cap B_{2L} \\ v_n = 0 & \text{on } (0,T)\times\Gamma\cup\{|x| = 2L\} \\ a(x)\,\partial_t v_n = 0 & \text{on } (0,T)\times\Omega. \end{cases}$$

We have $(\omega\cap B_{2L}, T)$ geometrically controls $\Omega\cap B_{2L}$ and

$$v_n \in C^0\left([0,T]; H_0^1(\Omega\cap B_{2L})\right)\cap C^1\left([0,T]; L^2(\Omega\cap B_{2L})\right),$$

thus using the observability estimate for the wave equation in bounded domain (see e.g. [7]), we end up with

$$E_{v_n}(s) = 0, \text{ for all } s\in[0,T].$$

On the other hand,

$$v_n \underset{n\to+\infty}{\longrightarrow} \partial_t w \text{ in } D'((0,T)\times\Omega).$$

We deduce that $\partial_t w = 0$. Recalling that $\chi = 1$ on $\{|x|\leq L\}$, hence

$$\partial_t z(t,x) = 0, \text{ on } \{|x|\leq L\}.$$

Using $a(x)\,\partial_t z = 0$ on $(0,T)\times\Omega$ along with $a(x) > \epsilon_0 > 0$ for $|x| \geq L$, we infer that $\partial_t z \equiv 0$ on $[0,T]\times\Omega$. This means that $z(t,x) = z(x)$ is independent of $t$. Therefore, we have

$$\Delta z = 0 \text{ and } z\in H_D(\Omega),$$

we conclude from this that $z\equiv 0$ on $[0,T]\times\Omega$. □

In view of the fact that the energy doesn't control the $L^2$ norm of the solution, we do not expect to prove an observability estimate for the global energy and this is the essential difference with the equation in a bounded domain or the Klein–Gordon equation.

We remind that under our assumptions we have the following Poincaré inequality (see [6] and [15])

$$\|f\|_{L^2(\Omega\cap B_R)} \leq C_R\,\|\nabla f\|_{L^2(\Omega)}, \text{ for every } f\in H_D(\Omega) \text{ and } R\geq r_0. \tag{2.6}$$

Next we show a weighted observability estimate for the local energy of solutions of the system (1.1).

**Proposition 2.** *We assume that Hyp A holds and $\omega$ satisfies GCC. Let $\delta > 0$ and $R_0 \geq L$. Let $\varphi$ be a positive function in $C^2(\mathbb{R}_+)$ such that $\varphi'$ in $L^\infty(\mathbb{R}_+)$. We suppose that there exists a positive constant $K$ such that*

$$\sup_{\mathbb{R}_+} \left| \frac{\varphi''(t)}{\varphi'(t)} \right| \leq K.$$

*Moreover we assume that the function $t \longmapsto \left|\frac{\varphi'(t)}{\varphi(t)}\right|$ is monotone decreasing and $\lim_{t\to+\infty} \left|\frac{\varphi'(t)}{\varphi(t)}\right| = 0$. There exist $T, t_0 > 0$ and $C_{T,\delta} = C(T, \delta, R_0) > 0$, such that the following inequality*

$$\begin{aligned}
&\int_t^{t+T} \int_{\Omega\cap B_{R_0}} \varphi(q(x)+s)\left(|u|^2 + |\nabla u|^2 + |\partial_t u|^2\right) dxds \\
&\leq C_{T,\delta} \int_t^{t+T} \int_\Omega a(x)\varphi(q(x)+s)|\partial_t u|^2 dxds \\
&+ C_{T,\delta} \int_t^{t+T} \int_\Omega \varphi(q(x)+s)|g(s,x)|^2 dxds \\
&+ C_{T,\delta} \int_t^{t+T} \int_\Omega \frac{(\varphi'(q(x)+s))^2}{\varphi(q(x)+s)} a(x)|u|^2 dxds \\
&+ \delta \int_t^{t+T} \int_\Omega \varphi(q(x)+s)\left(|\nabla u|^2 + |\partial_t u|^2\right) dxds,
\end{aligned} \tag{2.7}$$

*holds for every*

$$g \text{ such that } \sqrt{\varphi} g \in L^2_{loc}\left(\mathbb{R}_+, L^2(\Omega)\right),$$

*for all*

$$u \in C^0\left(\mathbb{R}_+, H^1_0(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right),$$

*solution of*

$$\begin{cases} \partial_t^2 u - \Delta u = g & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(0,x) = u_0 \quad \text{and} \quad \partial_t u(0,x) = u_1, \end{cases} \tag{2.8}$$

*such that $E_\varphi(u)(0) < \infty$.*

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**Proof.** Let $T > 0$ such that $(\omega, T)$ geometrically controls $\Omega$.

To prove this result we argue by contradiction: If (2.7) were false, there would exist sequences $(t_n)$, $(g_n)$ such that $\sqrt{\varphi} g_n \in L^2_{loc}\left(\mathbb{R}_+, L^2(\Omega)\right)$ and a sequence of solutions $(u_n)$ in $C^0\left(\mathbb{R}_+, H^1_0(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right)$ with $E_\varphi(u_n)(0) < \infty$ and such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and

$$\begin{aligned} &\int_{t_n}^{t_n+T} \int_{\Omega\cap B_{R_0}} \varphi(q(x)+s)\left(|u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2\right) dx ds \\ &\geq n\left(\int_{t_n}^{t_n+T}\int_\Omega a(x)\varphi(q(x)+s)|\partial_t u_n|^2 dx ds\right) \\ &+ n\int_{t_n}^{t_n+T}\int_\Omega \varphi(q(x)+s)|g_n(s,x)|^2 dx ds \\ &+ n\left(\int_{t_n}^{t_n+T}\int_\Omega \frac{\left(\varphi'(q(x)+s)\right)^2}{\varphi(q(x)+s)} a(x)|u_n|^2 dx ds\right) \\ &+ \delta\int_{t_n}^{t_n+T}\int_\Omega \varphi(q(x)+s)\left(|\nabla u_n|^2 + |\partial_t u_n|^2\right) dx ds. \end{aligned} \tag{2.9}$$

We set

$$\sigma_n^2 = \int_{t_n}^{t_n+T}\int_{\Omega\cap B_{R_0}} \varphi(q(x)+s)\left(|u_n|^2 + |\nabla u_n|^2 + |\partial_t u_n|^2\right) dx ds$$

and $v_n(t,x) = \frac{\left(\varphi(q(x)+t_n+t)\right)^{\frac{1}{2}} u_n(t_n+t,x)}{\sigma_n}$.

From (2.9), we infer that

$$\begin{aligned} &\frac{1}{\sigma_n^2}\int_{t_n}^{t_n+T}\int_\Omega \varphi(q(x)+t)\left(|\nabla u_n(t)|^2 + |\partial_t u_n(t)|^2\right) dx dt \leq \frac{1}{\delta} \\ &\text{and} \int_{t_n}^{t_n+T}\int_{\Omega\cap B_{R_0}} |v_n(t)|^2 dx dt \leq 1, \end{aligned} \tag{2.10}$$

and

$$\frac{1}{\sigma_n^2}\int_{t_n}^{t_n+T}\int_\Omega a(x)\varphi(q(x)+s)|\partial_t u_n|^2 dx ds \underset{n\to+\infty}{\longrightarrow} 0$$

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$$\frac{1}{\sigma_n^2}\int_{t_n}^{t_n+T}\int_{\Omega}\varphi\left(q\left(x\right)+s\right)\left|g_n\left(s,x\right)\right|^2dxds\underset{n\to+\infty}{\longrightarrow}0 \tag{2.11}$$

$$\frac{1}{\sigma_n^2}\int_{t_n}^{t_n+T}\int_{\Omega}\frac{\left(\varphi'\left(q\left(x\right)+s\right)\right)^2}{\varphi\left(q\left(x\right)+s\right)}a\left(x\right)\left|u_n\right|^2dxds\underset{n\to+\infty}{\longrightarrow}0.$$

It is clear that $v_n$ is a solution of the following system

$$\begin{cases}\partial_t^2 v_n-\Delta v_n=f_n\left(t,x\right) & \text{in } \mathbb{R}_+\times\Omega,\\ v_n\left(t,x\right)=0 & \text{on } \mathbb{R}_+\times\Gamma,\\ \left(v_n\left(0\right),\partial_t v_n\left(0\right)\right)\in H_0^1\left(\Omega\right)\times L^2\left(\Omega\right), & \end{cases}$$

where

$$\begin{aligned}f_n\left(t,x\right)&=\frac{1}{2\sigma_n}\left[\left(\varphi''\left(\varphi\right)^{-\frac{1}{2}}-\frac{1}{2}\left(\varphi'\right)^2\varphi^{-3/2}\right)\frac{|x|^2}{q^2}\right]u_n\left(t_n+t\right)\\ &+\frac{1}{2\sigma_n}\left[\left(\frac{d}{q}-\frac{|x|^2}{q^3}\right)\varphi'\left(\varphi\right)^{-\frac{1}{2}}\right]u_n\left(t_n+t\right)\\ &+\frac{1}{2\sigma_n}\left[\varphi''\left(\varphi\right)^{-\frac{1}{2}}-\frac{1}{2}\left(\varphi'\right)^2\varphi^{-3/2}\right]u_n\left(t_n+t\right)-\frac{1}{\sigma_n}\varphi^{\frac{1}{2}}g_n\left(t_n+t,x\right)\\ &+\frac{\varphi'\left(\varphi\right)^{-\frac{1}{2}}}{\sigma_n}\left(\partial_t u_n\left(t_n+t\right)+\frac{x\cdot\nabla u_n\left(t_n+t\right)}{q}\right),\end{aligned}$$

where $\varphi^{(j)}\left(t,x\right)=\varphi^{(j)}\left(q\left(x\right)+t+t_n\right)$, for $j=0,1,2$. Now we will show that

$$\int_0^T\int_{\Omega}\left|f_n\left(s,x\right)\right|^2dxds\underset{n\to+\infty}{\longrightarrow}0. \tag{2.12}$$

Using (2.11) and the fact that $\lim_{t\to+\infty}\left|\frac{\varphi'(t)}{\varphi(t)}\right|=0$, we obtain

$$\begin{aligned}&\int_0^T\int_{\Omega}\left|\frac{1}{2\sigma_n}\left[\left(\varphi''\left(\varphi\right)^{-\frac{1}{2}}-\frac{1}{2}\left(\varphi'\right)^2\varphi^{-3/2}\right)\frac{|x|^2}{q^2}\right]u_n\left(t_n+t\right)\right|^2dxdt\\ &+\int_0^T\int_{\Omega}\left|\frac{1}{2\sigma_n}\left[\left(\frac{d}{q}-\frac{|x|^2}{q^3}\right)\varphi'\left(\varphi\right)^{-\frac{1}{2}}\right]u_n\left(t_n+t\right)\right|^2dxdt\\ &+\int_0^T\int_{\Omega}\left|\frac{1}{2\sigma_n}\left[\varphi''\left(\varphi\right)^{-\frac{1}{2}}-\frac{1}{2}\left(\varphi'\right)^2\varphi^{-3/2}\right]u_n\left(t_n+t\right)\right|^2dxdt\end{aligned}$$

$$\leq \frac{C}{\sigma_n^2} \int_{t_n}^{t_n+T} \int_{\Omega} \frac{\left(\varphi'(q(x)+s)\right)^2}{\varphi(q(x)+s)} \left(1 + \left(\frac{\varphi'(t_n)}{\varphi(t_n)}\right)^2\right) |u_n|^2 \, dxds$$

$$\leq \frac{C}{\sigma_n^2} \left(\frac{\varphi'(t_n)}{\varphi(t_n)}\right)^2 \int_{t_n}^{t_n+T} \int_{\Omega\cap B_L} \varphi\left(q\left(x\right)+s\right) |u_n|^2 \, dxds + \frac{C}{\epsilon_0 \sigma_n^2} \int_{t_n}^{t_n+T} \int_{\Omega} \frac{\left(\varphi'(q(x)+s)\right)^2}{\varphi(q(x)+s)} a\left(x\right) |u_n|^2 \, dxds$$

$$\leq C \left(\frac{\varphi'(t_n)}{\varphi(t_n)}\right)^2 + \frac{C}{\epsilon_0 \sigma_n^2} \int_{t_n}^{t_n+T} \int_{\Omega} \frac{\left(\varphi'(q(x)+s)\right)^2}{\varphi(q(x)+s)} a\left(x\right) |u_n|^2 \, dxds \underset{n\to+\infty}{\longrightarrow} 0.$$

Now we estimate the remaining term of $f_n$. Turn into account of (2.10), we get

$$\int_0^T \int_{\Omega} \left| \frac{\varphi'(\varphi)^{-\frac{1}{2}}}{\sigma_n} \left(\partial_t u_n\left(t_n+t\right) + \frac{x\cdot\nabla u_n(t_n+t)}{q}\right) \right|^2 dxdt$$

$$\leq \frac{C}{\sigma_n^2} \left(\frac{\varphi'(t_n)}{\varphi(t_n)}\right)^2 \int_0^T \int_{\Omega} \varphi\left(q\left(x\right)+\left(t_n+t\right)\right) \left(\left|\partial_t u_n\left(t_n+t\right)\right|^2 + \left|\nabla u_n\left(t_n+t\right)\right|^2\right) dxdt$$

$$\leq \frac{C}{\delta} \left(\frac{\varphi'(t_n)}{\varphi(t_n)}\right)^2 \underset{n\to+\infty}{\longrightarrow} 0.$$

The results above combined with (2.11), give (2.12).

The next step is to show the boundedness of the energy of $v_n$. It is easy to see that

$$\int_0^T E_{v_n}\left(t\right) dt \leq \frac{c}{\sigma_n^2} \int_{t_n}^{t_n+T} \int_{\Omega} \varphi\left(q\left(x\right)+t\right) \left(\left|\nabla u_n\left(t\right)\right|^2 + \left|\partial_t u_n\left(t\right)\right|^2\right) dxdt$$

$$+ \frac{c}{\sigma_n^2} \int_{t_n}^{t_n+T} \int_{\Omega} \frac{\left(\varphi'(q(x)+t)\right)^2}{\varphi(q(x)+t)} \left|u_n\left(t\right)\right|^2 dxdt.$$

Now using (2.10) and (2.11) we infer that there exists a positive constant $C_\delta$ such that

$$\int_0^T E_{v_n}\left(t\right) dt \leq C_\delta, \text{ for } n \text{ large enough.} \tag{2.13}$$

On the other hand, we have

$$E_{v_n}\left(t\right) \leq \frac{c}{t} \left(\int_0^T \left(E_{v_n}\left(s\right) + s\int_{\Omega} \left|f_n\left(s,x\right)\right|^2 dx\right) ds\right),$$

for all $0 < t \leq T$. Turn into account of the estimate above along with (2.13) and (2.12), we obtain

$$E_{v_n}(T) \leq C_{T,\delta}, \text{ for } n \text{ large enough.} \tag{2.14}$$

On the other hand, from the energy identity, we see that

$$E_{v_n}(t) \leq E_{v_n}(T) + \int_0^T \left( E_{v_n}(s) + \int_\Omega |f_n(s,x)|^2 \, dx \right) ds,$$

for all $0 \leq t \leq T$. The estimate above combined with (2.13) and (2.14) gives

$$\sup_{[0,T]} E_{v_n}(s) \leq C_{T,\delta}, \text{ for } n \text{ large enough.} \tag{2.15}$$

The last step is to show that

$$\int_0^T \int_\Omega a(x) |\partial_t v_n|^2 \, dx dt \underset{n \to +\infty}{\longrightarrow} 0. \tag{2.16}$$

We have

$$\int_0^T \int_\Omega a(x) |\partial_t v_n|^2 \, dx dt \leq \frac{2}{\sigma_n^2} \int_{t_n}^{t_n+T} \int_\Omega \frac{(\varphi'(q(x)+s))^2}{\varphi(q(x)+s)} a(x) |u_n(s)|^2 \, dx ds$$
$$+ \frac{2}{\sigma_n^2} \int_{t_n}^{t_n+T} \int_\Omega \varphi(q(x)+s) a(x) |\partial_t u_n|^2 \, dx ds.$$

Using (2.11), we get (2.16). For the rest of the proof we have only to argue as in [8, Proof of proposition 2] by taking into account Lemma 1. □

## 3. Proof of Theorem 1

### 3.1. Preliminary results

Throughout this section we use the following notations:
Let $\beta$ be a real number such that

$$\begin{array}{ll} \beta > -1 & \text{if } 1 < r < 1 + \frac{2}{d} \\ -1 < \beta < \frac{3-r}{r-1} & \text{if } r = 1 + \frac{2}{d}. \end{array}$$

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ 0 & \text{for } |x| \geq 2L. \end{cases}$$

Finally we set

$$\varphi(s) = \ln^{\beta+1}(b+s), \quad f(s) = \frac{\ln^{\beta}(b+s)}{b+s}, \quad f_1(s) = \frac{\ln^{\beta}(b+s)}{(b+s)^2}$$
$$\text{and } f_2(s) = \frac{\ln^{\beta-r+1}(b+s)}{(b+s)^r},$$

with

$$\ln b = \max\left((2(r+1))^{r+1}, \frac{\beta+1-r}{r-1}, (8(r+1)(\beta+1))^{r+1}\right).$$

**Lemma 2.** *We assume that Hyp A holds and* $(\omega, T)$ *geometrically controls* $\Omega$. *Let* $\beta > -1$. *Let* $\delta > 0$ *and* $R_0 > L$. *There exists* $C_{T,\delta} = C(T, \delta, R_0) > 0$, *such that the following inequality*

$$\begin{aligned}
&\int_t^{t+T}\int_{\Omega\cap B_{R_0}} f(q(x)+s)\left(|u|^2 + |\nabla u|^2 + |\partial_t u|^2\right)dxds \\
&\leq C_{T,\delta}\int_t^{t+T}\int_{\Omega} a(x) f(q(x)+s)\left(|\partial_t u|^2 + |\partial_t u|^{2r}\right)dxds \\
&+ C_{T,\delta}\int_t^{t+T}\int_{\Omega} a(x) f_1'(q(x)+s)|u|^2 dxds \\
&+ \delta\int_t^{t+T}\int_{\Omega} f(q(x)+s)\left(|\nabla u(s)|^2 + |\partial_t u(s)|^2\right)dxds,
\end{aligned} \tag{3.1}$$

*holds for every* $t \geq t_0$ *and for all* $u$ *solution of* (1.1) *with initial data* $(u_0, u_1)$ *in* $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$.

**Proof.** In view of $f \in L^\infty(\mathbb{R}_+)$, we have $E_f(u)(0) < \infty$. On the other hand, it is clear that $f' \in L^\infty(\mathbb{R}_+)$ and there exists a positive constant $K$, such that

$$\sup_{\mathbb{R}_+}\left|\frac{f''(t)}{f'(t)}\right| \leq K.$$

In addition the function $t \longmapsto \left|\frac{f'(t)}{f(t)}\right|$ is decreasing and $\lim_{t\to+\infty}\left|\frac{f'(t)}{f(t)}\right| = 0$. Moreover, there exists $C > 0$, such that

$$\frac{(f'(t))^2}{f(t)} \leq C\left(-f_1'(t)\right), \text{ for all } t \geq 0.$$

Since

$$\partial_t u \in L^\infty\left(\mathbb{R}_+, H_0^1(\Omega)\right),$$

therefore, from Sobolev imbedding, we deduce that

$$\sqrt{a(x) f(q(x)+s)} |\partial_t u|^r \in L^2_{loc}\left(\mathbb{R}_+, L^2(\Omega)\right).$$

By taking into account of the results above, we can use Proposition 2 and we obtain (3.1). This finishes the proof of the proposition. □

In order to prove Theorem 1 we need the following result.

**Lemma 3.** *Let $T > 0$ and $u$ be the solution of (1.1) with initial data in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ such that*

$$E_\varphi(u)(0) = \int_\Omega \varphi(q(x)) \left(|\nabla u_0|^2 + |u_1|^2\right) dx < \infty. \tag{3.2}$$

*We set $\chi = 1 - \psi$ and*

$$\begin{aligned} X(t) &= \int_\Omega f(q(x)+t) \chi^2(x) u(t) \partial_t u(t) dx + \tfrac{k_1}{2} \int_\Omega a(x) f_1(q(x)+t) |u(t)|^2 dx \\ &+ \int_\Omega a(x) f_2(q(x)+t) |u(t)|^{r+1} dx + \tfrac{k}{2} \int_\Omega \ln^{\beta+1}(b+q(x)+t) \left(|\nabla u|^2 + |\partial_t u|^2\right) dx, \end{aligned} \tag{3.3}$$

*where*

$$k = \frac{1}{4(\beta+1)}, \ k_1 > 0.$$

*We have*

$$\begin{aligned} & X(t+T) - X(t) + \tfrac{1}{4} \int_t^{t+T} \int_\Omega f(q(x)+s) \left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds \\ & - \left(\tfrac{k_1}{4} - \tfrac{2(1+|\beta|)}{\epsilon_0}\right) \int_t^{t+T} \int_\Omega a(x) f_1'(q(x)+s) |u|^2 dx ds \\ & - \tfrac{1}{2} \int_t^{t+T} \int_\Omega a(x) f_2'(q(x)+s) |u|^{r+1} dx ds \\ & + \tfrac{1}{8(\beta+1)} \int_t^{t+T} \int_\Omega a(x) \ln^{\beta+1}(b+q(x)+s) |\partial_t u|^{r+1} dx ds \\ & \le \left(3 + \tfrac{1}{2} \left\|\nabla \chi^2\right\|_\infty\right) \int_t^{t+T} \int_{\Omega \cap B_{2L}} f(q(x)+s) \left(|u|^2 + |\nabla u|^2 + |\partial_t u|^2\right) dx ds \end{aligned}$$

$$+2\left(\frac{1}{\epsilon_0}+\frac{4(1+|\beta|)}{\epsilon_0^2 k_1}+4k_1\right)\int_t^{t+T}\int_\Omega a(x)\, f(q(x)+t)\,|\partial_t u|^2\,dxds. \tag{3.4}$$

**Proof.** First (3.2) allows us to apply (2.3) and to obtain

$$\begin{aligned}&E_\varphi(u)(t+T)+\int_t^{t+T}\int_\Omega a(x)\,\varphi(q(x)+s)\,|\partial_t u|^{r+1}\,dxds\\&\le E_\varphi(u)(t)+(\beta+1)\int_t^{t+T}\int_\Omega f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right)dxds.\end{aligned}$$

We set

$$\begin{aligned}X_0(t)&=\int_\Omega f(q(x)+t)\,\chi^2(x)\,u(t)\,\partial_t u(t)\,dx+\frac{k_1}{2}\int_\Omega a(x)\, f_1(q(x)+t)\,|u(t)|^2\,dx\\&+\int_\Omega a(x)\, f_2(q(x)+t)\,|u(t)|^{r+1}\,dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}X_0(t)&=\int_\Omega\left(|\partial_t u(t)|^2-|\nabla u(t)|^2-a(x)\,|\partial_t u(t)|^{r-1}\,u\partial_t u(t)\right)\chi^2(x)\, f(q(x)+t)\,dx\\&-\int_\Omega \chi^2(x)\, f'(q(x)+t)\,u(t)\,\frac{x\cdot\nabla u(t)}{q(x)}+f(q(x)+t)\,\nabla\chi^2(x)\,\nabla u(t)\,dx\\&+\int_\Omega f'(q(x)+t)\,\chi^2(x)\,u(t)\,\partial_t u(t)\,dx\\&+k_1\left(\int_\Omega a(x)\, f_1(q(x)+t)\,u(t)\,\partial_t u(t)\,dx+\frac{1}{2}\int_\Omega a(x)\, f_1'(q(x)+t)\,|u(t)|^2\,dx\right)\\&+\int_\Omega a(x)\, f_2'(q(x)+t)\,|u|^{r+1}\,dx+(r+1)\int_\Omega a(x)\, f_2(q(x)+t)\,|u|^{r-1}\,u\partial_t u dx.\end{aligned} \tag{3.5}$$

A direct computation gives

$$\frac{(f'(s))^2}{f(s)}\le(1+|\beta|)\,\frac{\ln^\beta(b+s)}{(b+s)^3}\le-(1+|\beta|)\, f_1'(s)$$

and

$$\frac{(f_1(s))^2}{f(s)}=\frac{\ln^\beta(b+s)}{(b+s)^3}\le -f_1'(s).$$

We note that $\|\chi\|_\infty \le 1$. Using Young's inequality and the fact that the support of $\chi$ is contained in $\{|x| \ge L\}$ and

$$a(x) > \epsilon_0 > 0 \text{ for } |x| \ge L,$$

we deduce that

$$\left| \int_\Omega f'(q(x)+t)\,\chi^2(x)\,u(t)\,\partial_t u(t)\,dx \right|$$
$$\le -\tfrac{k_1}{8} \int_\Omega a(x)\,f_1'(q(x)+t)\,|u(t)|^2\,dx + \tfrac{8(1+|\beta|)}{\epsilon_0^2 k_1} \int_\Omega a(x)\,f(q(x)+t)\,|\partial_t u(t)|^2\,dx,$$

and

$$\left| k_1 \int_\Omega a(x)\,f_1(q(x)+t)\,u(t)\,\partial_t u(t)\,dx \right|$$
$$\le -\tfrac{k_1}{8} \int_\Omega a(x)\,f_1'(q(x)+t)\,|u(t)|^2\,dx + 8k_1 \int_\Omega a(x)\,f(q(x)+t)\,|\partial_t u(t)|^2\,dx.$$

Using the same arguments we also deduce that

$$\int_\Omega \chi^2(x)\,f'(q(x)+t)\,u(t)\,\tfrac{x\cdot\nabla u(t)}{q(x)}\,dx$$
$$\le \tfrac{1}{2} \int_\Omega f(q(x)+t)\,|\nabla u(t)|^2\,dx - \tfrac{2(1+|\beta|)}{\epsilon_0} \int_\Omega a(x)\,f_1'(q(x)+t)\,|u(t)|^2\,dx.$$

Since the support of $\psi$ is contained in $\{|x| \le 2L\}$ and

$$a(x) > \epsilon_0 \text{ for } |x| \ge L,$$

therefore we see that

$$\int_\Omega \left(|\partial_t u(t)|^2 - |\nabla u(t)|^2\right) \chi^2(x)\,f(q(x)+t)\,dx$$
$$= \int_\Omega f(q(x)+t)\left(1 - 2\psi(x) + \psi^2(x)\right)\left(|\partial_t u(t)|^2 - |\nabla u(t)|^2\right) dx$$
$$\le \tfrac{2}{\epsilon_0} \int_\Omega a(x)\,f(q(x)+t)\,|\partial_t u(t)|^2\,dx$$

$$-\int_{\Omega} f\left(q\left(x\right)+t\right)\left(\left|\partial_{t}u\left(t\right)\right|^{2}+\left|\nabla u\left(t\right)\right|^{2}\right)dx$$
$$+3\int_{\Omega\cap B_{2L}} f\left(q\left(x\right)+t\right)\left(\left|\partial_{t}u\left(t\right)\right|^{2}+\left|\nabla u\left(t\right)\right|^{2}\right)dx.$$

We note that the support of $\nabla\chi^2$ is contained in $\{|x|\leq 2L\}$, using Young's inequality, we deduce that

$$\left|-\int_{\Omega} f\left(q\left(x\right)+t\right)u\left(t\right)\nabla\chi^{2}\left(x\right)\nabla u\left(t\right)dx\right|$$
$$\leq\frac{1}{2}\left\|\nabla\chi^{2}\right\|_{\infty}\int_{\Omega\cap B_{2L}} f\left(q\left(x\right)+t\right)\left(\left|u\left(t\right)\right|^{2}+\left|\nabla u\left(t\right)\right|^{2}\right)dx.$$

Since

$$\ln b\geq\frac{\beta+1-r}{r-1},$$

therefore a direct computation gives

$$\begin{aligned}
&-f_{2}\left(s\right)\geq\tfrac{\ln^{\beta-r+1}(b+s)}{(b+s)^{r+1}}\\
&\left(f\left(s\right)\right)^{r+1}\ln^{-r\left(\beta+1\right)}\left(b+s\right)\leq\tfrac{-f_{2}'(s)}{\ln(b+s)}\\
&\left(f_{2}\left(s\right)\right)^{\frac{r+1}{r}}\ln^{-\frac{\beta+1}{r}}\left(b+s\right)\leq\tfrac{-f_{2}'(s)}{\ln(b+s)}.
\end{aligned}$$

Now we can estimate the last term of the RHS of (3.5). Hölder's inequality along with Young's inequality, leads to

$$\int_{\Omega} a\left(x\right)f\left(q\left(x\right)+s\right)\left|\partial_{t}u\left(t\right)\right|^{r-1}u\partial_{t}u\,dx$$
$$\leq\left(\ln b\right)^{-\frac{1}{r+1}}\left(\int_{\Omega} a\left(x\right)\ln^{\beta+1}\left(b+q\left(x\right)+s\right)\left|\partial_{t}u\right|^{r+1}dx\right)^{\frac{r}{r+1}}$$
$$\times\left(-\int_{\Omega} a\left(x\right)f_{2}'\left(q\left(x\right)+s\right)\left|u\right|^{r+1}dx\right)^{\frac{1}{r+1}}$$
$$\leq\left(\ln b\right)^{-\frac{1}{r+1}}\int_{\Omega} a\left(x\right)\ln^{\beta+1}\left(b+q\left(x\right)+s\right)\left|\partial_{t}u\right|^{r+1}dx$$
$$-\left(\ln b\right)^{-\frac{1}{r+1}}\int_{\Omega} a\left(x\right)f_{2}'\left(q\left(x\right)+s\right)\left|u\right|^{r+1}dx,$$

and

$$
\begin{aligned}
&(r+1)\int\limits_{\Omega} a(x)\, f_2(q(x)+s)\,|u|^{r-1}\, u\partial_t u dx \\
&\leq (r+1)(\ln b)^{-\frac{r}{r+1}} \left(\int\limits_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dx\right)^{\frac{1}{r+1}} \\
&\times \left(-\int\limits_{\Omega} a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\, dx\right)^{\frac{r}{r+1}} \\
&\leq (\ln b)^{-\frac{1}{r+1}} \int\limits_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dx \\
&- r(\ln b)^{-\frac{1}{r+1}} \int\limits_{\Omega} a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\, dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int\limits_t^{t+T}\int\limits_{\Omega} a(x)\, f(q(x)+s)\,|\partial_t u|^{r-1}\, u\partial_t u dx ds \\
&+(r+1)\int\limits_t^{t+T}\int\limits_{\Omega} a(x)\, f_2(q(x)+s)\,|u|^{r-1}\, u\partial_t u dx ds \\
&\leq (r+1)(\ln b)^{-\frac{1}{r+1}} \int\limits_t^{t+T}\int\limits_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dx ds \\
&-(r+1)(\ln b)^{-\frac{1}{r+1}} \int\limits_t^{t+T}\int\limits_{\Omega} a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\, dx ds.
\end{aligned}
$$

Collecting the inequalities above, making some arrangement in (3.5) and integrating the result between $t$ and $t+T$, we end up with

$$
\begin{aligned}
&X(t+T)-X(t)+\left(\tfrac{1}{2}-(1+\beta)k\right)\int\limits_t^{t+T}\int\limits_{\Omega} f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right) dx ds \\
&-\left(\tfrac{k_1}{4}-\tfrac{2(1+|\beta|)}{\epsilon_0}\right)\int\limits_t^{t+T}\int\limits_{\Omega} a(x)\, f_1'(q(x)+s)\,|u(s)|^2\, dx ds
\end{aligned}
$$

$$-\left(1-(r+1)(\ln b)^{-\frac{1}{r+1}}\right)\int_t^{t+T}\int_\Omega a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\,dxds$$

$$+\left(k-(r+1)(\ln b)^{-\frac{1}{r+1}}\right)\int_t^{t+T}\int_\Omega a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\,dxds$$

$$\leq\left(3+\tfrac{1}{2}\left\|\nabla\chi^2\right\|_\infty\right)\left(\int_t^{t+T}\int_{\Omega\cap B_{2L}} f(q(x)+s)\left(|u|^2+|\nabla u|^2+|\partial_t u|^2\right)dxds\right)$$

$$+\left(\tfrac{2}{\epsilon_0}+\tfrac{8(1+|\beta|)}{\epsilon_0^2 k_1}+8k_1\right)\int_t^{t+T}\int_\Omega a(x)\, f(q(x)+s)\,|\partial_t u|^2\,dxds.$$

Using the fact that $k=\frac{1}{4(\beta+1)}$ and

$$\ln b\geq\max\left((2(r+1))^{r+1},(8(r+1)(\beta+1))^{r+1}\right),$$

we obtain (3.4). □

### 3.2. Proof of Theorem 1

We assume that Hyp A holds and $\omega$ satisfies the GCC. We set $\gamma=\beta+1$. Let $u$ be a solution of (1.1) with initial data in $H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)$ such that

$$E_\varphi(u)(0)=\int_\Omega \ln^{\beta+1}(1+q(x))\left(|\nabla u_0|^2+|u_1|^2\right)dx<\infty.$$

First we note that there exists a positive constant $c$ such that

$$\int_\Omega \ln^{\beta+1}(b+q(x))\left(|\nabla u_0|^2+|u_1|^2\right)dx\leq cE_\varphi(u)(0).$$

Let $T,t_0>0$ such that the observability estimate (3.1) holds. First we estimate the first term of the RHS of (3.4). Using the observability estimate (3.1), we see that

$$X(t+T)-X(t)$$

$$+\left(\tfrac{1}{4}-\left(3+\left\|\nabla\chi^2\right\|_\infty\right)\delta\right)\int_t^{t+T}\int_\Omega f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right)dxds$$

$$-\left(\tfrac{k_1}{4}-\tfrac{2(1+|\beta|)}{\epsilon_0}-\left(3+\left\|\nabla\chi^2\right\|_\infty\right)C_{T,\delta}\right)\int_t^{t+T}\int_\Omega a(x)\, f_1'(q(x)+s)\,|u|^2\,dxds$$

$$
\begin{aligned}
& -\frac{1}{2}\int_t^{t+T}\int_\Omega a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\,dxds \\
& +\frac{1}{8(\beta+1)}\int_t^{t+T}\int_\Omega a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\,dxds \\
& \leq k_3\int_t^{t+T}\int_\Omega a(x)\, f(q(x)+s)\left(|\partial_t u|^2+|\partial_t u|^{2r}\right)dxds,
\end{aligned}
\tag{3.6}
$$

for every $t \geq 0$, where $k_3 = 2\left(\frac{1}{\epsilon_0}+\frac{4(1+|\beta|)}{\epsilon_0^2 k_1}+4k_1+2\left(3+\left\|\nabla\chi^2\right\|_\infty\right)C_{T,\delta}\right)$.

On the other hand, using Young's inequality we get

$$
\begin{aligned}
X(t) \leq & \left(\frac{k_1}{2}+\frac{1}{\epsilon_0\epsilon}\right)\int_\Omega a(x)\, f_1(q(x)+t)\,|u(t)|^2\,dx \\
& +(k+\epsilon)\int_\Omega \ln^{\beta+1}(b+q(x)+t)\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx \\
& +\int_\Omega a(x)\, f_2(q(x)+t)\,|u(t)|^{r+1}\,dx
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
X(t) \geq & \left(\frac{k_1}{2}-\frac{1}{\epsilon_0\epsilon}\right)\int_\Omega a(x)\, f_1(q(x)+t)\,|u(t)|^2\,dx \\
& +(k-\epsilon)\int_\Omega \ln^{\beta+1}(b+q(x)+t)\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx \\
& +\int_\Omega a(x)\, f_2(q(x)+t)\,|u(t)|^{r+1}\,dx,
\end{aligned}
\tag{3.8}
$$

for all $\epsilon > 0$. We choose (by taking into account of the order below)

$\delta$ such that $\frac{1}{4}-\left(3+\left\|\nabla\chi^2\right\|_\infty\right)\delta = \frac{1}{8}$,

$\epsilon$ such that $k-\epsilon \geq \frac{1}{16(\beta+1)}$,

$k_1$ such that $\frac{k_1}{2}-\frac{1}{\epsilon_0\epsilon} \geq 1$ and $\frac{k_1}{4}-\frac{2(1+|\beta|)}{\epsilon_0}-\left(3+\left\|\nabla\chi^2\right\|_\infty\right)C_{T,\delta} \geq 1$.

Therefore

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$$X(t) \geq \int_\Omega a(x) f_1(q(x)+t)|u(t)|^2 dx + \frac{1}{16(\beta+1)} \int_\Omega \ln^{\beta+1}(b+q(x)+t)\left(|\nabla u(t)|^2 + |\partial_t u(t)|^2\right) dx + \int_\Omega a(x) f_2(q(x)+t)|u(t)|^{r+1} dx, \tag{3.9}$$

and

$$\begin{aligned} &X(t+T) - X(t) + \frac{1}{8} \int_t^{t+T} \int_\Omega f(q(x)+s)\left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds \\ &- \int_t^{t+T} \int_\Omega a(x) f_1'(q(x)+s)|u|^2 dx ds - \frac{1}{2} \int_t^{t+T} \int_\Omega a(x) f_2'(q(x)+s)|u|^{r+1} dx ds \\ &+ \frac{1}{8(\beta+1)} \int_t^{t+T} \int_\Omega a(x) \ln^{\beta+1}(b+q(x)+s)|\partial_t u|^{r+1} dx ds \\ &\leq k_3 \int_t^{t+T} \int_\Omega a(x) f(q(x)+s)\left(|\partial_t u|^2 + |\partial_t u|^{2r}\right) dx ds, \end{aligned} \tag{3.10}$$

for every $t \geq t_0$. Let $n_0$ be the ceiling of $\left(\frac{T}{t_0}\right)$. Thus

$$\begin{aligned} &X(nT) + \frac{1}{8} \int_{n_0 T}^{nT} \int_\Omega f(q(x)+s)\left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds \\ &- \int_{n_0 T}^{nT} \int_\Omega a(x) f_1'(q(x)+s)|u|^2 dx ds - \frac{1}{2} \int_{n_0 T}^{nT} \int_\Omega a(x) f_2'(q(x)+s)|u|^{r+1} dx ds \\ &+ \frac{1}{8(\beta+1)} \int_{n_0 T}^{nT} \int_\Omega a(x) \ln^{\beta+1}(b+q(x)+s)|\partial_t u|^{r+1} dx ds \\ &\leq k_3 \int_{n_0 T}^{nT} \int_\Omega a(x) f(q(x)+s)\left(|\partial_t u|^2 + |\partial_t u|^{2r}\right) dx ds + X(n_0 T), \text{ for all } n \geq n_0. \end{aligned} \tag{3.11}$$

Using Proposition 1, we deduce that there exists a positive constant $C = C(n_0, T)$

$$X(n_0 T) \leq C I_0 \tag{3.12}$$

where $I_0$ is defined in the statement of Theorem 1.

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Combining (3.11) and (3.12), we obtain

$$\begin{aligned}
&X(nT)+\tfrac{1}{8}\int_{n_0T}^{nT}\int_{\Omega} f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right)dxds\\
&-\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f_1'(q(x)+s)\,|u|^2\,dxds-\tfrac{1}{2}\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\,dxds\\
&+\tfrac{1}{8(\beta+1)}\int_{n_0T}^{nT}\int_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\,dxds\\
&\leq k_4\left(\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f(q(x)+s)\left(|\partial_t u|^2+|\partial_t u|^{2r}\right)dxds+I_0\right),
\end{aligned}\tag{3.13}$$

for all $n\geq n_0$ and for some $k_4>0$. The next step is to control the first term of the RHS of the estimate above by the last term of the LHS. We remind that

$$p=\begin{cases}2(r+1) & \text{if } d\leq 2\\ \frac{2d}{d-2} & \text{if } d\geq 3.\end{cases}$$

We have $r+1<2r<p$, using interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}
&\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f(q(x)+s)\,|\partial_t u|^{2r}\,dxds\\
&\leq \int_{n_0T}^{nT} f(s)\int_{\Omega} a(x)\,|\partial_t u|^{2r}\,dxds\\
&\leq \int_{n_0T}^{nT} f(s)\left(\int_{\Omega} a(x)\,|\partial_t u|^{r+1}\,dx\right)^{\frac{p-2r}{p-r-1}}\left(\int_{\Omega} a(x)\,|\partial_t u|^{p}\,dx\right)^{\frac{r-1}{p-r-1}}ds\\
&\leq\left(\|a\|_{L^\infty}\|\partial_t u\|^p_{L^\infty(\mathbb{R}_+,L^p(\Omega))}\int_{n_0T}^{nT}(f(s))^{\frac{p-r-1}{r-1}}(\ln(b+s))^{-\frac{(\beta+1)(p-2r)}{r-1}}ds\right)^{\frac{r-1}{p-r-1}}\\
&\times\left(\int_{n_0T}^{nT}\ln^{\beta+1}(b+s)\int_{\Omega} a(x)\,|\partial_t u|^{r+1}\,dxds\right)^{\frac{p-2r}{p-r-1}}
\end{aligned}$$

$$\leq \frac{\epsilon^{-\frac{p-2r}{r-1}}(r-1)\|a\|_{L^\infty}\|\partial_t u\|^p_{L^\infty(\mathbb{R}_+,L^p(\Omega))}}{p-r-1}\int_0^{+\infty}(b+s)^{-\frac{p-r-1}{r-1}}(\ln(b+s))^{\beta-\frac{p-2r}{r-1}}ds$$

$$+\frac{\epsilon(p-2r)}{p-r-1}\int_{n_0T}^{nT}\int_\Omega a(x)\ln^{\beta+1}(b+q(x)+s)|\partial_t u|^{r+1}dxds,$$

for all $\epsilon > 0$. Thus using (1.3) and Sobolev imbedding $H^1 \hookrightarrow L^p$, we get

$$\begin{aligned}&\int_{n_0T}^{nT}\int_\Omega a(x)f(q(x)+s)|\partial_t u|^{2r}dxds\\&\leq C\|a\|_{L^\infty}\epsilon^{-\frac{p-2r}{r-1}}\left(\|u_0\|^2_{H^2}+\|u_1\|^2_{H^1}+\|u_1\|^{2r}_{H^1}\right)^{\frac{p}{2}}\\&+\frac{\epsilon(p-2r)}{p-r-1}\int_{n_0T}^{nT}\int_\Omega a(x)(\ln(b+q(x)+s))^{\beta+1}|\partial_t u|^{r+1}dxds,\end{aligned}\tag{3.14}$$

for all $\epsilon > 0$. To estimate the last term, first we use Hölder's inequality

$$\int_{n_0T}^{nT}\int_\Omega a(x)f(q(x)+s)|\partial_t u|^2dxds$$

$$\leq\left(\|a\|_{L^\infty}\int_{n_0T}^{nT}\int_\Omega(f(q(x)+s))^{\frac{r+1}{r-1}}\ln^{-\frac{2(\beta+1)}{r-1}}(b+q(x)+s)dxds\right)^{\frac{r-1}{r+1}}$$

$$\times\left(\int_{n_0T}^{nT}\int_\Omega a(x)\ln^{\beta+1}(b+q(x)+s)|\partial_t u|^{r+1}dxds\right)^{\frac{2}{r+1}}$$

$$\leq\left(\|a\|_{L^\infty}\int_0^{+\infty}\int_\Omega(b+q(x)+s)^{-\frac{r+1}{r-1}}\ln^{\beta-\frac{2}{r-1}}(b+q(x)+s)dxds\right)^{\frac{r-1}{r+1}}$$

$$\times\left(\int_{n_0T}^{nT}\int_\Omega a(x)\ln^{\beta+1}(b+q(x)+s)|\partial_t u|^{r+1}dxds\right)^{\frac{2}{r+1}}.$$

By Young's inequality, we end up with

$$\begin{aligned}
&\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f(q(x)+s)\, |\partial_t u|^2\, dxds \\
&\leq \frac{(r-1)\epsilon^{-\frac{2}{r-1}}\|a\|_{L^\infty}}{r+1}\int_0^{+\infty}\int_{\Omega}(b+q(x)+s)^{-\frac{r+1}{r-1}}\ln^{\beta-\frac{2}{r-1}}(b+q(x)+s)\, dxds \\
&+\frac{2\epsilon}{r+1}\int_0^{nT}\int_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dxds \\
&\leq C\|a\|_{L^\infty}\frac{(r-1)\epsilon^{-\frac{2}{r-1}}}{r+1}\int_0^{+\infty}\int_0^{+\infty}\ln^{\beta-\frac{2}{r-1}}(b+y+s)\,(b+y+s)^{-\frac{r+1}{r-1}+d-1}\, dyds \\
&+\frac{2\epsilon}{r+1}\int_{n_0T}^{nT}\int_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dxds,
\end{aligned}$$

for all $\epsilon > 0$. In view of the fact that

$$\begin{aligned}
&-\tfrac{r+1}{r-1}+d<-1 && \text{if } 1<r<1+\tfrac{2}{d} \\
&\beta-\tfrac{2}{r-1}<-1 \text{ and } -\tfrac{r+1}{r-1}+d=-1 && \text{if } r=1+\tfrac{2}{d},
\end{aligned} \tag{3.15}$$

we see that

$$\begin{aligned}
&\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f(q(x)+s)\,|\partial_t u|^2\, dxds \\
&\leq C\epsilon^{-\frac{2}{r-1}}\|a\|_{L^\infty}+\frac{2\epsilon}{r+1}\int_{n_0T}^{nT}\int_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\,|\partial_t u|^{r+1}\, dxds,
\end{aligned} \tag{3.16}$$

for all $\epsilon > 0$. We choose $\epsilon$ such that

$$\frac{1}{8(\beta+1)}-k_4\epsilon\left(\frac{p-2r}{p-r-1}+\frac{2}{r+1}\right)\geq\frac{1}{16(\beta+1)}.$$

We conclude that there exists a positive constant $C_1$ such that

$$\begin{aligned}
&X(nT)+\frac{1}{8}\int_{n_0T}^{nT}\int_{\Omega} f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right)dxds \\
&-\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f_1'(q(x)+s)\,|u|^2\, dxds-\frac{1}{2}\int_{n_0T}^{nT}\int_{\Omega} a(x)\, f_2'(q(x)+s)\,|u|^{r+1}\, dxds
\end{aligned}$$

$$+\frac{1}{16(\beta+1)}\int\limits_{n_0T}^{nT}\int\limits_{\Omega} a(x)\ln^{\beta+1}(b+q(x)+s)\left|\partial_t u\right|^{r+1} dxds \leq C_1 I_0,$$

for all natural numbers $n \geq n_0$. Therefore we obtain

$$\int\limits_0^\infty\int\limits_\Omega f(q(x)+s)\left(|\nabla u|^2+|\partial_t u|^2\right)dxds \leq C_1 I_0.$$

Now using the weighted energy estimate (2.3), we infer that

$$\begin{aligned}E_\varphi(u)(t) &= \int\limits_\Omega \varphi(q(x)+s)\left(|\nabla u(s)|^2+|\partial_t u(s)|^2\right)\\ &\leq E_\varphi(u)(0)+(\beta+1)\int\limits_0^\infty\int\limits_\Omega f(q(x)+s)\left(|\nabla u(s)|^2+|\partial_t u(s)|^2\right)dxds\\ &\leq C_0 I_0,\end{aligned}$$

for some positive constant $C_0$. The sought estimate follows from the estimate above and the fact that

$$\ln^{\beta+1}(2+t)\,E_u(t) \leq E_\varphi(u)(t).$$

## 4. Proof of Theorem 2

### 4.1. Preliminary results

Throughout this section we use the following notations: We set

$$\tau(r,\lambda)=\frac{r\delta_0^{r-1}(\lambda+1)^{r-1}(r+1)^r}{1+\delta_0^{r-1}(\lambda+1)^{r-1}(r+1)^r\left(r\delta_0^{\frac{r-1}{r}}(\lambda+1)(r+1)+1\right)},$$

$\lambda$ is any positive constant and

$$\delta_0=(\lambda+1)^{\frac{r^2}{r^2-1}}(r+1)^{-\frac{r}{r-1}}.$$

We take

$$0<\gamma=1+\beta<\min\left(\frac{r}{r+2},\frac{d+2-dr}{r-1}\right),$$

and

$$k=(1+\lambda)(r+1)\delta_0.$$

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We set $\varphi(s) = (1+\alpha s)^{\beta+1}$ where

$$\alpha(r,\lambda) = \frac{r\delta_0^{\frac{r^2-1}{r}}(1+\lambda)^r(r+1)^{r+1}+1}{\delta_0^r(r-\tau)(1+\lambda)^r(r+1)^{r+1}}.$$

Finally, let $\psi \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $0 \le \psi \le 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \le L \\ 0 & \text{for } |x| \ge 2L. \end{cases}$$

In order to obtain an explicit decay rate we need the following result.

**Lemma 4.**

(1) *We have*

$$\min\left(\frac{r}{r+2}, \frac{d+2-dr}{r-1}\right) = \min\left(\frac{r}{r+2}, \frac{d+2-dr}{r-1}, \frac{p-2r}{r-1}\right).$$

(2) *There exists $\lambda$ positive and close to zero such that $\gamma < \tau(r,\lambda)$.*
(3) *For all $\lambda$ positive and close to zero and $r \in (1,3]$,*

$$\alpha(r,\lambda) \ge 1 \text{ and } \lim_{r\to 1}\alpha(r,\lambda) = \infty.$$

**Proof.**

(1) For $d = 1, 2$, it easy to see that $\frac{d+2-dr}{r-1} \le \frac{2}{r-1} = \frac{p-2r}{r-1}$. When $d \ge 3$, we have

$$\frac{d+2-dr}{r-1} \le \frac{p-2r}{r-1} \quad \text{if } r \ge 1 + \frac{2(d-4)}{(d-2)^2}$$
$$\frac{d+2-dr}{r-1} > \frac{p-2r}{r-1} \quad \text{if } r < 1 + \frac{2(d-4)}{(d-2)^2}.$$

And the result follows from the fact that, for $1 < r < 1 + \frac{2(d-4)}{(d-2)^2}$, we have $\frac{r}{r+2} \le \frac{p-2r}{r-1}$.
(2) We have $\lim_{\lambda\to 0}\tau(r,\lambda) = \frac{r}{r+2}$. Then if $\gamma < \frac{r}{r+2}$, then there exists $\lambda$ positive and close to zero such that $\gamma < \tau(r,\lambda)$.
(3) A direct computation gives

$$\alpha(r,\lambda) = \frac{1}{r}(\lambda+1)^{\frac{r-2r^3}{r^2-1}}(r+1)^{\frac{1}{r-1}}\left((\lambda+1)^{\frac{2r^2-1}{r+1}} + r(\lambda+1)^{2r} + 1\right).$$

For $\lambda$ positive and close to zero, we have $\lim_{r\to 1}\alpha(r,\lambda) = \infty$ and $\alpha(r,\lambda) \ge 1$. □

**Proposition 3.** *We assume that Hyp A holds and $(\omega, T)$ geometrically controls $\Omega$. Let $\delta > 0$, $R_0 > L$ and $-1 < \beta \le 0$. There exists $C_{T,\delta} = C(T,\delta,R_0,\alpha,\beta) > 0$ and $t_0 > 0$, such that the following inequality*

$$\begin{aligned}
&\int_t^{t+T} \int_{\Omega\cap B_{R_0}} (1+\alpha(q(x)+s))^\beta \left(|u|^2+|\nabla u|^2+|\partial_t u|^2\right) dxds \\
&\le C_{T,\delta} \int_t^{t+T} \int_\Omega a(x)(1+\alpha(q(x)+s))^\beta \left(|\partial_t u|^2+|\partial_t u|^{2r}\right) dxds \\
&+ C_{T,\delta} \int_t^{t+T} \int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta-2} |u|^2 dxds \\
&+ \delta \int_t^{t+T} \int_\Omega (1+\alpha(q(x)+s))^\beta \left(|\nabla u|^2+|\partial_t u|^2\right) dxds,
\end{aligned} \tag{4.1}$$

*holds for every $t \ge t_0$ and for all $u$ solution of (1.1) with initial data $(u_0, u_1)$ in* $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$.

**Proof.** We set

$$f(s) = (1+\alpha s)^\beta .$$

In view of $f \in L^\infty(\mathbb{R}_+)$, we have $E_f(u)(0) < \infty$. On the other hand, it is clear that $f' \in L^\infty(\mathbb{R}_+)$ and there exists a positive constant $K$, such that

$$\sup_{\mathbb{R}_+} \left|\frac{f''(t)}{f'(t)}\right| \le K.$$

In addition the function $t \longmapsto \left|\frac{f'(t)}{f(t)}\right|$ is decreasing and $\lim_{t\to+\infty} \left|\frac{f'(t)}{f(t)}\right| = 0$. Moreover, there exists $C > 0$, such that

$$\frac{(f'(t))^2}{f(t)} \le C\left(-f'(t)\right), \text{ for all } t \ge 0.$$

Since

$$\partial_t u \in L^\infty\left(\mathbb{R}_+, H_0^1(\Omega)\right),$$

then from Sobolev imbedding, we deduce that

$$\sqrt{a(x)(1+\alpha(q(x)+s))^\beta}\, |\partial_t u|^r \in L^2_{loc}\left(\mathbb{R}_+, L^2(\Omega)\right).$$

By taking into account of the results above, we can use Proposition 2 and we obtain (4.1). This finishes the proof of the proposition. □

In order to prove Theorem 2 we need the following result.

**Lemma 5.** *Let $u$ be a solution of (1.1) with initial data in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ such that*

$$E_\varphi(u)(0) = \left\| (1+\alpha q)^{\frac{1+\beta}{2}} \nabla u_0 \right\|_{L^2}^2 + \left\| (1+\alpha q)^{\frac{1+\beta}{2}} u_1 \right\|_{L^2}^2 < +\infty.$$

*We set $\chi = 1 - \psi$ and*

$$\begin{aligned} X(t) &= \int_\Omega (1+\alpha(q(x)+t))^\beta \chi^2(x) u(t) \partial_t u(t)\, dx \\ &+ \tfrac{k_1}{2} \int_\Omega (1+\alpha(q(x)+t))^{\beta-1} a(x) |u(t)|^2 dx \\ &+ \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r+1} |u(t)|^{r+1} dx \\ &+ \tfrac{k}{2} \int_\Omega (1+\alpha(q(x)+t))^{\beta+1} \left(|\nabla u|^2 + |\partial_t u|^2\right) dx, \end{aligned} \tag{4.2}$$

*where $k_1 > 0$. Then*

$$\begin{aligned} &X(t+T) - X(t) + \tfrac{1-k\alpha(1+\beta)}{2} \int_t^{t+T} \int_\Omega (1+\alpha(q(x)+s))^\beta \left(|\nabla u|^2 + |\partial_t u|^2\right) dxds \\ &+ \left(\tfrac{k_1\alpha(1-\beta)}{4} - \tfrac{\beta^2\alpha^2}{\epsilon_0\epsilon}\right) \int_t^{t+T} \int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta-2} |u(t)|^2 dxds \\ &+ \lambda\delta_0 \int_t^{t+T} \int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta+1} |\partial_t u|^{r+1} dxds \\ &\leq \left(3 + \left\|\nabla\chi^2\right\|_\infty\right) \int_t^{t+T} \int_{\Omega\cap B_{2L}} (1+\alpha(q(x)+s))^\beta \left(|u|^2 + |\nabla u|^2 + |\partial_t u|^2\right) dxds \\ &+ \left(\tfrac{2}{\epsilon_0} + \tfrac{8k_1}{\alpha(1-\beta)} + \tfrac{8\beta^2\alpha}{\epsilon_0^2 k_1(1-\beta)}\right) \int_t^{t+T} \int_\Omega a(x)(1+\alpha(q(x)+s))^\beta |\partial_t u|^2 dxds, \end{aligned} \tag{4.3}$$

*for all $t \geq 0$, where $\lambda$ is any positive constant.*

**Proof.** We have

$$\int_\Omega \varphi(q(x)) \left(|\nabla u_0|^2 + |u_1|^2\right) dx < \infty.$$

Then from (2.3), we infer

$$\begin{aligned}
&E_{\varphi}(u)(t+T)+\int_{t}^{t+T}\int_{\Omega} a(x)(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1}\,dxds \\
&\leq E_{\varphi}(u)(t)+(\beta+1)\alpha\int_{t}^{t+T}\int_{\Omega}(1+\alpha(q(x)+s))^{\beta}\left(|\nabla u|^2+|\partial_t u|^2\right)dxds.
\end{aligned}$$

We set

$$\begin{aligned}
X_0(t)&=\int_{\Omega}(1+\alpha(q(x)+t))^{\beta}\chi^2(x)\,u(t)\,\partial_t u(t)\,dx \\
&+\frac{k_1}{2}\int_{\Omega}(1+\alpha(q(x)+t))^{\beta-1}a(x)\,|u(t)|^2\,dx \\
&+\int_{\Omega}a(x)(1+\alpha(q(x)+t))^{\beta-r+1}|u(t)|^{r+1}\,dx.
\end{aligned}$$

Using the fact that $u$ is a solution of (1.1), we deduce that

$$\begin{aligned}
&\frac{d}{dt}X_0(t)=\int_{\Omega}\left(|\partial_t u(t)|^2-|\nabla u(t)|^2-a(x)|\partial_t u(t)|^{r-1}u(t)\,\partial_t u(t)\right) \\
&\times\chi^2(x)(1+\alpha(q(x)+t))^{\beta}\,dx \\
&-\int_{\Omega}(1+\alpha(q(x)+t))^{\beta}u(t)\,\nabla\chi^2(x)\,\nabla u(t)+\beta\alpha(1+\alpha(q(x)+t))^{\beta-1} \\
&\times\chi^2(x)\,u(t)\,\frac{x\cdot\nabla u(t)}{q(x)}dx \\
&+\beta\alpha\int_{\Omega}(1+\alpha(q(x)+t))^{\beta-1}\chi^2(x)\,u(t)\,\partial_t u(t)\,dx \\
&+k_1\left(\int_{\Omega}a(x)(1+\alpha(q(x)+t))^{\beta-1}u(t)\,\partial_t u(t)\,dx\right. \\
&\left.+\frac{\beta-1}{2}\alpha\int_{\Omega}a(x)(1+\alpha(q(x)+t))^{\beta-2}|u|^2\,dx\right) \\
&+(\beta+1-r)\alpha\int_{\Omega}a(x)(1+\alpha(q(x)+t))^{\beta-r}|u|^{r+1}\,dx \\
&+(r+1)\int_{\Omega}a(x)(1+\alpha(q(x)+t))^{\beta-r+1}|u|^{r-1}u\partial_t u\,dx.
\end{aligned} \tag{4.4}$$

We note that $\|\chi\|_{\infty}\leq 1$. Using Young's inequality and the fact that the support of $\chi$ is contained in $\{|x|\geq L\}$ and

$$a(x) > \epsilon_0 > 0 \text{ for } |x| \geq L,$$

we infer that

$$\begin{aligned}
&\left|\alpha\beta \int_\Omega (1+\alpha q(x)+\alpha t)^{\beta-1} \chi^2(x) u(t) \partial_t u(t)\, dx\right| \\
&\leq \tfrac{k_1\alpha(1-\beta)}{8} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-2} |u(t)|^2\, dx \\
&+ \tfrac{8\beta^2\alpha}{\epsilon_0^2 k_1(1-\beta)} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta} |\partial_t u(t)|^2\, dx
\end{aligned}$$

and

$$\begin{aligned}
&k_1\left|\int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-1} u(t) \partial_t u(t)\, dx\right| \\
&\leq \tfrac{k_1\alpha(1-\beta)}{8} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-2} |u(t)|^2\, dx \\
&+ \tfrac{8k_1}{\alpha(1-\beta)} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta} |\partial_t u(t)|^2\, dx.
\end{aligned}$$

Using the same arguments, we also deduce that

$$\begin{aligned}
&\left|\int_\Omega \beta\alpha(1+\alpha(q(x)+t))^{\beta-1} \chi^2(x) u(t) \tfrac{x\cdot\nabla u(t)}{q(x)} dx\right| \\
&\leq \tfrac{\beta^2\alpha^2}{\epsilon_0\epsilon} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-2} |u(t)|^2\, dx \\
&+\epsilon \int_\Omega (1+\alpha(q(x)+t))^{\beta} |\nabla u(t)|^2\, dx,
\end{aligned}$$

for all $\epsilon > 0$. Using the fact that the support of $\psi$ is contained in $\{|x| \leq 2L\}$ and

$$a(x) > \epsilon_0 > 0 \text{ for } |x| \geq L,$$

we get

$$\begin{aligned}
&\int_\Omega \chi^2(x)\left(|\partial_t u(t)|^2 - |\nabla u(t)|^2\right)(1+\alpha(q(x)+t))^{\beta}\, dx \\
&= \int_\Omega \left(1-2\psi(x)+\psi^2(x)\right)(1+\alpha(q(x)+t))^{\beta}\left(|\partial_t u(t)|^2 - |\nabla u(t)|^2\right) dx
\end{aligned}$$

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$$\leq \frac{2}{\epsilon_0} \int_\Omega a(x)(1+\alpha(q(x)+t))^\beta |\partial_t u(t)|^2 dx$$
$$- \int_\Omega (1+\alpha(q(x)+t))^\beta \left(|\partial_t u(t)|^2 + |\nabla u(t)|^2\right) dx$$
$$+ 3 \int_{\Omega \cap B_{2L}} (1+\alpha(q(x)+t))^\beta \left(|\partial_t u(t)|^2 + |\nabla u(t)|^2\right) dx.$$

We note that the support of $\nabla \chi^2$ is contained in $\{|x| \leq 2L\}$, using Young's inequality, we deduce that

$$\left| - \int_\Omega (1+\alpha(q(x)+t))^\beta u(t) \nabla \chi^2(x) \nabla u(t) dx \right|$$
$$\leq \frac{1}{2} \left\| \nabla \chi^2 \right\|_\infty \int_{\Omega \cap B_{2L}} (1+\alpha(q(x)+t))^\beta \left(|u(t)|^2 + |\nabla u(t)|^2\right) dx.$$

Young's inequality combined with the fact that $\|\chi\|_\infty \leq 1$, gives

$$\left| \int_\Omega a(x)(1+\alpha(q(x)+t))^\beta \chi^2(x) |\partial_t u(t)|^{r-1} u \partial_t u(t) dx \right|$$
$$\leq \frac{rk}{r+1} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta+1} |\partial_t u(t)|^{r+1} dx$$
$$+ \frac{k-r}{r+1} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r} |u(t)|^{r+1} dx$$

and

$$(r+1) \left| \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r+1} |u(t)|^{r-1} u \partial_t u(t) dx \right|$$
$$\leq \delta_0 \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta+1} |\partial_t u(t)|^{r+1} dx$$
$$+ r\delta_0^{-\frac{1}{r}} \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r} |u(t)|^{r+1} dx.$$

By taking into account of the estimates above, making some arrangement in (4.4) and integrating the result between $t$ and $t+T$, we obtain

$$X(t+T) - X(t) + (1-\epsilon-(1+\beta)k\alpha) \int_t^{t+T} \int_\Omega (1+\alpha(q(x)+s))^\beta \left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds$$

$$+\left(\tfrac{k_1\alpha(1-\beta)}{4}-\tfrac{\beta^2\alpha^2}{\epsilon_0\epsilon}\right)\int_t^{t+T}\int_\Omega a(x)\,(1+\alpha(q(x)+s))^{\beta-2}|u|^2\,dxds$$

$$+\left(\left(\alpha-\delta_0^{-\frac{1}{r}}\right)r-(\beta+1)\alpha-\tfrac{k^{-r}}{r+1}\right)\int_t^{t+T}\int_\Omega a(x)\,(1+\alpha(q(x)+s))^{\beta-r}|u|^{r+1}\,dxds$$

$$+\left(k-\tfrac{kr}{r+1}-\delta_0\right)\int_t^{t+T}\int_\Omega a(x)\,(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1}\,dxds$$

$$\leq\left(3+\left\|\nabla\chi^2\right\|_\infty\right)\int_t^{t+T}\int_{\Omega\cap B_{2L}}(1+\alpha(q(x)+s))^{\beta}\left(|u|^2+|\nabla u|^2+|\partial_t u|^2\right)dxds$$

$$+\left(\tfrac{2}{\epsilon_0}+\tfrac{8k_1}{\alpha(1-\beta)}+\tfrac{8\beta^2\alpha}{\epsilon_0^2k_1(1-\beta)}\right)\int_t^{t+T}\int_\Omega a(x)\,(1+\alpha(q(x)+s))^{\beta}|\partial_t u|^2\,dxds,$$

for all $\epsilon>0$.

We have

$$1-(\beta+1)k\alpha>1-\tau k\alpha=0.$$

So we can choose $\epsilon=\frac{1-(1+\beta)k\alpha}{2}$. It is easy to see that

$$\begin{aligned}&\left(\alpha-\delta_0^{-\frac{1}{r}}\right)r-(\beta+1)\alpha-\tfrac{k^{-r}}{r+1}\\&>\left(\alpha-\delta_0^{-\frac{1}{r}}\right)r-\tau\alpha-\tfrac{k^{-r}}{r+1}=0\end{aligned}$$

and

$$k-\tfrac{kr}{r+1}-\delta_0=\lambda\delta_0.$$

Collecting the estimates above, we get (4.3). □

### *4.2. Proof of Theorem 2*

We assume that Hyp A holds and $\omega$ satisfies the GCC. Let $u$ be a solution of (1.1) with initial data in $H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)$ such that

$$\left\|(1+q)^{\frac{\gamma}{2}}\nabla u_0\right\|_{L^2}^2+\left\|(1+q)^{\frac{\gamma}{2}}u_1\right\|_{L^2}^2<+\infty.$$

First we note that from Lemma 4 we have

$$0<\gamma=1+\beta<\min\left(\tau(r,\lambda),\tfrac{d+2-dr}{r-1},\tfrac{p-2r}{r-1}\right).$$

It is easy to see that

$$\left\|(1+\alpha q)^{\frac{\gamma}{2}} \nabla u_0\right\|_{L^2}^2 + \left\|(1+\alpha q)^{\frac{\gamma}{2}} u_1\right\|_{L^2}^2 < +\infty.$$

Then, using (4.3) and (4.1) and arguing as in the proof of Theorem 1 we obtain

$$\begin{aligned}
&X(t+T) - X(t) \\
&+ \left(\tfrac{1-k\alpha(1+\beta)}{2} - \left(3 + \left\|\nabla\chi^2\right\|_\infty\right)\delta\right) \int_t^{t+T}\int_\Omega (1+\alpha(q(x)+s))^\beta \left(|\nabla u|^2 + |\partial_t u|^2\right) dxds \\
&+ \left(\tfrac{k_1\alpha(1-\beta)}{4} - \tfrac{\beta^2\alpha^2}{\epsilon_0\epsilon} - \left(3 + \left\|\nabla\chi^2\right\|_\infty\right) C_{T,\delta}\right) \int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta-2}|u|^2 dxds \\
&+ \lambda\delta_0 \int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1} dxds \\
&\le k_2 \int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^\beta \left(|\partial_t u|^{2r} + |\partial_t u|^2\right) dxds,
\end{aligned} \tag{4.5}$$

for all $t \ge t_0$, and for some $k_2 > 0$.

Using Young's inequality we get

$$\begin{aligned}
X(t) \le &\left(\tfrac{k_1}{2} + \tfrac{1}{2\epsilon_0\epsilon}\right) \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-1}|u(t)|^2 dx \\
&+ (k+\epsilon)\int_\Omega (1+\alpha(q(x)+t))^{\beta+1}\left(|\nabla u(t)|^2 + |\partial_t u(t)|^2\right) dx \\
&+ \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r+1}|u(t)|^{r+1} dx
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
X(t) \ge &\left(\tfrac{k_1}{2} - \tfrac{1}{\epsilon_0\epsilon}\right) \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-1}|u(t)|^2 dx \\
&+ (k-\epsilon)\int_\Omega (1+\alpha(q(x)+t))^{\beta+1}\left(|\nabla u(t)|^2 + |\partial_t u(t)|^2\right) dx \\
&+ \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r+1}|u(t)|^{r+1} dx,
\end{aligned} \tag{4.7}$$

for all $\epsilon > 0$. We choose (by taking into account of the order below)

$\epsilon$ such that $k-\epsilon \geq \delta_0$
$\delta$ such that $\frac{1-k\alpha(1+\beta)}{2} - \left(3+\left\|\nabla\chi^2\right\|_\infty\right)\delta \geq \frac{1-k\alpha(1+\beta)}{4}$
$k_1$ such that $\frac{k_1}{2} - \frac{1}{2\epsilon_0\epsilon} \geq \delta_0$ and $\frac{k_1(1-\beta)}{4} - \frac{2\beta^2}{\epsilon_0\delta_0} - \left(3+\left\|\nabla\chi^2\right\|_\infty\right)C_{T,\delta} \geq \delta_0$.

Therefore

$$\begin{aligned} X(t) &\geq \delta_0 \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-1}|u(t)|^2\,dx \\ &+ \delta_0 \int_\Omega (1+\alpha(q(x)+t))^{\beta+1}\left(|\nabla u(t)|^2+|\partial_t u(t)|^2\right)dx \\ &+ \int_\Omega a(x)(1+\alpha(q(x)+t))^{\beta-r+1}|u(t)|^{r+1}\,dx, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &X(t+T)-X(t)+\tfrac{1-k\alpha(1+\beta)}{4}\int_t^{t+T}\int_\Omega (1+\alpha(q(x)+s))^\beta\left(|\nabla u|^2+|\partial_t u|^2\right)dxds \\ &+\delta_0\int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta-2}|u|^2\,dxds \\ &+\lambda\delta_0\int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1}\,dxds \\ &\leq k_2\int_t^{t+T}\int_\Omega a(x)(1+\alpha(q(x)+s))^\beta\left(|\partial_t u|^{2r}+|\partial_t u|^2\right)dxds, \end{aligned}$$

for all $t \geq t_0$. Let $n_0$ be the ceiling of $\left(\frac{T}{t_0}\right)$. Proceeding as in the proof of Theorem 1 and using the fact that

$$1+\beta < \min\left(\tfrac{d+2-dr}{r-1}, \tfrac{p-2r}{r-1}\right),$$

we obtain

$$\begin{aligned} &\int_{n_0T}^{nT}\int_\Omega a(x)(1+\alpha(q(x)+s))^\beta|\partial_t u|^{2r}\,dxds \\ &\leq C\epsilon^{-\frac{p-2r}{r-1}}\|a\|_{L^\infty}\left(\|u_0\|_{H^2}^2+\|u_1\|_{H^1}^2+\|u_1\|_{H^1}^{2r}\right)^{\frac{p}{2}} \\ &+\tfrac{\epsilon(p-2r)}{p-r-1}\int_{n_0T}^{nT}\int_\Omega a(x)(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1}\,dxds, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}&\int_{n_0T}^{nT}\int_{\Omega} a(x)(1+\alpha(q(x)+s))^{\beta}|\partial_t u|^2\,dxds\\&\le C\epsilon^{-\frac{2}{r-1}}\|a\|_{L^\infty}+\frac{2\epsilon}{r+1}\int_{n_0T}^{nT}\int_{\Omega} a(x)(1+\alpha(q(x)+s))^{\beta+1}|\partial_t u|^{r+1}\,dxds\end{aligned} \tag{4.10}$$

for all $\epsilon>0$. To finish the proof we have to proceed as in the proof of Theorem 1.

## 5. Proof of Theorem 3

### 5.1. Preliminary results

Throughout this section we use the following notations: We set

$$\tau_1(r,\lambda)=\frac{2r\delta_0^{r-1}(\lambda+1)^{r-1}(r+1)^r}{1+\delta_0^{r-1}(\lambda+1)^{r-1}(r+1)^r\left(r\delta_0^{\frac{r-1}{r}}(\lambda+1)(r+1)+2\right)},$$

$\lambda$ is any positive constant and

$$\delta_0=(\lambda+1)^{\frac{r^2}{r^2-1}}(r+1)^{-\frac{r}{r-1}}.$$

We take

$$0<\gamma=1+\beta<\min\left(\frac{2r}{r+3},\frac{d+2-dr}{r-1}\right),$$

and

$$k=(1+\lambda)(r+1)\delta_0.$$

We set $\varphi(s)=(1+\alpha s)^{\beta+1}$ where

$$\alpha(r,\lambda)=\frac{r\delta_0^{\frac{r^2-1}{r}}(1+\lambda)^r(r+1)^{r+1}+1}{\delta_0^r(r-\tau_1)(1+\lambda)^r(r+1)^{r+1}}.$$

Finally, let $\psi\in C_c^\infty(\mathbb{R}^d)$ such that $0\le\psi\le1$ and

$$\psi(x)=\begin{cases}1 & \text{for } |x|\le L\\ 0 & \text{for } |x|\ge 2L.\end{cases}$$

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**Lemma 6.**

(1) *We have*

$$\min\left(\tfrac{2r}{r+3},\tfrac{d+2-dr}{r-1}\right)=\min\left(\tfrac{2r}{r+3},\tfrac{d+2-dr}{r-1},\tfrac{p-2r}{r-1}\right).$$

(2) *There exists $\lambda>0$ and close to zero such that $\gamma<\tau_1(r,\lambda)$.*
(3) *For all $\lambda$ positive and close to zero and $r\in(1,3]$,*

$$\alpha(r,\lambda)\geq 1 \text{ and } \lim_{r\to 1}\alpha(r,\lambda)=\infty.$$

For the proof of the lemma above we have to use the same arguments of the proof of Lemma 4. From Proposition 2 we deduce the following result.

**Proposition 4.** *We assume that Hyp A holds and $(\omega,T)$ geometrically controls $\Omega$. Let $\delta>0$, $R$, $R_0>L$ and $-1<\beta\leq 0$. There exists $C_{T,\delta}=C(T,\delta,R_0,R,\alpha,\beta)>0$, such that the following inequality*

$$\begin{aligned}&\int_t^{t+T}\int_{\Omega\cap B_{R_0}}(R+\alpha s)^\beta\left(|u|^2+|\nabla u|^2+|\partial_t u|^2\right)dxds\\&\leq C_{T,\delta}\int_t^{t+T}\int_\Omega a(x)(R+\alpha s)^\beta\left(|\partial_t u|^2+|\partial_t u|^{2r}\right)dxds\\&+C_{T,\delta}\int_t^{t+T}\int_\Omega a(x)(R+\alpha s)^{\beta-2}|u|^2dxds\\&+\delta\int_t^{t+T}\int_\Omega(R+\alpha s)^\beta\left(|\nabla u|^2+|\partial_t u|^2\right)dxds,\end{aligned}\tag{5.1}$$

*holds for every $t\geq 0$ and for all $u$ solution of (1.1) with initial data $(u_0,u_1)$ in $H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)$.*

As in the proof of Theorem 2 we need to define and to show an estimate for an auxiliary function $X(t)$.

**Lemma 7.** *Let $u$ be a solution of (1.1) with initial data in $H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)$ such that. We set $\chi=1-\psi$ and*

$$\begin{aligned}X(t)=&\int_\Omega(R+\alpha t)^\beta\chi^2(x)u(t)\partial_t u(t)dx+\tfrac{k_1}{2}\int_\Omega(R+\alpha t)^{\beta-1}a(x)|u(t)|^2dx\\&+\int_\Omega a(x)(R+\alpha t)^{\beta-r+1}|u(t)|^{r+1}dx+\tfrac{k}{2}\int_\Omega(R+\alpha t)^{\beta+1}\left(|\nabla u|^2+|\partial_t u|^2\right)dx,\end{aligned}\tag{5.2}$$

*where $k_1 > 0$. Then*

$$
\begin{aligned}
&X(t+T) - X(t) + \tfrac{2-k\alpha(1+\beta)}{2} \int_t^{t+T} \int_\Omega (R+\alpha s)^\beta \left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds \\
&+ \left(\tfrac{k_1\alpha(1-\beta)}{4} - \tfrac{\beta^2\alpha^2}{\epsilon_0\epsilon}\right) \int_t^{t+T} \int_\Omega a(x) (R+\alpha s)^{\beta-2} |u(t)|^2 dx ds \\
&+ \lambda\delta_0 \int_t^{t+T} \int_\Omega a(x) (R+\alpha s)^{\beta+1} |\partial_t u|^{r+1} dx ds \qquad (5.3) \\
&\le \left(3 + \left\|\nabla\chi^2\right\|_\infty\right) \int_t^{t+T} \int_{\Omega\cap B_{2L}} (R+\alpha s)^\beta \left(|u|^2 + |\nabla u|^2 + |\partial_t u|^2\right) dx ds \\
&+ \left(\tfrac{2}{\epsilon_0} + \tfrac{8k_1}{\alpha(1-\beta)} + \tfrac{8\beta^2\alpha}{\epsilon_0^2 k_1(1-\beta)}\right) \int_t^{t+T} \int_\Omega a(x) (R+\alpha s)^\beta |\partial_t u|^2 dx ds,
\end{aligned}
$$

*for all $t \ge 0$ and any $\lambda > 0$.*

**Proof.** For the proof we have to argue as in the proof of Lemma 5 and to use the fact that

$$
\begin{aligned}
&E_\varphi(u)(t+T) + \int_t^{t+T} \int_\Omega a(x) (R+\alpha s)^{\beta+1} |\partial_t u|^{r+1} dx ds \\
&\le E_\varphi(u)(t) + \tfrac{(\beta+1)\alpha}{2} \int_t^{t+T} \int_\Omega (R+\alpha s)^\beta \left(|\nabla u|^2 + |\partial_t u|^2\right) dx ds. \qquad \square
\end{aligned}
$$

### *5.2. Proof of Theorem 3*

For the proof we have to proceed as in the proof of Theorem 2 and to use the finite speed propagation property and the fact that the support of the initial data is contained in $B_R$ and

$$0 < 1 + \beta < \min\left(\tfrac{d+2-dr}{r-1}, \tfrac{p-2r}{r-1}\right),$$

to show that

$$
\begin{aligned}
&\int_{n_0T}^{nT} (R+\alpha s)^\beta \int_\Omega a(x) |\partial_t u|^{2r} dx ds \le C\epsilon^{-\frac{p-2r}{r-1}} \|a\|_{L^\infty} \left(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|u_1\|_{H^1}^{2r}\right)^{\frac{p}{2}} \\
&+ \epsilon \int_{n_0T}^{nT} \int_\Omega a(x) (R+\alpha s)^{\beta+1} |\partial_t u|^{r+1} dx ds,
\end{aligned}
$$

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and

$$\int_{n_0T}^{nT} (R+\alpha s)^{\beta} \int_{\Omega} a(x) |\partial_t u|^2 \, dx ds$$
$$\leq C \|a\|_{L^\infty} \epsilon^{-\frac{2}{r-1}} + \epsilon \int_{n_0T}^{nT} (R+\alpha s)^{\beta+1} \int_{\Omega} a(x) |\partial_t u|^{r+1} \, dx ds,$$

for some positive constant $C$ and for all $\epsilon > 0$.

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## References

[1] L. Aloui, S. Ibrahim, M. Khenissi, Energy decay for linear dissipative wave equations in exterior domains, J. Differential Equations 259 (5) (2015) 2061–2079.
[2] J. Bae, M. Nakao, Energy decay for the wave equation with boundary and localized dissipations in exterior domains, Math. Nachr. 278 (7–8) (2005) 771–783.
[3] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (5) (1992) 1024–1065.
[4] N. Burq, P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, C. R. Acad. Sci. Ser. I Math. 325 (7) (October 1997) 749–752.
[5] Marcelo M. Cavalcanti, Flavio R. Dias Silva, Valéria N. Domingos Cavalcanti, Uniform decay rates for the wave equation with nonlinear damping locally distributed in unbounded domains with finite measure, SIAM J. Control Optim. 52 (1) (2014) 545–580.
[6] W. Dan, Y. Shibata, On a local energy decay of solutions of a dissipative wave equation, Funkcial. Ekvac. 38 (3) (1995) 545–568.
[7] M. Daoulatli, Behaviors of the energy of solutions of the wave equation with damping and external force, J. Math. Anal. Appl. 389 (1) (2012) 205–225.
[8] M. Daoulatli, Energy decay rates for solutions of the wave equation with linear damping in exterior domain, Evol. Equ. Control Theory 5 (1) (2016) 37–59.
[9] M. Daoulatli, Behaviors of the energy of solutions of two coupled wave equations with nonlinear damping on a compact manifold with boundary, arXiv:1703.00172.
[10] R. Ikehata, T. Matsuyama, L2-behaviour of solutions to the linear heat and wave equations in exterior domains, Sci. Math. Jpn. 55 (2002) 33–42.
[11] R. Ikehata, Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain, J. Differential Equations 188 (2) (2003) 390–405.
[12] R. Ikehata, G. Todorova, B. Yordanov, Critical exponent for semilinear wave equations with space-dependent potential, Funkcial. Ekvac. 52 (2009) 411–435.
[13] R. Ikehata, Energy decay of solutions for the semilinear dissipative wave equations in an exterior domain, Funkcial. Ekvac. 44 (2001) 487–499.
[14] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995) 617–653.
[15] P.D. Lax, R.S. Phillips, Scattering Theory, Pure and Applied Mathematics, vol. 26, Academic Press, Inc., Boston, MA, 1989.
[16] J.L. Lions, W. Strauss, Some non-linear evolution equations, Bull. Soc. Math. France 93 (1965) 43–96.
[17] Kiyoshi Mochizuki, Takahiro Motai, On energy decay–nondecay problems for wave equations with nonlinear dissipative term in $R^N$, J. Math. Soc. Japan 47 (3) (1995) 405–421.
[18] M. Nakao, Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations, Math. Z. 238 (2001) 781–797.

[19] M. Nakao, Decay of solutions to the Cauchy problem for the Klein–Gordon equation with a localized nonlinear dissipation, Hokkaido Math. J. 27 (1998) 245–271.

[20] M. Nakao, Energy decay to the Cauchy problem for a generalised nonlinear Klein–Gordon equation with a nonlinear dissipative term, Int. J. Dyn. Syst. Differ. Equ. 3 (3) (2011) 349–362.

[21] M. Nakao, Il Hyo Jung, Energy decay for the wave equation in exterior domains with some half-linear dissipation, Differential Integral Equations 16 (8) (2003) 927–948.

[22] K. Ono, The time decay to the Cauchy problem for semilinear dissipative wave equations, Adv. Math. Sci. Appl. 9 (1999) 243–262.

[23] K. Ono, $L^1$ estimates for dissipative wave equations in exterior domains, J. Math. Anal. Appl. 333 (2) (2007) 1079–1092.

[24] R. Racke, Non-homogeneous non-linear damped wave equations in unbounded domains, Math. Methods Appl. Sci. 13 (1990) 481–491.

[25] G. Todorova, B. Yordanov, The energy decay problem for wave equations with nonlinear dissipative terms in $R^n$, Indiana Univ. Math. J. 56 (1) (2007) 389–416.

[26] K. Wakasa, B. Yordanov, On the energy decay for dissipative nonlinear wave equations in one space dimension, J. Math. Anal. Appl. 455 (2) (2017) 1317–1322.

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