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A new variational inequality in the calculus of variations and Lipschitz regularity of minimizers

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Dedicated to Francis Clarke and Richard Vinter

Abstract

We consider a problem of the calculus of variations of the form

$$\begin{cases} \text{Minimize} & I(x) := \displaystyle\int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b)) \\ \text{Subject to:} & x \in W^{1,m}([a,b]; \mathbb{R}^n), \\ & x'(t) \in C \ \text{ a.e., } x(t) \in \Sigma \quad \forall t \in [a,b], \end{cases} \tag{P}$$

where $\Lambda : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is Borel measurable, C is a cone, Σ is a nonempty subset of $\mathbb{R}^n$ and Ψ is an arbitrary extended valued function. We prove that if $t \mapsto \Lambda(t, x, \xi)$ satisfies a nonsmooth version of Cesari's Condition (S), then any local minimizer of (P) in the AC norm is Lipschitz whenever Λ is coercive. The proof is obtained via a new variational inequality formulated here, that holds under the extended Condition (S) (just Borel measurability if Λ is autonomous): For a.e. t in $[a,b]$,

$$\Lambda\Big(t, x_*(t), \frac{x'_*(t)}{v}\Big)v - \Lambda(t, x_*(t), x'_*(t)) \geq p(t)(v-1), \ \ \forall v > 0, \tag{W}$$

where $p(t)$ is an absolutely continuous arc, the derivative of which belongs to a suitable subdifferential of $\Lambda(\cdot, x_*(t), x'_*(t))$ a.e. on $[a,b]$. The proof of the directional Weierstrass type condition (W) is based on Clarke's nonsmooth recent versions of the Maximum Principle.

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1. Introduction

We consider the classical problem of minimizing a functional

$$I(x) := \int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b))$$

among the absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ satisfying a given state constraint Σ. The map $\Lambda : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is referred to as the *Lagrangian*, and generic notation for its three variables is (t, x, ξ). The function Ψ takes its values in $\mathbb{R} \cup \{+\infty\}$ and this allows to cover any kind of endpoint constraints. Our main interest in this paper is establishing conditions needed to derive the Lipschitz regularity of the minimizers for (P), when we have a superlinear growth assumption, which is one of Tonelli-type growth hypotheses used for the existence of minimizers. It is well known that, in the general nonautonomous case, Tonelli existence hypotheses alone cannot guarantee, for a minimizer, the Lipschitz regularity or the validity of standard first order necessary conditions of optimality. This issue is clearly illustrated by the Ball-Mizel's example, [3], and we refer to [26, Chapter 11] for a related discussion. However, if one can prove that a minimizer possesses the Lipschitz continuity, then an established procedure allows to derive Euler-Lagrange type conditions.

Lipschitz regularity results for minimizers are also the first step in order to obtain further regularity properties of minimizers, to study the Hamilton-Jacobi equation associated with the problem (P), and they allow to avoid the occurrence of the Lavrentiev phenomenon, i.e., the fact that the infimum of I among the absolutely continuous functions is strictly less than the infimum among Lipschitz functions that share the same boundary values [2,6,7,18,27].

Most of the results in the literature concern the case of fixed boundary constraints for **autonomous** Lagrangians. In this context, the superlinearity of $\xi \mapsto \Lambda(x, \xi)$ guarantees the Lipschitz regularity of the minimizers. This result was first established in [16] under some regularity assumptions on Λ, and was subsequently refined allowing Λ to be just Borel [18]. The interest in looking for results that take into account such a wide class of Lagrangians is not just a taste of abstract generality. Indeed, in spite of Tonelli's existence result, some variational problems with a Lagrangian $\Lambda(t, x, \xi)$ that is not convex in the variable ξ may have a minimizer; moreover, discontinuous Lagrangians arise often in real life Engineering problems (e.g., fuel consumption, combustion in nonhomogeneous media). Much less is known on the Lipschitz regularity of minimizers for the nonautonomous case. In this framework, the main obstruction is given by the existence of polynomial Lagrangians $\Lambda(t, x, \xi)$ that are superlinear and convex in ξ, for which the Lavrentiev phenomenon occurs (see [3]). Among the few results in this framework ([16, §3], [12, §4.3], [23]), none gives back [18, Theorem 2.1] when the Lagrangian is autonomous, i.e., all require some extra regularity hypothesis on the variables x or ξ.

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We show here in Theorem 6.3 that, if $\Lambda(t,x,\xi)$ is Borel, superlinear in ξ, and bounded on bounded sets, then every minimizer of (P) is Lipschitz whenever Λ satisfies a suitable regularity assumption w.r.t. t, called Hypothesis (S_{x_*}). The conclusion concerns also a class of Lagrangians allowed, in addition, to take the value $+\infty$, under Hypothesis ($S^{\infty}_{x_*}$) described in § 3. Theorem 6.3 below improves previous work on the subject for several aspects:

- In the autonomous case, for global minimizers, it gives back the more general result [18, Theorem 2.1] and even extends its validity for some extended valued Lagrangians and a weaker local boundedness assumption;
- We consider minimizers that are local in the sense of the norm of the absolute continuous functions, not just global ones;
- We include a class of Lagrangians with extended values.

Hypothesis (S_{x_*}) is a nonsmooth extension of Cesari's Condition (S), used in [9, §2.7 A] to derive the Erdmann – DuBois-Reymond equation for differentiable Lagrangians.

A standard approach in proving the Lipschitz regularity of a minimizer $x_*(\cdot)$ is to apply the classical Weierstrass inequality for $\xi \mapsto \Lambda(t, x_*(t), \xi)$ to a suitable value of ξ (see [12, Theorem 4.5.2]. The Weierstrass inequality is valid for absolutely continuous minimizers under some regularity assumptions on $(x,\xi) \mapsto \Lambda(t,x,\xi)$ ([20, Theorem1], [12, Theorem 4.3.1]), not required in the Hypotheses (S_{x_*}) or ($S^{\infty}_{x_*}$) employed here. The proof of Theorem 6.3 relies instead on a **new variational inequality** that holds under the above mentioned hypotheses, formulated in Theorems 4.1 and 4.3: There exists an absolutely continuous arc $p \in W^{1,1}([a,b];\mathbb{R})$ such that, for almost every $t \in [a,b]$:

$$\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p(t)(v-1) \qquad \forall v > 0, \tag{W}$$

and p' belongs a.e. to a suitable subgradient of $s \mapsto \Lambda(s, x_*(t), x_*'(t))$ at $s = t$. Condition (W) is, to our knowledge, new even for the autonomous case, when it is valid taking $p(t)$ equal to a suitable constant, and for **every Borelian autonomous Lagrangian**.

We call (W) a **directional Weierstrass type condition** since it is equivalent to the following convexity condition of Λ along the directions of $x_*'(t)$: for a.e. t in $[a,b]$,

$$\Lambda(t, x_*, rx_*') - \Lambda(t, x_*, x_*') \geq \big(\Lambda(t, x_*, x_*') - p(t)\big)(r-1) \quad r > 0. \tag{W_r}$$

Condition (W) is related to the classical necessary conditions in the Calculus of Variations: when Λ is smooth it implies easily the Erdmann – Du Bois-Reymond (EDBR) equation in its full generality, and the validity of the Weierstrass inequality, together with the (EDBR), imply (W). Note, however, that the validity of the Weierstrass inequality requires either the Lipschitz continuity of the minimizer, or some Lipschitz regularity on the x-variable in $\Lambda(t,x,\xi)$, that we do not require for the validity of (W).

Most of the paper is devoted to the proof of the new variational inequality (W). When Λ takes values in $\mathbb{R}$ (Theorem 4.1) the first step is inspired by the proof of [26, Theorem 11.5.1] for the autonomous case assuming that $\Lambda(x,\cdot)$ is convex, and that of [19, Theorem 2.1] concerning the Lipschitz continuity for derivatives of higher order: We look at the necessary conditions for a suitable control problem obtained from the reference Problem (P) via reparametrization of the time variable. The result follows by applying Clarke's Maximum Principle: The local Lipschitz

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continuity assumed on t in Hypothesis (S_{x_*}) is thus shifted into a local Lipschitz condition on the extra state variable y appearing in an auxiliary Lagrangian $\ell(t, y, v)$ derived from the reference one Λ. Convexity in the velocity variable is no more needed, since as in [18, Lemma 2.3] we allow the parameter v to vary in the upper unbounded interval $[1/2, +\infty[$. The validity of (W) with $v > 0$ is obtained employing a compactness argument.

When the Lagrangian is autonomous, the proof of (W) when x_* is a strong local minimum may be carried out in a more elementary way by means of the Kuhn-Tucker multiplier rule for a suitable convex optimization problem as described in [13] for the case of Lagrangians which are convex in ξ. The condition that the derivative of the arc p in (W) belongs to a suitable subgradient of $s \mapsto \Lambda(s, x_*(t), x_*'(t))$ at $\tau = t$ is obtained via the Euler inclusion for the auxiliary Lagrangian ℓ. This approach is displayed in Section 8 and furnishes a straightforward alternative to the final part of the proof of [18, Theorem 2.2], though still relying on [18, Lemma 2.4].

The proof of (W) when Λ has extended values (Theorem 4.3) follows the path of that of [12, Theorem 4.5.5], which deals with the simpler autonomous case. The lower semicontinuity of $(t, \xi) \mapsto \Lambda(t, x, \xi)$ guarantees the same regularity of $(y, v) \mapsto \ell(t, y, v)$. Condition (iii) of Hypotheses $(S^\infty_{x_*})$ implies that ℓ satisfies the generalized Tonelli-Morrey condition [12, Definition 4.3.1]: The Euler-Weierstrass necessary conditions of [12, Theorem 4.3.2] then yield (W), since Condition (ii) of Hypotheses $(S^\infty_{x_*})$ avoids the abnormal case.

In the autonomous framework, Lipschitz regularity results were obtained (see [8,21]) imposing growth assumptions that are weaker than the superlinear one and allow to cover important cases which do not satisfy the Tonelli existence hypotheses. In this paper, stepping into the nonautonomous context, we are mainly interested in establishing the validity of the variational inequality (W), showing that a direct consequence is the Lipschitz continuity of minimizers in the superlinear case. We also provide an application deriving (under the Tonelli-type hypotheses) standard (nonsmooth) necessary optimality conditions for state constrained problems in the Calculus of Variations. The discussion of further possible consequences of the variational inequality (W), taking into account weaker growth conditions is the object of subsequent papers [4,5].

2. Notation

We recall that if X is a subset of $\mathbb{R}^n$, given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point x with $f(x) < +\infty$, the **proximal subdifferential** $\partial^P f(x)$ of f at x is the set of elements $\zeta \in \mathbb{R}^n$ such that there exists $\sigma \geq 0$ satisfying

$$f(x') - f(x) + \sigma|x' - x|^2 \geq \zeta \cdot (x' - x)$$

for x' in a neighborhood of x. If f is of class $\mathcal{C}^2$ near x then $\partial^P f(x) = \{\nabla f(x)\}$ (see [14, Proposition 11.1]).

The **convex subdifferential** of f at x is

$$\partial f(x) = \{\zeta \in \mathbb{R}^n : f(x') - f(x) \geq p \cdot (x' - x) \quad \forall x' \in \mathbb{R}^n\}.$$

The **limiting subdifferential** of f at $x \in X$ is the set

$$\partial^L f(x) := \{\lim \zeta_i : \zeta_i \in \partial^P f(x_i),\ x_i \to x,\ f(x_i) \to f(x)\}.$$

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If, moreover, f is real valued and is locally Lipschitz at x, the **Clarke's generalized gradient** $\partial^C f(x)$ of f at x is the convex hull co $\partial^L f(x)$ of $\partial^L f(x)$.

If f is of class $\mathcal{C}^1$ near x then $\partial^L f(x) = \partial^C f(x) = \{\nabla f(x)\}$ (see [14, Proposition 11.12]).

- If $x := (x_1, ..., x_m) \mapsto f(x)$ is a function then $D_{x_i} f$ denotes the classical partial derivative of f with respect to x_i. The symbols

$$\partial_{x_i} f,\ \partial^P_{x_i} f,\ \partial^L_{x_i} f,\ \partial^C_{x_i} f$$

  denote, respectively, the convex/proximal/limiting/Clarke subdifferential of $x_i \mapsto f(x_1, ..., x_i, ..., x_m)$.
- The L^∞-norm (resp. L^m-norm) on $[a, b]$ is denoted by $\|\cdot\|_\infty$ (resp. $\|\cdot\|_m$).

Definition 2.1 *(LB-measurability).* Let A, B be two Borelian subsets of, respectively, $\mathbb{R}^m$ and $\mathbb{R}^n$. A function $f : A \times B \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ of two variables (x, ξ) is said to be Lebesgue-Borel measurable if it is measurable with respect to the σ-algebra $\mathcal{L}_m \times \mathcal{B}_n$ generated by the products of Lebesgue measurable subsets of A and Borel measurable subsets of B.

We will use the fact that if $f(\cdot, \cdot)$ is as in Definition 2.1 and $u : \mathbb{R}^m \to \mathbb{R}^n$ is Lebesgue measurable then $f(t, u(t))$ is Lebesgue measurable (see [14, Proposition 6.34]).

We refer to [26,14] for a thorough description of the notions of limiting and generalized gradients: We just mention that the limiting subdifferential of a convex (resp. $\mathcal{C}^1$) function coincides with the subdifferential in the sense of convex analysis (resp. gradient).

The **convex hull** (resp. **closed convex hull**) of a subset X of $\mathbb{R}^k$ is denoted by co X (resp. $\overline{\text{co}}\ X$).

3. Main assumptions

We will consider the following variational problem (P) with *state and derivative constraints*.

Problem (P). Let $m \geq 1$. We consider a prescribed nonempty subset Σ of $\mathbb{R}^n$, a cone C in $\mathbb{R}^n$, and function $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\Psi \not\equiv +\infty$.

The **Lagrangian** $\Lambda : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is Borel.

The problem (P) is described as follows:

$$\begin{cases} \text{Minimize} & I(x) := \displaystyle\int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b)) \\ \text{Subject to:} & x \in W^{1,m}([a, b]; \mathbb{R}), \\ & x'(t) \in C \ \text{ a.e., } x(t) \in \Sigma \quad \forall t \in [a, b]. \end{cases} \tag{P}$$

For the sake of brevity we will often write $\Lambda(t, x_*, x'_*)$ instead of $\Lambda(t, x_*(t), x'_*(t))$.

We underline the fact that we do not impose any further assumption on the set Σ and the function Ψ.

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Definition 3.1. A function $x_* \in W^{1,m}([a,b];\mathbb{R})$ is a (global) **minimum** of (P) if $I(x_*)$ is finite and $I(x_*) \le I(x)$ for all admissible x. The function x_* is said to be a $W^{1,m}$ **local minimum** if the above inequality holds just as

$$\|x - x_*\|_{W^{1,m}} := \|x - x_*\|_\infty + \|x' - x_*'\|_m \le \varepsilon,$$

for some $\varepsilon > 0$. In such a situation, we will also refer to x_* as a $W^{1,m}$ ε-local minimum.

Remark 3.2. Clearly a **strong local minimum** i.e., a minimum among the *local* competitors with respect to the L^∞ norm, is a $W^{1,m}$ local minimum for every $m \ge 1$.

3.1. Finite valued Lagrangians

Given a minimizer x_*, we will suppose the validity of either the following Hypothesis (S_{x_*}) for real valued Lagrangians, or Hypothesis $(\mathrm{S}^\infty_{x_*})$ below that allows the Lagrangian to take the value $+\infty$.

Consider an arc x_* in $W^{1,1}([a,b];\mathbb{R}^n)$. Hypothesis (S_{x_*}) is a local Lipschitz condition of $y \mapsto \Lambda(y, x_*(t), \sigma x_*'(t))$ for a.e. $t \in [a,b]$ and for all $\sigma > 0$; a similar condition was considered in [19].

Hypothesis (S_{x_*}). Λ takes values in $\mathbb{R}$. There is $\varepsilon_* > 0$ and a LB-measurable map $k : [a,b]\times]0,+\infty[\to \mathbb{R}$ such that

$$k(t,1) \in L^1[a,b] \tag{3.1}$$

and, for a.e. $t \in [a,b]$, for all $\sigma > 0$

$$|\Lambda(t_2, x_*(t), \sigma x_*'(t)) - \Lambda(t_1, x_*(t), \sigma x_*'(t))| \le k(t,\sigma)\,|t_2 - t_1| \tag{3.2}$$

whenever $t_1, t_2 \in [t - \varepsilon_*, t + \varepsilon_*] \cap [a,b]$.

Remark 3.3. A natural candidate for $k(t,\sigma)$ in (3.2) is

$$k(t,\sigma) = \beta\big(\Lambda(t, x_*(t), \sigma x_*'(t)) + \sigma A|x_*'(t)| + \gamma(t)\big),$$

for suitable constants $A, \beta \ge 0$, and $\gamma \in L^1[a,b]$. Note that Hypothesis (S_{x_*}) is obviously fulfilled with $k(t,\sigma) := 0$ if $\Lambda(x,\xi)$ is autonomous.

The following characterization of Hypothesis (S_{x_*}) is straightforward.

Proposition 3.4 *(A proximal characterization of (S_{x_*})). Assume that the map $s \mapsto \Lambda(s, x_*(t), \sigma x_*'(t))$ is lower semicontinuous for a.e. $t \in [a,b]$ and $\sigma > 0$. Then Λ satisfies (3.2) of Hypothesis (S_{x_*}) if and only if, for a.e. $t \in [a,b]$ and every $\sigma > 0$,*

$$|\partial^P_\tau \Lambda(\tau, x_*(t), \sigma x_*'(t))| \le k(t,\sigma) \quad \forall |\tau - t| < \varepsilon_*. \tag{3.3}$$

If $s \mapsto \Lambda(s, x_*(t), \xi)$ is of class $\mathcal{C}^2$, then (3.3) becomes,

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$$|D_\tau \Lambda(\tau, x_*(t), \sigma x'_*(t))| \le k(t,\sigma) \quad \forall |\tau - t| < \varepsilon_*. \tag{3.4}$$

Proof. Condition (3.2) holds true if and only if the term $k(t,\sigma)$ is a Lipschitz constant for $s \mapsto \Lambda(s, x_*(t), \sigma x'_*(t))$ on $[t-\varepsilon_*, t+\varepsilon_*] \cap [a,b]$. It follows from [26, Theorem 4.9.1] that the latter is equivalent to (3.3). If $s \mapsto \Lambda(s, x_*(t), \xi)$ is of class $\mathcal{C}^2$ then (3.4) follows from the fact that $\partial^P_\tau \Lambda(\tau, x_*(t), \sigma x'_*(t)) = D_\tau \Lambda(\tau, x_*(t), \sigma x'_*(t))$. □

Remark 3.5 *(Hypothesis (S_{x_*}) and Cesari's Condition (S)).* Hypothesis (S_{x_*}) is fulfilled whenever Λ takes values in $\mathbb{R}$ and there is $\varepsilon_* > 0$ and a LB-measurable map $S : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$S(t, x'_*(t)) \in L^1[a,b]$$

and, for a.e. $t \in [a,b]$, for all $\xi \in \mathbb{R}^n$

$$|\Lambda(t_2, x_*(t), \xi) - \Lambda(t_1, x_*(t), \xi)| \le S(t,\xi)\,|t_2 - t_1| \tag{3.5}$$

whenever $t_1, t_2 \in [t-\varepsilon_*, t+\varepsilon_*] \cap [a,b]$. Indeed, if (3.5) holds, then for all $\sigma > 0$ and t such that (3.5) holds we have

$$|\Lambda(t_2, x_*(t), \sigma x'_*(t)) - \Lambda(t_1, x_*(t), \sigma x'_*(t))| \le k(t,\sigma)\,|t_2 - t_1|,$$

with

$$k(t,\sigma) := S(t, \sigma x'_*(t)).$$

The $(\mathcal{L}_1 \times \mathcal{B}_1)$-measurability of k follows from [14, Proposition 6.34], showing the validity of Hypothesis (S_{x_*}). If $s \mapsto \Lambda(s, x_*(t), \xi)$ is of class $\mathcal{C}^2$, Condition (3.5) is equivalent to Cesari's Condition (S) [9, §2.7], i.e.,

$$|D_\tau \Lambda(\tau, x_*(t), \xi)| \le S(t,\xi) \quad \forall |\tau - t| < \varepsilon_*. \tag{3.6}$$

The above implies the validity of the Erdmann – Du Bois-Reymond equation for $W^{1,1}$ minimizers, when Λ is smooth.

Proposition 3.6 *(A proximal sufficient condition for Hypotheses (S_{x_*})). Assume that Λ is real valued and that:*

(i) *The map $s \mapsto \Lambda(s, x_*(t), \sigma x'_*(t))$ is lower semicontinuous for a.e. $t \in [a,b]$ and $\sigma > 0$;*
(ii) *There are $A, \beta \ge 0$, and $\gamma \in L^1[a,b]$ and $\varepsilon_* > 0$ such that for a.e. $t \in [a,b]$ and $\sigma > 0$:*

$$|\partial^P_\tau \Lambda(\tau, x_*(t), \sigma x'_*(t))| \le \beta\big(\Lambda(\tau, x_*(t), \sigma x'_*(t)) + \sigma A |x'_*(t)| + \gamma(t)\big), \tag{3.7}$$

for all $\tau \in [t-\varepsilon_*, t+\varepsilon_*] \cap [a,b]$.

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Then Λ satisfies Hypothesis (S_{x_*}) with

$$k(t,\sigma) := \beta'(\Lambda(t, x_*(t), \sigma x_*'(t)) + A\sigma|x_*'(t)| + \gamma(t)),$$

for a suitable β' depending on ε_*.

Proof. Write $E \subset [a,b]$ the set of full measure such the properties (i) and (3.7) of (ii) are satisfied. Fix $t \in E$ and $\sigma > 0$. Consider the function

$$g_{t,\xi}(\tau) := \Lambda(\tau, x_*(t), \xi) + A|\xi| + \gamma(t), \quad \tau \in I := [t-\varepsilon_*, t+\varepsilon_*] \cap [a,b].$$

From (3.7), we deduce that $|\partial^P g_{t,\xi}(\tau)| \le \beta\, g_{t,\xi}(\tau)$ for all $\tau \in I$. Proposition 9.1 below is then applicable establishing that $g_{t,\xi}$ is Lipschitz continuous on I. Moreover, if $\beta = 0$, then

$$g_{t,\xi} = \text{constant on } I. \tag{3.8}$$

On the other hand, whenever $\beta > 0$, we obtain $g_{t,\xi}(\tau) \ge 0$ for all $\tau \in I$ and

$$|g_{t,\xi}(t_2) - g_{t,\xi}(t_1)| \le \alpha\, g_{t,\xi}(t)\, |t_2 - t_1| \quad \forall t_1, t_2 \in I, \tag{3.9}$$

where

$$\alpha := (e^{2\beta\varepsilon_*} + 1)\frac{e^{2\beta\varepsilon_*} - 1}{2\varepsilon_*}.$$

Define the function $k : [a,b]\times]0, +\infty[\to \mathbb{R}$:

$$k(t,\sigma) := \begin{cases} 0 & \text{if } \beta = 0 \\ \alpha(\Lambda(t, x_*(t), \sigma x_*'(t)) + A\sigma|x_*'(t)| + \gamma(t)) & \text{otherwise.} \end{cases}$$

Clearly $k(t,1) \in L^1[a,b]$, that is (3.1) of Hypothesis (S_{x_*}). Take now any $t \in E$ (which is of full measure); observing that

$$|g_{t,\xi}(t_2) - g_{t,\xi}(t_1)| = |\Lambda(t_2, x_*(t), \xi) - \Lambda(t_1, x_*(t), \xi)|,$$

from (3.8) and (3.9) we immediately deduce that Λ satisfies (3.2) of Hypothesis (S_{x_*}). □

Remark 3.7. Condition (3.7) is satisfied if, for instance, for every compact subset K of $\mathbb{R}^n$, there are constants $\beta_K, A_K \ge 0$ satisfying, for a.e. $t \in [a,b]$ and for every $(x,\xi) \in K \times \mathbb{R}^n$,

$$|\partial_\tau^P \Lambda(\tau, x, \xi)| \le \beta_K\big(\Lambda(\tau, x, \xi) + A_K|\xi| + 1\big). \tag{3.10}$$

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3.2. Extended valued Lagrangians

Next Hypothesis $(S^{\infty}_{x_*})$ covers situations in which a Lagrangian might take the value $+\infty$. Differently from Hypothesis (S_{x_*}), Hypothesis $(S^{\infty}_{x_*})$ subsumes also some additional conditions on the map $(t,\xi)\mapsto \Lambda(t,x_*(t),\xi)$.

Hypothesis $(S^{\infty}_{x_*})$. Given an absolutely continuous function x_* in $W^{1,1}([a,b];\mathbb{R}^n)$, the following conditions hold:

(i) The map $(s,\xi)\mapsto \Lambda(s,x_*(t),\xi)$ is lower semicontinuous for each $t\in[a,b]$.
(ii) There exists a non negligible subset E of $[a,b]$ such that for all $t\in E$ there are $0<\sigma_1<1<\sigma_2$ such that

$$\Lambda(t,x_*(t),\sigma_1 x_*'(t))<+\infty,\quad \Lambda(t,x_*(t),\sigma_2 x_*'(t))<+\infty.$$

(iii) There exist $\varepsilon_*>0$, $\beta, A\geq 0$ and a positive function $\gamma\in L^1[a,b]$ such that, for a.e. $t\in[a,b]$:

$$\begin{aligned}&\Lambda(\tau,x_*(t),\sigma x_*'(t))+A\sigma|x_*'(t)|+\gamma(t)\geq 0,\\ &|\partial_\tau^P\Lambda(\tau,x_*(t),\sigma x_*'(t))|\leq \beta\big(\Lambda(\tau,x_*(t),\sigma x_*'(t))+A\sigma|x_*'(t)|+\gamma(t)\big)\end{aligned}\tag{3.11}$$

for all $\tau\in[t-\varepsilon_*,t+\varepsilon_*]\cap[a,b]$ and $\sigma>0$ with $\Lambda(\tau,x_*(t),\sigma x_*'(t))<+\infty$.

Remark 3.8.

- Condition (ii) of Hypothesis $(S^{\infty}_{x_*})$ is fulfilled if for almost every t in $[a,b]$, $\sigma=1$ belongs to the interior of the effective domain of $0<\sigma\mapsto\Lambda(t,x_*(t),\sigma x_*'(t)))$.
- If $\Lambda(t,x,\xi)=\Lambda(x,\xi)$ is autonomous, Condition (iii) of Hypothesis $(S^{\infty}_{x_*})$ holds if just $\Lambda(x,\xi)\geq -A|\xi|-c$ for some positive constants A,c.
- Condition (iii) of Hypothesis $(S^{\infty}_{x_*})$ is fulfilled if, for a.e. $t\in[a,b]$, for every $\xi\in\mathbb{R}^n$, $\Lambda(t,x,\xi)\geq -A|\xi|-\gamma(t)$ and $\tau\mapsto\Lambda(\tau,x_*(t),\xi)$ is finite valued and locally Lipschitz around τ of constant $\beta\big(\Lambda(\tau,x_*(t),\sigma x_*'(t))+A\sigma|x_*'(t)|+\gamma(t)\big)$.

In the case where Λ takes values in $\mathbb{R}$, Proposition 3.6 immediately gives the following link between Hypotheses $(S^{\infty}_{x_*})$ and (S_{x_*}).

Corollary 3.9 *(Λ real valued: $(S^{\infty}_{x_*})$ implies (S_{x_*})). Let x_* be an absolutely continuous arc. Let Λ be real valued. Assume that Λ satisfies Hypothesis $(S^{\infty}_{x_*})$ and that the map $(s,\xi)\mapsto\Lambda(s,x_*(t),\xi)$ is lower semicontinuous for each $t\in[a,b]$. Then Λ fulfills Hypothesis (S_{x_*}).*

4. A Weierstrass-type variational inequality

Theorem 4.1 *(Directional Weierstrass type condition under (S_{x_*})). Let x_* be a $W^{1,m}$ local minimum of (P). Assume that Λ satisfies Hypothesis (S_{x_*}). There exists an **absolutely continuous** function $p\in W^{1,1}([a,b];\mathbb{R})$ such that, for almost every $t\in[a,b]$:*

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$$\Lambda\Big(t, x_*(t), \frac{x'_*(t)}{v}\Big)v - \Lambda(t, x_*(t), x'_*(t)) \geq p(t)(v-1) \quad \forall v > 0. \tag{W}$$

Moreover, the following inclusion and estimate hold:

$$p'(t) \in \partial_t^C \Lambda(t, x_*(t), x'_*(t)) \text{ a.e. } t \in [a,b], \tag{D}$$
$$|p'(t)| \leq k(t,1) \text{ a.e. } t \in [a,b].$$

Remark 4.2. If, for a.e. $t \in [a,b]$, $s \mapsto \Lambda(s, x_*(t), x'_*(t))$ is $\mathcal{C}^1$ near t then (D) becomes

$$p'(t) = D_t \Lambda(t, x_*(t), x'_*(t)) \text{ a.e. } t \in [a,b].$$

In Theorem 4.3 below, once an absolutely continuous arc x_* is given, we consider points $t \in [a,b]$ such that $\Lambda(t, x_*(t), x'_*(t)) < +\infty$, and we denote by

$$\partial^L_{(s,v)} \Big(\Lambda\Big(s, x_*(t), \frac{x'_*(t)}{v}\Big)v\Big)_{\substack{s=t \\ v=1}}$$

the limiting subdifferential of the function

$$(s,v) \in [a,b]\times]0,+\infty[\longmapsto \Lambda\Big(s, x_*(t), \frac{x'_*(t)}{v}\Big)v$$

at $s = t$ and $v = 1$.

Theorem 4.3 *(Directional Weierstrass type condition under $(S^\infty_{x_*})$). Let x_* be a $W^{1,m}$ local minimum of (P). Assume that Λ satisfies Hypothesis $(S^\infty_{x_*})$. Then there exists an* ***absolutely continuous*** *function $p \in W^{1,1}([a,b];\mathbb{R})$ such that, for almost every $t \in [a,b]$:*

$$\Lambda\Big(t, x_*(t), \frac{x'_*(t)}{v}\Big)v - \Lambda(t, x_*(t), x'_*(t)) \geq p(t)(v-1) \qquad \forall v > 0. \tag{W}$$

In addition, the following inclusion and estimate hold:

$$p'(t) \in co\left\{\omega : (\omega, p(t)) \in \partial^L_{(s,v)} \Big(\Lambda\Big(s, x_*(t), \frac{x'_*(t)}{v}\Big)v\Big)_{\substack{s=t \\ v=1}}\right\} \text{ a.e. } t. \tag{D$^\infty$}$$

Moreover,

$$|p'(t)| \leq \beta(\Lambda(t, x_*(t), x'_*(t)) + A|x'_*(t)| + \gamma(t)) \quad \text{a.e. } t \in [a,b].$$

Remark 4.4. Unless one assumes further regularity assumptions on Λ, it is well known, from the properties of the (convexified) limiting partial subdifferentials, that the inclusion (D$^\infty$) does not imply (D) (see, for instance [11, Example 2.5.2] and [11, Proposition 2.3.15]).

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Remark 4.5. If, for a.e. $t \in [a, b[$, the map $(s, \xi) \mapsto \Lambda(s, x_*(t), \xi)$ is continuously differentiable around the point $(t, x_*(t), x'_*(t))$ then (D^∞) becomes

$$p'(t) = D_t \Lambda(t, x_*(t), x'_*(t)) \text{ a.e. } t \in [a, b].$$

Indeed in this case, for a.e. t,

$$\partial^L_{(s,v)} \Big(\Lambda \Big(s, x_*(t), \frac{x'_*(t)}{v} \Big) v \Big)_{\substack{s=t \\ v=1}} = \{ D_t \Lambda(t, x_*(t), x'_*(t)), D_\xi \Lambda(t, x_*(t), x'_*(t)) \}.$$

If the Lagrangian Λ is convex w.r.t. the velocity variable ξ, then we can consider a weaker condition than (S_{x_*}), imposing just a local (in σ) version of it.

Proposition 4.6. *Let x_* be a $W^{1,m}$ local minimizer for (P). Assume that Λ takes values in $\mathbb{R}$ and that the following assumptions hold:*

(H') *$\xi \mapsto \Lambda(t, x_*(t), \xi)$ is convex for a.e. t;*
(S'_{x_*}) *There exist $\varepsilon_* > 0$ and a LB-measurable map*

$$k : [a, b] \times [1 - \sigma_*, 1 + \sigma_*] \to \mathbb{R}_+$$

such that $k(t, 1) \in L^1[a, b]$ and, for a.e. $t \in [a, b]$, for all $\sigma \in [1 - \sigma_*, 1 + \sigma_*]$

$$|\Lambda(t_2, x_*(t), \sigma x'_*(t)) - \Lambda(t_1, x_*(t), \sigma x'_*(t))| \le k(t, \sigma)\, |t_2 - t_1|$$

whenever $t_1, t_2 \in [t - \varepsilon_*, t + \varepsilon_*] \cap [a, b]$.

Then, the same conclusions of Theorem 4.1 are valid.

Remark 4.7. Consider the optimization problem

$$\begin{cases} \text{Minimize} & \Psi(x(a), x(b)) \\ \text{Subject to:} & x \in W^{1,1}([a, b]; \mathbb{R}^n), \\ & x'(t) \in F(t, x(t)) \text{ a.e., } x(t) \in \Sigma \quad \forall t \in [a, b], \end{cases} \tag{P_1}$$

where Ψ and Σ are the same of problem (P), and $F : [a, b] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction. Employing a standard argument problem (P_1) can be expressed as a problem (P): replace the differential inclusion $x'(t) \in F(t, x(t))$ by an integral term involving the Lagrangian

$$\Lambda_F(t, x, \xi) = I_{\mathrm{Gr}\, F(\cdot,x)}(t, \xi)$$

where $I_{\mathrm{Gr}\ F(\cdot,x)}$ is the indicator function of the graph of $F(\cdot, x)$. Under unrestrictive assumptions Λ_F satisfies conditions (i) and (iii) of ($S^\infty_{x_*}$), but one can not expect that (ii) holds. Nevertheless, it is easy to see that the conclusions of Theorem 4.3 are still valid for this particular class of Lagrangians derived from optimization problems such as (P_1).

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Consider now an autonomous Lagrangian $\Lambda(x,\xi)$. In this case, Condition (iii) of Hypothesis $(S^{\infty}_{x_*})$ and, whenever $\Lambda(x,\xi)$ is real valued, Hypothesis (S_{x_*}) are satisfied. Moreover, it follows from (D) that $p' = 0$, so that $p(t)$ is a constant function. The conclusion of Theorem 4.1 and Theorem 4.3 can thus be summarized in the Corollary below.

Corollary 4.8 *(The directional Weierstrass type condition, autonomous case). Assume that the Lagrangian $\Lambda(t,x,\xi) = \Lambda(x,\xi)$ is Borel, autonomous and let x_* be a $W^{1,m}$ local minimum of (P). Assume moreover that one of the following conditions holds:*

(a) *Λ takes values in $\mathbb{R}$, or*
(b) *(i) and (ii) of Hypothesis $(S^{\infty}_{x_*})$ are satisfied.*

Then there exists $\lambda \in \mathbb{R}$ such that, for almost every $t \in [a,b]$:

$$\Lambda\Big(x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(x_*(t), x_*'(t)) \geq \lambda(v-1) \quad \forall v > 0. \tag{4.1}$$

When Λ takes values in $\mathbb{R}$ and x_* is a global minimum, an alternative and self-contained proof of Corollary 4.8, is based on a convex optimization argument and is provided in Section 8.

Remark 4.9. Inequality (4.1), with $v \geq 1/2$ instead of $v > 0$, appears as a technical tool in the proof of [12, Theorem 4.4.5], where Λ is supposed to be real valued and lower semicontinuous, and x_* is a *strong* local minimizer.

5. Condition (W), Weierstrass inequality and the Erdmann – DuBois-Reymond equation

We discuss here how the new Condition (W) is related to some classical necessary conditions in the Calculus of Variations.

5.1. An interpretation of Condition (W)

Our Condition (W) implies the convexity at $r = 1$ of the function

$$0 < r \mapsto \Lambda(t, x_*(t), r x_*'(t)).$$

Proposition 5.1 *(An equivalent formulation of (W)). Condition (W) is equivalent to: For a.e. $t \in [a,b]$,*

$$\Lambda(t, x_*, r x_*') - \Lambda(t, x_*, x_*') \geq \big(\Lambda(t, x_*, x_*') - p(t)\big)(r-1) \quad \forall r > 0. \tag{W_r}$$

Proof. Fix t such that the inequality (W) holds. The change of variable $r = \dfrac{1}{v}$ gives

$$\Lambda(t, x_*, r x_*')\frac{1}{r} - \Lambda(t, x_*, x_*') \geq p(t)\Big(\frac{1}{r} - 1\Big), \quad r > 0. \tag{5.1}$$

By multiplying both terms of (5.1) by r yields

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$$\Lambda(t, x_*, rx'_*) - r\Lambda(t, x_*, x'_*) \geq p(t)(1-r) \quad \forall r > 0,$$

whence (W_r), once one adds $(r-1)\Lambda(t, x_*, x'_*)$ to both terms of the latter inequality. The converse does similarly follow. □

5.2. Relations between (W)+ (D), Weierstrass and (EDBR) conditions

We assume here that Λ is smooth and that x_* is a smooth minimizer of (P). The Weierstrass condition states that, for all $\xi \in \mathbb{R}^n$,

$$\Lambda(t, x_*(t), \xi) - \Lambda(t, x_*(t), x'_*(t)) \geq D_\xi \Lambda(t, x_*(t), x'_*(t)) \cdot (\xi - x'_*(t)). \tag{W}$$

It is known to hold if, for instance, x_* is Lipschitz or if $\Lambda(t, \cdot, \cdot)$ satisfies a suitable regularity condition [12, Theorem 4.3.2]. The Erdmann – Du Bois-Reymond equation states that there is a constant c such that

$$\Lambda(t, x_*, x'_*) - x'_* \cdot D_\xi \Lambda(t, x_*, x'_*) = c + \int_a^t D_t \Lambda(s, x_*, x'_*)\, ds. \tag{EDBR}$$

It is known to hold if, for instance, Λ satisfies Cesari's Condition (S) recalled in (3.6).

Proposition 5.2 *(W) and (EDBR) ⇒ (W) and (D). The Weierstrass condition (W), together with the Erdmann – Du Bois-Reymond equation (EDBR), imply (W) with* $p'(t) = D_t \Lambda(t, x_*(t), x'_*(t))$ *a.e. in* $[a, b]$.

Proof. Assume that $(t, \xi) \mapsto \Lambda(t, x, \xi)$ is smooth for each x in $\mathbb{R}^n$, and that x_* is a smooth minimizer for (P). Assume that (W) holds.

For $r > 0$ set $\xi = rx'_*(t)$. From (W) we get,

$$\Lambda(t, x_*, rx'_*) - \Lambda(t, x_*, x'_*) \geq D_\xi \Lambda(t, x_*(t), x'_*(t)) \cdot x'_*(t)(r-1). \tag{5.2}$$

Now, (EDBR) implies that

$$D_\xi \Lambda(t, x_*(t), x'_*(t)) \cdot x'_*(t) = \Lambda(t, x_*, x'_*) - p(t),$$

with

$$p'(t) = D_t \Lambda(t, x_*(t), x'_*(t)).$$

Therefore, (5.2) gives (W_r), which, from Proposition 5.1, turns out to be equivalent to (W). □

Remark 5.3. For Lagrangians $\Lambda(x, \xi)$ that are extended valued, autonomous and convex in ξ, the validity of (W) would follow similarly as in the Proof of Proposition 5.2 from [1, Theorem 4.1] under the assumption, stronger than (ii) of Hypothesis ($S^\infty_{x_*}$), that $x'_*(t)$ belongs to the interior of the effective domain of $\Lambda(x_*(t), \cdot)$ for a.e. $t \in [a, b]$.

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Remark 5.4. In spite of Proposition 5.2, our new variational inequality (W) is not in general a trivial consequence of (W) and (EDBR), even in the smooth setting. Indeed the validity of (W) requires either the Lipschitzianity of x_*, or some regularity on $(x,\xi)\mapsto\Lambda(t,x,\xi)$ (see [20, Theorem 1], [12, Theorem 4.3.1]), which are not assumed in Theorem 4.1.

Conditions (W) and (D) immediately give the Erdmann – Du Bois-Reymond (EDBR) equation. Here, we do not necessarily assume that Λ is finite valued.

Proposition 5.5 *(Erdmann – Du Bois-Reymond equation). Let x_* be a $W^{1,m}$ local minimizer of (P) and suppose the validity of either of Hypothesis (S_{x_*}) or Hypothesis $(S^\infty_{x_*})$. Assume, moreover, that $(y,\xi)\mapsto\Lambda(y,x_*(t),\xi)$ is finite and of class $\mathcal{C}^1$ in a neighbourhood of $(t,x_*'(t))$, for a.e. $t\in[a,b]$. Then*

$$p(t):=\Lambda(t,x_*,x_*')-x_*'\cdot D_\xi\Lambda(t,x_*,x_*')$$

is absolutely continuous and

$$p'(t)=D_t\Lambda(t,x_*(t),x_*'(t))\ \text{a.e. } t\in[a,b].$$

Proof. Theorem 4.1 or Theorem 4.3 imply the validity of (W). For t and $p(\cdot)$ such that (W) holds, $v=1$ minimizes

$$\varphi(v):=\Lambda\Big(t,x_*(t),\frac{x_*'(t)}{v}\Big)v-p(t)v,\qquad v>0.$$

Therefore

$$0=\varphi'(1)=\Lambda(t,x_*,x_*')-x_*'\cdot D_\xi\Lambda(t,x_*,x_*')-p(t).$$

Now, p is absolutely continuous and both (D), (D^∞) imply that $p'(t)=D_t\Lambda(t,x_*,x_*')$ a.e. on $[a,b]$. □

Remark 5.6. When Λ has values in $\mathbb{R}$, the conclusion of Corollary 5.5 is well known, at least when x_* is a strong local minimizer (cf. [17] and [9, §2.7 A]). The case where Λ may have extended values seems new to us. Investigations about possible generalizations of the (EDBR) equation in a nonsmooth setting will be the object of a forthcoming paper [4].

6. Lipschitz regularity in the superlinear case

As an application of our condition (W), we formulate a new Lipschitz regularity result for the minimizers of (P). Given a continuous function x_* in $[a,b]$ we consider the following conditions.

Superlinearity. *There exists $\Theta:[0,+\infty[\to\mathbb{R}$ such that, for almost every $t\in[a,b]$,*

$$\Lambda(t,x_*(t),\xi)\geq\Theta(|\xi|)\quad\forall\xi\in\mathbb{R}^n,\qquad\lim_{r\to+\infty}\frac{\Theta(r)}{r}=+\infty.\tag{G_{x_*}}$$

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Remark 6.1. Assumption (G_{x_*}) is satisfied if, for instance, there are:

- A positive function $h : \mathbb{R}^n \to \mathbb{R}$, bounded on bounded sets,
- An increasing function $\Theta' : [0, +\infty[\to \mathbb{R}$ with $\lim_{r\to+\infty} \frac{\Theta'(r)}{r} = +\infty$;
- A constant $\alpha \geq 0$,

such that for every $x, \xi \in \mathbb{R}^n$ with $|\xi| \geq \alpha|x|$,

$$\Lambda(t, x, \xi) \geq \Theta'(|\xi| - \alpha|x|) - h(x) \quad \forall x, \xi \quad |\xi| \geq \alpha|x|. \tag{6.1}$$

Indeed, let $M = \|x_*\|_\infty$ and $C = \|h(x_*(t))\|_\infty$. Then (6.1) gives

$$\Lambda(t, x_*(t), \xi) \geq \Theta'(|\xi| - M\alpha) - C \quad \forall |\xi| \geq M\alpha.$$

Therefore Λ fulfills (G_{x_*}) with

$$\Theta(r) = \begin{cases} \Theta'(r - M\alpha) - C \text{ if } r \geq M\alpha, \\ 0 \text{ otherwise.} \end{cases}$$

Condition (6.1) was considered, for instance, in [24].

The next condition requires that $\Lambda(t, x, \cdot)$ is uniformly bounded on an annulus along the graph of x_*.

Boundedness on some bounded set. *There exist $\rho > 0$ and $M \geq 0$ such that, for almost every $t \in [a, b]$,*

$$\Lambda(t, x_*(t), \xi) \leq M \qquad \forall \xi \in \mathbb{R}^n, \ |\xi| = \rho. \tag{B_{x_*}}$$

Remark 6.2. Assumption (B_{x_*}) is satisfied if $\Lambda(t, x, \xi)$ is bounded above in a neighbourhood of each point $(t_0, x_0, 0)$, i.e., if

$$\forall (t_0, x_0) \in [a, b] \times \mathbb{R}^n \quad \exists M, r > 0 \quad \forall (t, x, \xi) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$$
$$|t - t_0| \leq r, \ |x - x_0| \leq r, \ |\xi| \leq r \Rightarrow L(t, x, \xi) \leq M.$$

Such a condition was considered, in the autonomous case, in [18].

A celebrated result asserts that if $\Lambda(x, \xi)$ is real valued, **autonomous**, with $\Lambda(x, \cdot)$ uniformly bounded on bounded sets and superlinear, then any $W^{1,1}$ **global** minimizer of

$$I(x) := \int_a^b \Lambda(t, x(t), x'(t))\, dt$$

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with prescribed boundary conditions $x(a) = A, x(b) = B$, is Lipschitz [16,1,18]. If Λ is nonautonomous, not only $W^{1,1}$ minimizers of I may not be Lipschitz, but even the Lavrentiev phenomenon may occur [3]. For the nonautonomous case, only a few results guarantee the Lipschitz regularity of the minimizers, and all of them require some extra regularity assumptions on (x, ξ), when the Lagrangian $\Lambda(t, x, \xi)$ depends on t. We show here that Condition (W), together with superlinearity, ensure the Lipschitzianity of minimizers.

Theorem 6.3 *(Lipschitz regularity in the superlinear case). Let x_* be a $W^{1,m}$ local minimum of (P), where Λ is Borel.*

Assume that Condition (W) holds for some bounded function $p(\cdot)$ and for $v \geq \overline{v} > 0$, e.g., that Λ satisfies Hypothesis (S_{x_*}) or $(S^{\infty}_{x_*})$. Suppose in addition that Λ is superlinear and satisfies the local boundedness Condition (B_{x_*}). Then x_* is Lipschitz.

Proof. Let t be such that (W), (G_{x_*}) and (B_{x_*}) hold. It is not restrictive to suppose that $\overline{v} \geq 1$. Either $|x'_*(t)| \leq \rho\overline{v}$, or $\dfrac{|x'_*(t)|}{\rho} > \overline{v}$. In the latter case, by applying (W) to $v := \dfrac{|x'_*(t)|}{\rho}$, we obtain

$$\Lambda\left(t, x_*(t), \rho\frac{x'_*(t)}{|x'_*(t)|}\right)\frac{|x'_*(t)|}{\rho} \geq \Lambda(t, x_*(t), x'_*(t)) + p(t)\left(\frac{|x'_*(t)|}{\rho} - 1\right), \tag{6.2}$$

so that, multiplying across by $\dfrac{\rho}{|x'_*(t)|}$ and taking into account (G_{x_*}) we get

$$\begin{aligned}\Lambda\left(t, x_*(t), \rho\frac{x'_*(t)}{|x'_*(t)|}\right) &\geq \rho\frac{\Theta(|x'_*(t)|)}{|x'_*(t)|} + p(t)\left(1 - \frac{\rho}{|x'_*(t)|}\right)\\ &\geq \rho\frac{\Theta(|x'_*(t)|)}{|x'_*(t)|} - \|p\|_{\infty}.\end{aligned} \tag{6.3}$$

Owing to (B_{x_*}), we get $M + \|p\|_{\infty} \geq \rho\dfrac{\Theta(|x'_*(t)|)}{|x'_*(t)|}$. However, from (G_{x_*}), there exists $R > 0$ such that

$$|x'_*(t)| \geq R \Rightarrow \rho\frac{\Theta(|x'_*(t)|)}{|x'_*(t)|} > M + \|p\|_{\infty}. \tag{6.4}$$

It follows that $\|x_*\|_{\infty} \leq \max\{R, \rho\overline{v}\}$. □

Remark 6.4. In the **autonomous** case Hypothesis (S_{x_*}) is fulfilled when $\Lambda(x, \xi)$ is Borel and real valued: Theorem 6.3 gives back [18, Theorem 2.1]. If $\Lambda(x, \xi)$ is **autonomous** and takes extended values, then iii) of Hypothesis $(S^{\infty}_{x_*})$ is fulfilled: in this situation the conclusion of Theorem 6.3 seems new to us.

There are examples, even in the autonomous case, of variational problems whose minimizers fail to be Lipschitz if (B_{x_*}) or (G_{x_*}) are not assumed (see [1]), though weaker growth conditions have been considered (cf. [8], [21]). The next example shows that the conclusion of Theorem 6.3 may fail without assuming Hypothesis (S_{x_*}) on $t \mapsto \Lambda(t, x, \xi))$ or Hypothesis $(S^{\infty}_{x_*})$.

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Example 6.5. Consider the innocent looking polynomial Lagrangian presented by Ball and Mizel in [3], given by

$$\Lambda(t,x,\xi) = (t^4 - x^6)^2\xi^{28} + \epsilon\xi^2.$$

Clearly $\Lambda(t,x,\cdot)$ is convex and superlinear. For suitable values of ϵ, k_1, k_2, it is shown in [3, Theorem 5.1] that there is a minimizer x_* of I among the absolutely continuous functions on $[-1,1]$ satisfying $x(-1) = k_1, x(1) = k_2$ such that the Lavrentiev phenomenon occurs, i.e.,

$$I(x_*) < \inf\{I(x) : x \in \mathrm{Lip}[-1,1],\ x(-1) = k_1,\ x(1) = k_2\}.$$

A further important feature of this example is that the minimizer fails to satisfy Euler-Lagrange type conditions expressed in terms of an absolutely continuous co-state arc. In view of Theorem 6.3, there are no bounded functions p for which (W) holds at x_*. In particular Λ does not satisfy Hypothesis (S_{x_*}).

Remark 6.6. In the autonomous and superlinear case, for global minimizers, Theorem 6.3 gives back the more general result [18, Theorem 2.1], where Λ is required to be just Borel. It extends it in several directions: we allow Λ to take the value $+\infty$, minimizers to be just local, and we consider state constraints in the minimization problem. Also, our assumption (B_{x_*}) is weaker than the more common requirement that $\Lambda(t,x_*,\cdot)$ is uniformly bounded on bounded sets.

The conclusion of Theorem 6.3 is established in literature for finite valued Lagrangians assuming (G_{x_*}), (B_{x_*}) and conditions, other than Hypothesis (S_{x_*}) or ($\mathrm{S}^{\infty}_{x_*}$), that involve the regularity of $\Lambda(t,x,\xi)$ with respect to x or ξ.

1. Clarke – Vinter [16, Corollary 3.2]: x_* is a global minimum, the Lagrangian $\Lambda(t,x,\xi)$ is real valued, **convex** in ξ, **locally Lipschitz on** $[a,b]\times\mathbb{R}^n\times\mathbb{R}^n$ and for every bounded subset K of $\mathbb{R}^n$ there are $\beta_K \geq 0$ and $\gamma_K \in L^1[a,b]$ satisfying

$$|D_t\Lambda(t,x,\xi)| \leq \beta_K|\Lambda(t,x,\xi)| + \gamma_K(t) \tag{6.5}$$

   at every $(t,x,\xi) \in [a,b]\times K\times\mathbb{R}^n$ where $(s,z) \mapsto \Lambda(s,z,\xi)$ is differentiable. Notice that (6.5) implies the validity of (3.10) and thus that of (3.7): [16, Corollary 3.2] is therefore a consequence of Theorem 4.1.
2. Vinter [26, Proposition 11.4.3] and [19]: The Lagrangian $\Lambda(t,x,\xi)$ is real valued **convex** in (x,ξ).
3. Clarke [12, Theorem 4.5.2]: Λ satisfies the **generalized Tonelli-Morrey** assumption: **The map** $(x,\xi) \mapsto \Lambda(t,x,\xi)$ **is lower semicontinuous for each** t; for every bounded subset K of $\mathbb{R}^n$ there exist a constant c_K and a summable function $\gamma_K \in L^1[a,b]$ such that for almost every $t \in [a,b]$, for every $(x,\xi) \in K\times\mathbb{R}^n$, for every $(\zeta,\psi) \in \partial^P_{x,\xi}\Lambda(t,x,\xi)$, one has

$$\frac{|\zeta|}{1+|\psi|} \leq c_K\,(\Lambda(t,x,\xi) + |\xi|) + \gamma_K(t). \tag{6.6}$$

4. Clarke [12, Theorem 4.5.4]: $\Lambda(t,x,\xi)$ is **differentiable in** (x,ξ) for each t, $\Lambda(t,x,\cdot)$ is convex for every (t,x) and Λ satisfies the following **uniform estimate on** $D_x\Lambda(t,x,x'_*(t))$:

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There exists a summable function $k \in L^1[a,b]$ and $\varepsilon > 0$ such that, for almost every $t \in [a,b]$,

$$\forall x \in \mathbb{R}^n \quad |x - x_*(t)| \le \varepsilon, \qquad |D_x \Lambda(t, x, x'_*(t))| \le k(t). \tag{6.7}$$

5. Palladino-Vinter [23, Proposition 4.1]: x_* is a **strong local minimum**, the Lagrangian $\Lambda(t,x,\xi)$ is **locally Lipschitz in** (x,ξ), convex in ξ, and $t \mapsto \Lambda(t, x_*(t), \xi)$ is **of bounded variation**, uniformly with respect to $\xi \in \mathbb{R}^n$.

All of the above conditions require more than just the Borel measurability of the Lagrangian $\Lambda(t,x,\xi)$ with respect to the variables (x,ξ) and thus do not give back, in the autonomous case, the regularity result established in [18] for discontinuous, non convex Lagrangians.

We provide here an example of a Lagrangian that fulfills the assumptions of Theorem 6.3, but none of the conditions formulated in Remark 6.6.

Example 6.7. Take a real positive and bounded Borel function φ that is not lower semicontinuous. Then

$$\Lambda(t,x,\xi) = e^t(\varphi(x)\xi^4 + \xi^2), \quad (t,x,\xi) \in [0,1] \times \mathbb{R} \times \mathbb{R}$$

does not satisfy any of the conditions formulated in Remark 6.6, but nevertheless satisfies Hypothesis (S_{x_*}). Moreover, Λ satisfies both Conditions (G_{x_*}) and (B_{x_*}), so that it fulfills the assumptions of Theorem 6.3.

It is worth noticing that, if φ equals the identity around 0, then Λ furnishes an example of function that does not even satisfy the generalized Tonelli-Morrey condition recalled in (6.6). Indeed, taking $K = \{0\}$, if $(\zeta,\psi) \in \partial^P_{x,\xi} \Lambda(t,0,\xi)$, then $\zeta = e^t\xi^4$, $\psi = 2e^t\xi$ so that

$$\frac{|\zeta|}{1+|\psi|} = \frac{e^t|\xi|^4}{1+2e^t|\xi|}.$$

The validity of (6.6) would imply the existence of a constant c and a function $\gamma \in L^1[0,1]$ satisfying, for almost every $t \in [0,1]$,

$$\frac{e^t|\xi|^4}{1+2e^t|\xi|} \le c(e^t\xi^2 + |\xi|) + \gamma(t) \quad \forall \xi \in \mathbb{R},$$

leading to a contradiction, since for any t the function $\dfrac{e^t|\xi|^4}{1+2e^t|\xi|}$ is asymptotic to $\dfrac{|\xi|^3}{2}$ as $|\xi| \to +\infty$.

7. Proofs of Theorems 4.1, 4.3 and Proposition 4.6

The section is devoted to the proofs of Theorem 4.1, Theorem 4.3 and Proposition 4.6; these are carried out in several steps. A preliminary section provides some basic analytical tools which are in common for the two proofs. The initial sections are shared by the two proofs.

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7.1. Basic steps

The next Lemma is a slight improvement of [1, Lemma 4.8]: the difference with respect to the original formulation is that the assumption of the convergence of the sequence $(y_h)_h$ in $W^{1,\infty}([a,b];\mathbb{R})$ in [1, Lemma 4.8] is replaced here with a $W^{1,m}([a,b];\mathbb{R})$ convergence together with a uniform bound from below of the derivatives $(y_h')_h$.

Lemma 7.1. *Let $(y_h)_h$ be a sequence of functions in $W^{1,1}([a,b];\mathbb{R})$ satisfying the following conditions:*

(i) *For all $h\in\mathbb{N}$, $y_h(a)=a$ and $y_h(b)=b$;*
(ii) *There is $\omega>0$ such that, for every $h\in\mathbb{N}$, $y_h'\geq\omega$ almost everywhere in $[a,b]$;*
(iii) *The sequence $(y_h)_h$ converges to $y_*(t):=t$ in $W^{1,1}([a,b];\mathbb{R})$.*

Then, for every $x_*\in W^{1,m}([a,b];\mathbb{R})$, the sequence $(x_*\circ y_h^{-1})_h$ admits a subsequence that converges to x_* in $W^{1,m}([a,b];\mathbb{R})$.

Proof. The sequence of the inverse functions $(y_h^{-1})_h$ converges to $y_*(t)=t$ in $W^{1,1}([a,b];\mathbb{R})$: Indeed, by means of the change of variable $t=y_h(s)$, for each h we obtain:

$$\begin{aligned}\|y_h^{-1}(t)-t\|_{W^{1,1}} &= \|y_h^{-1}(t)-t\|_\infty+\|(y_h^{-1})'(t)-1\|_1\\ &= \|y_h(t)-t\|_\infty+\int_a^b\left|\frac{1-y_h'(y_h^{-1}(t))}{y_h'(y_h^{-1}(t))}\right|dt\\ &= \|y_h(t)-t\|_\infty+\int_a^b\left|\frac{1-y_h'(s)}{y_h'(s)}\right||y_h'(s)|\,ds\\ &= \|y_h(t)-t\|_\infty+\|y_h'-1\|_1=\|y_h(t)-t\|_{W^{1,1}}.\end{aligned}$$

Modulo a subsequence we may assume that $((y_h^{-1})')_h$ converges to 1 almost everywhere. For every $h\in\mathbb{N}$ let T_h be the linear operator $T_h: W^{1,m}([a,b];\mathbb{R})\to W^{1,m}([a,b];\mathbb{R})$ defined by

$$T_h(u):=u\circ y_h^{-1}\qquad\forall h\in\mathbb{N}.$$

The claim is equivalent to show that $(T_h(x_*))_h$ converges to x_* in $W^{1,m}([a,b];\mathbb{R})$.

Notice first that the operators $(T_h)_h$ are equi-bounded, since it follows from [1, Lemma 4.7] that, for all $u\in W^{1,m}([a,b];\mathbb{R})$,

$$T_h(u)'=(u'\circ y_h^{-1})(y_h^{-1})'$$

whence the change of variable $s=y_h^{-1}(t)$ yields

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$$\|T_h(u)'\|_m^m = \int_a^b |u'(y_h^{-1}(t))|^m |(y_h^{-1})'(t)|^m \, dt$$
$$= \int_a^b \frac{|u'(s)|^m}{|y_h'(s)|^{m-1}} \, ds \le \frac{1}{\omega^{m-1}} \|u'\|_m$$

and, consequently,

$$\|T_h(u)\|_{W^{1,m}} = \|T_h(u)\|_\infty + \|T_h(u)'\|_m \le c\|u\|_{W^{1,m}}$$

for some positive constant c. Now, if $u \in \mathcal{C}^2[a,b]$, then $(T_h(u))_h$ converges to u in $W^{1,m}([a,b];\mathbb{R})$. Indeed $(T_h(u))_h$ converges uniformly to u; moreover, the functions $(y_h^{-1})_h$ are equi-Lipschitz of rank $1/\omega$, so that

$$\lim_{h\to+\infty} T_h(u)' = u', \qquad |T_h(u)'| \le \frac{1}{\omega}\|u'\|_\infty \quad \text{a.e.}$$

The dominated convergence theorem yields the convergence of $(T_h(u)')_h$ to u' in $L^m[a,b]$.

Now, let $(u_k)_k$ be a sequence of $\mathcal{C}^2$ functions that converge to x_* in $W^{1,m}([a,b];\mathbb{R})$. For each $h, m \in \mathbb{N}$ we have:

$$T_h(x_*) - x_* = (T_h(x_*) - T_h(u_k)) + (T_h(u_k) - u_k) + (u_k - x_*).$$

The above arguments imply that:

- For any $h \in \mathbb{N}$, $(T_h x_* - T_h u_k)_k$ converges to 0 in $W^{1,m}([a,b];\mathbb{R})$;
- For any $k \in \mathbb{N}$, $(T_h u_k - u_k)_h$ converges to 0 in $W^{1,m}([a,b];\mathbb{R})$.

The conclusion follows. □

In what follows we fix a natural $j \ge 2$. The aim now is to prove first the validity of (W) just for $v \ge 1/j$ instead of $v > 0$.

Lemma 7.2. *Let $x_* \in W^{1,m}([a,b];\mathbb{R})$. For every $\varepsilon > 0$ there exists $\rho > 0$ such that, for every $y \in W^{1,1}([a,b];\mathbb{R})$ satisfying the boundary conditions $y(a) = a$, $y(b) = b$, the following property holds:*

$$\|y(t) - t\|_{W^{1,1}} \le \rho, \ y' \ge \omega := 1/j \Longrightarrow \|x_* \circ y^{-1} - x_*\|_{W^{1,m}} \le \varepsilon. \tag{7.1}$$

Proof. Assume that (7.1) does not hold. There are $\varepsilon_0 > 0$ and a sequence $(y_h)_h$ of functions in $W^{1,1}([a,b];\mathbb{R})$ with $y_h' \ge 1/j$ for all $h \in \mathbb{N}$ and

$$y_h(a) = a, \ y_h(b) = b, \ \|y_h(t) - t\|_{W^{1,1}} \le \frac{1}{h+1} \text{ a.e.}, \quad \|x_* \circ y_h^{-1} - x_*\|_{W^{1,m}} > \varepsilon_0.$$

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Lemma 7.1 shows that a subsequence of $(x_* \circ y_h^{-1})_h$ converges to x_* in $W^{1,m}([a,b];\mathbb{R})$, a contradiction. □

We consider the following Borel extension $\widetilde{\Lambda}$ of Λ to $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ defined by

$$\forall x,\xi\in\mathbb{R}^n \qquad \widetilde{\Lambda}(t,x,\xi):=\begin{cases}\Lambda(a,x,\xi) & \text{if } t\le a;\\ \Lambda(t,x,\xi) & \text{if } t\in[a,b],\\ \Lambda(b,x,\xi) & \text{if } t\ge b.\end{cases}$$

Definition 7.3 *(An auxiliary Lagrangian).* For $t\in[a,b]$, $y\in\mathbb{R}$ and $v\in\mathbb{R}$ we set

$$\ell(t,y,v):=\begin{cases}\widetilde{\Lambda}\Big(y,x_*(t),\dfrac{x_*'(t)}{v}\Big)v & \text{if } v\ge 1/j \text{ and } x_*'(t) \text{ is defined,}\\ 0 & \text{otherwise.}\end{cases}$$

We consider the following auxiliary optimal control problem.

Definition 7.4 *(An auxiliary control problem).*

$$\begin{cases}\text{Minimize} & J(y,v):=\displaystyle\int_a^b \ell(t,y(t),v(t))\,dt\\ \text{Subject to:} & y\in W^{1,1}([a,b];\mathbb{R}),\ y'=v\ge 1/j \text{ almost everywhere,}\\ & y(a)=a,\ y(b)=b.\end{cases}\tag{OC}$$

Remark 7.5. The function $\ell(t,y,v)$ is $(\mathcal{L}_1[a,b]\times\mathcal{B}_1(\mathbb{R})\times\mathcal{B}_1(\mathbb{R}))$-measurable. It follows from [14, Proposition 6.34] that the map $t\mapsto\ell(t,y(t),v(t))$ is measurable for every $y(\cdot),v(\cdot)$ measurable.

The next Lemma relates the minimality for x_* to (P) with that one for $(y_*(t):=t, v_*(t):=1)$ for (OC), and is crucial in the forthcoming proofs.

Lemma 7.6. *Let $y_*(t):=t$, $v_*(t):=1$, $t\in[a,b]$.*

1. *If x_* is a minimizer of (P) then (y_*,v_*) is a minimizer for (OC).*
2. *If x_* is a $W^{1,m}$ ε-local minimizer of (P), then there exists $\rho>0$ such that (y_*,v_*) is a $W^{1,1}$ ρ-local minimizer for (OC), i.e., a minimizer of (OC) among the admissible pairs (y,v) with $\|y-y_*\|_{W^{1,1}}\le\rho$.*

Proof. Let $y\in W^{1,1}([a,b];\mathbb{R})$ be such that $y(a)=a$, $y(b)=b$. We assume that $v:=y'\ge 1/j$ almost everywhere on $[a,b]$, otherwise trivially one has $J(y,v)\ge J(y_*,v_*)$.

Since $y'>0$ almost everywhere then $y:[a,b]\to[a,b]$ is increasing and

$$|y(t_2)-y(t_1)|=\left|\int_{t_1}^{t_2}y'(t)\,dt\right|\ge\frac{1}{j}|t_2-t_1|\quad\forall t_1,t_2\in[a,b].$$

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Therefore y is invertible and its inverse y^{-1} is Lipschitz of rank j. We consider the absolutely continuous function

$$x(\tau) := x_*(y^{-1}(\tau)), \quad \tau \in [a, b].$$

Notice that $x(a) = x_*(a)$ and $x(b) = x_*(b)$; also $x(\tau) \in x_*([a, b]) \subset \Sigma$. Moreover, $x'_*(y^{-1}(\tau))$, $(y^{-1})'(\tau)$ exist for almost every $\tau \in [a, b]$; for such values of τ we have

$$(y^{-1})'(\tau) = \frac{1}{y'(y^{-1}(\tau))}, \quad x'(\tau) = \frac{x'_*(y^{-1}(\tau))}{y'(y^{-1}(\tau))}.$$

The fact that x is absolutely continuous and the validity of the chain rule for the derivative follows from the fact that y^{-1} is monotonic (see, for instance, [25, Corollary 5] and [22, Chapter IX, Theorem 5]).

Thus x', as well as x'_*, belongs almost everywhere to the cone C.

If x_* is a $W^{1,m}$ ε-local minimum, let ρ be as in Lemma 7.2 so that, if $\|y(t) - t\|_{W^{1,1}} \leq \rho$, then $\|x - x_*\|_{W^{1,m}} \leq \varepsilon$.

The minimality property of x_* thus yields

$$\int_a^b \Lambda(\tau, x(\tau), x'(\tau))\, d\tau + \Psi(x(a), x(b)) \geq$$

$$\geq I(x_*) = \int_a^b \Lambda(t, x_*(t), x'_*(t))\, dt + \Psi(x_*(a), x_*(b)). \tag{7.2}$$

Since $(x(a), x(b)) = (x_*(a), x_*(b))$ from (7.2) we deduce that

$$\int_a^b \Lambda(\tau, x(\tau), x'(\tau))\, d\tau \geq \int_a^b \Lambda(t, x_*(t), x'_*(t))\, dt. \tag{7.3}$$

Now, the change of variables $t = y^{-1}(\tau)$ gives

$$\begin{aligned}
\int_a^b \Lambda\big(\tau, x(\tau), x'(\tau)\big)\, d\tau &= \int_a^b \Lambda\Big(\tau, x_*(y^{-1}(\tau)), \frac{x'_*(y^{-1}(\tau))}{y'(y^{-1}(\tau))}\Big)\, d\tau \\
&= \int_a^b \Lambda\Big(y(t), x_*(t), \frac{x'_*(t)}{y'(t)}\Big) y'(t)\, dt \\
&= \int_a^b \ell(t, y(t), y'(t))\, dt.
\end{aligned}$$

Thus (7.3) may be rewritten as $J(y, v) \geq J(y_*, v_*)$, which proves the claim. □

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7.2. Proof of Theorem 4.1

Consider a minimizer x_* for (P). We assume here that Λ fulfills Hypothesis (S_{x_*}). Notice that, since Λ is assumed here to have values in $\mathbb{R}$, then $\ell(t, y, v) \in \mathbb{R}$ for almost every $t \in [a, b]$ and all $y \in \mathbb{R}$, $v \geq 1/j$, thus (OC) is well defined. By Lemma 7.6 the couple $(y_*(t), v_*(t)) := (t, 1)$ minimizes (OC) among the feasible (y, v) that satisfy $\|y - y_*\|_{W^{1,1}} \leq \rho$. The next result allows us to apply a suitable Maximum Principle.

Lemma 7.7. *For almost every $t \in [a, b]$, we have*

$$y_1, y_2 \in [t - \varepsilon_*, t + \varepsilon_*],\ v \geq 1/j \Longrightarrow \\ |\ell(t, y_2, v) - \ell(t, y_1, v)| \leq k^*(t, v)|y_2 - y_1|, \tag{7.4}$$

where $k^*(t, v) := k\left(t, \frac{1}{v}\right)v$ is $(\mathcal{L}_1[a, b] \times \mathcal{B}_1(]0, +\infty[))$-measurable and $k^*(t, 1)$ is summable.

Proof. Let t be such that (3.2) holds. If $y_1 \leq y_2 \in [t - \varepsilon_*, t + \varepsilon_*]$ and $v \geq 1/j$ we obtain

$$|\ell(t, y_2, v) - \ell(t, y_1, v)| = \left|\widetilde{\Lambda}\left(y_2, x_*(t), \frac{x_*'(t)}{v}\right) - \widetilde{\Lambda}\left(y_1, x_*(t), \frac{x_*'(t)}{v}\right)\right| v.$$

We claim that

$$\left|\widetilde{\Lambda}\left(y_2, x_*(t), \frac{x_*'(t)}{v}\right) - \widetilde{\Lambda}\left(y_1, x_*(t), \frac{x_*'(t)}{v}\right)\right| \leq k\left(t, \frac{1}{v}\right)|y_2 - y_1|. \tag{7.5}$$

Indeed, the claim is trivial if $y_1 \leq y_2 \leq a$ or $b \leq y_1 \leq y_2$. Assuming without restriction that $2\varepsilon_* < b - a$, the following cases may occur.

1. If $y_1 \leq a \leq y_2$, then

$$\begin{aligned}&\left|\widetilde{\Lambda}\left(y_2, x_*(t), \frac{x_*'(t)}{v}\right) - \widetilde{\Lambda}\left(y_1, x_*(t), \frac{x_*'(t)}{v}\right)\right| \\ &= \left|\Lambda\left(y_2, x_*(t), \frac{x_*'(t)}{v}\right) - \Lambda\left(y_1, x_*(t), \frac{x_*'(t)}{v}\right)\right| \\ &\leq k\left(t, \frac{1}{v}\right)|y_2 - a| \leq k\left(t, \frac{1}{v}\right)|y_2 - y_1|.\end{aligned}$$

2. If $a \leq y_1 \leq y_2 \leq b$ then $\widetilde{\Lambda}\left(y_i, x_*(t), \frac{x_*'(t)}{v}\right) = \Lambda\left(y_i, x_*(t), \frac{x_*'(t)}{v}\right)$ $(i = 1, 2)$ and (7.5) follows directly from (3.2).
3. If $y_1 \leq b \leq y_2$ the claim follows as in Point 1.

The validity of (7.4) is a direct consequence (7.5). The integrability of $k^*(t, 1)$ follows from that of $k(t, 1)$, assumed in Hypothesis (S_{x_*}). □

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Claim. Given $j \in \mathbb{N}$, $j \geq 2$, there is an absolutely continuous function $p_j \in [a,b]$ such that

$$\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p_j(t)(v-1) \quad \forall v \geq 1/j, \tag{7.6}$$

$$p_j'(t) \in \partial_t^C \Lambda(t, x_*(t), x_*'(t)) \text{ a.e. } t \in [a,b], \tag{7.7}$$

$$|p_j'(t)| \leq k(t,1) \text{ a.e. } t \in [a,b]. \tag{7.8}$$

Proof. Lemma 7.7 shows that the function $\ell(t,y,v)$ ($t \in [a,b]$, $y \in \mathbb{R}$, $v \geq 1/j$) and the optimal solution ($y_*(t) := t$, $v_*(t) := 1$) for (OC) satisfy the assumptions [14, Hypothesis 22.25] of Clarke's Maximum Principle (cf. [10,14], see also [26, Theorem 6.2.1 and Lemma 6.3.1]).

Write the Hamiltonian

$$H^\eta(t,y,p,v) := pv - \eta\ell(t,y,v), \quad \eta \geq 0$$

We deduce the existence of an absolutely continuous function $p_j : [a,b] \to \mathbb{R}$ and a scalar $\eta \in \{0,1\}$ satisfying the nontriviality condition

$$(\eta, p_j(t)) \neq 0 \quad \forall t \in [a,b], \tag{7.9}$$

and the adjoint inclusion

$$-p_j'(t) \in \partial_y^C H^\eta(t, y_*(t), p_j(t), v_*(t)) = -\eta\partial_y^C \ell\Big(t, y_*(t), v_*(t)\Big) \text{ a.e.}, \tag{7.10}$$

as well as the maximum condition

$$H^\eta(t, y_*(t), p_j(t), v_*(t)) = \max\left\{H^\eta(t, y_*(t), p_j(t), v) : v \geq 1/j\right\}, \tag{7.11}$$

valid for almost every $t \in [a,b]$. Taking into account that $y_*(t) = t$ and $v_*(t) = 1$, (7.11) becomes

$$p_j(t) - \eta\ell(t,t,1) \geq p_j(t)v - \eta\ell\Big(t,t,v\Big) \quad \forall v \geq 1/j,$$

for almost every $t \in [a,b]$; in terms of the reference Lagrangian Λ, the above gives

$$p_j(t) - \eta\Lambda(t, x_*, x_*') \geq p_j(t)v - \eta\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v \ \ \forall v \geq 1/j \tag{7.12}$$

for almost every $t \in [a,b]$.

If $\eta = 0$, then by (7.12), we would obtain that for almost every $t \in [a,b]$,

$$p_j(t) \geq p_j(t)v \quad \forall v \geq 1/j,$$

which would imply that $p_j(t) = 0$, contradicting (7.9). Thus $\eta = 1$, so that (7.10) yields, for almost every $t \in [a,b]$,

$$p_j'(t) \in \partial_t^C \Lambda(t, x_*(t), x_*'(t)).$$

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It follows from Proposition 3.6 that

$$|p_j'(t)| \le k(t,1) \text{ a.e. } t \in [a,b],$$

confirming the statement. In order to conclude the proof of Theorem 4.1, set

$$\Gamma(t) := \partial_t^C \Lambda(t, x_*(t), x_*'(t)) \times \mathbb{R}.$$

Then (7.7) implies that

$$p_j'(t) \in \text{co}\,\{\omega : (\omega, p_j(t)) \in \Gamma(t)\} = \partial_t^C \Lambda(t, x_*(t), x_*'(t)) \quad \text{a.e. } t \in [a,b].$$

Conditions (7.6), (7.8) and Lemma 7.8 below, with $h(t) := k(t,1)$, yields the conclusion. □

7.3. *Proof of Proposition 4.6*

The proof of Proposition 4.6 is similar to the proof of Theorem 4.1, but simpler: it is not necessary to invoke a compactness argument to obtain the validity of the Weierstrass condition for all $v > 0$, indeed, when we apply the maximum principle to the auxiliary optimal control problem, the Weierstrass condition holds only for $v \in \left[\frac{1}{1+\sigma_*}, \frac{1}{1-\sigma_*}\right]$ (from (S'_{x_*})), and its extension to the set $(0, +\infty)$ is an immediate consequence of (H') owing to a well know convexity argument (cf. [26]).

7.4. *Proof of Theorem 4.3*

It is enough to prove the validity of (W) for every $v > 0$ such that $\Lambda\left(t, x_*(t), \frac{x_*'(t)}{v}\right)$ is finite.

Claim. For every $j \in \mathbb{N}$, $j \ge 2$, there exists an **absolutely continuous** function $p_j \in W^{1,1}([a,b];\mathbb{R})$ such that, for almost every $t \in [a,b]$:

$$\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \ge p_j(t)(v-1) \qquad \forall v \in [1/j, j] \tag{7.13}$$

In addition, the following inclusion and estimate hold:

$$p_j'(t) \in \text{co}\left\{\omega : (\omega, p_j(t)) \in \partial^L_{(s,v)}\Big(\Lambda\Big(s, x_*(t), \frac{x_*'(t)}{v}\Big)v\Big)_{\substack{s=t\\ v=1}}\right\} \text{ a.e. } t, \tag{7.14}$$

$$|p_j'(t)| \le \beta(\Lambda(t, x_*(t), x_*'(t)) + A|x_*'(t)| + \gamma(t)) \quad \text{a.e. } t \in [a,b]. \tag{7.15}$$

It follows from Lemma 7.6 that $y_*(t) := t$ is a $W^{1,1}$ local minimizer for

$$\int_a^b \ell(t, y(t), y'(t))\,dt$$

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among the functions $y \in W^{1,1}([a,b];\mathbb{R})$ satisfying $y(a)=a$, $y(b)=b$, where, for $t\in[a,b]$, $y\in\mathbb{R}$ and $v\in\mathbb{R}$, we set

$$\ell(t,y,v):=\begin{cases}\widetilde{\Lambda}\Big(y,x_*(t),\dfrac{x_*'(t)}{v}\Big)v & \text{if } v\in[1/j,j] \text{ and } x_*'(t) \text{ is defined,}\\ +\infty & \text{otherwise.}\end{cases}$$

In order to apply some suitable necessary conditions, we prove now that the function ℓ satisfies the generalized Tonelli-Morrey growth condition [12, Definition 4.3.1]. First observe that, using a standard 'truncation argument' (cf. [26]), we can assume that (3.11) is valid for all $\tau\in[a,b]$. Take t be in the subset of full measure of $[a,b]$ such that (3.11) holds and $(y,v)\in\mathbb{R}\times\mathbb{R}$ with $\ell(t,y,v)<+\infty$. Consider any (ζ,ψ) in $\partial^P_{(y,v)}\ell(t,y,v)$. Then $\zeta\in\partial^P_y\ell(t,y,v)$ and there exists $\zeta'\in\partial^P_y\widetilde{\Lambda}\Big(y,x_*(t),\dfrac{x_*'(t)}{v}\Big)$ with $\zeta=v\zeta'$. Then

$$|\zeta'|\le\beta\Big(\widetilde{\Lambda}\Big(y,x_*(t),\frac{x_*'(t)}{v}\Big)+A\left|\frac{x_*'(t)}{v}\right|+\gamma(t)\Big). \tag{7.16}$$

Indeed:

- if $y\in[a,b]$ then (7.16) follows directly from (3.11).
- Assume $y<a$. Since $\ell(t,y,v)<+\infty$ then, for all $y'<a$,

$$\begin{aligned}\widetilde{\Lambda}\Big(y',x_*(t),\frac{x_*'(t)}{v}\Big)&=\widetilde{\Lambda}\Big(y,x_*(t),\frac{x_*'(t)}{v}\Big)\\&=\Lambda\Big(a,x_*(t),\frac{x_*'(t)}{v}\Big)<+\infty\end{aligned}$$

  and therefore $\zeta'=0$. The claim follows from Condition (3.11) and the fact that, for a.e. t and all $v>0$,

$$\Lambda\Big(a,x_*(t),\frac{x_*'(t)}{v}\Big)+A\left|\frac{x_*'(t)}{v}\right|+\gamma(t)\ge 0.$$

- If $y>b$ the claim follows as above.

Therefore

$$\frac{|\zeta|}{1+|\psi|}\le|\zeta|=v|\zeta'|\le\beta\Big(\ell(t,y,v)+A|x_*'(t)|+v\gamma(t)\Big) \tag{7.17}$$

$$\le\beta\Big(\ell(t,y,v)+A|x_*'(t)|+j\gamma(t)\Big). \tag{7.18}$$

This proves that ℓ fulfills the generalized Tonelli-Morrey condition. Notice that the validity of $(S^\infty_{x_*})$ implies that the map $(y,v)\mapsto\ell(t,y,v)$ is lower semicontinuous for every $t\in[a,b]$. Indeed, the claim is trivial at $(\overline{y},\overline{v})$ if $\overline{y}\in[a,b]$ or $\overline{v}\le 1/j$. If $\overline{y}<a$ and $\overline{v}>1/j$ then

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$$\ell(t, y, v) = \Lambda\Big(a, x_*(t), \frac{x'_*(t)}{v}\Big)v$$

in a neighborhood of $(\overline{y}, \overline{v})$: if $(y_k, v_k)_k$ is a sequence converging to $(\overline{y}, \overline{v})$ then, for k big enough,

$$\ell(t, y_k, v_k) = \Lambda\Big(a, x_*(t), \frac{x'_*(t)}{v_k}\Big)v_k$$

and

$$\Lambda\Big(a, x_*(t), \frac{x'_*(t)}{\overline{v}}\Big)\overline{v} \leq \liminf_{k\to+\infty} \Lambda\Big(a, x_*(t), \frac{x'_*(t)}{v_k}\Big)v_k.$$

We are thus in the position of applying [12, Theorem 4.3.2] to the (possibly local) minimizer $y_*(t) = t$. Consequently, there exist a function $p_j \in W^{1,1}([a, b]; \mathbb{R})$ and $\eta \in \{0, 1\}$ satisfying the Weierstrass condition: For almost every $t \in [a, b]$ and for all v such that $\ell(t, y_*(t), v) < +\infty$, we have

$$(\eta, p_j(t)) \neq 0 \quad \forall t \in [a, b], \tag{7.19}$$

$$\eta\ell(t, y_*(t), v) - \eta\ell(t, y_*(t), y'_*(t)) \geq p_j(t)(v - 1) \quad \forall v \in [1/j, j]. \tag{7.20}$$

Let t be such that (7.20) holds. If $\eta = 0$, then

$$p_j(t)(v - 1) \leq 0$$

for all $v \in [1/j, j]$ with $\ell(t, y_*(t), v) < +\infty$, so that (ii) of Hypothesis $(S^\infty_{x_*})$ implies $p_j(t) = 0$ for every $t \in E$, contradicting (7.19). The abnormal case being excluded, it turns out that $\eta = 1$, thus (7.20) gives (7.13). The Euler inclusion in [12, Theorem 4.3.2] now becomes

$$p'_j(t) \in \text{co}\left\{\omega : (\omega, p_j(t)) \in \partial^L_{(y,v)}\ell(t, t, 1)\right\} \text{ a.e. in } [a, b],$$

proving (7.14). Invoking Proposition 9.1, from (iii) of Hypothesis $(S^\infty_{x_*})$ we deduce that, for a.e. $t \in [a, b]$, the map $s \to \Lambda\Big(s, x_*(t), \frac{x'_*(t)}{v}\Big)$ is Lipschitz continuous (uniformly on $v \in [1/j, j]$). As a consequence, using standard properties of the limiting subdifferential applied to the differential inclusion (7.14), Condition (iii) of Hypothesis $(S^\infty_{x_*})$ guarantee also that estimate (7.15) is satisfied.

Set, for a.e. $t \in [a, b]$,

$$\Gamma(t) := \partial^L_{(s,v)}\Big(\Lambda\Big(s, x_*(t), \frac{x'_*(t)}{v}\Big)v\Big)_{\substack{s=t\\ v=1}}.$$

Conditions (7.13), (7.14), (7.15) allow to apply Lemma 7.8 below, with

$$h(t) := \beta(\Lambda(t, x_*(t), x'_*(t)) + A|x'_*(t)| + \gamma(t)).$$

The conclusion follows.

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7.5. Passing from $v \geq 1/j$ to $v > 0$ in (W)

Lemma 7.8 allows to pass from the validity of (W) from $v \geq 1/j$, $j \geq 1$, to $v > 0$.

Lemma 7.8. *Let x_* be a minimizer for (P). Assume that Λ satisfies Hypothesis (S_{x_*}). Let $h \in L^1[a,b]$ and $\Gamma : [a,b] \rightsquigarrow \mathbb{R} \times \mathbb{R}$ be a multifunction with closed values. Take a sequence of absolutely continuous functions $(p_j)_j$ in $W^{1,1}([a,b];\mathbb{R})$, satisfying*

$$\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p_j(t)(v-1) \quad \forall v \geq 1/j, \tag{7.21}$$

$$p_j'(t) \in co\,\{\omega : (\omega, p_j(t)) \in \Gamma(t)\} \text{ a.e. } t \in [a,b], \quad |p_j'| \leq h.$$

Then, there exists an absolutely continuous function $p \in W^{1,1}([a,b];\mathbb{R})$ such that, for a.e. $t \in [a,b]$,

$$\Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p(t)(v-1) \quad \forall v > 0, \tag{7.22}$$

$$p'(t) \in co\,\{\omega : (\omega, p(t)) \in \Gamma(t)\} \text{ a.e. } t \in [a,b], \quad |p'| \leq h. \tag{7.23}$$

The same conclusion is valid if Λ satisfies Hypothesis ($S^\infty_{x_*}$) and (7.21) holds true for all $v \in [1/j, j]$ ($j \geq 2$).

Proof. The integrable bound yields the existence of an integrable function $w(t)$ such that, modulo a suitable subsequence, $(p_j')_j$ converges weakly to w; notice that $|w| \leq h$.

By a standard countable additivity argument there is a set of full measure Ω of $[a,b]$ such that (7.21) holds for every $t \in \Omega$ and for every $j \geq 2$. Fix $t_0 \in \Omega$. Let $v_1 < 1 < v_2$ be such that

$$\Lambda\Big(t_0, x_*(t_0), \frac{x_*'(t_0)}{v_i}\Big)v_i < +\infty, \quad i = 1, 2;$$

the existence of v_1, v_2 is obvious under Hypothesis (S_{x_*}); otherwise it is guaranteed by (ii) of Hypothesis ($S^\infty_{x_*}$), for some t_0 in E. By applying (7.21) with v_i, $i = 1, 2$ we obtain the following estimates, valid for *every* $j \geq 2$:

$$\frac{1}{1-v_1}\Big(\Lambda(t_0, x_*(t_0), x_*'(t_0)\Big) - \Lambda\Big(t_0, x_*(t_0), \frac{x_*'(t_0)}{v_1}\Big)v_1\Big) \leq p_j(t_0);$$

$$p_j(t_0) \leq \frac{1}{v_2 - 1}\Big(\Lambda\Big(t_0, x_*(t_0), \frac{x_*'(t_0)}{v_2}\Big)v_2 - \Lambda(t_0, x_*(t_0), x_*'(t_0))\Big).$$

Therefore the sequence $(p_j(t_0))_j$ is bounded: modulo a subsequence we may assume that it converges to $\bar{p} \in \mathbb{R}$. Set

$$p(t) := \bar{p} + \int_{t_0}^{t} w(s)\,ds, \quad t \in [a,b].$$

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Then p is absolutely continuous and p_j converges weakly to p in $W^{1,1}([a,b];\mathbb{R})$.

Fix $t \in \Omega$ and let $v > 0$. Let $N \geq 1$ with $1/N < v$. Since (7.21) holds for every $j \geq 2$, by passing to the limit for $j \to +\infty$ we obtain (7.22) for every $t \in \Omega$.

It remains to check the validity of (7.23). The sequence p_j' converges weakly to p' in $L^1[a,b]$. Mazur's Lemma and implies that there is a sequence

$$q_j \in \operatorname{co}\{p_m' : m \geq j\}$$

converging to p' in $L^1[a,b]$: we will assume, modulo a subsequence, that $(q_j)_j$ converges a.e. to p'. Carathódory theorem implies that, for a.e. t in $[a,b]$ and every j in $\mathbb{N}$,

$$\begin{aligned} &q_j(t) = \pi_{1,j}(t)\, p'_{m_{1,j}(t)}(t) + \pi_{2,j}(t)\, p'_{m_{2,j}(t)}(t); \\ &|p'_{m_{i,j}(t)}| \leq h(t), \quad i = 1,2, \qquad \lim_{j\to+\infty} q_j(t) = p'(t) \\ &p'_{m_{i,j}(t)}(t) \in \operatorname{co}\{\omega : (\omega, p_{m_{i,j}(t)}(t)) \in \Gamma(t)\} \quad i = 1,2, \end{aligned} \tag{7.24}$$

for some $m_{1,j}(t), m_{2,j}(t) \geq j$ in $\mathbb{N}$ and positive $\pi_{1,j}(t), \pi_{2,j}(t)$ with $\pi_{1,j}(t) + \pi_{2,j}(t) = 1$. Choose t such that (7.24) holds. Modulo a subsequence we may assume that

$$\lim_{j\to+\infty} p'_{m_{i,j}(t)}(t) := \omega_i(t) \in \mathbb{R}, \quad i = 1,2,$$

and

$$\lim_{j\to+\infty} \pi_{i,j}(t) := \pi_i(t) \in [0,1],$$

with $\pi_1(t) + \pi_2(t) = 1$. The closure of $\Gamma(t)$ implies that

$$\omega_i(t) \in \operatorname{co}\{\omega : (\omega, p(t)) \in \Gamma(t)\} \quad i = 1,2. \tag{7.25}$$

Therefore, by passing to the limit in (7.24) for $j \to +\infty$, from (7.25) we obtain

$$p'(t) = \pi_1(t)\omega_1(t) + \pi_2(t)\omega_2(t) \in \operatorname{co}\{\omega : (\omega, p(t)) \in \Gamma(t)\}. \quad \square$$

8. An alternative proof of (W) in the autonomous case

In the autonomous case, when $\Lambda(x,\xi)$ has values in $\mathbb{R}$, inequality (W) appears in the proofs of [12, Theorem 4.5.5] and of [26, Theorem 11.5.1], under some additional assumptions: lower semicontinuity of Λ and strong local minima in [12], convexity of $\Lambda(x,\cdot)$ and global minimizers in [26]. We show here that it may be deduced without recurring to the Maximum Principle, by adapting the proof of [13, Theorem 1.1], where it is assumed that $\xi \mapsto \Lambda(x,\xi)$ be convex, a condition that is not required here. Consider a strong local minimum x_* of (P). Let $j \in \mathbb{N}$, $j \geq 2$. We introduce the auxiliary Lagrangian L as a function of two variables (t,v) defined on $[a,b] \times \mathbb{R}$ by

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$$L(t,v) := \begin{cases} \Lambda\Big(x_*(t), \dfrac{x_*'(t)}{v}\Big)v & \text{if } v \geq 1/j, \\ +\infty & \text{if } v < 1/j. \end{cases}$$

Denoting by $L^{**}(t,v)$ the **bipolar** of $v \mapsto L(t,v)$, it follows from [18, Lemma 2.4] that $v \equiv 1$ minimizes

$$\Phi(v) := \int_a^b L^{**}(t, v(t))\, dt$$

among the measurable functions v belonging to

$$V := \{v : [a,b] \to [1/j, +\infty[\text{ measurable}, \ \Phi(v) < +\infty\},$$

and satisfying the affine constraint

$$H(v) := \int_a^b v(t)\, dt = b - a.$$

Now the set V is convex, and the convexity of Φ follows from that of the bipolar $L^{**}(t,\cdot)$ of L. We may thus invoke Kuhn-Tucker's Theorem [14, Theorem 9.4]: There exists $(\eta_j, \lambda_j) \in \mathbb{R}^2$ with either $\eta_j = 0, \lambda_j \neq 0$ or $\eta_j = 1$, satisfying

$$\eta_j \Phi(v) + \lambda_j H(v) \geq \eta_j \Phi(1) + \lambda_j H(1) \quad \forall v \in V.$$

Clearly the above implies $\eta_j \neq 0$, and thus becomes

$$\Phi(v) + \lambda_j H(v) \geq \Phi(1) + \lambda_j H(1) \quad \forall v \in V$$

or, equivalently,

$$\int_a^b L^{**}(t, v(t)) + \lambda_j v(t)\, dt \geq \int_a^b L^{**}(t,1) + \lambda_j\, dt \quad \forall v \in V.$$

By [14, Theorem 6.32] we obtain, for almost every $t \in [a,b]$,

$$L^{**}(t,v) + \lambda_j v \geq L^{**}(t,1) + \lambda_j \quad \forall v \geq 1/j. \tag{8.1}$$

Again by [18, Lemma 2.4], we know that $L^{**}(t,1) = L(t,1)$ (the argument used there for global minima actually works if the minimizer is strong local); taking into account that $L(t,v) \geq L^{**}(t,v)$ for all $v \geq 1/j$, the above variational inequality (8.1) yields, for almost every $t \in [a,b]$,

$$L(t,v) + \lambda_j v \geq L(t,1) + \lambda_j \quad \forall v \geq 1/j$$

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or, equivalently,

$$\Lambda\Big(x_*(t), \frac{x'_*(t)}{v}\Big)v - \Lambda(x_*(t), x'_*(t)) \geq \lambda_j(v-1) \quad \forall v \geq 1/j.$$

Set, for each $j \in \mathbb{N}$, $p_j(t) := \lambda_j$. Then

$$(p'_j(t), p_j(t)) \in \Gamma(t) := \{0\} \times \mathbb{R} \quad \text{a.e. } t \in [a,b].$$

Lemma 7.8, with $h \equiv 0$, yields the conclusion.

9. A proximal Gronwall inequality

We consider here a Gronwall type inequality where a proximal gradient replaces the classical derivative: it plays a central role to prove Proposition 3.6, Corollary 3.9 and estimate (7.15) in the proof of Theorem 4.3. The new fact, with respect of the existing versions in the literature (e.g., [15, Exercise 11.29]), is that the Lipschitzianity of the function involved in the inequality follows as a consequence instead of being an assumption: Point 1 of Proposition 9.1 was suggested to us by Francis Clarke in a personal communication.

Proposition 9.1 *(A proximal Gronwall inequality). Let $c < d$ in $\mathbb{R}$, $g : [c,d] \to \mathbb{R}$ be lower semicontinuous. Assume that there is $\beta \geq 0$ satisfying*

$$|\partial^P g(\tau)| \leq \beta\, g(\tau) \quad \forall \tau \in [c,d]. \tag{9.1}$$

Then:

1. *g is Lipschitz on $[c,d]$ and, if $\beta > 0$, $g \geq 0$.*
2. *For all $\tau, \tau' \in [c,d]$*

$$|g(\tau') - g(\tau)| \leq g(\tau)\Big(e^{\beta|\tau'-\tau|} - 1\Big). \tag{9.2}$$

3. *There exists a constant $\alpha \leq (e^{\beta(d-c)} + 1)\dfrac{e^{\beta(d-c)} - 1}{d-c}$ such that*

$$|g(t_2) - g(t_1)| \leq \alpha\, g(t)|t_2 - t_1| \quad \forall t, t_1, t_2 \in [c,d]. \tag{9.3}$$

Proof. If $\beta = 0$, then the only possibility to obtain a proximal subgradient $\zeta \in \partial^P g(\tau)$ is $\zeta = 0$. Invoking the Mean Value Inequality Theorem (cf. [14, Theorem 11.2]) we obtain that $g \equiv$ constant on $[c,d]$, so that Points 1,2,3 are fulfilled with $\alpha = 0$.

We continue the proof assuming that $\beta > 0$.

1. It follows from the Product Rule in Lemma 9.3 that

$$\partial^P(g(\tau)\, e^{-\beta\tau}) \leq 0 \quad \text{and} \quad \partial^P(g(\tau))e^{\beta\tau}) \geq 0, \quad \forall \tau \in [c,d].$$

Owing to the Mean Value Inequality Theorem, we deduce that the function $\tau \to g(\tau)\, e^{-\beta\tau}$ is decreasing and $\tau \to g(\tau)\, e^{\beta\tau}$ is increasing on $[c,d]$. Then, for all $\tau \in [c,d]$,

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$$g(c)\,e^{-\beta c} \geq g(\tau)\,e^{-\beta\tau} \geq g(d)\,e^{-\beta d},$$

proving that g is bounded on $[c,d]$. Using again (9.1), we obtain that $\partial^P g$ is bounded on $[c,d]$: [14, Corollary 11.7] shows that g is Lipschitz on $[c,d]$. The Proximal Density Theorem (cf. [14, Theorem 7.34]) allows us to conclude that $g \geq 0$ on $[c,d]$.

2. Assume $\tau' \leq \tau$. Since $s \mapsto g(s)\,e^{-\beta s}$ is decreasing we have

$$g(\tau)\,e^{-\beta\tau} \leq g(\tau')\,e^{-\beta\tau'},$$

implying

$$(g(\tau) - g(\tau'))e^{-\beta\tau'} \leq g(\tau)\,(e^{-\beta\tau'} - e^{-\beta\tau})$$

from which we deduce that (recall that, from Point 1, $g \geq 0$)

$$g(\tau) - g(\tau') \leq g(\tau)\,(1 - e^{-\beta(\tau-\tau')}) \leq g(\tau)\,(e^{\beta|\tau-\tau'|} - 1). \tag{9.4}$$

Analogously, since $s \mapsto g(s)\,e^{\beta s}$ is increasing, we obtain

$$g(\tau') - g(\tau) \leq g(\tau)\,)(e^{\beta|\tau-\tau'|} - 1), \tag{9.5}$$

thus (9.2) follows from (9.4) and (9.5). The case $\tau \leq \tau'$ is analogous.

3. Let $\tau', \tau \in [c,d]$ with $\tau' \neq \tau$. Since and $0 < |\tau' - \tau| \leq d - c$ we have

$$e^{\beta|\tau'-\tau|} - 1 = \frac{e^{\beta|\tau'-\tau|} - 1}{\beta|\tau' - \tau|}\beta|\tau' - \tau| \leq \frac{e^{\beta(d-c)} - 1}{\beta(d-c)}\beta|\tau' - \tau|.$$

Thus, if we set $\beta' := \dfrac{e^{\beta(d-c)} - 1}{d-c}$, the inequality (9.2) gives:

$$|g(\tau') - g(\tau)| \leq \beta'\,g(\tau)\,|\tau' - \tau|. \tag{9.6}$$

Moreover, from the monotonicity properties of functions $\tau \to g(\tau)\,e^{\pm\beta\tau}$, we immediately deduce that

$$g(\tau') \leq C\,g(\tau) \quad \forall \tau' \in [c,d], \tag{9.7}$$

with $C := e^{\beta(d-c)}$. Now let t_1, t_2 in $[c,d]$, we may assume without restriction that $t_1 \leq t_2$. Two cases may occur:

- If $t_1 \leq t \leq t_2$, by applying successively (9.6) with $\tau' = t_1$ and with $\tau' = t_2$, by the triangular inequality we have

$$|g(t_2) - g(t_1)| \leq \beta'\,g(t)\,|t_2 - t_1|.$$

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- If $t_2 \le t_1 \le t$ or $t \le t_1 \le t_2$, by applying (9.6) with $\tau' = t_2$ and $\tau = t_1$ we obtain

$$|g(t_2) - g(t_1)| \le \beta' g(t_1)\, |t_2 - t_1|. \tag{9.8}$$

  Since, from (9.7),

$$|g(t_1)| \le C\, g(t),$$

  it follows from (9.8) that

$$|g(t_2) - g(t_1)| \le \beta'' g(t)\, |t_2 - t_1|,$$

  with $\beta'' := \beta' C = e^{\beta(d-c)} \dfrac{e^{\beta(d-c)} - 1}{d - c}$.

Thus in any case we obtain (9.3), with $\alpha := \max\{\beta', \beta''\} = \beta''$. □

Remark 9.2.

- The conclusion of Proposition 9.1 may fail if (9.1) is assumed to hold just for a.e. τ. For instance, let

$$g(t) = \begin{cases} 1 & t \ne 0, \\ 0 & t = 0. \end{cases}$$

  Then

$$\partial^P g(t) = \begin{cases} 0 & t \ne 0, \\ \mathbb{R} & t = 0. \end{cases}$$

  Thus $|\partial^P g(t)| \le g(t)$ for all $t \ne 0$, nevertheless g is not even continuous.
- Point 2 and Point 3 of Proposition 9.1 are well established when g is locally Lipschitz and, at every point t where g is differentiable,

$$|g'(t)| \le \beta\, g(t).$$

The classical Product Rule for the subdifferential of a product of two functions g and h requires their local Lipschitz continuity (e.g., [26, Theorem 5.5.3]). In the next Lemma 9.3 we show that no assumptions are needed for g, when h is differentiable with locally Lipschitz continuous derivative.

Lemma 9.3 *(A proximal product rule). Let $g : [c, d] \to \mathbb{R}$ be a function, and $h : [c, d] \to]0, +\infty[$ be differentiable with locally Lipschitz continuous derivative. Then*

$$\partial^P (g\, h) = h \partial^P g + g\, h'.$$

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Proof. We first prove that

$$\partial^P(g\,h) \subseteq h\,\partial^P g + g\,h'. \tag{9.9}$$

Let $t \in [c,d]$ and $\zeta \in \partial^P(g\,h)(t)$. Then, there are $\varepsilon > 0$, $\sigma \in \mathbb{R}$ such that, for all $\tau \in [c,d]$ with $|\tau - t| \le \varepsilon$,

$$g(\tau)\,h(\tau) - g(t)\,h(t) \ge \zeta\,(\tau - t) - \sigma\,|\tau - t|^2. \tag{9.10}$$

Since h is positive, multiplying both sides of (9.10) by $h(\tau)^{-1} = \dfrac{1}{h(\tau)}$ we obtain

$$g(\tau) - g(t)\,h(t)\,h(\tau)^{-1} \ge \zeta\,h(\tau)^{-1}\,(\tau - t) - \sigma\,h(\tau)^{-1}\,|\tau - t|^2$$

or, equivalently,

$$g(\tau) - g(t) \ge g(t)\big(h(t)\,h(\tau)^{-1} - 1\big) + \zeta\,h(\tau)^{-1}\,(\tau - t) - \sigma\,h(\tau)^{-1}\,|\tau - t|^2. \tag{9.11}$$

Now, owing to the regularity of h, for $\tau \in [c,d]$ we have

$$h(\tau)^{-1} - h(t)^{-1} + \frac{h'(t)}{h^2(t)}(\tau - t) = O\big(|\tau - t|^2\big) \quad \tau \to t,$$

so that

$$h(t)h(\tau)^{-1} - 1 = -\frac{h'(t)}{h(t)}(\tau - t) + O\big(|\tau - t|^2\big) \quad \tau \to t.$$

Since h is bounded from below by a strictly positive constant, it follows from (9.11) that, for $|\tau - t| \le \varepsilon' \le \varepsilon$,

$$g(\tau) - g(t) \ge \left(-\frac{g(t)\,h'(t)}{h(t)} + \zeta\,h(t)^{-1}\right)(\tau - t) - \sigma'\,|\tau - t|^2,$$

for a suitable $\sigma' > 0$, proving that

$$-g(t)\,h'(t)h(t)^{-1} + \zeta\,h(t)^{-1} \in \partial^P g(t). \tag{9.12}$$

Multiplying both terms of (9.12) by $h(t)$ we obtain that

$$-g(t)\,h'(t) + \zeta \in h(t)\,\partial^P g(t),$$

which proves (9.9). In order to conclude, we write g as the product of $g\,h$ with $(h)^{-1}$. Since $(h)^{-1} = \dfrac{1}{h}$ is strictly positive and differentiable with locally Lipschitz continuous derivative, the application of (9.9) yields

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$$\partial^P g = \partial^P\big((g\,h)(h)^{-1}\big)$$
$$\subseteq (h)^{-1}\partial^P(g\,h) + (g\,h)\big((h)^{-1}\big)' = \frac{1}{h}\Big(\partial^P(g\,h) - g\,h'\Big),$$

which gives $h\,\partial^P g + g\,h' \subseteq \partial^P(g\,h)$. □

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