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A global characterization of the Fučík spectrum for a system of ordinary differential equations

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Abstract

The Fučík spectrum for systems of second order ordinary differential equations with Dirichlet or Neumann boundary values is considered: it is proved that the Fučík spectrum consists of global $\mathcal{C}^1$ surfaces, and that through each eigenvalue of the linear system pass exactly two of these surfaces. Further qualitative, asymptotic and symmetry properties of these spectral surfaces are given. Finally, related problems with nonlinearities which cross asymptotically some eigenvalues, as well as linear–superlinear systems are studied.
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1. Introduction

The notion of Fučík spectrum was introduced for the scalar Laplace problem in [1] and [2]. It is defined as the set $\Sigma_{\text{scal}} \subseteq \mathbb{R}^2$ of the points (λ^+, λ^-) for which there exists a nontrivial solution of the problem

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- & \text{in } (0,1), \\ Bu = 0 & \text{in } \{0;1\}; \end{cases} \tag{1.1}$$

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here $u^{\pm}(x) = \max\{0, \pm u(x)\}$ and $Bu = 0$ represents Dirichlet or Neumann boundary conditions. The Fučík spectrum Σ_{scal} for Eq. (1.1) is explicitly known and consists of unbounded curves in $\mathbb{R}^2$: more precisely, from every diagonal point $(\lambda_k, \lambda_k) \in \mathbb{R}^2$ (where λ_k, $k \in \mathbb{N}$, denote the eigenvalues of $-u''$ with boundary conditions B) emanate two curves $C_k^{\pm}$, with the property that the corresponding solutions have exactly $k-1$ nodal points and are positive (respectively negative) near zero. The two curves may coincide, and in fact they always do for the Neumann problem, and for k even in the Dirichlet problem.

The corresponding partial differential equation

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & \text{in } \Omega, \\ Bu = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ denotes a bounded domain, is more difficult to handle and the corresponding Fučík spectrum is only partially known, namely:

- the so-called trivial part of the spectrum, corresponding to positive or negative solutions;
- if λ_k is a simple eigenvalue, it was proved in [3,4], that the Fučík spectrum in $(\lambda_{k-1}, \lambda_{k+1})^2 \subset \mathbb{R}$ consists of two curves (maybe coincident) which pass through the point (λ_k, λ_k); this result was extended to a larger area in [5];
- if λ_k is a multiple eigenvalue, it can be proved that there still exist these two curves, but there can be more points belonging to the spectrum between them: see [6,7], and in particular [5] (which seems to be the most general result to this date);
- the first nontrivial curve, passing through the point (λ_2, λ_2), see [8].

In [9] the following generalization of the Fučík spectrum to *nonlinear elliptic systems* was introduced:

$$\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & \text{in } \Omega, \\ -\Delta v = \mu^+ u^+ - \mu^- u^- & \text{in } \Omega, \\ Bu = Bv = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

It was shown there that, due to intrinsic symmetries of the system, one may assume that two of the four values $\lambda^+, \lambda^-, \mu^+, \mu^-$ coincide. Thus, setting $\lambda^+ = \mu^+$, the Fučík spectrum of (1.3) can be defined as: $\widehat{\Sigma} = \{(\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3;\ (1.3) \text{ has a nontrivial solution}\}$. It was also proved that near a simple eigenvalue of the system the Fučík spectrum consists of exactly two (maybe coincident) 2-dimensional surfaces, which can be parametrized as $\lambda^+_{1,2}(\lambda^-, \mu^-)$. Furthermore, by a variational characterization, the existence of an unbounded continuum of points in $\widehat{\Sigma}$ containing the point $(\lambda_2, \lambda_2, \lambda_2)$ was given.

Finally, it was shown that many of the properties of the scalar Fučík spectrum continue to hold for its system counterpart (1.5), in particular, the properties related to the solvability of problems with "jumping nonlinearities" (that is nonlinearities with asymptotically linear, but different, growths at $+\infty$ and $-\infty$). In short, in dependence of the position of these asymptotic slopes with respect to the Fučík spectrum $\widehat{\Sigma}$ one has: if these slopes lie in the regions in $\mathbb{R}^3 \setminus \widehat{\Sigma}$ which contain points $\lambda^+ = \lambda^- = \mu^-$ of the diagonal, then one has solvability for every forcing term; on the other hand, if these slopes lie between two spectral surfaces emanating from the same eigenvalue $(\lambda_k, \lambda_k, \lambda_k)$ (in the case they are distinct), one obtains existence or nonexistence of solutions in dependence of the forcing terms.

These properties show the importance of a good knowledge of the shape of the Fučík spectrum.

In this paper we consider the Fučík spectrum for a *system of second order ordinary differential equations*: we consider the problem

$$\begin{cases} -u'' = \lambda^+ v^+ - \lambda^- v^- & \text{in } (0,1), \\ -v'' = \lambda^+ u^+ - \mu^- u^- & \text{in } (0,1), \\ Bu = Bv = 0 & \text{in } \{0;1\}, \end{cases} \tag{1.4}$$

and we call *Fučík spectrum* the set

$$\widehat{\Sigma} = \left\{ \left(\lambda^+, \lambda^-, \mu^-\right) \in \mathbb{R}^3 \text{such that } \lambda^\pm, \mu^- \geqslant 0 \text{ and (1.4) has nontrivial solutions} \right\}. \tag{1.5}$$

Throughout the paper we will always denote by $0 \leqslant \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ the eigenvalues of $-\frac{d^2}{dx^2}$ in $H_0^1(0,1)$, when dealing with the Dirichlet problem, and in $H^1(0,1)$ for the Neumann problem; moreover $\{\phi_k\}_{k=1,2,\ldots}$ will be the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_k\|_{L^2} = 1$ and $\phi_1 > 0$.

We recall here that the linear spectrum of a system coupled like (1.4), namely

$$\begin{cases} -u'' = \lambda v & \text{in } (0,1), \\ -v'' = \lambda u & \text{in } (0,1), \\ Bu = Bv = 0 & \text{in } \{0;1\}, \end{cases} \tag{1.6}$$

is composed of the eigenvalues λ_k and $-\lambda_k$ $(k = 1, 2, \ldots)$, with corresponding eigenfunctions the pairs (ϕ_k, ϕ_k) and $(\phi_k, -\phi_k)$, respectively. As a consequence, the points $(\lambda_k, \lambda_k, \lambda_k)$ belong to $\widehat{\Sigma}$ for $k \geqslant 1$.

When restricting to dimension one, the techniques for ordinary differential equations become available, and this will allow to obtain a much more detailed description of $\widehat{\Sigma}$ than was possible for (1.3). In particular, the techniques which we use here are inspired from [10], where the interest was the Fučík spectrum for a fourth order problem in dimension one with so-called Navier boundary conditions. Indeed, we remark that this equation is a special case of system (1.4) which is obtained by setting $\mu^+ = \mu^-$.

The results presented here will be in different directions. First (in Section 3) we will derive some qualitative properties of the nontrivial solutions of (1.4):

Theorem 1.1. *If* $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}$ *and* (u, v) *is a corresponding nontrivial solution of problem* (1.4)*, then u and v have only simple zeros, of the same number, and of the same sign both in a neighborhood of* 0 *and of* 1.

The main result of the paper will be to obtain a good description of the Fučík spectrum $\widehat{\Sigma}$: first, through the implicit function theorem, we will obtain a local description of $\widehat{\Sigma}$ in a neighborhood of one of its points (see Lemma 4.1), and then, as a consequence, also a global description (see Proposition 4.4).

Finally, by joining the global description with the results in Theorem 1.1 and with the knowledge of the linear spectrum (1.6) and of the scalar Fučík spectrum Σ_{scal}, we will be able to establish

Theorem 1.2. *Through any point $(\lambda_k, \lambda_k, \lambda_k) \in \mathbb{R}^3$, with $k \geqslant 2$, pass two $\mathcal{C}^1$ surfaces in $\widehat{\Sigma}$:*

- *the two surfaces corresponding to $k \geqslant 2$ are always coincident with Neumann boundary conditions, and they are coincident if and only if k is even with Dirichlet boundary conditions*;
- *if the two surfaces corresponding to $k \geqslant 2$ do not coincide, then their intersection contains an unbounded curve which passes through $(\lambda_k, \lambda_k, \lambda_k)$*;
- *the surfaces corresponding to different values of k are disjoint*;
- *each of the surfaces is characterized by the number of (simple) interior zeros (in fact, $k-1$) of the corresponding nontrivial solutions and by their sign in a neighborhood of zero*; *they will therefore be denoted by $\widehat{\Sigma}_k^+$ and $\widehat{\Sigma}_k^-$*;
- *these surfaces may be represented expressing the variable λ^+ in terms of the other two, and they are monotone decreasing in the two variables and unbounded in the three directions*;
- *all points in $\widehat{\Sigma}$ belong to one of these surfaces (except for those corresponding to solutions which do not change sign, see relation (2.8))*;
- *in the Neumann case, if $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^{\pm}$ with $k \geqslant 2$, then*

$$\sqrt{\lambda^- \mu^-} > \lambda_k/4 \quad \text{and} \quad \lambda^+ > \lambda_k/4;$$

in the Dirichlet case: if $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^+$ (respectively $\widehat{\Sigma}_k^-$) with $k \geqslant 2$, then

$$\sqrt{\lambda^- \mu^-} > \lambda_{[k/2]} \quad \text{and} \quad \lambda^+ > \lambda_{[(k+1)/2]}$$

respectively

$$\sqrt{\lambda^- \mu^-} > \lambda_{[(k+1)/2]} \quad \text{and} \quad \lambda^+ > \lambda_{[k/2]}$$

(here $[\cdot]$ is the integer part of its argument).

In Section 4.3, we also present a result about the symmetries of the nontrivial solutions:

Proposition 1.3. *The nontrivial solutions corresponding to points in $\widehat{\Sigma}_k^{\pm}$ with k odd are symmetric, in the sense that $(u, v)(x) = (u, v)(1-x)$.*

Moreover, for Neumann boundary conditions, one has that if $j \geqslant 0$ and $k \equiv 1$ (MOD 2^{j+1}), then (u, v) are $(1/2)^j$ periodic in $[0, 1]$.

Finally, in Section 5, we will analyze some properties of the set $(\mathbb{R}^+)^3 \setminus \widehat{\Sigma}$, motivated by the fact that this set is related to the solvability of nonlinear problems with jumping nonlinearities (see there for more details). Indeed, Proposition 5.3 presents regions where we may guarantee solvability, as given in Theorem 5.4.

We remark that also problems with nonlinearities which are asymptotically linear at $-\infty$ and superlinear at $+\infty$ may be treated by exploiting the properties of the Fučík spectrum, as done for the scalar equation in [11] and in [12]: what is important for these applications is the knowledge of the asymptotic behavior of the curves in Σ_{scal}; in particular, the key point in order to have the possibility to obtain existence results for such a nonlinearity, is the presence of a "gap at infinity" between the surfaces in $\widehat{\Sigma}$ corresponding to consecutive eigenvalues; the results in Proposition 5.5 will guarantee the existence of such gaps, at least for Neumann boundary conditions.

A result for the first of these gaps was already obtained in [13], that is, existence of solution when the slopes at $-\infty$ are positive and satisfy $\sqrt{\lambda^-\mu^-} < \lambda_2/4$. We are confident that the better insight on $\widehat{\Sigma}$ obtained here will prove useful to improve this result.

To conclude this introduction, we remark that, although the results obtained here are much more detailed than for the multidimensional case [9], we cannot obtain an explicit description of $\widehat{\Sigma}$ as is possible for the scalar problem in dimension one. This is due to the fact that the interaction of the two functions u and v gives rise to more complicated phenomena: just to give an idea, observe that the nontrivial solutions of the scalar Fučík problem in dimension one can be built by suitably gluing bumps of the form $\sin(\sqrt{\lambda^+}\,x)$ and $-\sin(\sqrt{\lambda^-}\,x)$; on the other hand, for the system, there appear intervals where u and v have opposite sign, and there the solutions have a component of the form of the hyperbolic functions sinh and cosh.

2. Some useful results about the Fučík spectrum

For the scalar problem in dimension one, it is known that the Fučík spectrum Σ_{scal} may be explicitly calculated: for Dirichlet boundary conditions it is composed by the curves

$$\Sigma_{2i}^+ \equiv \Sigma_{2i}^-: \quad \frac{i\pi}{\sqrt{\lambda^+}} + \frac{i\pi}{\sqrt{\lambda^-}} = 1, \quad i \geqslant 1, \tag{2.1}$$

$$\Sigma_{2i-1}^+: \quad \frac{i\pi}{\sqrt{\lambda^+}} + \frac{(i-1)\pi}{\sqrt{\lambda^-}} = 1, \qquad \Sigma_{2i-1}^-: \quad \frac{(i-1)\pi}{\sqrt{\lambda^+}} + \frac{i\pi}{\sqrt{\lambda^-}} = 1, \quad i \geqslant 1; \tag{2.2}$$

for Neumann boundary conditions it is given by the curves

$$\Sigma_1^+: \quad \lambda^+ = 0, \qquad \Sigma_1^-: \quad \lambda^- = 0, \tag{2.3}$$

$$\Sigma_j^+ \equiv \Sigma_j^-: \quad \frac{(j-1)\pi}{2\sqrt{\lambda^+}} + \frac{(j-1)\pi}{2\sqrt{\lambda^-}} = 1, \quad j \geqslant 2, \tag{2.4}$$

where the curve Σ_k^+ (respectively Σ_k^-) corresponds to nontrivial solutions having $k-1$ internal zeros and starting positive (respectively negative); observe that $\Sigma_k^\pm$ passes through the point (λ_k, λ_k) (recall that $\lambda_k = (k\pi)^2$ in the Dirichlet case and $\lambda_k = ((k-1)\pi)^2$ in the Neumann case). Observe also that the asymptotes of these curves are located, for the Dirichlet case, at the values

$$\lambda^- = \lambda_i \quad \text{for } \Sigma_{2i-1}^-,\ \Sigma_{2i},\ \Sigma_{2i+1}^+, \tag{2.5}$$

$$\lambda^+ = \lambda_i \quad \text{for } \Sigma_{2i-1}^+,\ \Sigma_{2i},\ \Sigma_{2i+1}^-, \tag{2.6}$$

and for the Neumann case at

$$\lambda^- = \frac{\lambda_j}{4} \quad \text{and} \quad \lambda^+ = \frac{\lambda_j}{4} \quad \text{for } \Sigma_j. \tag{2.7}$$

These simple computations may be found in [1], for Dirichlet boundary conditions, and in [11], for the periodic case, which is very similar to the Neumann case.

We also refer to [2], where the most important properties of the (scalar) Fučík spectrum are presented for what concerns its application to the solvability of related nonlinear equations.

A survey of all these results was given in [14].

Since we are going to use several results from [9], we devote the last part of this section to briefly summarize them.

First, we remark that in (1.4) (and hence in $\widehat{\Sigma}$) we are considering only nonnegative coefficients, and identical coefficients for u^+ and v^+; the reasons for this choice are widely discussed in [9] and rely on the observation that, even if we use four parameters, the "interesting" points of the spectrum have coefficients which are all of the same sign and which, through a suitable rescaling, may be transformed into an equivalent point of this form (i.e. lying in $\widehat{\Sigma}$).

Furthermore, for points in $\widehat{\Sigma}$, the following lemma was proved:

Lemma 2.1. *Let* $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}$ *and* (u, v) *be a corresponding nontrivial solution, then*:

- *both* u *and* v *change sign or none of the two*;
- *if none of the two changes sign then* $\lambda^+ = \lambda_1$ *or* $\sqrt{\lambda^- \mu^-} = \lambda_1$;
- *if both change sign, then* $\lambda^+ > \lambda_1$ *and* $\sqrt{\lambda^- \mu^-} > \lambda_1$.

This lemma implies that $\widehat{\Sigma}$ is composed of a trivial part

$$\widehat{\Sigma}_t = \bigl\{\bigl(\lambda^+, \lambda^-, \mu^-\bigr) \in \mathbb{R}^3 \text{ such that: } \lambda^+ = \lambda_1 \text{ or } \sqrt{\lambda^- \mu^-} = \lambda_1\bigr\}, \tag{2.8}$$

corresponding to solutions which do not change sign, and a nontrivial part $\widehat{\Sigma}_{nt}$ (to which we may concentrate our attention) which we define as

$$\begin{aligned} \widehat{\Sigma}_{nt} = \bigl\{&\bigl(\lambda^+, \lambda^-, \mu^-\bigr) \in \mathbb{R}^3 \text{ such that:} \\ &\lambda^\pm, \mu^- > 0 \text{ and (1.4) has nontrivial solutions which (both) change sign}\bigr\}. \end{aligned} \tag{2.9}$$

Finally, we remark that $\widehat{\Sigma}_{nt}$ has several symmetries, in particular we have

Lemma 2.2. *If* $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_{nt}$ *with corresponding nontrivial solutions* (u, v)*, then*

1. $(\lambda^+, \mu^-, \lambda^-) \in \widehat{\Sigma}_{nt}$ *with corresponding nontrivial solutions* (v, u);
2. $(\sqrt{\lambda^- \mu^-}, \lambda^+\sqrt{\frac{\mu^-}{\lambda^-}}, \lambda^+\sqrt{\frac{\lambda^-}{\mu^-}}) \in \widehat{\Sigma}_{nt}$, *with corresponding nontrivial solutions* $(-u, -\sqrt{\frac{\lambda^-}{\mu^-}}v)$.

For what concerns the relationship between Σ_{scal} and $\widehat{\Sigma}$ we recall

Lemma 2.3. *If* $(\lambda^+, \lambda^-) \in \Sigma_{\mathrm{scal}}$ *with corresponding nontrivial solution* u*, then* $(\lambda^+, \lambda^-, \lambda^-) \in \widehat{\Sigma}$ *with corresponding nontrivial solutions* (u, u).

3. Some properties of the nontrivial solutions

In this section we will obtain some preliminary lemmas about the nontrivial solutions of a system like (1.4); by joining these results, we obtain Theorem 1.1.

Lemma 3.1. *Weak solutions of problem* (1.4) *are in fact classical solutions.*

Proof. Weak solutions of problem (1.4) are by definition in $H^1(\Omega)$, then the right-hand side in the equations is in H^1 too; so one gets, by a boot strap argument, that the solutions are in $H^3(\Omega)$; this, in dimension one, gives solutions in $\mathcal{C}^2(\overline{\Omega})$, and so they are classical solutions. □

Lemma 3.2. *Let $c, d \in L^\infty(0,1)$ with $c, d > 0$ a.e. and (u, v) be a nontrivial solution of the boundary value problem (BVP for short)*

$$\begin{cases} -u'' = c(x)v & \text{in } (0,1), \\ -v'' = d(x)u & \text{in } (0,1), \\ Bu = Bv = 0 & \text{in } \{0;1\}. \end{cases} \tag{3.1}$$

Then for no point $x^* \in [0,1]$ *may hold*:

$$u(x^*) \geqslant 0, \quad u'(x^*) \geqslant 0, \quad v(x^*) \leqslant 0, \quad v'(x^*) \leqslant 0, \tag{3.2}$$

$$\text{nor} \quad u(x^*) \geqslant 0, \quad u'(x^*) \leqslant 0, \quad v(x^*) \leqslant 0, \quad v'(x^*) \geqslant 0, \tag{3.3}$$

$$\text{nor} \quad u(x^*) \leqslant 0, \quad u'(x^*) \leqslant 0, \quad v(x^*) \geqslant 0, \quad v'(x^*) \geqslant 0, \tag{3.4}$$

$$\text{nor} \quad u(x^*) \leqslant 0, \quad u'(x^*) \geqslant 0, \quad v(x^*) \geqslant 0, \quad v'(x^*) \leqslant 0. \tag{3.5}$$

Moreover:

- *for $\bar{x} \in [0,1]$,*
  (Ai) *if $u(\bar{x}) = 0$ (or $v(\bar{x}) = 0$) then $u'(\bar{x})v'(\bar{x}) > 0$,*
  (Aii) *if $u'(\bar{x}) = 0$ (or $v'(\bar{x}) = 0$) then $u(\bar{x})v(\bar{x}) > 0$;*
- *for $x_1, x_2 \in [0,1]$, $x_1 < x_2$,*
  (Bi) *if $u'(x_1) = u'(x_2) = 0$ and $u'(x) \neq 0$ for $x \in (x_1, x_2)$ then $u(x_1)u(x_2) < 0$,*
  (Bii) *if $u(x_1) = u(x_2) = 0$ and $u(x) \neq 0$ for $x \in (x_1, x_2)$ then $u'(x_1)u'(x_2) < 0$,*
  and the same holds for v.

Proof. For the system (3.1), uniqueness of solution for the corresponding initial value problem (IVP for short) holds and then we immediately see that a nontrivial solution may have $u(x^*) = v(x^*) = u'(x^*) = v'(x^*) = 0$ in no point $x^* \in [0,1]$.

By integrating the equations in (3.1) we get, for $x, x_0 \in [0,1]$,

$$u(x) = u(x_0) + u'(x_0)(x - x_0) - \int_{x_0}^{x} d\xi_1 \int_{x_0}^{\xi_1} c(\xi_2)v(\xi_2)\,d\xi_2, \tag{3.6}$$

$$v(x) = v(x_0) + v'(x_0)(x - x_0) - \int_{x_0}^{x} d\xi_1 \int_{x_0}^{\xi_1} d(\xi_2)u(\xi_2)\,d\xi_2, \tag{3.7}$$

$$u'(x) = u'(x_0) - \int_{x_0}^{x} c(\xi_1)v(\xi_1)\,d\xi_1, \tag{3.8}$$

$$v'(x) = v'(x_0) - \int_{x_0}^{x} d(\xi_1)u(\xi_1)\,d\xi_1. \tag{3.9}$$

We first prove (3.2) with strict inequalities: suppose the inequalities hold strictly in $x^* < 1$ and let $x_1 > x^*$ be the first point (if any) where one of the two functions has zero derivative; observe that u and v maintain their sign in $(x^*, x_1]$ and then by (3.8) $u'(x_1) > 0$ and by (3.9) $v'(x_1) < 0$: this implies that u and v may never satisfy the boundary condition at 1.

By the same argument one sees that if (3.3) holds with strict inequalities in $x^* > 0$ then u and v may never satisfy the boundary condition at 0.

Then observe that if $u(x^*) = 0$, then in a right neighborhood of x^* it assumes the sign of u' in the same neighborhood; if $u'(x^*) = 0$, then in a right neighborhood of x^* it assumes, by (3.8), the opposite sign of v in the same neighborhood, and so on; then one may relax all inequalities in (3.2) since even if only one of the four is strict then in a right neighborhood all become strict; the same considerations for left neighborhoods of x^* allow to do the same with (3.3).

Finally, Eqs. (3.4) and (3.5) follow by linearity (or by exchanging the role of u and v).

The proof of (3.2) and (3.4) in 1 and of (3.3) and (3.5) in 0 is superfluous by virtue of the boundary conditions: as an example, observe that in the Dirichlet case (3.2) is equivalent to (3.5) in 0 and 1.

Now, (Ai) and (Aii) follow by combining the four relations; we just give an example of this (all the functions in the following will be evaluated in $\bar{x}$): let $u = 0$; if $v \leqslant 0$ use (3.2) and (3.3) and obtain that $u' = 0$ gives contradiction for any v', while $u' > 0$ gives $v' > 0$ by (3.2) and $u' < 0$ gives $v' < 0$ by (3.3); for $v \geqslant 0$ the same is obtained by using (3.4) and (3.5).

To prove (Bi) observe that, by (Aii), $v(x_1) \neq 0 \neq v(x_2)$ and, by (3.8),

$$0 = u'(x_2) = -\int_{x_1}^{x_2} c(\xi_1)v(\xi_1)\,d\xi_1, \tag{3.10}$$

implying that v changes sign in (x_1, x_2); if it changes sign once then $v(x_1)v(x_2) < 0$ and then, by (Aii), $u(x_1)u(x_2) < 0$ as claimed. Now observe that u may have no more than one zero in (x_1, x_2) since otherwise there would exist also a point $\bar{x} \in (x_1, x_2)$ with $u'(\bar{x}) = 0$; then we just need to show that it may not happen that v changes sign more then once in (x_1, x_2) while u does not change sign: actually in this case, since $uv > 0$ in x_1 and x_2 by (Aii), there would exist an interval $[x_3, x_4] \subseteq (x_1, x_2)$ such that $uv < 0$ in (x_3, x_4) and $v(x_3) = v(x_4) = 0$; but this contradicts (Aii) since then there would exist also a point $\bar{x} \in (x_3, x_4)$ with $v'(\bar{x}) = 0$ and $u(\bar{x})v(\bar{x}) < 0$.

Finally, (Bii) is trivial since by (Ai) the zeros of u are simple and in the given hypotheses u has the same sign in a right neighborhood of x_1 and in a left neighborhood of x_2. □

Proposition 3.3. *If $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}$ and (u, v) is a corresponding nontrivial solution of problem* (1.4)*, then the same conclusions of the above lemma hold for* (u, v); *in particular u and v have only simple zeros and the same sign both in a neighborhood of* 0 *and of* 1.

Proof. The nontrivial solution satisfies problem (3.1) with $c(x) = \lambda^+ \chi_{\{v \geqslant 0\}}(x) + \lambda^- \chi_{\{v<0\}}(x)$ and $d(x) = \lambda^+ \chi_{\{u \geqslant 0\}}(x) + \mu^- \chi_{\{u<0\}}(x)$. (Here and in the following $\chi_A(x)$ is the characteristic function of the set A, and then satisfies the hypotheses of Lemma 3.2.)

In particular, (Ai) in Lemma 3.2 implies that the zeros are always simple, while (Ai) and (Aii) imply that $uv > 0$ both in a neighborhood of 0 and of 1. □

The results in the previous lemmas allow to obtain the following important result:

Proposition 3.4. *If* $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_{nt}$ *and* (u, v) *is a corresponding nontrivial solution of problem* (1.4)*, then* u *and* v *have the same number of* (*simple*) *zeros.*

Proof. Let x_1 be the first stationary point of u: for Neumann boundary conditions this is 0, while for Dirichlet boundary conditions we may see that there is no zero of u or v in $(0, x_1]$: actually u and u' maintain their sign in $(0, x_1)$, by (Ai) in Lemma 3.2 $u'(0)v'(0) > 0$, and then if v had a zero $\bar{x} \in (0, x_1]$ we would have $u'(\bar{x})v'(\bar{x}) \leqslant 0$ contradicting (Ai) in Lemma 3.2.

Now let x_2 be the second stationary point of u: by (Bi) in Lemma 3.2 $u(x_1)u(x_2) < 0$ and then u has exactly one zero $x_u \in (x_1, x_2)$: we claim that v has a unique zero in $[x_1, x_2]$ too: actually it may not have an even number of zeros since by (Aii) in Lemma 3.2 $v(x_1)v(x_2) < 0$, and if it had three or more, then it would also have two stationary points and for at least one of them $vu \leqslant 0$, contradicting (Aii) in Lemma 3.2. The same argument may be applied to the following intervals between stationary points and, finally, we have no zero between the last stationary point of u and 1 by the same argument used above.

Then, we conclude that the zeros of u and v are exactly one for each interval between two stationary points of u, and so we proved the claim. □

Remark 3.5. We observe that the results in this section may be described in terms of the phase plane: actually, the equations from (3.2) to (3.5) simply say that u and v may never lie in opposite (closed) quadrants of the phase plane.

With this interpretation, it is simple to deduce intuitively the properties in Lemma 3.2 and also Proposition 3.4.

4. Construction of $\widehat{\Sigma}_{nt}$

In this section, we will build the nontrivial part of the Fučík spectrum $\widehat{\Sigma}_{nt}$; in order to do this, we will first construct a related set in $\mathbb{R}^4$. We consider the initial value problem

$$\text{(IVP)} \quad \begin{cases} -u'' = \lambda^+ v^+ - \lambda^- v^-, \\ -v'' = \lambda^+ u^+ - \mu^- u^-, \\ (u, v, u', v')(0) = (u_0, v_0, u'_0, v'_0), \end{cases} \qquad \text{with } \lambda^+, \lambda^-, \mu^- \geqslant 0.$$

Then we define, for Dirichlet boundary conditions, the sets

$$\widetilde{\Sigma}^\pm = \left\{ \begin{array}{l} (\lambda^+, \lambda^-, \mu^-, s) \in (\mathbb{R}^+)^3 \times \mathbb{R}\text{: the solution } (u, v) \text{ of IVP with} \\ (u_0, v_0, u'_0, v'_0) = (0, 0, \pm 1, s) \text{ satisfies } u(1) = v(1) = 0 \end{array} \right\} \tag{4.1}$$

and for Neumann boundary conditions

$$\widetilde{\Sigma}^\pm = \left\{ \begin{array}{l} (\lambda^+, \lambda^-, \mu^-, s) \in (\mathbb{R}^+)^3 \times \mathbb{R}\text{: the solution } (u, v) \text{ of IVP with} \\ (u_0, v_0, u'_0, v'_0) = (\pm 1, s, 0, 0) \text{ satisfies } u'(1) = v'(1) = 0 \end{array} \right\}. \tag{4.2}$$

Then we will denote by

$$\widehat{\Sigma}^{\pm} = \{(\lambda^+, \lambda^-, \mu^-) \in (\mathbb{R}^+)^3 \colon \exists s \in \mathbb{R} \text{ such that } (\lambda^+, \lambda^-, \mu^-, s) \in \widetilde{\Sigma}^{\pm}\}, \tag{4.3}$$

and so $\widehat{\Sigma} = \widehat{\Sigma}^+ \cup \widehat{\Sigma}^-$.

In particular, we only need to study one of the two components (say $\widetilde{\Sigma}^+$), since the other one may be found by exploiting the symmetry 2 in Lemma 2.2.

4.1. Local study of $\widetilde{\Sigma}^+$

In the following lemma we will use the implicit function theorem to describe $\widetilde{\Sigma}^+$ in a neighborhood of one of its points.

Lemma 4.1. *Given* $(\bar\lambda^+, \bar\lambda^-, \bar\mu^-, \bar s) \in \widetilde{\Sigma}^+$ *such that the corresponding nontrivial solutions* $(\bar u, \bar v)$ *change sign* (*both*), *then* $\widetilde{\Sigma}^+$ *is locally of the form* $(\lambda^+(\lambda^-, \mu^-), \lambda^-, \mu^-, s(\lambda^-, \mu^-))$, *where* (*for a suitable* $\varepsilon > 0$)

$$(\lambda^+, s)\colon \quad (\bar\lambda^- - \varepsilon, \bar\lambda^- + \varepsilon) \times (\bar\mu^- - \varepsilon, \bar\mu^- + \varepsilon) \to \mathbb{R}^2 \tag{4.4}$$

is a $\mathcal{C}^1$ *function of* λ^- *and* μ^-; *moreover*

$$\frac{\partial\lambda^+}{\partial\lambda^-}(\bar\lambda^-, \bar\mu^-) = \frac{-\int_0^1 (\bar v^-)^2}{\int_0^1 (\bar u^+)^2 + (\bar v^+)^2} < 0, \tag{4.5}$$

$$\frac{\partial\lambda^+}{\partial\mu^-}(\bar\lambda^-, \bar\mu^-) = \frac{-\int_0^1 (\bar u^-)^2}{\int_0^1 (\bar u^+)^2 + (\bar v^+)^2} < 0. \tag{4.6}$$

Finally, the related nontrivial solutions have all the same number of (*simple*) *zeros* (*hence both change sign*) *and the both have same sign in a neighborhood of* 0 *and of* 1.

Proof. We will give the proof for Dirichlet boundary conditions.

We will denote by $(u, v)[\lambda^+, \lambda^-, \mu^-, s](x)$ the solution of the IVP

$$\begin{cases} -u'' = \lambda^+ v^+ - \lambda^- v^- & \text{in } (0,1), \\ -v'' = \lambda^+ u^+ - \mu^- u^- & \text{in } (0,1), \\ (u, v, u', v')(0) = (0, 0, 1, s) \end{cases} \tag{4.7}$$

and we will apply the implicit function theorem to the system

$$(u, v)[\lambda^+, \lambda^-, \mu^-, s](1) = (0, 0), \tag{4.8}$$

that is, we want to solve locally the set of its zeros with respect to the variables λ^-, μ^-.

We remark that $(u, v)[\lambda^+, \lambda^-, \mu^-, s](x)$ is a $\mathcal{C}^1$ function of the five variables $(\lambda^+, \lambda^-, \mu^-, s) \in \overline{N}$ and $x \in [0, 1]$, where $\overline{N}$ is a suitable neighborhood of the point $(\bar\lambda^+, \bar\lambda^-, \bar\mu^-, \bar s)$; actually the derivatives may be calculated through the differential equation, where the nonlinearities $\lambda^+ v^+ - \lambda^- v^-$ and $\lambda^+ u^+ - \mu^- u^-$ are $\mathcal{C}^1$ functions of the variables $\lambda^\pm$, μ^-.

Denote by $(\bar u, \bar v) = (u, v)[\bar\lambda^+, \bar\lambda^-, \bar\mu^-, \bar s]$; since the zeros of $\bar u$ and $\bar v$ are simple by Lemma 3.2, we may restrict the neighborhood $\overline{N}$ such that this property is maintained for all the functions $(u, v)[\lambda^+, \lambda^-, \mu^-, s]$ with $(\lambda^+, \lambda^-, \mu^-, s) \in \overline{N}$ and such that the internal zeros do not disappear and the sign in a neighborhood of 0 and of the last zero does not change. We remark also that since $\bar u$ and $\bar v$ change sign then this property too is maintained in $\overline{N}$.

Now let $c(x) = \bar\lambda^+ \chi_{\{\bar v>0\}} + \bar\lambda^- \chi_{\{\bar v<0\}}$ and $d(x) = \bar\lambda^+ \chi_{\{\bar u>0\}} + \bar\mu^- \chi_{\{\bar u<0\}}$: then $(\bar u, \bar v)$ also satisfies the IVP

$$\begin{cases} -\bar u'' = c(x)\bar v & \text{in } (0,1), \\ -\bar v'' = d(x)\bar u & \text{in } (0,1), \\ (\bar u, \bar v, \bar u', \bar v')(0) = (0, 0, 1, \bar s). \end{cases} \tag{4.9}$$

Then let $(u_s, v_s)(x) = \frac{\partial}{\partial s}(u, v)[\bar\lambda^+, \bar\lambda^-, \bar\mu^-, \bar s](x)$: differentiating (4.9) with respect to s we get (the dependence on s is just in the initial condition):

$$\begin{cases} -u_s'' = c(x)v_s & \text{in } (0,1), \\ -v_s'' = d(x)u_s & \text{in } (0,1), \\ (u_s, v_s, u_s', v_s')(0) = (0, 0, 0, 1); \end{cases} \tag{4.10}$$

note here that by using Eqs. (3.6) and (3.7) one obtains that $v_s(x) > 0$ and $u_s(x) < 0$ for $x \in (0, 1]$.

Now, multiplying the first equation in (4.10) by $\bar v$, the second by $\bar u$ and summing we have $\int -u_s''\bar v - v_s''\bar u = \int c(x)v_s\bar v + d(x)u_s\bar u$; integrating by parts two times we obtain

$$\int_0^1 -u_s\bar v'' - v_s\bar u'' + [-u_s'\bar v - v_s'\bar u + u_s\bar v' + v_s\bar u']_0^1 = \int_0^1 c(x)v_s\bar v + d(x)u_s\bar u. \tag{4.11}$$

Since $(\bar u, \bar v)$ is solution of the BVP too, it satisfies

$$\bar u(0) = \bar v(0) = \bar u(1) = \bar v(1) = 0 \tag{4.12}$$

and so in Eq. (4.11) only the following term survives:

$$(u_s\bar v' + v_s\bar u')(1) = 0. \tag{4.13}$$

In the same way, let $(u_{\lambda^+}, v_{\lambda^+})(x)$, $(u_{\lambda^-}, v_{\lambda^-})(x)$ and $(u_{\mu^-}, v_{\mu^-})(x)$ be defined like $(u_s, u_s)(x)$ above: differentiating (4.9) with respect to λ^+, λ^- and μ^- respectively, we get (the dependence on λ^+, λ^- and μ^- is in the coefficients $c(x)$ and $d(x)$):

$$\begin{cases} -u_{\lambda^+}'' = c(x)v_{\lambda^+} + \bar v^+ & \text{in } (0,1), \\ -v_{\lambda^+}'' = d(x)u_{\lambda^+} + \bar u^+ & \text{in } (0,1), \\ (u_{\lambda^+}, v_{\lambda^+}, u_{\lambda^+}', v_{\lambda^+}')(0) = (0, 0, 0, 0), \end{cases} \tag{4.14}$$

$$\begin{cases} -u_{\lambda^-}'' = c(x)v_{\lambda^-} - \bar v^- & \text{in } (0,1), \\ -v_{\lambda^-}'' = d(x)u_{\lambda^+} & \text{in } (0,1), \\ (u_{\lambda^-}, v_{\lambda^-}, u_{\lambda^-}', v_{\lambda^-}')(0) = (0, 0, 0, 0), \end{cases} \tag{4.15}$$

$$\begin{cases} -u''_{\mu^-} = c(x)v_{\mu^-} & \text{in } (0,1), \\ -v''_{\mu^-} = d(x)u_{\mu^-} - \bar{u}^- & \text{in } (0,1), \\ (u_{\mu^-}, v_{\mu^-}, u'_{\mu^-}, v'_{\mu^-})(0) = (0,0,0,0); \end{cases} \tag{4.16}$$

then, proceeding as for (4.13), we get

$$(u_{\lambda^+}\bar{v}' + v_{\lambda^+}\bar{u}')(1) = \int_0^1 \left(\bar{u}^+\right)^2 + \int_0^1 \left(\bar{v}^+\right)^2 > 0, \tag{4.17}$$

$$(u_{\lambda^-}\bar{v}' + v_{\lambda^-}\bar{u}')(1) = \int_0^1 (\bar{v}^-)^2 > 0, \tag{4.18}$$

$$(u_{\mu^-}\bar{v}' + v_{\mu^-}\bar{u}')(1) = \int_0^1 (\bar{u}^-)^2 > 0. \tag{4.19}$$

We deduce by the above computations that the vector $(u_s(1), v_s(1))$ is not zero, and by (4.13) it is orthogonal to $(\bar{v}'(1), \bar{u}'(1))$, while $(u_{\lambda^+}(1), v_{\lambda^+}(1))$ is not orthogonal to it; then

$$\det \begin{bmatrix} u_{\lambda^+}(1) & v_{\lambda^+}(1) \\ u_s(1) & v_s(1) \end{bmatrix} \neq 0, \tag{4.20}$$

which is indeed the condition we need to apply the implicit function theorem to system (4.8), that is to obtain the claimed function $(\lambda^+, s)(\lambda^-, \mu^-)$.

Now we may also obtain the derivatives $\frac{\partial\lambda^+}{\partial\lambda^-}(\bar{\lambda}^-, \bar{\mu}^-)$ and $\frac{\partial\lambda^+}{\partial\mu^-}(\bar{\lambda}^-, \bar{\mu}^-)$: by differentiating (4.8) we get

$$\left[\frac{\partial(u,v)}{\partial(s,\lambda^+)}\right]\left[\frac{\partial(s,\lambda^+)}{\partial(\lambda^-,\mu^-)}\right] + \left[\frac{\partial(u,v)}{\partial(\lambda^-,\mu^-)}\right] = 0, \tag{4.21}$$

that is

$$\begin{bmatrix} u_s(1) & u_{\lambda^+}(1) \\ v_s(1) & v_{\lambda^+}(1) \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial\lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial s}{\partial\mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \\ \frac{\partial\lambda^+}{\partial\lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial\lambda^+}{\partial\mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \end{bmatrix} = -\begin{bmatrix} u_{\lambda^-}(1) & u_{\mu^-}(1) \\ v_{\lambda^-}(1) & v_{\mu^-}(1) \end{bmatrix}; \tag{4.22}$$

by multiplying on the left by the vector $[\bar{v}'(1), \bar{u}'(1)]$ we get

$$\begin{aligned} &\left[\, (u_s\bar{v} + v_s\bar{u})(1) \quad (u_{\lambda^+}\bar{v} + v_{\lambda^+}\bar{u})(1) \,\right] \begin{bmatrix} \frac{\partial s}{\partial\lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial s}{\partial\mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \\ \frac{\partial\lambda^+}{\partial\lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial\lambda^+}{\partial\mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \end{bmatrix} \\ &\quad = -\left[\, (u_{\lambda^-}\bar{v}' + v_{\lambda^-}\bar{u}')(1) \quad (u_{\mu^-}\bar{v}' + v_{\mu^-}\bar{u}')(1) \,\right], \end{aligned} \tag{4.23}$$

that is (by Eqs. (4.13), (4.17)–(4.19))

$$\begin{bmatrix} 0 & \int_0^1 (\bar{u}^+)^2 + (\bar{v}^+)^2 \end{bmatrix} \begin{bmatrix} \frac{\partial s}{\partial \lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial s}{\partial \mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \\ \frac{\partial \lambda^+}{\partial \lambda^-}(\bar{\lambda}^-, \bar{\mu}^-) & \frac{\partial \lambda^+}{\partial \mu^-}(\bar{\lambda}^-, \bar{\mu}^-) \end{bmatrix}$$

$$= -\begin{bmatrix} \int_0^1 (\bar{v}^-)^2 & \int_0^1 (\bar{u}^-)^2 \end{bmatrix} \tag{4.24}$$

and then we get Eqs. (4.5) and (4.6).

The same results may be obtained for Neumann boundary conditions by applying the implicit function theorem to the system

$$(u', v')\big[\lambda^+, \lambda^-, \mu^-, s\big](1) = (0, 0), \tag{4.25}$$

where now (u, v) is the solution of the IVP with $(u, v, u', v')(0) = (1, s, 0, 0)$. □

We remark that the hypothesis that the nontrivial solutions change sign was used to guarantee the condition (4.20): in fact, the implicit function theorem (at least in this form) may not be applied to $\widehat{\Sigma}_t$.

4.2. *Global study of* $\widetilde{\Sigma}^{\pm}$

Now, we want to exploit the local information obtained in Lemma 4.1, in order to obtain a qualitative description of the set $\widetilde{\Sigma}^+$; to do this we will need the following lemma and its Corollary 4.3:

Lemma 4.2. *Given* $\{(\lambda_n^+, \lambda_n^-, \mu_n^-, s_n)\} \subseteq \widetilde{\Sigma}^+$ *with* $(\lambda_n^+, \lambda_n^-, \mu_n^-) \to (\lambda_0^+, \lambda_0^-, \mu_0^-) \in \mathbb{R}^3$*, there exists a subsequence* $s_n \to s_0 \in \mathbb{R}$ *such that* $(\lambda_0^+, \lambda_0^-, \mu_0^-, s_0) \in \widetilde{\Sigma}^+$.

Moreover, if the sequence (u_n, v_n) *of the corresponding nontrivial solutions is composed by functions all with the same number of (simple) zeros, then the functions* (u_0, v_0) *corresponding to the point* $(\lambda_0^+, \lambda_0^-, \mu_0^-, s_0)$ *have this number of (simple) zeros too.*

Proof. Consider the Dirichlet boundary conditions. As seen in Lemma 3.1, the functions $u_n, v_n \in H^2(0,1) \subseteq \mathcal{C}^{1,1/2}([0,1])$.

Let $(U_n, V_n) = (\frac{u_n}{\|u_n\|_{H^2}}, \frac{v_n}{\|v_n\|_{H^2}})$: then, up to a subsequence, $(U_n, V_n) \to (U_0, V_0)$ weakly in $[H^2(0,1)]^2$ and strongly in $[\mathcal{C}^1([0,1])]^2$.

The variational equation for (U_n, V_n) is $\int_0^1 U_n'\phi' + V_n'\psi' = \int_0^1 (\lambda_n^+ V_n^+ - \lambda_n^- V_n^-)\phi + (\lambda_n^+ U_n^+ - \mu_n^- U_n^-)\psi$ for all $\phi, \psi \in H_0^1$; taking the limit gives

$$\int_0^1 U_0'\phi' + V_0'\psi' = \int_0^1 \big(\lambda_0^+ V_0^+ - \lambda_0^- V_0^-\big)\phi + \big(\lambda_0^+ U_0^+ - \mu_0^- U_0^-\big)\psi \quad \text{for all } \phi, \psi \in H_0^1, \tag{4.26}$$

that is (U_0, V_0) is a solution of (1.4) with coefficients $(\lambda_0^+, \lambda_0^-, \mu_0^-)$.

Since the solutions are strong too by Lemma 3.1, we have

$$\int_0^1 \left(-U_n'' - U_0''\right)^2 = \int_0^1 \left[\left(\lambda_n^+ V_n^+ - \lambda_n^- V_n^-\right) - \left(\lambda_0^+ V_0^+ - \lambda_0^- V_0^-\right)\right]^2, \tag{4.27}$$

where the right-hand side goes to zero and so $U_n \to U_0$ and (by the same argument) $V_n \to V_0$ strongly in H^2; this implies that $\|U_0\|_{H^2} = \|V_0\|_{H^2} = 1$ and so (U_0, V_0) is a nontrivial solution, that is $(\lambda_0^+, \lambda_0^-, \mu_0^-, \frac{V_0'(0)}{U_0'(0)}) \in \widetilde{\Sigma}^+$ (observe that $U_0'(0), V_0'(0) > 0$ by Lemma 3.2 and the $\mathcal{C}^1$ convergence).

Moreover, since $s_n = v_n'(0) = \frac{V_n'(0)}{U_n'(0)}$, its limit exists (by the $\mathcal{C}^1$ convergence) and it is indeed $\frac{V_0'(0)}{U_0'(0)}$, so we proved the first part of the lemma.

To conclude, observe that $(U_n(x), V_n(x)) = (u_n(x), v_n(x))\, U_n'(0)$ and $U_n'(0)$ is (up to a subsequence) bounded away from zero; this implies that if the sequence (u_n, v_n) is composed by functions all with the same number of simple zeros then the same is true for (U_n, V_n) and, by the $\mathcal{C}^1$ convergence, also for (U_0, V_0).

For Neumann boundary conditions, the proof goes in the same way, by proving that (up to a subsequence) $s_n = v_n(0) = \frac{V_n(0)}{U_n(0)} \to \frac{V_0(0)}{U_0(0)}$ and that $(\lambda_0^+, \lambda_0^-, \mu_0^-, \frac{V_0(0)}{U_0(0)}) \in \widetilde{\Sigma}^+$. □

Corollary 4.3. *The set $\widehat{\Sigma}^+$ is closed.*

Proof. Actually, the previous lemma implies that if a sequence $\{(\lambda_n^+, \lambda_n^-, \mu_n^-)\} \subseteq \widehat{\Sigma}^+$ is such that $(\lambda_n^+, \lambda_n^-, \mu_n^-) \to (\lambda_0^+, \lambda_0^-, \mu_0^-) \in \mathbb{R}^3$, then $(\lambda_0^+, \lambda_0^-, \mu_0^-) \in \widehat{\Sigma}^+$. □

Now, a first qualitative description of the set $\widetilde{\Sigma}^+$ may be given:

Proposition 4.4. *Let C be a connected component of $\widetilde{\Sigma}^+$ such that the corresponding nontrivial solutions change sign; then there exists a connected open set $\Lambda \subseteq \{(\lambda^-, \mu^-)\colon \lambda^-, \mu^- > 0\}$ such that C is of the form $C = \{(\lambda^+(\lambda^-, \mu^-), \lambda^-, \mu^-, s(\lambda^-, \mu^-))\colon (\lambda^-, \mu^-) \in \Lambda\}$ and the following assertions hold:*

1. $(\lambda^+, s) : \Lambda \to \mathbb{R}^2$ *is a $\mathcal{C}^1$ function of λ^-, μ^- and for any $(\lambda^-, \mu^-) \in \Lambda$*

$$\frac{\partial \lambda^+}{\partial \lambda^-}(\lambda^-, \mu^-) < 0, \tag{4.28}$$

$$\frac{\partial \lambda^+}{\partial \mu^-}(\lambda^-, \mu^-) < 0. \tag{4.29}$$

2. *There exist a real $\rho \geqslant 0$ and a nonincreasing function $D : (\rho, +\infty) \to \mathbb{R}^+$ such that for any $\tilde{\lambda}^- > 0$,*

$$\Lambda \cap \left\{(\tilde{\lambda}^-, \mu^-)\colon \mu^- \in \mathbb{R}^+\right\} = \begin{cases} \emptyset & \textit{for } \tilde{\lambda}^- \leqslant \rho, \\ \{(\tilde{\lambda}^-, \mu^-)\colon \mu^- > D(\tilde{\lambda}^-)\} & \textit{for } \tilde{\lambda}^- > \rho; \end{cases} \tag{4.30}$$

in the case $\tilde{\lambda}^- > \rho$ *we have*

$$\lim_{\mu^- \to D(\tilde{\lambda}^-)} \lambda^+(\tilde{\lambda}^-, \mu^-) = +\infty, \tag{4.31}$$

and

$$\lim_{\mu^- \to +\infty} \lambda^+(\tilde{\lambda}^-, \mu^-) = E(\tilde{\lambda}^-) \tag{4.32}$$

defines a nonincreasing function $E : (\rho, +\infty) \to \mathbb{R}^+$.
Finally, the same holds if we exchange the role of the variables λ^- *and* μ^-.

3. *The corresponding nontrivial solutions of* (1.4) *have all the same number of* (*simple*) *zeros.*
4. $\exists!\, (\lambda_*^-, \mu_*^-) \in \Lambda$ *such that* $\lambda^+(\lambda_*^-, \mu_*^-) = \lambda_*^- = \mu_*^-$, *and then* $\lambda_*^- = \mu_*^- = \lambda_k$ *for some* $k \geqslant 2$.

Proof. From Lemma 4.1 we deduce that C may be expressed as a function of λ^-, μ^- defined on a open connected set and we obtain the point 3 and Eqs. (4.28) and (4.29); moreover, by point 3 and Lemma 2.1 we deduce that λ^-, μ^- and $\lambda^+(\lambda^-, \mu^-)$ are all strictly positive.

Now remark that, by Lemma 4.2, if $\lim_{(\lambda^-,\mu^-)\to(\lambda_0^-,\mu_0^-)} \lambda^+(\lambda^-, \mu^-) = \lambda_0^+ \in \mathbb{R}$ then for some $s_0 \in \mathbb{R}$ the point $(\lambda_0^+, \lambda_0^-, \mu_0^-, s_0) \in C$ and then (λ_0^-, μ_0^-) would be an interior point of Λ by Lemma 4.1.

Once fixed $\tilde{\lambda}^- > 0$, the variable μ^- has to take values in an open subset of $(0, +\infty)$, but the function $\lambda^+(\tilde{\lambda}^-, \mu^-)$ has to be positive and decreasing in μ^-; this and the above remark imply that this open set has to be a halfline, that the limit of $\lambda^+(\tilde{\lambda}_0^-, \cdot)$ at $+\infty$ exists and is nonnegative while the limit at the initial point $D(\tilde{\lambda}^-)$ of the halfline must be $+\infty$, proving the first part of point 2.

The fact that one may exchange the role of the variables λ^- and μ^- follows by the symmetries of the Fučík spectrum and this implies that the function D is nonincreasing (otherwise the sections in direction λ^- would not be halflines). Finally E is nonincreasing by (4.28).

To prove point 4, consider the halfline $\{(\lambda^-, \mu^-)\colon \lambda^- = \mu^- > 0\}$ (observe that by point 2 it may not have empty intersection with Λ): since the function λ^+ is $\mathcal{C}^1(\Lambda)$, by Eqs. (4.28) and (4.29) we have that it is decreasing along this halfline and so we may deduce as above that it has a nonnegative limit for $\lambda^- = \mu^- \to +\infty$ and an asymptote where the halfline enters Λ; this implies that there exists a (unique) point ξ where $\lambda^+(\xi, \xi) = \xi$, proving point 4 (in fact in this case problem (1.4) reduces to (1.6) and then $\xi \in \sigma(-\Delta) \setminus \{\lambda_1\}$). □

Proposition 4.4 allows us to characterize all the connected components of $\widetilde{\Sigma}^\pm$: actually, the complete knowledge of the linear spectrum (1.6) and property 4 in Proposition 4.4 imply the following:

Proposition 4.5. *There exists a one to one relation between the eigenvalues* $\{\lambda_k\}_{k\geqslant 2}$ *of the Laplacian and those connected components of* $\widetilde{\Sigma}^+$ (*respectively of* $\widetilde{\Sigma}^-$) *which correspond to nontrivial sign changing solutions.*

Proof. For any eigenvalue λ_k there exists a unique value s (namely $s = 1$) such that the point $(\lambda_k, \lambda_k, \lambda_k, s) \in \widetilde{\Sigma}^+$ and, by Lemma 4.1, through this point passes one connected component of $\widetilde{\Sigma}^+$; conversely, by point 4 in Proposition 4.4, we have that any connected component in $\widetilde{\Sigma}^+$ passes through a point of the form $(\lambda_k, \lambda_k, \lambda_k, s)$ for some $k \geqslant 2$, $s \in \mathbb{R}$.

These facts give the one to one relation. $\square$

Definition 4.6. In view of Proposition 4.5, we will use the notation $\widetilde{\Sigma}_k^+$ (respectively $\widetilde{\Sigma}_k^-$) for the component corresponding to λ_k and $\widehat{\Sigma}_k^+$ (respectively $\widehat{\Sigma}_k^-$) for its projection in $\{(\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3\}$.

Moreover, we will denote by $\Lambda_k^\pm$, $D_k^\pm$, $E_k^\pm$ and $\rho_k^\pm$, what was denoted by Λ, D, E and ρ for $\widetilde{\Sigma}_k^+$, and by $f_k^\pm$ the corresponding function $\lambda^+(\lambda^-, \mu^-)$ (see in Proposition 4.4).

Looking at the set $\widehat{\Sigma}$, Proposition 4.5 implies the following

Corollary 4.7. *Trough any point $(\lambda_k, \lambda_k, \lambda_k) \in \widehat{\Sigma}$ with $k \geqslant 2$ pass exactly two surfaces in $\widehat{\Sigma}_{nt}$, namely $\widehat{\Sigma}_k^+$ and $\widehat{\Sigma}_k^-$, which may or may not coincide.*

In particular, as happens for the eigenfunction corresponding to λ_k, the nontrivial solutions corresponding to a point in $\widehat{\Sigma}_k^\pm$ have always $k-1$ (simple) interior zeros.

Moreover, any point in $\widehat{\Sigma}_{nt}$ belongs to one of these surfaces.

Now we may also complete the result of Lemma 2.3 about the relationship between the Fučík spectrum for the system considered above and the Fučík spectrum for the scalar case Σ_{scal}: we may assert

Proposition 4.8. *There exists a bijection between Σ_{scal} and the subset of $\widehat{\Sigma}$ with $\lambda^- = \mu^-$; in particular, if u is a nontrivial solution of the scalar problem* (1.1) *corresponding to a point $(\lambda^+, \lambda^-) \in \Sigma_{\mathrm{scal}}$, then the couple (u, u) is a nontrivial solution of problem* (1.4) *corresponding to the point $(\lambda^+, \lambda^-, \lambda^-) \in \widehat{\Sigma}$ and vice versa.*

Proof. For $\lambda^+ = \lambda_1$ or $\lambda^- = \mu^- = \lambda_1$ the claim is trivial since we know explicitly the form of $\widehat{\Sigma}_t$.

In Lemma 2.3 it was already proved that to each point in Σ_{scal} corresponds a point in $\widehat{\Sigma}$ with the claimed relation.

In fact, this relation is a bijection: actually, also for the scalar problem we may identify each curve in the Fučík spectrum by the number of nodes and the sign near zero of the corresponding nontrivial solutions, and hence each surface $\widetilde{\Sigma}_k^+$ (respectively $\widetilde{\Sigma}_k^-$) contains the corresponding curve of Σ_{scal}; moreover, each surface $\widetilde{\Sigma}_k^+$ (respectively $\widetilde{\Sigma}_k^-$) contains a unique curve with $\lambda^- = \mu^-$ (since we saw indeed that we can express the surface as a function of these two variables) so that there are no points in $\widehat{\Sigma}$ with $\lambda^- = \mu^-$ other than those related to Σ_{scal}. $\square$

We remark that in Lemma 2.3 we only could say that Σ_{scal} is contained in the subset of $\widehat{\Sigma}$ with $\lambda^- = \mu^-$; here, for the one-dimensional problem, we can say that the converse is true: that is, whenever $\lambda^- = \mu^-$ in $\widehat{\Sigma}$ then the corresponding nontrivial solution is composed by a pair of functions which are also nontrivial solutions of the scalar Fučík problem with coefficients (λ^+, λ^-).

This immersion of Σ_{scal} in $\widehat{\Sigma}$ gives us even more information about the structure of $\widehat{\Sigma}$: by combining its knowledge with the symmetries in Lemma 2.2 and the monotonicity result in point 1 of Proposition 4.4, we obtain

Proposition 4.9. *In the Dirichlet case* (*here* $[\cdot]$ *denotes the integer part of a real*)

$$\left(\lambda^+,\lambda^-,\mu^-\right)\in\widehat{\Sigma}_k^+ \quad \textit{with } k\geqslant 2 \quad\Rightarrow\quad \sqrt{\lambda^-\mu^-}>\lambda_{[k/2]} \quad \textit{and} \quad \lambda^+>\lambda_{[(k+1)/2]}, \tag{4.33}$$

$$\left(\lambda^+,\lambda^-,\mu^-\right)\in\widehat{\Sigma}_k^- \quad \textit{with } k\geqslant 2 \quad\Rightarrow\quad \sqrt{\lambda^-\mu^-}>\lambda_{[(k+1)/2]} \quad \textit{and} \quad \lambda^+>\lambda_{[k/2]}. \tag{4.34}$$

In the Neumann case,

$$\left(\lambda^+,\lambda^-,\mu^-\right)\in\widehat{\Sigma}_k^\pm \quad \textit{with } k\geqslant 2 \quad\Rightarrow\quad \sqrt{\lambda^-\mu^-}>\lambda_k/4 \quad \textit{and} \quad \lambda^+>\lambda_k/4. \tag{4.35}$$

In terms of the functions D, E defined in Proposition 4.4, (4.35) implies that $\sqrt{\lambda^- D_k^\pm(\lambda^-)}\geqslant \lambda_k/4$ and $E_k^\pm(\lambda^-)\geqslant\lambda_k/4$ for all $\lambda^->\rho_k^\pm$; analogous relations come from (4.33) and (4.34).

Proof. First observe that the inequalities in the claim are exactly those which hold for the corresponding curves in Σ_{scal} (see in Section 2), and hence the lemma is trivially true if we restrict at the points with $\lambda^-=\mu^-$.

The inequalities for λ^+ hold since $f_k^\pm$ is decreasing in both variables and then it may not assume a value lower than those assumed for $\lambda^-=\mu^-$.

The inequalities for $\sqrt{\lambda^-\mu^-}$ are obtained by the symmetry 2 in Lemma 2.2: actually if $(\lambda^+,\lambda^-,\mu^-)\in\widehat{\Sigma}_k^\pm$ then $(\sqrt{\lambda^-\mu^-},\lambda^+\sqrt{\frac{\mu^-}{\lambda^-}},\lambda^+\sqrt{\frac{\lambda^-}{\mu^-}})\in\widehat{\Sigma}_k^\mp$, and then by the previously proven inequalities for λ^+ we obtain the analogous ones for $\sqrt{\lambda^-\mu^-}$, observing that the value for $\widehat{\Sigma}_k^+$ passes to $\widehat{\Sigma}_k^-$ and vice versa, since the considered symmetry changes the sign of the nontrivial solutions. □

4.3. *Relationship between the curves $\widehat{\Sigma}_k^\pm$ and symmetries of the nontrivial solutions*

In this section we will prove two propositions dealing with the possible intersections between the surfaces $\widehat{\Sigma}_k^\pm$, and we will also obtain (in Proposition 4.12) the result about the symmetries of the nontrivial solutions stated in Proposition 1.3.

Propositions 4.10 and 4.11, joined with the results of the previous section, will complete the proof of Theorem 1.2.

Proposition 4.10. *If $h>k\geqslant 2$, then $(\widehat{\Sigma}_k^+\cup\widehat{\Sigma}_k^-)\cap(\widehat{\Sigma}_h^+\cup\widehat{\Sigma}_h^-)=\emptyset$.*
Moreover $(\widehat{\Sigma}_k^+\cup\widehat{\Sigma}_k^-)\cap\widehat{\Sigma}_t=\emptyset$.

Proof. Let the function $\lambda^+(\lambda^-,\mu^-)$ describe $\widehat{\Sigma}_k^+$ (or $\widehat{\Sigma}_k^-$): then $\lambda^+(\lambda_k,\lambda_k)=\lambda_k$ and since it is strictly decreasing in both variables we get $\lambda^+(\lambda_{k+1},\lambda_{k+1})<\lambda_k<\lambda_{k+1}$; then in $\lambda^-=\mu^-=\lambda_{k+1}$ the surfaces $\widehat{\Sigma}_k^\pm$ are lower than the surfaces $\widehat{\Sigma}_{k+1}^\pm$.

Then, it is enough to prove that $(\widehat{\Sigma}_k^+\cup\widehat{\Sigma}_k^-)\cap(\widehat{\Sigma}_{k+1}^+\cup\widehat{\Sigma}_{k+1}^-)=\emptyset$ to imply the claim for any $k\geqslant 2$.

By contradiction, suppose $(\lambda^+,\lambda^-,\mu^-)\in(\widehat{\Sigma}_k^+\cup\widehat{\Sigma}_k^-)\cap(\widehat{\Sigma}_{k+1}^+\cup\widehat{\Sigma}_{k+1}^-)$, then we have the corresponding nontrivial solutions (u_k,v_k) and (u_{k+1},v_{k+1}), where the second ones change sign once more than the first ones and so in one of the two endpoints (suppose in 0) the sign is the same and we may choose a rescalement such that $u_k(0)=u_{k+1}(0)$ and $u_k'(0)=u_{k+1}'(0)$.

Then let $y=u_k-u_{k+1}$ and $z=v_k-v_{k+1}$: we have $y(0)=y'(0)=0$ and

$$\begin{aligned}-y''&=-(u_k-u_{k+1})''=\lambda^+\left(v_k^+-v_{k+1}^+\right)-\lambda^-\left(v_k^--v_{k+1}^-\right)\\&=\left[\lambda^+\chi_{v++}(x)+\lambda^-\chi_{v--}(x)+c_{v1}(x)\chi_{v+-}(x)+c_{v2}(x)\chi_{v-+}(x)\right]z,\end{aligned}$$

$$
\begin{aligned}
-z'' = -(v_k - v_{k+1})'' &= \lambda^+\big(u_k^+ - u_{k+1}^+\big) - \mu^-(u_k^- - u_{k+1}^-) \\
&= \big[\lambda^+ \chi_{u++}(x) + \mu^- \chi_{u--}(x) + c_{u1}(x)\chi_{u+-}(x) + c_{u2}(x)\chi_{u-+}(x)\big]y \qquad (4.36)
\end{aligned}
$$

where

$$\chi_{u\pm_1,\pm_2}(x) = \chi_{\{\pm_1 u_k>0, \pm_2 u_{k+1}>0\}}(x), \tag{4.37}$$

$$c_{u1}(x) = \frac{\lambda^+ u_k^+ + \mu^- u_{k+1}^-}{u_k^+ + u_{k+1}^-}\chi_{u+-}(x), \tag{4.38}$$

$$c_{u2}(x) = \frac{-\lambda^+ u_{k+1}^+ - \mu^- u_k^-}{-u_{k+1}^+ - u_k^-}\chi_{u-+}(x); \tag{4.39}$$

and analogous definitions for $\chi_{v\,\pm_1,\pm_2}$, c_{v1} and c_{v2}.

Since the functions in square brackets in (4.36) are $L^\infty(0,1)$ and positive a.e., by Lemma 3.2 we get $y \equiv z \equiv 0$, a contradiction since in a neighborhood of 1 the signs of u_k and u_{k+1} are different and then $y \neq 0$.

To conclude, observe that the same argument works also with $\widehat{\Sigma}_2^\pm$ and $\widehat{\Sigma}_t$ in place of $\widehat{\Sigma}_{k+1}^\pm$ and $\widehat{\Sigma}_k^\pm$. $\square$

Proposition 4.11. *In the case of Neumann boundary conditions $\widehat{\Sigma}_k^+ \equiv \widehat{\Sigma}_k^-$ for all $k \geqslant 2$.*

In the case of Dirichlet boundary conditions

- $\widehat{\Sigma}_k^+ \equiv \widehat{\Sigma}_k^-$ *for all even $k \geqslant 2$,*
- $\widehat{\Sigma}_k^+ \not\equiv \widehat{\Sigma}_k^-$ *for all odd $k \geqslant 3$.*

The proof of this proposition is relatively simple except for the Neumann case with odd k: to obtain this result we will need some symmetry properties of the nontrivial solutions: in particular, the proof of Proposition 4.11 will be given together with that of the following

Proposition 4.12. *The nontrivial solutions corresponding to points in $\widehat{\Sigma}_k^\pm$ with k odd are symmetric, that is,*

$$(u,v)(x) = (u,v)(1-x). \tag{4.40}$$

Moreover, for Neumann boundary conditions, one has that if $j \geqslant 0$ and $k \equiv 1$ (MOD 2^{j+1}), then

$$(u,v)(x) = (u,v)\big((1/2)^j - x\big) \quad \textit{for } x \in \big[0,(1/2)^j\big] \tag{4.41}$$

and then u, v are $(1/2)^j$ periodic in $[0,1]$.

Proof of Propositions 4.11 and 4.12. Let $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^+$ and (u_*, v_*) be a corresponding nontrivial solution (then $(\lambda^+, \lambda^-, \mu^-, s) \in \widetilde{\Sigma}_k^+$ where $s = \frac{v_*'(0)}{u_*'(0)}$ for Dirichlet boundary conditions and $s = \frac{v_*(0)}{u_*(0)}$ for Neumann boundary conditions); then define the new functions $(U(x), V(x)) = (u_*(1-x), v_*(1-x))$.

If k is even, the nontrivial solutions which start positive end negative and vice versa, then $(U(x), V(x))$ both start negative, have $k-1$ internal zeros and satisfy the Fučík problem with coefficients $(\lambda^+, \lambda^-, \mu^-)$ too, that is $(\lambda^+, \lambda^-, \mu^-, s) \in \widetilde{\Sigma}_k^-$ where $s = -\frac{v_*'(1)}{u_*'(1)}$ (respectively $s = -\frac{v_*(1)}{u_*(1)}$ for the Neumann case) which implies $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^-$ too.

This gives $\widehat{\Sigma}_k^+ \subseteq \widehat{\Sigma}_k^-$ for the case k even; the inverse inclusion follows in the same way.

If k is odd, we have that the nontrivial solutions which start positive end positive and those which start negative end negative. By Eq. (4.5) $\frac{\partial\lambda^+}{\partial\lambda^-}(\lambda_k, \lambda_k) = \frac{-\int_0^1 (\phi_k^-)^2}{\int_0^1 2(\phi_k^+)^2}$; for Dirichlet boundary conditions this ratio is different if we consider ϕ_k starting positive or starting negative, since it has a different number of positive and negative congruent bumps; this implies that $\widehat{\Sigma}_k^+$ and $\widehat{\Sigma}_k^-$ are different in a neighborhood of $(\lambda_k, \lambda_k, \lambda_k)$ and then $\widehat{\Sigma}_k^+ \not\equiv \widehat{\Sigma}_k^-$.

Now we prove the first symmetry claimed in Proposition 4.12. We consider initially positive solutions (the argument is the same for initially negative solutions): in this case also $(U(x), V(x))$ start positive, have $k-1$ internal zeros as u_*, v_* and are nontrivial solutions corresponding to $(\lambda^+, \lambda^-, \mu^-)$. This implies that in fact $(U, V) \equiv (u_*, v_*)$ (that is, they are symmetric), since otherwise they would give rise to another branch of $\widetilde{\Sigma}_k^+$, which is excluded by Propositions 4.4 and 4.5.

Now we restrict to the case of Neumann boundary conditions: we just proved that, for k odd, u_*, v_* are symmetric and so

$$u_*'(1/2) = v_*'(1/2) = 0, \quad u_*(1) = u_*(0) \quad \text{and} \quad v_*(1) = v_*(0); \tag{4.42}$$

moreover, both have $(k-1)/2$ zeros in $(0, 1/2)$.

Then we define the functions

$$(\tilde{u}, \tilde{v}) : [0,2] \to \mathbb{R}^2: \quad x \mapsto \begin{cases} (u_*(x), v_*(x)) & \text{for } x \in [0,1], \\ (u_*(x-1), v_*(x-1)) & \text{for } x \in (1,2], \end{cases} \tag{4.43}$$

$$\mathbf{w}_1 = (u_1, v_1) : [0,1] \to \mathbb{R}^2: \quad x \mapsto \big(\tilde{u}(x+1/2), \tilde{v}(x+1/2)\big). \tag{4.44}$$

By (4.42), $\mathbf{w}_1$ is another nontrivial solution of the Fučík problem; now we have two possibilities (remember that $\tilde{u}(1/2)\tilde{v}(1/2) \leqslant 0$ is excluded by Lemma 3.2 since $\tilde{u}'(1/2) = 0$):

(a) if $(k-1)/2 \equiv 1$ (MOD 2), then the number of zeros in $(0, 1/2)$ is odd and then $\tilde{u}(1/2) < 0$, $\tilde{v}(1/2) < 0$; this implies that $\mathbf{w}_1(x)$ is a nontrivial solution starting negative and so $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^-$;

(b) if $(k-1)/2 \equiv 0$ (MOD 2), then the number of zeros in $(0, 1/2)$ is even and then $\tilde{u}(1/2) > 0$, $\tilde{v}(1/2) > 0$; then consider the new functions $(U(x), V(x)) = (\tilde{u}(3/2 - x), \tilde{v}(3/2 - x))$ with $x \in [0,1]$: they start positive, have $k-1$ internal zeros and are nontrivial solutions corresponding to $(\lambda^+, \lambda^-, \mu^-)$ as u_*, v_*; then as before they must be the same and so we prove that $(u_*, v_*)(1/2 - x) = (u_*, v_*)(x)$ for $x \in [0, 1/2]$.

This now implies

$$u_*'(1/4) = v_*'(1/4) = 0, \quad u_*(1/2) = u_*(0) \quad \text{and} \quad v_*(1/2) = v_*(0) \tag{4.45}$$

and both have $(k-1)/4$ zeros in $(0, 1/4)$, which corresponds to what we had in Eq. (4.42), and so allows us to repeat the argument with the function

$$\mathbf{w}_2 = (u_2, v_2) : [0, 1/2] \to \mathbb{R}^2: \quad x \mapsto \big(\tilde{u}(x+1/4), \tilde{v}(x+1/4)\big). \tag{4.46}$$

The procedure continues by splitting the interval, until we get i^* for which $(k-1)/2^{i^*} \equiv 1$ (MOD 2) and so we fall into case (a), implying that $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^-$ and so $\widehat{\Sigma}_k^+ \subseteq \widehat{\Sigma}_k^-$.

However, for $j < i^*$, we have $(k-1)/2^j \equiv 0$ (MOD 2) (or, which is the same, $k \equiv 1$ (MOD 2^{j+1})), then we fall into case (b) and so we prove $(u, v)(x) = (u, v)((1/2)^j - x)$ for $x \in [0, (1/2)^j]$.

Finally, observe that the relations $(u, v)(x) = (u, v)((1/2)^{j-1} - x)$ for $x \in [0, (1/2)^{j-1}]$ and $(u, v)(x) = (u, v)((1/2)^j - x)$ for $x \in [0, (1/2)^j]$ imply (just put $(1/2)^j - x$ in place of x in the first one) that $(u, v)(x) = (u, v)((1/2)^j + x)$ for $x \in [0, (1/2)^j]$ and then the functions are (by induction over j) also $(1/2)^{i^*-1}$-periodic (which is a minimal period since $u_*((1/2)^{i^*}) < 0$). □

Proof of Theorem 1.2 and Proposition 1.3. Theorem 1.2 follows by joining Propositions 4.4, 4.5, Corollary 4.7 and Propositions 4.9–4.11.

Proposition 1.3 comes from Proposition 4.12. □

Remark 4.13. Observe that for the scalar case it is always true that the nontrivial solutions have a periodicity and some sort of symmetry, since after a positive and a negative bump they start to repeat.

For the system, we are able to assert a periodicity, by exploiting the above symmetries, only for Neumann boundary conditions and for k of the form given in Proposition 4.12.

To conclude this section, we exhibit a curve in $(\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-)$, whose existence is a consequence of the symmetries of $\widehat{\Sigma}$:

Proposition 4.14. *$(\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-)$ contains an unbounded curve which passes through $(\lambda_k, \lambda_k, \lambda_k)$; moreover, this is the unique point in $(\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-)$ with $\lambda^- = \mu^-$.*

Proof. The last claim is a trivial consequence of Proposition 4.8.

First observe that by joining the two symmetries in Lemma 2.2, we obtain that if $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^+$ with corresponding nontrivial solutions (u, v), and $\lambda^+ = \sqrt{\lambda^-\mu^-}$, then $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}$ with corresponding nontrivial solutions $(-\sqrt{\frac{\lambda^-}{\mu^-}}v, -u)$, and then it belongs to $\widehat{\Sigma}_k^-$.

Now, arguing as for point 4 in Proposition 4.4, one gets that for any assigned ratio $\xi > 0$, there exists a unique $(\lambda^+, \lambda^-, \mu^-) \in \widehat{\Sigma}_k^+$ such that $\frac{\lambda^-}{\mu^-} = \xi$ and $\lambda^+ = \sqrt{\lambda^-\mu^-}$.

By continuity, this provides a curve in $(\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-)$ parameterized by ξ, passing through $(\lambda_k, \lambda_k, \lambda_k)$ for $\xi = 1$. □

Remark 4.15. In the scalar case with Dirichlet boundary conditions, it is known that the branches of Σ passing trough a point (λ_k, λ_k) with k odd intersect only in (λ_k, λ_k). For the system, we were not able to prove an analogous result; that is, $\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-$ (with k odd) may contain other points than $(\lambda_k, \lambda_k, \lambda_k)$ and those given in Proposition 4.14.

5. Systems with "eigenvalue intersecting nonlinearities"

In this section we study the solvability of systems with so-called "jumping nonlinearities," i.e. nonlinearities which asymptotically intersect some of the eigenvalue of the linear system. We consider

$$\begin{cases} -u'' = g_1(x, v) + h_1(x) & \text{in } (0, 1), \\ -v'' = g_2(x, u) + h_2(x) & \text{in } (0, 1), \\ Bu = Bv = 0 & \text{in } \{0; 1\}, \end{cases} \tag{5.1}$$

where we assume $h_{1,2} \in L^2(0, 1)$, and $g_{1,2} \in \mathcal{C}^0([0, 1] \times \mathbb{R})$ with

$$\lim_{s\to\pm\infty} \frac{g_1(x, s)}{s} = \lambda^{\pm}, \qquad \lim_{t\to\pm\infty} \frac{g_2(x, t)}{t} = \mu^{\pm}, \qquad \lambda^+ = \mu^+, \tag{5.2}$$

where the limits are uniform with respect to $x \in [0, 1]$. As mentioned in the introduction, the solvability properties of (5.1) depend on the location of λ^+, λ^- and μ^- with respect to the Fučík system. In particular, we need to analyze some properties of the complement of the Fučík spectrum $\widehat{\Sigma}$ in $(\mathbb{R}^+)^3$.

Throughout this section, when considering the complement of a set, it will be intended with respect to $(\mathbb{R}^+)^3$.

Also, we will denote by $\mathcal{D} = \{(\lambda^+, \lambda^-, \mu^-) \in (\mathbb{R}^+)^3\colon \lambda^+ = \lambda^- = \mu^-\}$ the diagonal in $(\mathbb{R}^+)^3$ and, in order to simplify the statement of the claims, we will consider the functions $f_k^{\pm}$, $D_k^{\pm}$ and $E_k^{\pm}$ (see in Definition 4.6) to be defined in all $(\mathbb{R}^+)^2$ (respectively $\mathbb{R}^+$), assigning the value $+\infty$ where they are not naturally defined; we also define, for uniformity of notation,

$$f_1^{\pm} : \left(R^+\right)^2 \to \mathbb{R} \cup \{+\infty\}: \quad f_1^+(x, y) \equiv \lambda_1; \ f_1^-(x, y) = \begin{cases} 0 & \text{for } \sqrt{xy} > \lambda_1, \\ +\infty & \text{otherwise.} \end{cases}$$

First, we determine the connected components of $\widehat{\Sigma}^c$ which contain a segment of $\mathcal{D}$: for this we define

$$R_1 = \bigcup \{\mathcal{O}\colon \mathcal{O} \text{ is a connected component of } \widehat{\Sigma}^c \text{ and } \mathcal{O} \cap \mathcal{D} \neq \emptyset\},$$
$$R_2 = \bigcup \{\mathcal{O}\colon \mathcal{O} \text{ is a connected component of } \widehat{\Sigma}^c \text{ and } \mathcal{O} \cap \mathcal{D} = \emptyset\}.$$

We start this study with the following two lemmas:

Lemma 5.1. *For $k \geqslant 2$, $(\widehat{\Sigma}_k^+)^c$ (respectively $(\widehat{\Sigma}_k^-)^c$) has exactly two connected components.*

Proof. All the points $(\lambda^+, \lambda^-, \mu^-)$ such that $\lambda^+ > f_k^+(\lambda^-, \mu^-)$ are in the same connected component, since Λ_k^+ is connected by definition.

Also, the points such that $\lambda^+ < f_k^+(\lambda^-, \mu^-)$ are in the same connected component, by the shape of Λ_k^+ and the behavior of f_k^+ at its boundary.

Finally, consider a path starting from a point P with $\lambda^+ > f_k^+(\lambda^-, \mu^-)$ and reaching a point Q with $\lambda^+ < f_k^+(\lambda^-, \mu^-)$: since the path is a compact set, we have that λ^+ is bounded, say $\lambda^+ \leqslant d$ along this path. Since f_k^+ is continuous one has for all points in the path with $(\lambda^-, \mu^-) \in \Lambda_k^+$ that

$\lambda^+ > f_k^+(\lambda^-, \mu^-)$, but since the limit of f_k^+ at the boundary of Λ_k^+ is $+\infty$, for (λ^-, μ^-) to exit from Λ_k^+ one would get $\lambda^+ > d$ somewhere, which is impossible; so Q may not be connected to P and then the connected components are exactly two.

The same proof works for $(\widehat{\Sigma}_k^-)^c$. $\square$

Lemma 5.2. *For $k \geqslant 2$, $(\widehat{\Sigma}_k^+ \cup \widehat{\Sigma}_k^-)^c$ has exactly two connected components when $\widehat{\Sigma}_k^+ \equiv \widehat{\Sigma}_k^-$ and at least four when $\widehat{\Sigma}_k^+ \not\equiv \widehat{\Sigma}_k^-$.*

Proof. The case $\widehat{\Sigma}_k^+ \equiv \widehat{\Sigma}_k^-$ is trivial by Lemma 5.1.

When $\widehat{\Sigma}_k^+ \not\equiv \widehat{\Sigma}_k^-$, that is for Dirichlet boundary conditions and k odd, we know that when $\lambda^- = \mu^-$ we have the same structure as Σ_{scal}; in particular, there exist points above $\widehat{\Sigma}_k^+$ and below $\widehat{\Sigma}_k^-$ as well as points in $(\widehat{\Sigma}_k^+ \cup \widehat{\Sigma}_k^-)^c$ which lie above $\widehat{\Sigma}_k^-$ or below $\widehat{\Sigma}_k^+$; moreover, it is a consequence of the above Lemma 5.1 that the points which are above one of the surfaces and those below the same surface may not be in the same connected component, so we get at least four components (there may be more than four, depending on the structure of the set $\widehat{\Sigma}_k^+ \cap \widehat{\Sigma}_k^-$). $\square$

We now state the main result concerning the structure of $\widehat{\Sigma}^c$:

Proposition 5.3. *For Neumann boundary conditions,*

$$R_1 = \widehat{\Sigma}^c \quad \textit{and} \quad R_2 = \emptyset.$$

For Dirichlet boundary conditions, both R_1 and R_2 are not empty; in fact,

$$\begin{aligned} R_1 = &\left\{(x, y, z) \in \left(\mathbb{R}^+\right)^3\colon \max\left\{f_k^+, f_k^-\right\}(y, z) < x < \min\left\{f_{k+1}^+, f_{k+1}^-\right\}(y, z) \textit{ for some } k \geqslant 1\right\} \\ &\cup \left\{(x, y, z) \in \left(\mathbb{R}^+\right)^3\colon 0 < x < \min\left\{f_1^+, f_1^-\right\}(y, z)\right\}, \end{aligned} \tag{5.3}$$

while

$$R_2 = \left\{(x, y, z) \in \left(\mathbb{R}^+\right)^3\colon \min\left\{f_k^+, f_k^-\right\}(y, z) < x < \max\left\{f_k^+, f_k^-\right\}(y, z) \textit{ for some } k \geqslant 1\right\}.$$

Proof. By Proposition 4.10 and Lemma 5.2, the connected components which are delimited by the surfaces $\widehat{\Sigma}_k^+ \cup \widehat{\Sigma}_k^-$ on one side and $\widehat{\Sigma}_{k+1}^+ \cup \widehat{\Sigma}_{k+1}^-$ on the other (or, by $\widehat{\Sigma}_t$ and $\widehat{\Sigma}_2^+ \equiv \widehat{\Sigma}_2^-$ for $k = 1$), contain the nonvoid segment of the diagonal $\mathcal{D}$ with $\lambda^+ = \lambda^- = \mu^- \in (\lambda_k, \lambda_{k+1})$; then these components are in R_1.

In the Neumann case these regions cover the whole of $\widehat{\Sigma}^c$ by Lemma 5.2. In the Dirichlet case we have in addition the set $\{(x, y, z) \in (\mathbb{R}^+)^3\colon 0 < x < \min\{f_1^+, f_1^-\}(y, z)\}$ which covers the segment of the diagonal $\mathcal{D}$ with $\lambda^+ = \lambda^- = \mu^- \in [0, \lambda_1)$ and hence lies in R_1; all the other connected components are in R_2 since the above regions already cover all the points in $\mathcal{D} \cap \widehat{\Sigma}^c$. $\square$

We now consider the nonlinear system (5.1), where $g_1, g_2 \in \mathcal{C}^0([0, 1] \times \mathbb{R})$ satisfy (5.2).

Theorem 5.4. *Suppose that $(\lambda^+, \lambda^-, \mu^-) \in R_1$, where λ^+, λ^- and μ^- are given by* (5.2). *Then system* (5.1) *has solution for any forcing term $(h_1, h_2) \in (L^2(0, 1))^2$.*

Proof. The proof is the same as in [9] (Theorem 4.5) and it is similar to the proof in [1] for the scalar case. See also Corollary 4.6 in [9] for a more general result.

In short, the idea of the proof is to find a solution of (5.1) as a zero of a suitable map $S:[L^2]^2 \to [L^2]^2$. This is obtained by use of the Leray–Schauder degree, after having proved an a priori estimate for the possible solutions; this allows to relate via homotopy the degree of S to the degree of another map associated to the Fučík problem with coefficients $(\lambda^+, \lambda^-, \mu^-)$. Then, since this last degree is 1 whenever $(\lambda^+, \lambda^-, \mu^-)$ belongs to a connected component like those in R_1, the claimed solution is obtained (see also Lemma 4.2 and Corollary 4.3 in [9], which extend to the case of systems a known property of the scalar Fučík spectrum). □

5.1. A remark about "linear-superlinear" systems

In this section we investigate what can be said about the shape "at infinity" of the sets $\widehat{\Sigma}$ and $\widehat{\Sigma}^c$. The motivation for this analysis is again system (5.1), but in the case in which one or more of the limits in (5.2) are infinite, in particular, when at least one of the two nonlinearities is asymptotically linear on one side and superlinear on the other side.

In order to clarify the kind of properties we want to investigate, we resume here briefly what is known for the scalar problems: consider the scalar analogue of system (5.1)

$$\begin{cases} -u'' = g(x,u) + h(x) & \text{in } (0,1), \\ Bu = 0 & \text{in } \{0;1\} \end{cases} \tag{5.4}$$

with $\lim_{s\to\pm\infty} \frac{g(x,s)}{s} = \lambda^{\pm}$, and let now $\lambda^+ = +\infty$: it is known that the solvability of (5.4) is related to the existence of "gaps at infinity" between the curves of Σ_{scal}.

In brief, in [11] and [12], an existence result with arbitrary $h \in L^2$ was obtained in the situation where λ^- admits a neighborhood $N(\lambda^-)$ such that, for a suitable $\bar{x}$, the set $\{x > \bar{x},\ y \in N(\lambda^-)\}$ is contained in the same connected component of Σ^c_{scal} and this component contains a segment of the diagonal $\{\lambda^+ = \lambda^-\}$ (that is, components like those in R_1 of the previous section). Also, if instead of $N(\lambda^-)$ one has just a half-neighborhood of λ^-, one still obtains existence results under an additional nonresonance condition.

On the other hand, in [14] it was shown that the existence of a solution with arbitrary $h \in L^2$ never holds for the case in which the set $\{x > \bar{x},\ y \in N(\lambda^-)\}$ is contained in a component of Σ^c_{scal} which does not contain a segment of the diagonal (the analogues of the regions in R_2 of the previous section).

By looking at the shape of Σ_{scal} (see Section 2), one may observe that the first situation is typical of the Neumann boundary conditions, while the second one is typical of the Dirichlet case.

Going back to the system, we observe that its structure gives rise to more possibilities: actually (see also in [13]), one may have the superlinearity in both equations or in just one of the two; since in this paper we are always considering the same coefficient for u^+ and v^+, the case of superlinearity in both equations will be analyzed through the case of finite (λ^-, μ^-) and $\lambda^+ = +\infty$, while the case of superlinearity in only one equation will be the case of finite (λ^+, λ^-) and $\mu^- = +\infty$.

Motivated by the above results, we define the following "neighborhoods of halflines"

$$S^x_\varepsilon(\lambda^-, \mu^-)\big[\lambda^+\big] = \big\{(x,y,z) \in \mathbb{R}^3\colon\ x \geqslant \lambda^+,\ |y - \lambda^-| \leqslant \varepsilon,\ |z - \mu^-| \leqslant \varepsilon\big\}, \tag{5.5}$$

$$S^z_\varepsilon\big(\lambda^+, \lambda^-\big)[\mu^-] = \big\{(x,y,z) \in \mathbb{R}^3\colon\ z \geqslant \mu^-,\ |y - \lambda^-| \leqslant \varepsilon,\ \big|x - \lambda^+\big| \leqslant \varepsilon\big\}, \tag{5.6}$$

and we will investigate the following sets ($i = 1, 2$):

$$A_i^x = \{(\lambda^-, \mu^-) \in (\mathbb{R}^+)^2 \colon \exists \lambda^+ > 0,\ \varepsilon > 0 \colon S_\varepsilon^x(\lambda^-, \mu^-)[\lambda^+] \subseteq R_i\}, \tag{5.7}$$

$$A_i^z = \{(\lambda^+, \lambda^-) \in (\mathbb{R}^+)^2 \colon \exists \mu^- > 0,\ \varepsilon > 0 \colon S_\varepsilon^z(\lambda^+, \lambda^-)[\mu^-] \subseteq R_i\}, \tag{5.8}$$

$$B_i^x = \{(\lambda^-, \mu^-) \in (\mathbb{R}^+)^2 \colon \exists \lambda^+ > 0 \colon S_0^x(\lambda^-, \mu^-)[\lambda^+] \subseteq R_i\}, \tag{5.9}$$

$$B_i^z = \{(\lambda^+, \lambda^-) \in (\mathbb{R}^+)^2 \colon \exists \mu^- > 0 \colon S_0^z(\lambda^+, \lambda^-)[\mu^-] \subseteq R_i\}. \tag{5.10}$$

In short, the set B_1^x will contain those points (λ^-, μ^-) for which problem (5.1) is solvable for λ^+ large enough, the set A_1^x will contain those for which problem (5.1) with $\lambda^+ = +\infty$ could still be solvable, in analogy with the scalar case; the sets B_1^z and A_1^z will be the analogues for the case where we fix (λ^+, λ^-) and we consider μ^- large or infinity; finally, the sets with index 2 will contain points for which the situation is analogous to that in which the scalar case is not solvable for arbitrary forcing term.

The topological result of this section is the following

Proposition 5.5. *For Neumann boundary conditions,*

- *the sets A_1^x and A_1^z are open and dense in $(\mathbb{R}^+)^2$,*
- $A_2^x = A_2^z = B_2^x = B_2^z = \emptyset$,
- $B_1^x = B_1^z = (\mathbb{R}^+ \setminus \{\lambda_1\})^2$.

For Dirichlet boundary conditions

- *$A_1^x \cup A_2^x$ and $A_1^z \cup A_2^z$ are open and dense in $(\mathbb{R}^+)^2$,*
- $B_1^x \cup B_2^x = B_1^z \cup B_2^z = (\mathbb{R}^+ \setminus \{\lambda_1\})^2$,
- *A_2^x and B_2^x are both not empty.*

Proof. Let us consider first the Neumann case.

It is trivial that A_2^x, A_2^z, B_2^x and B_2^z are empty since for Neumann boundary conditions R_2 is empty.

Now, given $(\lambda^-, \mu^-) \in (\mathbb{R}^+)^2$, for any $k \geqslant 2$,

(1) if $(\lambda^-, \mu^-) \in \Lambda_k^+$ then for $\tilde{\lambda}^+ > f_k^+(\lambda^-, \mu^-)$ there exists $\varepsilon > 0$ such that $S_\varepsilon^x(\lambda^-, \mu^-)[\tilde{\lambda}^+]$ does not intersect $\widehat{\Sigma}_k^+$,
(2) if $(\lambda^-, \mu^-) \notin \bar{\Lambda}_k^+$, then there exists $\varepsilon > 0$ such that for arbitrary $\tilde{\lambda}^+$, $S_\varepsilon^x(\lambda^-, \mu^-)[\tilde{\lambda}^+]$ does not intersect $\widehat{\Sigma}_k^+$,
(3) if $(\lambda^-, \mu^-) \in \partial\Lambda_k^+$, then for arbitrary $\tilde{\lambda}^+$, $S_0^x(\lambda^-, \mu^-)[\tilde{\lambda}^+]$ does not intersect $\widehat{\Sigma}_k^+$, but for any $\varepsilon > 0$ no $\tilde{\lambda}^+$ exists such that $S_\varepsilon^x(\lambda^-, \mu^-)[\tilde{\lambda}^+]$ does not intersect $\widehat{\Sigma}_k^+$,

and analogous results hold for the intersections with $\widehat{\Sigma}_k^+$.

However, cases (1) and (3) may hold only for a finite number of k by Proposition 4.9: this implies that one may choose a $\tilde{\lambda}^+$ which does not depend on k, that is, $(\lambda^-, \mu^-) \in B_1^x$.

Also, if case (3) never happens then $(\lambda^-, \mu^-) \in A_1^x$, while if it happens for some k, then for any $\delta > 0$ small enough $(\lambda^- - \delta, \mu^- - \delta) \in A_1^x$: this indeed implies that A_1^x is dense; finally, it is easy to see by its definition that it is an open set.

For A_1^z and B_1^z one proceeds in the same way, by comparing λ^+ with $E_k^{\pm}(\lambda^-)$ and again using Proposition 4.9.

For the Dirichlet case, we argue in the same way to obtain that $A_1^x \cup A_2^x$ and $A_1^z \cup A_2^z$ are open and dense and $B_1^x \cup B_2^x = B_1^z \cup B_2^z = (\mathbb{R}^+ \setminus \{\lambda_1\})^2$.

Finally, it is simple (by the knowledge of Σ_{scal} and the continuity of the surfaces in $\widehat{\Sigma}$) to see that if $\lambda^- = \mu^- \neq \lambda_k$, then $(\lambda^-, \mu^-) \in A_2^x \subseteq B_2^x$. □

6. Some interesting problems

We conclude the paper with some unanswered questions which we believe could be of some interest.

- Asymptotic behavior of the surfaces $\widehat{\Sigma}_k^{\pm}$: in Proposition 4.9 some bounds are given, but no exact value (except for the case $\lambda^- = \mu^-$, in which case the problem reduces to the scalar one).
  Is it possible to say more about these asymptotic values? In particular, it is known (see Eq. (2.5)) that in the scalar Dirichlet case the asymptotes of Σ_{2i-1}^-, Σ_{2i} and Σ_{2i+1}^+ coincide: does the same happen for the surfaces in $\widehat{\Sigma}$?
  Observe also that in Proposition 5.5, we obtained that in the Neumann case there indeed exist gaps at infinity between the surfaces of $\widehat{\Sigma}$, but we could not say exactly where these gaps are located, and also we could not guarantee that a gap exists between an arbitrary pair of consecutive surfaces $\widehat{\Sigma}_k$, $\widehat{\Sigma}_{k+1}$.
  We believe that some results in these directions could be achieved by the analysis of the behavior of the nontrivial solutions when one of the parameters goes to $+\infty$, as was done in [10] for the scalar fourth order problem.
- Do the nontrivial solutions have more symmetries than those proved in Proposition 1.3?

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