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The Cauchy problem for shallow water waves of large amplitude in Besov space

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Abstract

In this paper, we consider a nonlinear evolution equation modelling the propagation of surface waves in the shallow water regime of large amplitude, which is characterised by some cubical nonlinearities. First, we establish the local well-posedness in Besov space $B^{3/2}_{2,1}$. Then, we give a blow-up criterion. Finally, with a given analytic initial data, we establish the analyticity of the solutions in both variables, globally in space and locally in time.

MSC: 35Q53; 35B30; 35G25

Keywords: Besov spaces; Shallow water waves of large amplitude; Local well-posedness; Blow-up criterion; Analyticity

1. Introduction

This paper is concerned with an evolution equation which models the propagation of surface waves in the shallow water regime of large amplitude [32]

$$u_t + u_x + \frac{3\varepsilon}{2} u u_x - \frac{4\delta^2}{18} u_{xxx} - \frac{7\delta^2}{18} u_{xxt}$$

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$$= \frac{\varepsilon\delta^2}{6}(2u_x u_{xx} + uu_{xxx}) - \frac{\varepsilon^2\delta^2}{96}\left(398uu_x u_{xx} + 45u^2 u_{xxx} + 154u_x^3\right). \tag{1.1}$$

Here $u(t,x)$ is the horizontal velocity, ε is known as the amplitude parameter and $\delta \ll 1$ is the shallowness parameter. The equation (1.1) is characterised by the retainment of $\varepsilon^2\delta^2$-term, capturing the strong nonlinear effects induced by large amplitude waves, which once be neglected reduces to the following Camassa-Holm (CH) type equation

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x - \frac{4\delta^2}{18}u_{xxx} - \frac{7\delta^2}{18}u_{xxt} = \frac{\varepsilon\delta^2}{6}(2u_x u_{xx} + uu_{xxx}). \tag{1.2}$$

The prominent CH equation, for which the ratio of the nonlinear terms of (1.2) being $3:2:1$, was first derived formally by Fuchssteiner and Fokas [21] as a bi-Hamiltonian equation and later derived in the context of water waves as a model for unidirectional propagation of shallow water waves of moderate amplitude by Camassa and Holm [5] (see also the alternative derivation in [8,14,26]). The CH equation has been studied extensively in the last twenty years because of its many remarkable properties: infinitely many conservation laws and complete integrability [5,18], existence of peaked solitons and multi-peakons [1,5], well-posedness and breaking waves [3,4,9,11–13,15,16,20,25], just to mention a few.

In the spirit of [26], Ronald Quirchmayr derived equation (1.1), exhibiting additional terms with higher nonlinearities with the aim to describe large-amplitude waves in [32], along with a study on the local well-posedness of the corresponding Cauchy problem for initial data $u_0 \in H^3(\mathbb{R})$. Employing a semigroup approach due to Kato [27], the local well-posedness for equation (1.1) was established for a class of initial data comprising less regular data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ in [34]. Furthermore, a blow-up criterion for solutions was presented in [32] and the traveling wave solutions were studied in [23,28].

In this paper, we are devoted to investigate the local well-posedness of the Cauchy problem for (1.1) in Besov space $B^{3/2}_{2,1}$. By virtue of the Littlewood-Paley decomposition, nonhomogeneous Besov spaces and iterative method, the methods proposed in [15–17] have been applied with success when studying the well-posedness of various shallow water wave equations in Besov space (see for example [19,22,29–31,33]). While when dealing with the Cauchy problem for (1.1) in Besov space $B^{3/2}_{2,1}$, we encounter one main difficulty. That is the appearance of some higher-order nonlinearities makes the estimate of the uniform bound of the approximate solution difficult, which is well known to play a key role in the establishment of the local well-posedness of shallow water wave equations in Besov space $B^{3/2}_{2,1}$. Roughly speaking, when recasting the equation (1.1) in a form of nonlocal conservation law

$$\begin{aligned} &u_t + \left(\frac{4}{7} + \frac{3\varepsilon}{7}u - \frac{135\varepsilon^2}{112}u^2\right)u_x \\ &= -\partial_x\left(1 - \frac{7\delta^2}{18}\partial_x^2\right)^{-1}\left[\frac{3}{7}u + \frac{15\varepsilon}{28}u^2 + \frac{45\varepsilon^2}{112}u^3 + \frac{\varepsilon\delta^2}{12}u_x^2 + \frac{2\varepsilon^2\delta^2}{3}uu_x^2\right], \end{aligned} \tag{1.3}$$

we find that the "coefficient" of u_x is $\frac{4}{7} + \frac{3\varepsilon}{7}u - \frac{135\varepsilon^2}{112}u^2$. Then the usual method to estimate the term $\exp\{\int_\tau^t \|\partial_x(\frac{4}{7} + \frac{3\varepsilon}{7}u - \frac{135\varepsilon^2}{112}u^2)\|_{B^{1/2}_{2,1}} dt'\}$ is invalid as the simultaneous appearance of one power term u and quadratic term u^2. By a felicitous choice of the interval of the existent time

T of the solution and applying the mean-value theorem for the integral to $\exp\{\int_\tau^t \|u\|^2_{B^{3/2}_{2,1}} dt'\}$, we obtain the constant bound $e^{\frac{1}{3}}$ of $\exp\{C\int_\tau^t \|u\|^2_{B^{3/2}_{2,1}} dt'\}$, implying the usually fractional bound of $\exp\{\int_\tau^t \|\partial_x(\frac{4}{7} + \frac{3\varepsilon}{7}u - \frac{135\varepsilon^2}{112}u^2)\|_{B^{1/2}_{2,1}} dt'\}$, and thus overcome this difficulty. This technique has been adopted to estimate the other high-order nonlinearities as u^2, u^3, u_x^2 and uu_x^2. The other problematical issue is that the appearance of higher-order nonlinearities makes the proof of several required nonlinear estimates somewhat delicate.

Another goal of this paper is to give a more precise blow-up criterion compared with the one presented in [32], where it should be pointed that the equation (1.1) does not possess the H^1-norm conserved quantity. Finally, provided that the initial profile u_0 is an analytic function on the real line $\mathbb{R}$, we obtain the analyticity of the corresponding solutions in both variables, with $x \in \mathbb{R}$ and t in an interval around zero. Analyticity is inherent to traveling water waves (see [10]).

We supplement (1.1) with the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.4}$$

To introduce the main results, we define

$$E^{3/2}_{2,1}(T) = C([0,T]; B^{3/2}_{2,1}) \cap C^1([0,T]; B^{1/2}_{2,1}).$$

The main results of this paper are as follows:

Theorem 1.1. *Let $u_0 \in B^{3/2}_{2,1}$. There exists a time $T > 0$ such that the problem (1.1) and (1.4) has a unique solution in $E^{3/2}_{2,1}(T)$. Moreover, the solution depends continuously on the initial data, i.e., the mapping $\Phi : u_0 \mapsto u$ is continuous from a neighborhood of $u_0 \in B^{3/2}_{2,1}$ into $E^{3/2}_{2,1}(T)$.*

Remark 1.1. 1. We obtain the local well-posedness of equations (1.1) and (1.4) in the case $B^{3/2}_{2,1}$. However, this is not true in the case $B^{3/2}_{2,\infty}$ in view of the proof of Proposition 4 in [16]. Noting that $B^{3/2}_{2,1} \hookrightarrow H^{3/2} \hookrightarrow B^{3/2}_{2,\infty}$, one can see that $s = 3/2$ is the critical index. 2. Using the maximal time of existence T^* instead of T, we can improve the formulation of Theorem 1.1 with the time-interval of existence being $[0, T^*)$, $T^* \in [0,\infty) \cup \{\infty\}$.

Theorem 1.2. *Let $u_0 \in B^{3/2}_{2,1}$ and u be the corresponding solution to the problem (1.1) and (1.4). Assume that T^* is the maximal time of existence of the solution to the problem (1.1) and (1.4). If $T^* < +\infty$, then*

$$\int_0^{T^*} \|u_x\|^2_{L^\infty} d\tau = +\infty. \tag{1.5}$$

Referring to the definition of the space E_{s_0} in (5.1), we present the following analytic result.

Theorem 1.3. *If the initial data u_0 is real analytic on the line $\mathbb{R}$ and belongs to a space E_{s_0} for some $0 < s_0 \le 1$, then there exist an $\varepsilon > 0$ and a unique solution u to the problem (1.1) and (1.4) that is analytic on $\mathbb{R} \times [0, \varepsilon)$.*

Remark 1.2. A rereading of the proof of Theorem 1.1 and Theorem 1.3 yields that there exists a real analytic extension of u to $(-\varepsilon, \varepsilon)$.

In the sequel, we will, for notational convenience, deal with the following initial value problem with different coefficients, which implies the Theorem 1.1-Theorem 1.3 as the concrete values of the coefficients have no impact on the results.

$$\begin{cases} u_t + (1 + u - u^2)u_x = P(D)f(u, u_x), \\ u(0, x) = u_0(x), \end{cases} \tag{1.6}$$

with the operator $P(D) = -\partial_x(1 - \partial_x^2)^{-1}$, and

$$f(u, u_x) = u + u^2 + u^3 + u_x^2 + uu_x^2. \tag{1.7}$$

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove the local well-posedness for the Cauchy problem (1.1) and (1.4) in Besov space $B^{3/2}_{2,1}$. Section 4 is devoted to establishing a blow-up criterion. Finally, we study the analyticity of the Cauchy problem (1.1) and (1.4) based on a contraction type argument in a suitably chosen scale of the Banach spaces in Section 5.

2. Preliminaries

For convenience of the reader, we recall some conclusions on the properties of Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and the theory of the transport equation. One may check [2,15–17] for more details.

Lemma 2.1. *(Littlewood-Paley decomposition). There exist two smooth radial functions* (χ, ϕ) *valued in* $[0, 1]$, *such that* χ *is supported in the ball* $B = \{\xi \in \mathbb{R}^n, |\xi| \le \frac{4}{3}\}$ *and* ϕ *is supported in the ring* $C = \{\xi \in \mathbb{R}^n, \frac{4}{3} \le |\xi| \le \frac{8}{3}\}$. *Moreover,*

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{q \ge 0} \phi(2^{-q}\xi) = 1$$

and

$$\begin{aligned} Supp\, \phi(2^{-q}\cdot) \cap Supp\, \phi(2^{-q'}\cdot) = \emptyset, \quad if\ |q - q'| \ge 2, \\ Supp\, \chi(\cdot) \cap Supp\, \phi(2^{-q}\cdot) = \emptyset, \quad if\ |q| \ge 1. \end{aligned}$$

Then for $u \in \mathcal{S}'(\mathbb{R}^n)$, *the nonhomogeneous dyadic operators are defined as follows:*

$$\begin{aligned} &\triangle_q u = 0, \quad if\ q \le -2, \\ &\triangle_{-1} u = \chi(\mathcal{D})u = \mathscr{F}_x^{-1}\chi\mathscr{F}_x u, \\ &\triangle_q u = \phi(2^{-q}\mathcal{D}) = \mathscr{F}_x^{-1}\phi(2^{-q}\xi)\mathscr{F}_x u, \quad if q \ge 0. \end{aligned}$$

Thus $u = \sum_{q \ge 0} \triangle_q u$ *in* $\mathcal{S}'(\mathbb{R}^n)$.

Remark 2.1. The low frequency cut-off S_q is defined by

$$S_q u = \sum_{p=-1}^{q-1} \triangle_p u = \chi(2^{-q}\mathcal{D})u = \mathscr{F}_x^{-1}\chi(2^{-q}\xi)\mathscr{F}_x u, \quad \forall q \in N.$$

It is easily checked that

$$\triangle_p \triangle_q \equiv 0, \quad |p-q| \geq 2,$$
$$\triangle_q(S_{p-1}u\triangle_p v) \equiv 0, \quad |p-q| \geq 5, \ \forall u, v \in \mathcal{S}'(\mathbb{R}^n)$$

as well as

$$\|\triangle_q u\|_{L^p} \leq \|u\|_{L^p}, \qquad \|S_q u\|_{L^p} \leq C\|u\|_{L^p}, \quad \forall 1 \leq p \leq +\infty$$

with the aid of Young's inequality, where C is a positive constant independent of q.

Definition 2.1. *(*Besov spaces*). Let $s \in \mathbb{R}$, $1 \leq p \leq +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^n)$ is defined by*

$$B^s_{p,r}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,r}} = \|2^{qs}\triangle_q f\|_{l^r(L^P)} = \|(2^{qs}\|\triangle_q f\|_{L^p})_{q\geq -1}\|_{l^r} < \infty\}.$$

In particular, $B^\infty_{p,r} = \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Lemma 2.2. *Let $s \in \mathbb{R}$, $1 \leq p, r, p_j, r_j \leq \infty$, $j = 1, 2$, then:*

(1) *Topological properties: $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.*
(2) *Density: C^∞_c is dense in $B^s_{p,r} \Leftrightarrow 1 \leq p, r < \infty$.*
(3) *Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$, if $p_1 \leq p_2$ and $r_1 \leq r_2$.*
(4) *Algebraic properties: $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ is a Banach algebra. $B^s_{p,r}$ is a Banach algebra $\Leftrightarrow$ $B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{1}{p}$ or ($s \geq \frac{1}{p}$ and $r = 1$). In particular, $B^{1/2}_{2,1}$ is continuously embedded in $B^{1/2}_{2,\infty} \cap L^\infty$ and $B^{1/2}_{2,\infty} \cap L^\infty$ is a Banach algebra.*
(5) *1-D Moser-type estimates:*
   (i) *For $s > 0$,*

$$\|fg\|_{B^s_{p,r}} \leq C(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^s_{p,r}}\|f\|_{L^\infty}).$$

   (ii) *$\forall s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, we have*

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) *Complex interpolation:*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|f\|^\theta_{B^{s_1}_{p,r}}\|g\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \quad \forall \theta \in [0, 1].$$

(7) *Real interpolation:* $\forall\theta \in (0,1), s_1 > s_2, s = \theta s_1 + (1-\theta)s_2$, *there exists a constant* C *such that*

$$\|u\|_{B^s_{p,1}} \le \frac{C(\theta)}{s_1 - s_2}\|u\|^{\theta}_{B^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \forall u \in B^{s_1}_{p,\infty}.$$

In particular, for any $0 < \theta < 1$, *we have that*

$$\|u\|_{B^{1/2}_{2,1}} \le \|u\|_{B^{\frac{3}{2}-\theta}_{2,1}} \le C(\theta)\|u\|^{\theta}_{B^{1/2}_{2,\infty}}\|u\|^{1-\theta}_{B^{3/2}_{2,\infty}}. \tag{2.1}$$

(8) *Fatou lemma: if* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $B^s_{p,r}$ *and* $u_n \to u$ *in* $\mathcal{S}'$, *then* $u \in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}.$$

(9) *Let* $m \in \mathbb{R}$ *and* f *be an* s^m*-multiplier (i.e.,* $f : \mathbb{R}^n \to \mathbb{R}$ *is smooth and satisfies that* $\forall\alpha \in N^n$, $\exists$ *a constant* C_α, *s.t.* $|\partial_\alpha f(\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|}$ *for all* $\xi \in \mathbb{R}^n$*). Then the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.

(10) *The paraproduct is continuous from* $B^{-1/p}_{p,1} \times (B^{1/p}_{p,\infty} \cap L^\infty)$ *to* $B^{-1/p}_{p,1}$, *i.e.,*

$$\|fg\|_{B^{-1/p}_{p,\infty}} \le C\|f\|_{B^{-1/p}_{p,1}}\|g\|_{B^{1/p}_{p,\infty}\cap L^\infty}.$$

(11) *A logarithmic interpolation inequality*

$$\|f\|_{B^{1/p}_{p,1}} \le C\|f\|_{B^{1/p}_{p,\infty}} \ln\Big(e + \frac{\|f\|_{B^{1+1/p}_{p,\infty}}}{\|f\|_{B^{1/p}_{p,\infty}}}\Big).$$

Lemma 2.3. *Let* $1 \le p, r \le \infty$ *and* $s > -\min(\frac{1}{p}, 1-\frac{1}{p})$. *Assume that* $f_0 \in B^s_{p,r}$, $F \in L^1(0,T; B^s_{p,r})$ *and* $\partial_x v$ *belongs to* $L^1(0,T; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or to* $L^1(0,T; B^{1/p}_{p,r} \cap L^\infty)$ *otherwise. If* $f \in L^\infty(0,T; B^s_{p,r}) \cap C([0,T]; \mathcal{S}'(\mathbb{R}))$ *solves the following 1-D linear transport equation:*

$$\begin{cases} f_t + vf_x = F, \\ f|_{t=0} = f_0. \end{cases} \tag{T}$$

then there exists a constant C *depending only on* s, p, r *such that the following statements hold:*

(1) *If* $r = 1$ *or* $s \ne 1 + \frac{1}{p}$, *then*

$$\|f\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} d\tau,$$

or

$$\|f\|_{B^s_{p,r}} \leq e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} d\tau) \tag{2.2}$$

holds, where $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r}\cap L^\infty} d\tau$ *if* $s < 1 + \frac{1}{p}$ *and* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{s-1}_{p,r}} d\tau$ *else.*

(2) *If* $s \leq 1 + \frac{1}{p}$, $f_0' \in L^\infty$, $f_x \in L^\infty((0,t)\times\mathbb{R})$ *and* $F_x \in L^1(0,T;L^\infty)$, *then*

$$\begin{aligned}&\|f(t)\|_{B^s_{p,r}} + \|f_x(t)\|_{L^\infty}\\ &\leq e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \|f_{0x}\|_{L^\infty} + \int_0^t e^{-CV(\tau)}[\|F(\tau)\|_{B^s_{p,r}} + \|F_x(\tau)\|_{L^\infty}]d\tau)\end{aligned}$$

with $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r}\cap L^\infty} d\tau$.

(3) *If* $f = v$, *then for all* $s > 0$, *the estimate* (2.2) *holds with* $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty} d\tau$.

(4) *If* $r < \infty$, *then* $f \in C([0,T]; B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T]; B^{s'}_{p,r})$ *for all* $s' < s$.

Lemma 2.4. *(Existence and uniqueness) Let* p, r, s, f_0 *and* F *be as in the statement of Lemma 2.3. Assume that* $v \in L^\rho(0,t; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$ *and* $v_x \in L^1(0,T; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$ *and* $v_x \in L^1(0,T; B^{1/p}_{p,\infty} \cap L^\infty)$ *if* $< 1 + \frac{1}{p}$. *Then the transport equation (T) has a unique solution* $f \in L^\infty([0,T]; B^s_{p,r}) \cap (\bigcap_{s'<s} C([0,T]; B^{s'}_{p,1})$ *and the inequalities in Lemma 2.3 hold true. Moreover, if* $r < \infty$, *then we have* $f \in C([0,T]; B^s_{p,r})$.

Lemma 2.5. *([16]) Denote* $\overline{\mathbb{N}} = \mathbb{N} \cup \infty$. *Let* $(v^{(n)})_{n\in\overline{\mathbb{N}}}$ *be a sequence of functions belonging to* $C([0,T]; B^{1/2}_{2,1})$. *Assume that* $v^{(n)}$ *is the solution to*

$$\begin{cases}\partial_t v^{(n)} + a^{(n)}\partial_x v^{(n)} = f,\\ v^{(n)}|_{t=0} = v_0\end{cases} \tag{2.3}$$

with $v_0 \in B^{1/2}_{2,1}$, $f \in L^1(0,T; B^{1/2}_{2,1})$ *and that for some* $\alpha \in L^1(0,T)$,

$$\sup_{n\in\mathbb{N}} \|\partial_x a^{(n)}(t)\|_{B^{1/2}_{2,1}} \leq \alpha(t).$$

If in addition $a^{(n)}$ *tends to* $a^{(\infty)}$ *in* $L^1(0,T; B^{1/2}_{2,1})$ *then* $v^{(n)}$ *tends to* $v^{(\infty)}$ *in* $C(0,T; B^{1/2}_{2,1})$.

3. Proof of Theorem 1.1

In this section we aim to prove Theorem 1.1 with the aid of the following six steps.

First step: Approximate solution. We use a standard iterative process to build a solution. Starting from $u^{(0)} := 0$, by induction we define a sequence of smooth functions $(u^{(n)})_{n\in\mathbb{N}}$ by solving the following linear transport equation:

$$\begin{cases} u_t^{(n+1)} + \left(1 + u^{(n)} - (u^{(n)})^2\right) u_x^{(n+1)} = P(D) f(u^{(n)}, u_x^{(n)}), \\ u^{(n+1)}(0, x) = u_0^{(n+1)}(x) = S_{n+1} u_0. \end{cases} \tag{3.1}$$

Since $S_{n+1}u_0$ belongs to $B^{\infty}_{2,r}$, by using Lemma 2.4, with the aid of induction, we show that for all $n \in \mathbb{N}$, the above equation has a global solution which belongs to $C(\mathbb{R}^+, B^{\infty}_{2,r})$.

Second step: Uniform bounds. Applying (2.2) of Lemma 2.3 to (3.1), we obtain

$$\begin{aligned} \|u^{(n+1)}(t)\|_{B^{3/2}_{2,1}} &\le e^{C\int_0^t \|\partial_x(u^{(n)}) - \partial_x((u^{(n)})^2)\|_{B^{1/2}_{2,1}} dt'} \|u_0\|_{B^{3/2}_{2,1}} \\ &+ \int_0^t e^{C\int_\tau^t \|\partial_x(u^{(n)}) - \partial_x((u^{(n)})^2)\|_{B^{1/2}_{2,1}} dt'} \|P(D) f(u^{(n)}, u_x^{(n)})\|_{B^{3/2}_{2,1}} d\tau. \end{aligned} \tag{3.2}$$

As $P(D)$ is a S^{-1}-multiplier and by the fact that $B^{s-1}_{2,r}$ is a Banach algebra and $B^s_{2,r} \hookrightarrow B^{s-1}_{2,r}$, we have that

$$\begin{aligned} \|P(D) f(u^{(n)}, u_x^{(n)})\|_{B^{3/2}_{2,1}} &\le \|f(u^{(n)}, u_x^{(n)})\|_{B^{1/2}_{2,1}}, \\ &\le \frac{C}{2}\left(\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}} + \|u^{(n)}\|^3_{B^{3/2}_{2,1}}\right), \end{aligned} \tag{3.3}$$

and

$$\|\partial_x(u^{(n)}) - \partial_x((u^{(n)})^2)\|_{B^{1/2}_{2,1}} \le C\left(\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}}\right). \tag{3.4}$$

Inserting (3.3) and (3.4) into (3.2) yields for all $n \in \mathbb{N}$:

$$\begin{aligned} &\|u^{(n+1)}(t)\|_{B^{3/2}_{2,1}} \\ &\le e^{C\int_0^t \left(\|u^{(n)}(t')\|_{B^{3/2}_{2,1}} + \|u^{(n)}(t')\|^2_{B^{3/2}_{2,1}}\right) dt'} \|u_0\|_{B^{3/2}_{2,1}} + \frac{C}{2}\int_0^t e^{C\int_\tau^t \left(\|u^{(n)}(t')\|_{B^{3/2}_{2,1}} + \|u^{(n)}(t')\|^2_{B^{3/2}_{2,1}}\right) dt'} \\ &\times \left(\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}} + \|u^{(n)}\|^3_{B^{3/2}_{2,1}}\right) d\tau. \end{aligned} \tag{3.5}$$

Let us choose a $T > 0$ such that

$$T \le \frac{1}{3C\left(1 + 4\|u_0\|_{B^{3/2}_{2,1}} + 16\|u_0\|^2_{B^{3/2}_{2,1}}\right)}, \tag{3.6}$$

and suppose by induction that for all $t \in [0, T]$

$$\|u^{(n)}(t)\|_{B^{3/2}_{2,1}} \leq \frac{2\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t}. \tag{3.7}$$

Indeed, one obtains from (3.5) and (3.7) that

$$\begin{aligned}
&\exp\{C\int_{\tau}^{t}\left(\|(u^{(n)})(t')\|_{B^{3/2}_{2,1}}+\|(u^{(n)})(t')\|^2_{B^{3/2}_{2,1}}\right)dt'\} \\
&\leq \exp\{\int_{\tau}^{t}\left(\frac{2C\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t'}+\frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t')^2}\right)dt'\} \\
&= \exp\{-\frac{1}{2}\int_{\tau}^{t}\frac{d(1-4C\|u_0\|_{B^{3/2}_{2,1}}t')}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t')}+\frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}(t-\tau)}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\xi_1)^2}\} \\
&\leq \exp\{\frac{1}{2}\ln(\frac{1-4C\|u_0\|^{Q}_{B^{3/2}_{2,1}}\tau}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t})+\frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}T}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}T)^2}\} \\
&\leq \exp\{16C\|u_0\|^2_{B^{3/2}_{2,1}}T\}(\frac{1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t})^{\frac{1}{2}} \\
&\leq e^{\frac{1}{3}}(\frac{1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t})^{\frac{1}{2}},
\end{aligned} \tag{3.8}$$

where the mean-value theorem for the integral has been employed with $\tau < \xi_1 < t$. When $\tau = 0$, we have

$$e^{C\int_0^t\left(\|(u^{(n)})(t')\|_{B^{3/2}_{2,1}}+\|(u^{(n)})(t')\|^2_{B^{3/2}_{2,1}}\right)dt'} \leq \frac{e^{\frac{1}{3}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{\frac{1}{2}}}. \tag{3.9}$$

Then combining (3.7) and (3.8), we have

$$\begin{aligned}
&\frac{C}{2}\int_0^t e^{C\int_\tau^t\left(\|(u^{(n)})(t')\|_{B^{3/2}_{2,1}}+\|(u^{(n)})(t')\|^2_{B^{3/2}_{2,1}}\right)dt'}(\|u^{(n)}\|_{B^{3/2}_{2,1}}+\|u^{(n)}\|^2_{B^{3/2}_{2,1}}+\|u^{(n)}\|^3_{B^{3/2}_{2,1}})d\tau \\
&\leq \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}}\int_0^t[(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^{\frac{1}{2}} \\
&\times\left(\frac{C}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)}+\frac{2C\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^2}+\frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^3}\right)]d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}} \\
&\quad \times \int_0^t \left(\frac{C}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^{1/2}} + \frac{2C\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^{3/2}} + \frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^{5/2}} \right) d\tau \\
&= \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}} \\
&\quad \times \left(\frac{Ct}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\xi_2)^{1/2}} + (1-4C\|u_0\|_{B^{3/2}_{2,1}}\tau)^{-1/2}\Big|_0^t + \frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}t}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}\xi_3)^{5/2}} \right) \\
&\le \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}} \left(\frac{1}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}} - 1 \right) \\
&\quad + \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{(1-4C\|u_0\|_{B^{3/2}_{2,1}})^{1/2}} \left(\frac{CT}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{1/2}} + \frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}T}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^{5/2}} \right),
\end{aligned} \tag{3.10}
$$

where the mean-value theorem for the integral has been employed with $0 < \xi_2, \xi_3 < t$. Inserting (3.9) and (3.10) into (3.5), we get that

$$
\begin{aligned}
\|u^{(n+1)}\|_{B^{3/2}_{2,1}} &\le \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t} \left(1 + CT + \frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}T}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}t)^2} \right) \\
&\le \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t} \left(1 + CT + \frac{4C\|u_0\|^2_{B^{3/2}_{2,1}}T}{(1-4C\|u_0\|_{B^{3/2}_{2,1}}T)^2} \right) \\
&\le \frac{e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t} \left(1 + CT + 16CT\|u_0\|^2_{B^{3/2}_{2,1}} \right) \\
&\le \frac{\frac{4}{3}e^{\frac{1}{3}}\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t} \le \frac{2\|u_0\|_{B^{3/2}_{2,1}}}{1-4C\|u_0\|_{B^{3/2}_{2,1}}t}.
\end{aligned} \tag{3.11}
$$

Thus, $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^{3/2}_{2,1})$. Using equation (3.1), one can easily prove that $(\partial_t u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^{1/2}_{2,1})$. Consequently, $(u^{(n)})_{n\in\mathbb{N}} \subset C([0,T]; B^{3/2}_{2,1}) \cap C^1([0,T]; B^{1/2}_{2,1})$.

Third step: Convergence. We first show that $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\, B^{1/2}_{2,\infty})$, then by using (2.1) we prove that $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];\, B^{1/2}_{2,1})$. For $(m,n)\in\mathbb{N}^2$, we have

$$
\begin{aligned}
&\left[\partial_t + \left(1+u^{(n+m)}-(u^{(n+m)})^2\right)\partial_x\right](u^{(n+1+m)}-u^{(n+1)})\\
&= -\left[\left(u^{(n+m)}-u^{(n)}\right)-\left((u^{(n+m)})^2-(u^{(n)})^2\right)\right]u_x^{(n+1)}\\
&+P(D)[\left(u^{(n+m)}-u^{(n)}\right)+\left((u^{(n+m)})^2-(u^{(n)})^2\right)+\left((u^{(n+m)})^3-(u^{(n)})^3\right)\\
&+\left((u_x^{(n+m)})^2-(u_x^{(n)})^2\right)+\left(u^{(n+m)}(u_x^{(n+m)})^2-u^{(n)}(u_x^{(n)})^2\right)\\
&:= S_1(x,t)+S_2(x,t).
\end{aligned}
\tag{3.12}
$$

We define

$$
w_{n,m} = \|(u^{(n+m)}-u^{(n)})(t)\|_{B^{1/2}_{2,\infty}} \tag{3.13}
$$

and

$$
w_n(t) = \sup_{m\in\mathbb{N}} w_{n,m}(t) \tag{3.14}
$$

as well as

$$
\widetilde{w}(t) = \limsup_{n\to\infty} w_n(t). \tag{3.15}
$$

We will show $\widetilde{w}(t)=0$, for $t\in[0,T]$. By (3.6), (3.7) and (3.9), we have that

$$
\|u^{(n)}(t)\|_{B^{3/2}_{2,1}} \le 4\|u_0\|_{B^{3/2}_{2,1}} \tag{3.16}
$$

and

$$
e^{C\int_0^t\left(\|(u^{(n)})(t')\|_{B^{3/2}_{2,1}}+\|(u^{(n)})(t')\|^2_{B^{3/2}_{2,1}}\right)dt'} \le 2. \tag{3.17}
$$

Then, by using (2), (3), (4) and (10) of Lemma 2.2 as well as $B^{3/2}_{2,1}\hookrightarrow B^{3/2}_{2,\infty}\cap \mathrm{Lip}$ and the above inequality we obtain

$$
\begin{aligned}
\|(u^{(n+m)}-u^{(n)})u_x^{(n+1)}\|_{B^{1/2}_{2,\infty}} &\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|u_x^{(n+1)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\|u^{(n+1)}\|_{B^{3/2}_{2,\infty}\cap \mathrm{Lip}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\|u^{(n+1)}\|_{B^{3/2}_{2,1}},
\end{aligned}
$$

$$
\begin{aligned}
&\|\left((u^{(n+m)})^2-(u^{(n)})^2\right)u_x^{(n+1)}\|_{B^{1/2}_{2,\infty}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|u_x^{(n+1)}(u^{(n+m)}+u^{(n)})\|_{B^{1/2}_{2,1}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|u_x^{(n+1)}\|_{B^{1/2}_{2,1}}\|u^{(n+m)}+u^{(n)}\|_{B^{1/2}_{2,1}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\left(\|u^{(n+1)}\|_{B^{3/2}_{2,1}}\|u^{(n+m)}\|_{B^{3/2}_{2,1}}+\|u^{(n+1)}\|_{B^{3/2}_{2,1}}\|u^{(n)}\|_{B^{3/2}_{2,1}}\right),
\end{aligned}
$$

$$
\|P(D)\left(u^{(n+m)}-u^{(n)}\right)\|_{B^{1/2}_{2,\infty}}\le\|u^{(n+m)}-u^{(n)}\|_{B^{-1/2}_{2,\infty}}\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}},
$$

$$
\begin{aligned}
\|P(D)\left((u^{(n+m)})^2-(u^{(n)})^2\right)\|_{B^{1/2}_{2,\infty}}&\le\|(u^{(n+m)})^2-(u^{(n)})^2\|_{B^{-1/2}_{2,\infty}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|u^{(n+m)}+u^{(n)}\|_{B^{-1/2}_{2,1}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\left(\|u^{(n+m)}\|_{B^{3/2}_{2,1}}+\|u^{(n)}\|_{B^{3/2}_{2,1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\|P(D)\left((u_x^{(n+m)})^2-(u_x^{(n)})^2\right)\|_{B^{1/2}_{2,\infty}}&\le\|(u_x^{(n+m)})^2-(u_x^{(n)})^2\|_{B^{-1/2}_{2,\infty}}\\
&\le C\|u_x^{(n+m)}-u_x^{(n)}\|_{B^{-1/2}_{2,1}}\|u_x^{(n+m)}+u_x^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\le C\|u_x^{(n+m)}-u_x^{(n)}\|_{B^{-1/2}_{2,1}}\|u^{(n+m)}+u^{(n)}\|_{B^{3/2}_{2,\infty}\cap L^\infty}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\left(\|u^{(n+m)}\|_{B^{3/2}_{2,1}}+\|u^{(n)}\|_{B^{3/2}_{2,1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|P(D)\left((u^{(n+m)})^3-(u^{(n)})^3\right)\|_{B^{1/2}_{2,\infty}}\\
&\le\|(u^{(n+m)})^3-(u^{(n)})^3\|_{B^{-1/2}_{2,\infty}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|(u^{(n+m)})^2+u^{(n+m)}u^{(n)}+(u^{(n)})^2\|_{B^{-1/2}_{2,1}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\left(\|u^{(n+m)}\|^2_{B^{3/2}_{2,1}}+\|u^{(n)}\|_{B^{3/2}_{2,1}}\|u^{(n+m)}\|_{B^{3/2}_{2,1}}+\|u^{(n)}\|^2_{B^{3/2}_{2,1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|P(D)\left(u^{(n+m)}(u_x^{(n+m)})^2-u^{(n)}(u_x^{(n)})^2\right)\|_{B^{1/2}_{2,\infty}}\\
&\le\|u^{(n+m)}(u_x^{(n+m)})^2-u^{(n)}(u_x^{(n)})^2\|_{B^{-1/2}_{2,\infty}}\\
&\le\|\left(u^{(n+m)}-u^{(n)}\right)(u_x^{(n+m)})^2\|_{B^{-1/2}_{2,\infty}}+\|u^{(n)}\left((u_x^{(n+m)})^2-(u_x^{(n)})^2\right)\|_{B^{-1/2}_{2,\infty}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{-1/2}_{2,1}}\|(u_x^{(n+m)})^2\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\quad+C\|u^{(n)}\left(u_x^{(n+m)}-u_x^{(n)}\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|u_x^{(n+m)}-u_x^{(n)}\|_{B^{-1/2}_{2,1}}\\
&\le C\|u^{(n+m)}-u^{(n)}\|_{B^{1/2}_{2,1}}\|u_x^{(n+m)}\|^2_{B^{1/2}_{2,1}}
\end{aligned}
$$

$$+ C\|u^{(n+m)} - u^{(n)}\|_{B^{1/2}_{2,1}} \left(\|u^{(n)}\|_{B^{3/2}_{2,1}} \|u^{(n+m)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}} \right)$$
$$\leq C\|u^{(n+m)} - u^{(n)}\|_{B^{1/2}_{2,1}} \left(\|u^{(n+m)}\|^2_{B^{3/2}_{2,1}} + \|u^{(n)}\|_{B^{3/2}_{2,1}} \|u^{(n+m)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}} \right).$$

We define

$$M = (13\|u_0\|_{B^{3/2}_{2,1}} + 1)^2, \tag{3.18}$$

then we have from above inequalities and (3.16) that

$$\begin{aligned}
&\|S_1(x,t) + S_2(x,t)\|_{B^{1/2}_{2,\infty}} \\
&\leq C\|u^{(n+m)} - u^{(n)}\|_{B^{1/2}_{2,1}} \\
&\times (1 + \|u^{(n+m)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(n+1)}\|_{B^{3/2}_{2,1}} + \|u^{(n+1)}\|_{B^{3/2}_{2,1}} \|u^{(n+m)}\|_{B^{3/2}_{2,1}} \\
&+ \|u^{(n+1)}\|_{B^{3/2}_{2,1}} \|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(n+m)}\|^2_{B^{3/2}_{2,1}} + \|u^{(n)}\|_{B^{3/2}_{2,1}} \|u^{(n+m)}\|_{B^{3/2}_{2,1}} + \|u^{(n)}\|^2_{B^{3/2}_{2,1}}) \\
&\leq C\frac{M}{2}\|u^{(n+m)} - u^{(n)}\|_{B^{1/2}_{2,1}}.
\end{aligned} \tag{3.19}$$

Note that

$$\begin{aligned}
\|(u_0^{(n+1+m)} - u_0^{(n+1)})\|_{B^{1/2}_{2,\infty}} &= \|(S_{n+1+m}u_0 - S_{n+1}u_0)\|_{B^{1/2}_{2,\infty}} = \|\sum_{q=n+1}^{n+m} \triangle_q u_0\|_{B^{1/2}_{2,\infty}} \\
&= \sup_{k\geq 1} 2^{\frac{1}{2}k} \|\triangle_k (\sum_{q=n+1}^{n+m} \triangle_q u_0)\|_{L^2} \\
&= \sup_{n+1\leq k\leq n+m+1} 2^{-k} 2^{\frac{3}{2}k} \|\triangle_{k-1}\triangle_k u_0 + \triangle_{k+1}\triangle_k u_0\|_{L^2} \\
&\leq \sup_{n\leq k\leq n+m} 2^{-k} 2^{\frac{3}{2}k} \|\triangle_k u_0\|_{L^2} \leq C2^{-n}\|u_0\|_{B^{3/2}_{2,1}}.
\end{aligned} \tag{3.20}$$

Applying (2.2) of Lemma 2.3 and using (3.17)-(3.20), we have for $t \in [0,T]$,

$$\begin{aligned}
&\|(u^{(n+1+m)} - u^{(n+1)})(t)\|_{B^{1/2}_{2,\infty}} \\
&\leq e^{C\int_0^t \|(u^{(n+m)}(t') - (u^{(n+m)}(t'))^2)\|_{B^{3/2}_{2,\infty}\cap \mathrm{Lip}} dt'} \|(u_0^{(n+1+m)} - u_0^{(n+1)})\|_{B^{1/2}_{2,\infty}} \\
&\quad + \int_0^t e^{C\int_\tau^t \|(u^{(n+m)}(t') - (u^{(n+m)}(t'))^2)\|_{B^{3/2}_{2,\infty}\cap \mathrm{Lip}} dt'} \|S_1(x,t) + S_2(x,t)\|_{B^{1/2}_{2,\infty}} d\tau
\end{aligned}$$

$$\leq CM2^{-n} + CM\int_0^t \|(u^{(n+m)} - u^{(n)})\|_{B^{1/2}_{2,1}} d\tau. \tag{3.21}$$

Combining (3.20), (3.21), (3.13) and (11) of Lemma 2.2, we know that for $\forall (n,m) \in \mathbb{N}^2$

$$\begin{aligned} w_{n+1,m} &= \|(u^{(n+1+m)} - u^{(n+1)})\|_{B^{1/2}_{2,\infty}} \\ &\leq CM\left(2^{-n} + \int_0^t \|(u^{(n+m)} - u^{(n)})\|_{B^{1/2}_{2,1}} d\tau\right) \\ &\leq CM\left[2^{-n} + \int_0^t \|(u^{(n+m)} - u^{(n)})\|_{B^{1/2}_{2,\infty}} \ln\left(e + \frac{\|(u^{(n+m)} - u^{(n)})\|_{B^{3/2}_{2,\infty}}}{\|(u^{(n+m)} - u^{(n)})\|_{B^{1/2}_{2,\infty}}}\right) d\tau\right] \\ &\leq CM\left[2^{-n} + \int_0^t w_{n,m}(\tau) \ln\left(e + \frac{M}{w_{n,m}(\tau)}\right) d\tau\right]. \end{aligned} \tag{3.22}$$

By (3.14) and (3.22), we have

$$w_{n+1} \leq CM\left[2^{-n} + \int_0^t w_n(\tau) \ln\left(e + \frac{M}{w_n(\tau)}\right) d\tau\right]. \tag{3.23}$$

Letting $n \to +\infty$ in (3.23) yields

$$\widetilde{w}(t) \leq CM\int_0^t \widetilde{w}(\tau) \ln(e + \frac{M}{\widetilde{w}(\tau)}) d\tau. \tag{3.24}$$

Because for $x \in (0, 1]$ and $\alpha > 0$, we have

$$\ln(e + \frac{\alpha}{x}) \leq \ln(e + \alpha)(1 - \ln x), \tag{3.25}$$

inequality (3.24) can be rewritten as

$$\widetilde{w}(t) \leq CM\int_0^t \widetilde{w}(\tau) \ln(e + M)(1 - \ln \widetilde{w}(\tau)) d\tau \tag{3.26}$$

provided that $\widetilde{w}(t) \leq 1$ on $[0, T]$. Using a Gronwall type argument (see e.g. Lemma 5.2.1 in [7]) yields $\widetilde{w}(t) = 0$ for $t \in [0, T]$.

Now we claim that $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B^{1/2}_{2,1})$. Using (2.1) of Lemma 2.2, we have that

$$\begin{aligned}
&\|(u^{(n+1+m)}-u^{(n+1)})(t)\|_{B^{1/2}_{2,1}}\\
&\le C(\theta)\|(u^{(n+1+m)}-u^{(n+1)})(t)\|^{\theta}_{B^{1/2}_{2,\infty}}\|(u^{(n+1+m)}-u^{(n+1)})(t)\|^{1-\theta}_{B^{3/2}_{2,\infty}}\\
&\le C(\theta)\|(u^{(n+1+m)}-u^{(n+1)})(t)\|^{\theta}_{B^{1/2}_{2,\infty}}\left(\|u^{(n+1+m)}(t)\|_{B^{3/2}_{2,1}}+\|u^{(n+1)}(t)\|_{B^{3/2}_{2,1}}\right)^{1-\theta}\\
&\le C(\theta)(CM)^{1-\theta}\|(u^{(n+1+m)}-u^{(n+1)})(t)\|^{\theta}_{B^{1/2}_{2,\infty}}\\
&=C(\theta)(CM)^{1-\theta}w^{\theta}_{n+1,m}(t). \qquad (3.27)
\end{aligned}$$

For $\forall t\in[0,T]$, $m\in\mathbb{N}$, we get from (3.27) that

$$\limsup_{n\to\infty}\|(u^{(n+1+m)}-u^{(n+1)})(t)\|_{B^{1/2}_{2,1}}=0.$$

Thus, $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{1/2}_{2,1})$, whence $(u^{(n)})_{n\in\mathbb{N}}$ converges to some limit $u\in C([0,T];B^{1/2}_{2,1})$.

Fourth step: Existence and continuity of solution in $E^{3/2}_{2,1}(T)$. Now we have to check that u belongs to $E^{3/2}_{2,1}(T)$ and satisfies (1.6). Since $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $L^{\infty}([0,T];B^{3/2}_{2,1})$. From (8) of Lemma 2.2, we have that $u\in L^{\infty}([0,T];B^{3/2}_{2,1})$. From (1.6), we can easily prove that $u_t\in L^{\infty}([0,T];B^{1/2}_{2,1})$. It is easily checked that u is indeed a solution to (1.6) by passing to the limit in (3.1). Using similar proof to [15], we can obtain that $u\in E^{3/2}_{2,1}(T)$.

Fifth step: Uniqueness. Uniqueness is a corollary of the following result.

Proposition 3.1. *Let v, u be solutions to the problem (1.6) with initial data v_0, u_0, respectively. Let $w(t):=v-u$. Obviously, $w(0):=v_0-u_0$. There exists a constant C such that if for some $T_\star\le T$*

$$\sup_{t\in[0,T_\star]}\left(e^{-C\int_0^t\|\partial_x(u(\tau)-(u(\tau))^2)\|_{B^{1/2}_{2,\infty}\cap L^\infty}d\tau}\|w(t)\|_{B^{1/2}_{2,\infty}}\right)\le 1,$$

then the following inequality holds true for $t\in[0,T_\star]$:

$$\|w(t)\|_{B^{1/2}_{2,\infty}}\le e^{\left(1+C\int_0^t\|\partial_x(u(\tau)-(u(\tau))^2)\|_{B^{1/2}_{2,\infty}\cap L^\infty}d\tau\right)}\left(\frac{\|w(0)\|_{B^{1/2}_{2,\infty}}}{e}\right)^{\exp[-CtZ\ln(e+Z)]}, \qquad (3.28)$$

where Z is defined as

$$Z=\left(13\|u_0\|_{B^{3/2}_{2,1}}+13\|v_0\|_{B^{3/2}_{2,1}}+1\right)^2.$$

In particular, if

$$\|w(0)\|_{B^{1/2}_{2,\infty}} \le e^{1-\exp[CTZ\ln(e+Z)]}, \tag{3.29}$$

then (3.28) *is valid on* $[0,T]$.

Proof. Obviously, w solves the following Cauchy problem for the transport equation:

$$\begin{aligned}
&w_t + \left(1+u-(u)^2\right) w_x \\
&= -\left(w + w(v+u)\right) v_x + P(D)[w + w(v+u) + w(v^2+vu+u^2) \\
&+ w_x(v_x+u_x) + wv_x + uw_x(v_x+u_x)) \\
&:= \widetilde{S_1}(x,t) + \widetilde{S_2}(x,t).
\end{aligned} \tag{3.30}$$

Using (2.2) of Lemma 2.3 and (3.30), we have

$$\begin{aligned}
&\|w(t)\|_{B^{1/2}_{2,\infty}} \le \|w(0)\|_{B^{1/2}_{2,\infty}} e^{C\int_0^t \|\partial_x\left(u(\tau)-(u(\tau))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau} \\
&+ \int_0^t e^{C\int_\tau^t \|\partial_x\left(u(t')-(u(t'))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} dt'} \|\widetilde{S_1}(x,t) + \widetilde{S_2}(x,t)\|_{B^{1/2}_{2,\infty}}, d\tau.
\end{aligned} \tag{3.31}$$

Following the proof of (3.19), we obtain

$$\|\widetilde{S_1}(x,t) + \widetilde{S_2}(x,t)\|_{B^{1/2}_{2,\infty}} \le CZ\|w\|_{B^{1/2}_{2,1}}. \tag{3.32}$$

Inserting (3.32) into (3.31) yields

$$\begin{aligned}
\|w(t)\|_{B^{1/2}_{2,\infty}} &\le \|w(0)\|_{B^{1/2}_{2,\infty}} e^{C\int_0^t \|\partial_x\left(u(\tau)-(u(\tau))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau} \\
&+ CZ\int_0^t e^{C\int_\tau^t \|\partial_x\left(u(t')-(u(t'))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} dt'} \|w\|_{B^{1/2}_{2,1}} d\tau \\
&\le \|w(0)\|_{B^{1/2}_{2,\infty}} e^{C\int_0^t \|\partial_x\left(u(\tau)-(u(\tau))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau} \\
&+ CZ\int_0^t e^{C\int_\tau^t \|\partial_x\left(u(t')-(u(t'))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} dt'} \|w\|_{B^{1/2}_{2,\infty}} \ln\left(e + \frac{\|w\|_{B^{3/2}_{2,\infty}}}{\|w\|_{B^{1/2}_{2,\infty}}}\right) d\tau \\
&\le \|w(0)\|_{B^{1/2}_{2,\infty}} e^{C\int_0^t \|\partial_x\left(u(\tau)-(u(\tau))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau} \\
&+ CZ\int_0^t e^{C\int_\tau^t \|\partial_x\left(u(t')-(u(t'))^2\right)\|_{B^{1/2}_{2,\infty}\cap L^\infty} dt'} \|w\|_{B^{1/2}_{2,\infty}}
\end{aligned}$$

$$\times \ln\left(e + \frac{Z}{e^{-C\int_0^\tau \|\partial_x(u(t')-(u(t'))^2)\|_{B^{1/2}_{2,\infty}\cap L^\infty} dt'}\|w\|_{B^{1/2}_{2,\infty}}}\right) d\tau. \tag{3.33}$$

Denote

$$W(t) = e^{-C\int_0^t \|\partial_x(u(\tau)-(u(\tau))^2)\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau}\|w\|_{B^{1/2}_{2,\infty}}.$$

Inequality (3.33) can be rewritten as

$$\begin{aligned} W(t) &\le \|W(0)\|_{B^{1/2}_{2,\infty}} + CZ\int_0^t W(\tau)\ln(e + \frac{Z}{W(\tau)})d\tau \\ &\le \|W(0)\|_{B^{1/2}_{2,\infty}} + CZ\int_0^t W(\tau)\ln(e+Z)(1-\ln W(\tau))d\tau \end{aligned}$$

provided that $W(\tau) \le 1$ on $[0,t]$. In light of the hypothesis and using a Gronwall type argument yields

$$\frac{W(t)}{e} \le (\frac{W(0)}{e})^{\exp[-CtZ\ln(e+Z)]},$$

implying the desired result. (3.29) implies that (3.28) is valid with $T_\star = T$. $\square$

Sixth step: Continuity with respect to the initial data in $B^{3/2}_{2,1}$.

Proposition 3.2. *For any $u_0 \in B^{3/2}_{2,1}$, there exist a $T > 0$ and a neighborhood V of u_0 in $B^{3/2}_{2,1}$ such that the map*

$$\Phi : \begin{cases} V \subset B^{3/2}_{2,1} \to C([0,T]; B^{3/2}_{2,1}), \\ v_0 \to v \text{ solution to (1.6) with initial datum } v_0 \end{cases}$$

is continuous.

Proof. Motivated by [16], we prove Proposition 3.2 by using Lemma 2.5.

First step: Continuity in $C([0,T]; B^{1/2}_{2,1})$. For $u_0 \in B^{3/2}_{2,1}$ and $r > 0$, we claim that there exist a $T > 0$ and a $M > 0$ such that for any $u_0' \in B^{3/2}_{2,1}$ with $\|u_0 - u_0'\|_{B^{3/2}_{2,1}} \le r$, the solution $u' = \Phi(u_0')$ of (1.6) associated with u_0' belongs to $C([0,T]; B^{3/2}_{2,1})$ and satisfies

$$\|u'\|_{L^\infty(0,T;B^{3/2}_{2,1})} \le M.$$

Indeed, from

$$\|u'\|_{B^{3/2}_{2,1}} \le \frac{2\|u_0'\|_{B^{3/2}_{2,1}}}{1-4C\|u_0'\|_{B^{3/2}_{2,1}}t},$$

we know that $T < \frac{1}{4C\|u_0'\|_{B^{3/2}_{2,1}}}$. Thus we can choose

$$T = \frac{1}{8C\left((\|u_0\|_{B^{3/2}_{2,1}}+r)+r\right)}, \quad M = 4\|u_0\|_{B^{3/2}_{2,1}} + 4r.$$

Then

$$T \le \frac{1}{8C(\|u_0'\|_{B^{3/2}_{2,1}}+r)},$$

and

$$\|u'\|_{B^{3/2}_{2,1}} \le \frac{2\|u_0'\|_{B^{3/2}_{2,1}}}{1-\frac{\|u_0'\|_{B^{3/2}_{2,1}}}{2(\|u_0'\|_{B^{3/2}_{2,1}}+r)}} \le 4\|u_0'\|_{B^{3/2}_{2,1}} \le M.$$

Combining the above uniform bounds with Proposition 3.1, we infer that

$$\frac{\|\Phi(u_0')-\Phi(u_0)\|_{L^\infty(0,T;B^{1/2}_{2,\infty})}}{e} \le e^{C(M+M^2)T}\left(\frac{\|u_0'-u_0\|_{B^{1/2}_{2,\infty}}}{e}\right)^{\exp[-CTZ\ln(e+Z)]}$$

provided that

$$\|u_0'-u_0\|_{B^{1/2}_{2,\infty}} \le e^{1-\exp[CTZ\ln(e+Z)]}.$$

In view of the uniform bounds in $C([0,T];B^{3/2}_{2,1})$ and an interpolation argument, we infer the map Φ is continuous from $B^{3/2}_{2,1}$ into $C([0,T];B^{1/2}_{2,1})$.

Second step: Continuity in $C([0,T];B^{3/2}_{2,1})$. Let $u_0^{(\infty)} \in B^{3/2}_{2,1}$ and $(u_0^{(n)})_{n\in\mathbb{N}}$ tend to $u_0^{(\infty)}$ in $B^{3/2}_{2,1}$. We denote by $u^{(n)}$ the solution with the initial data $u_0^{(n)}$. From the first step, we can find $T, M>0$ such that for all $n \in \mathbb{N}$, $u^{(n)}$ is defined on $[0,T]$ and

$$\sup_{n\in\overline{\mathbb{N}}} \|u^{(n)}\|_{L^\infty(0,T;B^{3/2}_{2,1})} \le M.$$

Thanks to step one, proving that $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T]; B^{3/2}_{2,1})$ amounts to proving that $v^{(n)} = \partial_x u^{(n)}$ tends to $v^{(\infty)} = \partial_x u^{(\infty)}$ in $C([0,T]; B^{1/2}_{2,1})$. Notice that $v^{(n)}$ solves the following linear transport equations

$$\begin{cases} \partial_t v^{(n)} + \left(1 + u^{(n)} - (u^{(n)})^2\right) \partial_x v^{(n)} = \widetilde{f}^{(n)}, \\ v^{(n)}|_{t=0} = \partial_x u_0^{(n)}, \end{cases}$$

with

$$\begin{aligned} \widetilde{f}^{(n)} =& u^{(n)} + (u^{(n)})^2 + (u^{(n)})^3 + 3u^{(n)}(u_x^{(n)})^2 \\ & - G * \left(u^{(n)} + (u^{(n)})^2 + (u^{(n)})^3 + (u_x^{(n)})^2 + u^{(n)}(u_x^{(n)})^2\right), \end{aligned}$$

where $(1-\partial_x^2)^{-1} f = G * f$. Following the method in [27], we decompose $v^{(n)} = v_1^{(n)} + v_2^{(n)}$ with

$$\begin{cases} \partial_t v_1^{(n)} + \left(1 + u^{(n)} - (u^{(n)})^2\right) \partial_x v_1^{(n)} = \widetilde{f}^{(n)} - \widetilde{f}^{(\infty)}, \\ v_1^{(n)}|_{t=0} = \partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}, \end{cases}$$

and

$$\begin{cases} \partial_t v_2^{(n)} + \left(1 + u^{(n)} - (u^{(n)})^2\right) \partial_x v_2^{(n)} = \widetilde{f}^{(\infty)}, \\ v_2^{(n)}|_{t=0} = \partial_x u_0^{(\infty)}. \end{cases}$$

Since $B^{1/2}_{2,1}$ is a Banach algebra, $\forall n \in N$, by using the S^{-2}-multiplier property of $(1-\partial_x^2)^{-1}$ and $B^{3/2}_{2,1} \hookrightarrow B^{1/2}_{2,1}$, we can check that $(\widetilde{f}^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^{1/2}_{2,1})$. Moreover, we can obtain the following inequalities

$$\begin{aligned} \|\widetilde{f}^{(n)} - \widetilde{f}^{(\infty)}\|_{B^{1/2}_{2,1}} \le C & \left(1 + (\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(\infty)}\|_{B^{3/2}_{2,1}}) + (\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|u^{(\infty)}\|_{B^{3/2}_{2,1}})^2\right) \\ & \times \left(\|u^{(n)} - u^{(\infty)}\|_{B^{1/2}_{2,1}} + \|u_x^{(n)} - u_x^{(\infty)}\|_{B^{1/2}_{2,1}}\right). \end{aligned}$$

Applying Lemma 2.3, one can deduce that

$$\begin{aligned} & \|v_1^{(n)}(t)\|_{B^{1/2}_{2,1}} \\ & \le e^{C\int_0^t \|\partial_x\left(u(\tau) - (u(\tau))^2\right)\|_{B^{1/2}_{2,1}} d\tau} \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{1/2}_{2,1}} \\ & + C\int_0^t e^{C\int_\tau^t \|\partial_x\left(u(\tau') - (u(\tau'))^2\right)\|_{B^{1/2}_{2,1}} d\tau'} \|\widetilde{f}^{(n)} - \widetilde{f}^{(\infty)}\|_{B^{1/2}_{2,1}} d\tau \\ & \le e^{C\int_0^t \|\partial_x\left(u(\tau) - (u(\tau))^2\right)\|_{B^{1/2}_{2,1}} d\tau} \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{1/2}_{2,1}} \end{aligned}$$

$$
+C\int_0^t e^{C\int_\tau^t \|\partial_x\left(u(\tau')-(u(\tau'))^2\right)\|_{B^{1/2}_{2,1}}d\tau'}\left(\|u^{(n)}-u^{(\infty)}\|_{B^{1/2}_{2,1}}+\|u^{(n)}_x-u^{(\infty)}_x\|_{B^{1/2}_{2,1}}\right)
$$

$$
\times\left(1+(\|u^{(n)}\|_{B^{3/2}_{2,1}}+\|u^{(\infty)}\|_{B^{3/2}_{2,1}})+(\|u^{(n)}\|_{B^{3/2}_{2,1}}+\|u^{(\infty)}\|_{B^{3/2}_{2,1}})^2\right)d\tau. \tag{3.34}
$$

Applying similar arguments as in [16] on p. 441 to (3.34), we have

$$
\partial_x u^{(n)}\to\partial_x u^{(\infty)}\quad\text{in } B^{1/2}_{2,1}.
$$

We have completed the proof of Proposition 3.2. □

Summing up the above six steps, we get Theorem 1.1.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 needs the following Lemma.

Lemma 4.1. *[32] Let $u(x,t)\in B^{3/2}_{2,1}$ be a solution to the problem (1.1) and (1.4) and T^* be the maximal existence time of the corresponding solution, then it holds that*

$$
\|u\|_{H^1}\le\|u_0\|_{H^1}\exp\{C\int_0^t\|u_x\|^2_{L^\infty}d\tau\}\quad \textit{for } t\in[0,T^*) \tag{4.1}
$$

for some constant C.

With Lemma 4.1 in hand, we are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Applying $\triangle_q$ to the first equation of (1.6) yields

$$
\begin{aligned}
&\left(\partial_t+(1+u-u^2)\partial_x\right)\triangle_q u\\
&=[u-u^2,\triangle_q]\partial_x u+\triangle_q P(D)\left(u+u^2+u^3+u_x^2+uu_x^2\right).
\end{aligned} \tag{4.2}
$$

By virtue of (2.54) on p. 112 in [2], we have that

$$
\begin{aligned}
&\|2^{\frac{3}{2}q}\|[u-u^2,\triangle_q]\partial_x u\|_{L^2}\|_{l^1}\\
&\le C\left(\|(u-u^2)_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}+\|u_x\|_{L^\infty}\|u-u^2\|_{B^{3/2}_{2,1}}\right).
\end{aligned} \tag{4.3}
$$

By (5) of Lemma 2.2 and the fact that $P(D)$ is an S^{-1}-multiplier, we get that

$$\begin{aligned}
&\|(u-u^2)_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}+\|u_x\|_{L^\infty}\|u-u^2\|_{B^{3/2}_{2,1}}\\
&\le C\left(1+\|u\|_{L^\infty}\right)\|u_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}+C\|u_x\|_{L^\infty}\left(\|u\|_{B^{3/2}_{2,1}}+\|u\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}\right)\\
&\le C\left(1+\|u\|_{L^\infty}\right)\|u_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}},
\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}
&\|P(D)\left(u+u^2+u^3+u_x^2+uu_x^2\right)\|_{B^{3/2}_{2,1}}\\
&\le C\|\left(u+u^2+u^3+u_x^2+uu_x^2\right)\|_{B^{1/2}_{2,1}}\\
&\le C\left(1+\|u\|_{L^\infty}+\|u\|^2_{L^\infty}+\|u_x\|_{L^\infty}+\|u\|_{L^\infty}\|u_x\|_{L^\infty}\right)\|u\|_{B^{3/2}_{2,1}}.
\end{aligned}\tag{4.5}$$

Going along the lines of the proof of Proposition A.1 in [15] leads to

$$\begin{aligned}
\|u\|_{B^{3/2}_{2,1}}\le{}&\|u_0\|_{B^{3/2}_{2,1}}\\
&+C\int_0^t\left(1+\|u\|_{L^\infty}+\|u\|^2_{L^\infty}+\|u_x\|_{L^\infty}+\|u\|_{L^\infty}\|u_x\|_{L^\infty}\right)\|u\|_{B^{3/2}_{2,1}}d\tau
\end{aligned}\tag{4.6}$$

in view of (4.4) and (4.5). Employing Gronwall's inequality combined with (4.1) and $H^1\hookrightarrow L^\infty$, we obtain

$$\begin{aligned}
&\|u\|_{B^{3/2}_{2,1}}\\
&\le\|u_0\|_{B^{3/2}_{2,1}}\exp\left[C\int_0^t\left(1+\|u\|_{L^\infty}+\|u\|^2_{L^\infty}+\|u_x\|_{L^\infty}+\|u\|_{L^\infty}\|u_x\|_{L^\infty}\right)d\tau\right]\\
&\le\|u_0\|_{B^{3/2}_{2,1}}\exp\left[C\int_0^t\left(1+\|u\|_{H^1}+\|u\|^2_{H^1}+\|u_x\|_{L^\infty}+\|u\|_{H^1}\|u_x\|_{L^\infty}\right)d\tau\right]\\
&\le\|u_0\|_{B^{3/2}_{2,1}}\exp\Bigg[C\int_0^t\Bigg(1+\|u_0\|_{H^1}e^{C\int_0^\tau\|u_x\|^2_{L^\infty}dt'}+\|u_0\|^2_{H^1}e^{C\int_0^\tau\|u_x\|^2_{L^\infty}dt'}\\
&\quad+\|u_x\|_{L^\infty}+\|u_0\|_{H^1}e^{C\int_0^\tau\|u_x\|^2_{L^\infty}dt'}\|u_x\|_{L^\infty}\Bigg)d\tau\Bigg].
\end{aligned}\tag{4.7}$$

Now we prove the claim (1.5) by contradiction. If there exists a M such that

$$\int_0^{T^*}\|u_x\|^2_{L^\infty}d\tau\le M,\tag{4.8}$$

then we have

$$\int_0^{T^*} \|u_x\|_{L^\infty} d\tau \le M$$

in view of the embedding $L^2(0,T^*) \hookrightarrow L^1(0,T^*)$ for finite T^* and get from (4.7) that

$$\begin{aligned}
&\|u(T^*)\|_{B^{3/2}_{2,1}} \le \|u_0\|_{B^{3/2}_{2,1}} \times \\
&\exp\left[C\int_0^{T^*}\left(1+\|u_0\|_{H^1}e^{CM}+\|u_0\|^2_{H^1}e^{CM}+\|u_x\|_{L^\infty}+\|u_0\|_{H^1}e^{CM}\|u_x\|_{L^\infty}\right)d\tau\right] \\
&\le \|u_0\|_{B^{3/2}_{2,1}}\exp\Bigg[C\left(1+\|u_0\|_{B^{3/2}_{2,1}}e^{CM}+\|u_0\|^2_{B^{3/2}_{2,1}}e^{CM}\right)T^* \\
&\qquad +C\left(1+\|u_0\|_{B^{3/2}_{2,1}}e^{CM}\right)\int_0^{T^*}\|u_x\|_{L^\infty}d\tau\Bigg] < +\infty,
\end{aligned} \tag{4.9}$$

which contradicts the fact that T^* is the maximal time of the existence of the solution. This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

In this section, we are devoted to establishing the existence and uniqueness of analytic solutions to the system (1.6) on the line $\mathbb{R}$.

The proof of Theorem 1.3 needs a suitable scale of Banach spaces as follows. For any $s > 0$, we set

$$E_s = \left\{ u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k\in\mathbb{N}_0} \frac{s^k \|\partial^k u\|_{H^2}}{k!/(k+1)!} < \infty \right\}, \tag{5.1}$$

where $\mathbb{N}_0$ is the set of nonnegative integers. It is easy to verify that E_s equipped with the norm $|||\cdot|||$ is a Banach space and that for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \le |||u|||_s. \tag{5.2}$$

By this definition, one can easily get that u in E_s is a real analytic function on $\mathbb{R}$ and what is crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Lemma 5.1. *([24]) Let $0 < s < 1$. There is a constant $C > 0$, independent of s, such that for any u and v in E_s we have*

$$|||uv|||_s \le C|||u|||_s |||v|||_s. \tag{5.3}$$

Lemma 5.2. *([24]) There is a constant $C > 0$ such that for any $0 < s' < s < 1$, we have $|||\partial_x u|||_{s'} \leq \frac{C}{s-s'} |||u|||_s$, and $\left|\left|\left|(1-\partial_x^2)^{-1}u\right|\right|\right|_{s'} \leq |||u|||_s$, $\left|\left|\left|(1-\partial_x^2)^{-1}\partial_x u\right|\right|\right|_{s'} \leq |||u|||_s$.*

Theorem 5.1. *([6]) Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t, u(t)), \\ u(0) = 0. \end{cases} \tag{5.4}$$

Let T, R and C be positive constants and assume that F satisfies the following conditions:

(1) *If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is real analytic in $|t| < T$ and continuous on $|t| \leq T$ with values in X_s and*

$$\sup_{|t|\leq T} |||u|||_s < R, \tag{5.5}$$

then $t \mapsto F(t, u(t))$ is a real analytic function on $|t| < T$ with values in $X_{s'}$.

(2) *For any $0 < s' < s < 1$ and any $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,*

$$\sup_{|t|\leq T} |||F(t,u) - F(t,v)|||_{s'} \leq \frac{C}{s-s'} |||u - v|||_s. \tag{5.6}$$

(3) *There exists $M > 0$ such that for any $0 < s < 1$*

$$\sup_{|t|\leq T} |||F(t,0)|||_s \leq \frac{M}{1-s}. \tag{5.7}$$

Then there exists a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $0 < s < 1$ is real analytic in $|t| < (1-s)T_0$ with values in X_s, and is a solution to the Cauchy problem (5.4).

To prove Theorem 1.3, we need to show that all three conditions of the abstract version of the Cauchy-Kowalevski theorem (Theorem 5.1) hold for system (1.6) on the scale $\{X_s\}_{0<s<1}$. Towards this end, we restate the Cauchy problem (1.6) in a more convenient form. Let $u_1 = u$, $u_2 = u_x$, then the problem (1.6) is transformed in a system for u_1 and u_2.

$$\begin{cases} \partial_t u_1 = -u_2 - \frac{1}{2}\partial_x(u_1^2) + \frac{1}{3}\partial_x(u_1^3) - \partial_x G * (u_1 + u_1^2 \\ \qquad\qquad + u_1^3 + u_2^2 + u_1 u_2^2) = F_1(u_1, u_2), \\ \partial_t u_2 = \partial_x(-u_2 - u_1 u_2 + u_1^2 u_2) + (u_1 + u_1^2 + u_1^3 + u_2^2 + u_1 u_2^2) \\ \qquad\qquad - G * (u_1 + u_1^2 + u_1^3 + u_2^2 + u_1 u_2^2) = F_2(u_1, u_2), \\ u_1(x, 0) = u_0(x), u_2(x, 0) = u_0'(x). \end{cases} \tag{5.8}$$

Proof of Theorem 1.3. Let $u = (u_1, u_2)$, $F = (F_1, F_2)$ in (5.8) and X_s be a scale of decreasing Banach spaces defined as $X_s = E_s \times E_s$. Since the map $F(u_1, u_2)$ does not depend on t explicitly, we just need to verify the first two conditions of Theorem 5.1.

Obviously, $t \mapsto F(t,u(t)) = (F_1(u_1,u_2), F_2(u_1,u_2))$ is real analytic if $t \mapsto u_1(t)$ and $t \mapsto u_2(t)$ are both real analytic. Hence, the verification of the first condition of the abstract theorem needs only to show that for $s' < s$, $F_1(u_1,u_2)$ and $F_2(u_1,u_2)$ are in $E_{s'}$ for $u_1,u_2 \in E_s$. By Lemma 5.1 and Lemma 5.2, we can get the estimates of F_1 and F_2 as

$$\begin{aligned}
|||F_1(u_1,u_2)|||_{s'} &= \left|\left|\left| -u_2 - \frac{1}{2}\partial_x(u_1^2) + \frac{1}{3}\partial_x(u_1^3) - \partial_x G*(u_1+u_1^2+u_1^3+u_2^2+u_1u_2^2)\right|\right|\right|_{s'} \\
&\le \frac{C}{s-s'}\left(|||u_1|||_s + |||u_1|||_s^2 + |||u_1|||_s^3\right) \\
&\quad + C\left(|||u_1|||_s + |||u_1|||_s^2 + |||u_1|||_s^3 + |||u_2|||_s^2 + |||u_1|||_s|||u_2|||_s^3\right),
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
|||F_2(u_1,u_2)|||_{s'} &\le \left|\left|\left|\partial_x(-u_2 - u_1u_2 + u_1^2u_2)\right|\right|\right|_{s'} \\
&\quad + \left|\left|\left|(u_1+u_1^2+u_1^3+u_2^2+u_1u_2^2) - G*(u_1+u_1^2+u_1^3+u_2^2+u_1u_2^2)\right|\right|\right|_{s'} \\
&\le \frac{C}{s-s'}\left(|||u_2|||_s + |||u_1|||_s|||u_2|||_s + |||u_1|||_s^2|||u_2|||_s\right) \\
&\quad + C\left(|||u_1|||_s + |||u_1|||_s^2 + |||u_1|||_s^3 + |||u_2|||_s^2 + |||u_1|||_s|||u_2|||_s^3\right).
\end{aligned} \tag{5.10}$$

We proceed to verify the second condition of the abstract theorem. Employing the triangle inequality and Lemma 5.1 with Lemma 5.2, we have

$$\begin{aligned}
&|||F_1(u_1,u_2) - F_1(v_1,v_2)|||_{s'} \\
&\le \left|\left|\left|(v_2-u_2) + \frac{1}{2}\partial_x(v_1^2-u_1^2) + \frac{1}{3}\partial_x(u_1^3-v_1^3)\right|\right|\right|_{s'} \\
&+ \left|\left|\left|\partial_x G*\left((u_1-v_1)+(u_1^2-v_1^2)+(u_1^3-v_1^3)+(u_2^2-v_2^2)+(u_1u_2^2-v_1v_2^2)\right)\right|\right|\right|_{s'} \\
&\le C|||u_2-v_2|||_s + \frac{C}{s-s'}|||u_1-v_1|||_s|||u_1+v_1|||_s + \frac{C}{s-s'}|||u_1-v_1|||_s \\
&\times \left(|||u_1|||_s^2 + |||u_1|||_s|||v_1|||_s + |||v_1|||_s^2\right) + |||u_1-v_1|||_s + |||u_1-v_1|||_s|||u_1+v_1|||_s \\
&+ |||u_1-v_1|||_s\left(|||u_1|||_s^2 + |||u_1|||_s|||v_1|||_s + |||v_1|||_s^2\right) + |||u_2-v_2|||_s|||u_2+v_2|||_s \\
&+ |||u_1-v_1|||_s|||u_2|||_s^2 + |||u_2-v_2|||_s|||v_1|||_s|||u_2+v_2|||_s \\
&\le \frac{C}{s-s'}|||u-v|||_{X_s}.
\end{aligned} \tag{5.11}$$

Similarly, we can show that

$$|||F_2(u_1,u_2) - F_2(v_1,v_2)|||_{s'} \le \frac{C}{s-s'}|||u-v|||_{X_s} \tag{5.12}$$

holds. The conditions (1)-(3) are now easily verified once our system (5.8) is transformed into a new system with zero initial data as in (5.4). This completes the proof of Theorem 1.2. □

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