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Existence and orbital stability of standing waves for some nonlinear Schrödinger equations, perturbation of a model case

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ABSTRACT

The following nonlinear Schrödinger equation is studied

$$i\partial_t w + \Delta w + f(x,w) = 0, \quad w = w(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \ N \geqslant 3.$$

f is a nonlinearity that can be written in the form $f(x,s) = V(x)|s|^{p-1}s + r(x,s)$, where V decays at infinity like $|x|^{-b}$ for some $b \in (0,2)$ and r is a perturbation having the same qualitative behaviour as $V(x)|s|^{p-1}s$ for small $|s|$. f is possibly singular at the origin $0 \in \mathbb{R}^N$. A standing wave is a solution of the form $w(t,x) = e^{i\lambda t}u(x)$ where $\lambda > 0$ and $u : \mathbb{R}^N \to \mathbb{R}$. For $1 < p < 1 + (4-2b)/(N-2)$, the existence in $H^1(\mathbb{R}^N)$ of a C^1-branch of standing waves parametrized by frequencies λ in a right neighbourhood of $\lambda = 0$ is proven. These standing waves are shown to be orbitally stable if $1 < p < 1 + (4-2b)/N$ and unstable if $1 + (4-2b)/N < p < 1 + (4-2b)/(N-2)$.

1. Introduction

This paper is concerned with the nonlinear Schrödinger equation

$$i\partial_t w + \Delta w + f(x,w) = 0, \quad w = w(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \quad N \geqslant 3, \tag{1}$$

where Δ denotes the Laplacian with respect to the space variable, $x \in \mathbb{R}^N$. The nonlinearity f is supposed to have the following properties. First, the restriction $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory

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function such that $f(x,0)=0$ for almost every $x\in\mathbb{R}^N$. $f(x,\cdot)$ is then extended to the complex plane on setting

$$f(x,z)=\frac{z}{|z|}f\big(x,|z|\big)\quad\text{for all } z\in\mathbb{C}\setminus\{0\},\quad f(x,0)=0. \tag{2}$$

Thus f satisfies $f(x,e^{i\varphi}s)=e^{i\varphi}f(x,s)$ for almost every $x\in\mathbb{R}^N$, all $\varphi\in\mathbb{R}$ and all $s\geqslant 0$.

The problem in this general setting arises in various fields of mathematical physics, such as nonlinear optics or many-body quantum systems, and has been widely studied for a long time. The present paper focuses on the situation where the nonlinearity f is given as a perturbation of the nonautonomous, power-like nonlinearity $V(x)|w|^{p-1}w$ with $p>1$. The work on the latter issue was initiated in [3] and has been the subject of several recent papers, see [1,4,6]. (Actually [6] deals with a slightly more general nonlinearity, see Section 1.1 below.) These articles are mainly concerned with the existence and orbital stability of standing waves for the nonlinear Schrödinger equation

$$i\partial_t w+\Delta w+V(x)|w|^{p-1}w=0. \tag{3}$$

A *standing wave* is a solution of the form

$$w(t,x)=e^{i\lambda t}u(x)\quad\text{where } \lambda>0 \text{ and } u:\mathbb{R}^N\to\mathbb{R}$$

and for solutions of this form, the nonlinear Schrödinger equation reduces to a semilinear elliptic equation involving the parameter $\lambda>0$,

$$\Delta u-\lambda u+f(x,u)=0\quad\text{corresponding to (1)} \tag{4}$$

and

$$\Delta u-\lambda u+V(x)|u|^{p-1}u=0\quad\text{corresponding to (3).} \tag{5}$$

The notion of orbital stability is recalled in Section 5. The question of existence of solutions for (1) is exhaustively studied in [2], in very general situations. It is appropriate to seek solutions with

$$w\in C\big([0,T),H^1\big(\mathbb{R}^N,\mathbb{C}\big)\big)\cap C^1\big((0,T),H^{-1}\big(\mathbb{R}^N,\mathbb{C}\big)\big)\quad\text{for some } T>0$$

and correspondingly, solutions of (4) with $u\in H^1(\mathbb{R}^N,\mathbb{R})$. In Section 2, the conditions (f2) and (f3) are formulated, which ensure that the Cauchy problem associated with (1) is well-posed.

The results established in [1,4,6] are of local nature, in the sense that existence of standing waves is proven and stability/instability is studied for values of the frequency λ in a right neighborhood of $\lambda=0$. The arguments used to prove existence of standing wave solutions in [6] and [1] are purely variational, whereas the authors of [4] make use of a continuation method, involving an implicit function theorem. The introduction in [4] gives a more detailed discussion of these respective methods and of the hypotheses they require. Let us now precise the assumptions under which we get our results. We suppose that the nonlinearity f is of the form

(f1) $f(x,s)=V(x)|s|^{p-1}s+r(x,s)$ for $x\in\mathbb{R}^N\setminus\{0\}$, $s\in\mathbb{R}$,

where V satisfies the assumptions (V1)–(V3) of [4], namely:

(V1) $V\in C^1(\mathbb{R}^N\setminus\{0\})$.
(V2) There exists $b\in(0,2)$ such that $\lim_{|x|\to\infty}|x|^bV(x)=1$ and $\limsup_{|x|\to 0}|x|^b|V(x)|<\infty$. Also $1<p<1+\frac{4-2b}{N-2}$.
(V3) $\lim_{|x|\to\infty}|x|^bW(x)=0$ and $\limsup_{|x|\to 0}|x|^b|W(x)|<\infty$ where $W(x)=x\cdot\nabla V(x)+bV(x)$.

The hypotheses made on the perturbation r concern its regularity as well as its behaviour around $s=0$. Here and henceforth ∂_1 denotes the gradient with respect to $x \in \mathbb{R}^N$ and ∂_2 the derivative with respect to $s \in \mathbb{R}$. We first suppose

(r1) $r \in C^1([\mathbb{R}^N \setminus \{0\}] \times \mathbb{R})$, $r(x,\cdot) \in C^2(\mathbb{R}\setminus\{0\})$ and $\partial_1 r(x,\cdot) \in C^1(\mathbb{R})$ for all $x \neq 0$, $r(x,0) = \partial_2 r(x,0) = 0$ and $\partial_1 r(x,0) = 0$.

Next we assume that there exists $s_0 \in (0, \frac{1}{2}]$ such that, for $x \in \mathbb{R}^N \setminus \{0\}$ and for $|s| \leqslant 2s_0$, the following properties hold:

(r2) $|s\partial_{22}^2 r(x,s)| \leqslant C|x|^{-b}|s|^{p-1}$ for $s \neq 0$,
(r3) $|x \cdot \partial_{21}^2 r(x,s)| \leqslant C|x|^{-b}|s|^{p-1}$.

Finally, for $\theta = (2-b)/(p-1)$, we assume that for all $x \neq 0$ and all $s \in \mathbb{R}$ we have

(r4) $\lim_{k\to 0^+} k^{-2}\partial_2 r(\frac{x}{k}, k^\theta s) = 0$,
(r5) $\lim_{k\to 0^+} k^{-(2+\theta)}(\frac{x}{k}) \cdot \partial_1 r(\frac{x}{k}, k^\theta s) = 0$.

Under the assumptions above, our main result is the following.

Theorem 1. *Suppose that V satisfies* (V1)–(V3), *r satisfies* (r1)–(r5) *and let f be defined by* (f1).

(a) *Existence*: *There exist $\lambda^* > 0$ and $u \in C^1((0,\lambda^*), H^1(\mathbb{R}^N,\mathbb{R}))$ such that $(\lambda, u(\lambda))$ is a weak solution of* (4) *for all $\lambda \in (0,\lambda^*)$, $u_\lambda = u(\lambda) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u_\lambda > 0$ on $\mathbb{R}^N \setminus \{0\}$. Furthermore, the following limits exist and are finite*:

$$\lim_{\lambda\to 0} \lambda^{-\gamma}|\nabla u_\lambda|_{L^2(\mathbb{R}^N)} = L_1 > 0 \quad \textit{where } \gamma = \frac{4-2b-(N-2)(p-1)}{2(p-1)} > 0,$$
$$\lim_{\lambda\to 0} \lambda^{-\gamma+1}|u_\lambda|_{L^2(\mathbb{R}^N)} = L_2 > 0,$$
$$\lim_{\lambda\to 0} \lambda^{-\delta}|u_\lambda|_{L^\infty(\mathbb{R}^N)} = 0 \quad \textit{for any } \delta < \frac{2-b}{2(p-1)}.$$

(b) *Stability*: *Suppose that f satisfies the hypotheses* (f2) *and* (f3) *of Section* 2. *There exists $\tilde{\lambda} \in (0,\lambda^*)$ such that, for all $\lambda \in (0,\tilde{\lambda})$, the following statements hold. For $1 < p < 1 + (4-2b)/N$, the standing wave associated with u_λ is orbitally stable. For $1 + (4-2b)/N < p < 1 + (4-2b)/(N-2)$, the standing wave is not orbitally stable.*

To prove Theorem 1, we use the following strategy. A first step is to prove that the Cauchy problem is well-posed, in the sense of [2]. This is done in Section 2. Then, following the method developed in [4], we prove existence and stability of standing waves for a modified version of (1). Namely, we reduce our study of the time-independent Schrödinger equation (4) to the study of the auxiliary equation (8) in which the nonlinearity is truncated outside some neighborhood of $s=0$. The auxiliary problem is defined in Section 3.

Thanks to the behaviour of r for small $|s|$, we are able to prove existence of standing waves for (8). This is done in Section 4. As in [4], the proof of existence makes use of a rescaled version of (8). The assumptions (r4) and (r5) control the behaviour of the perturbation r under the change of variables (10). Since L^∞-small solutions of the modified problem are also solutions of (4), Theorem 1(a) is then easily proven in Section 4.1.

The orbital stability/instability is addressed in Section 5. The main ingredient is the general theory of stability presented in [5]. Section 5.1 is devoted to checking the basic assumptions of [5]. The stability/instability criterion (18) is then discussed in Section 5.2 and Theorem 1(b) is proven in Section 5.3.

We shall extensively refer to [4] and we will omit technical details, if they are very similar to arguments in [4].

1.1. Examples

Example 1. Let us first consider a nonlinearity of the form

$$f(x,s) = V(x)|s|^{p-1}s + h(x)g(s)$$

where V satisfies (V1)–(V3). The hypotheses (r1)–(r3) are satisfied if

(i) $h \in C^1(\mathbb{R}^N \setminus \{0\})$ and $g \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$,
(ii) $|x|^b h(x)$ and $|x|^b x \cdot \nabla h(x)$ are bounded,
(iii) $g(0) = g'(0) = 0$ and there exists $s_0 \in (0, \frac{1}{2}]$ such that $|sg''(s)| \leqslant C|s|^{q-1}$ for $0 < |s| \leqslant 2s_0$, for some $q \geqslant p$.

Next, it is not difficult to check that the conditions (r4) and (r5) require $q > p$. Of course, additional conditions on g must be made so that the Cauchy problem be well posed (see conditions (f2) and (f3) below).

$|x|^{-b}$, $\sin(|x|)/|x|^{b+1}$, $1/(1+|x|^2)^a$ for $a \geqslant b/2$ or $e^{-|x|}$ are examples of a function h satisfying (i) and (ii). Note that h may vanish faster than $|x|^{-b}$ as $|x| \to \infty$, whereas by assumption (V2), we must have $\lim_{|x|\to\infty} |x|^b V(x) = 1$. As for V, h is allowed to be singular at the origin as well.

Example 2. A typical nonlinearity that arises in various applications is a sum of power-type nonlinearities of the form

$$f(x,s) = V(x)|s|^{p-1}s + \sum_{i=1}^{m} Z_i(x)|s|^{q_i-1}s.$$

All our assumptions are satisfied if

(i) $Z_i \in C^1(\mathbb{R}^N \setminus \{0\})$ for $i = 1, \dots, m$,
(ii) $|x|^b Z_i(x)$ and $|x|^b x \cdot \nabla Z_i(x)$ are bounded for $i = 1, \dots, m$,
(iii) $q_i > p$ for $i = 1, \dots, m$.

We must also require that $q_i < 1 + \frac{4-2b}{N-2}$ to ensure time local well-posedness.

Example 3. Finally we would like to consider a nonlinearity of the form $f(x,s) = V(x)g(s)$. This is the case treated by Jeanjean and Le Coz [6]. We suppose that V satisfies (V1)–(V3). Such a nonlinearity is written in the form (f1) by setting $R(s) = g(s) - |s|^{p-1}s$ and $r(x,s) = V(x)R(s)$. Translating our conditions on r in terms of conditions on R and then on g, we obtain that (r1)–(r3) are satisfied if

(i) $g \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ with $g(0) = g'(0) = 0$, and
(ii) there exists $s_0 \in (0, \frac{1}{2}]$ such that $|sg''(s)| \leqslant C|s|^{p-1}$ for $0 < |s| \leqslant 2s_0$.

Furthermore, since

$$\begin{aligned}
k^{-2}\partial_2 r\left(\frac{x}{k}, k^\theta s\right) &= k^{-2}V(x/k)R'\left(k^\theta s\right) = k^{-2}V(x/k)\left\{g'\left(k^\theta s\right) - pk^{\theta(p-1)}|s|^{p-1}\right\} \\
&= |x|^{-b}|x/k|^b V(x/k) k^{b-2} k^{\theta(p-1)}\left\{k^{-\theta(p-1)}g'\left(k^\theta s\right) - p|s|^{p-1}\right\} \\
&= |x|^{-b}|x/k|^b V(x/k)\left\{k^{-\theta(p-1)}g'\left(k^\theta s\right) - p|s|^{p-1}\right\}
\end{aligned}$$

and $|x/k|^b V(x/k)$ is bounded, we see that (r4) holds if and only if

$$\lim_{k\to 0^+} k^{-\theta(p-1)} g'\big(k^\theta s\big) = p|s|^{p-1}$$

for all $s \in \mathbb{R}$. But this is equivalent to the condition

(iii) $\lim_{t\to 0} \frac{g'(t)}{|t|^{p-1}} = p$.

Similarly, since

$$\begin{aligned} k^{-(2+\theta)}(x/k)\cdot\partial_1 r\left(\frac{x}{k}, k^\theta s\right) &= k^{-(2+\theta)}(x/k)\cdot\nabla V(x/k) R\big(k^\theta s\big) \\ &= k^{-(2+\theta)}(x/k)\cdot\nabla V(x/k)\big\{g\big(k^\theta s\big) - k^{\theta p}|s|^{p-1}s\big\} \\ &= |x|^{-b}|x/k|^b(x/k)\cdot\nabla V(x/k)\big\{k^{-\theta p} g\big(k^\theta s\big) - |s|^{p-1}s\big\} \end{aligned}$$

and $|x/k|^b(x/k)\cdot\nabla V(x/k)$ is bounded, we have that (r5) holds if and only if

$$\lim_{k\to 0^+} k^{-\theta p} g\big(k^\theta s\big) = |s|^{p-1}s$$

for all $s \in \mathbb{R}$, which is equivalent to the condition

(iv) $\lim_{t\to 0} \frac{g(t)}{|t|^{p-1}t} = 1$.

Note that the conditions (iii) and (iv) correspond to the hypotheses (H7) and (H4) of [6], respectively. At this point, let us make a few comments to compare our work with that of [6]. Since we obtain a smooth branch of standing wave solutions of (1) by applying an implicit function theorem, whereas the authors in [6] prove the existence of solutions by purely variational arguments, we need more regularity on the functions V and g than what is assumed in [6]. Our method also requires the hypothesis (ii) to ensure smoothness in the implicit function theorem argument, namely the property (iii) of the function $\widetilde{F}$ in Lemma 7. However, except for these smoothness restrictions, the previous discussion shows that our method covers the same kind of situations as in [6], as well as more general nonlinearities like that of Example 2.

Notation. Throughout the paper, we shall work in several function spaces. In Sections 2 and 5, we use spaces of *complex* functions. We write H^1 for the Sobolev space $H^1(\mathbb{R}^N,\mathbb{C})$ and H^{-1} for its dual space. In Section 4, we work in the *real* Sobolev space $H^1(\mathbb{R}^N,\mathbb{R})$ which we denote by H. Its dual space is also denoted by H^{-1}. The usual Lebesgue spaces $L^p(\mathbb{R}^N)$ are assumed to consist of complex/real-valued functions depending on the context and are also simply written as L^p.

The symbol C denotes various positive constants which depend on the parameters in a way that is not relevant for the analysis.

2. The Cauchy problem

In this section we explain what hypotheses must be satisfied by the nonlinearity in (1) to ensure time local/global well-posedness, as well as conservation of charge and energy. Since we shall not go into the very details, we invite the reader to consult Example 3.2.4 in [2], which deals with the simpler case $b = 0$. In our context, the results of [2] can be formulated as follows. We consider $L^2 := L^2(\mathbb{R}^N,\mathbb{C})$ and $H^1 := H^1(\mathbb{R}^N,\mathbb{C})$, equipped with their usual norms $|\cdot|_{L^2}$ and $\|\cdot\|_{H^1}$, respectively. In this section, we denote by H^{-1} the dual space of H^1.

Theorem 2. *Let $g \in C(H^1, H^{-1})$ be a nonlinearity such that $g = \sum_{i=1}^4 g_i$, where $g_i \in C(H^1, H^{-1})$ satisfy the following hypotheses. For $i = 1, \ldots, 4$:*

(a) *$g_i(0) = 0$ and there exists $G_i \in C^1(H^1, \mathbb{R})$ such that $G_i(0) = 0$ and $g_i = G_i'$.*
(b) *There exist $r_i, \rho_i \in [2, 2^*)$ such that for every $M > 0$ there exists $C_i(M) > 0$ such that*

$$\bigl|g_i(u) - g_i(v)\bigr|_{L^{\rho_i'}} \leqslant C_i(M)|u - v|_{L^{r_i}}$$

for all $u, v \in H^1$ satisfying $\|u\|_{H^1} + \|v\|_{H^1} \leqslant M$, where $1/\rho_i + 1/\rho_i' = 1$.

Consider the initial-value problem

$$i\partial_t w + \Delta w + g(w) = 0, \quad w(0) = \varphi \in H^1. \tag{6}$$

Denote $G = \sum_{i=1}^4 G_i$ and define the charge and energy respectively by

$$Q(u) = \frac{1}{2}|u|_{L^2} \quad \text{and} \quad E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - G(u) \quad \text{for all } u \in H^1.$$

Then, for all $\varphi \in H^1$, there exist $T = T(\varphi) > 0$ and a unique solution

$$w \in C\bigl([0, T), H^1\bigr) \cap C^1\bigl((0, T), H^{-1}\bigr)$$

of (6)*. Furthermore, there is conservation of charge and energy, that is,*

$$Q\bigl(w(t)\bigr) = Q(\varphi) \quad \text{and} \quad E\bigl(w(t)\bigr) = E(\varphi) \quad \text{for all } t \in [0, T).$$

If in addition we have that

(c) *there exist $\varepsilon \in (0, 1)$ and a function $\eta \in C([0, \varepsilon), [0, \infty))$ with $\eta(0) = 0$, such that*

$$\sum_{i=1}^4 G_i(u) \leqslant \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + \eta\bigl(|u|_{L^2}\bigr) \quad \text{for all } u \in H^1 \text{ with } \|u\|_{H^1} < \varepsilon,$$

then there exists $\delta > 0$ such that, for every $\varphi \in H^1$ with $\|\varphi\|_{H^1} \leqslant \delta$, we can set $T(\varphi) = \infty$ and, furthermore, $\sup\{\|w(t)\|_{H^1}\colon t \geqslant 0\} \leqslant \varepsilon$.

Finally, if we have that

(d) *there exist $\varepsilon \in (0, 1)$ and a function $\eta \in C([0, \infty), [0, \infty))$ with $\eta(0) = 0$, such that*

$$\sum_{i=1}^4 G_i(u) \leqslant \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + \eta\bigl(|u|_{L^2}\bigr) \quad \text{for all } u \in H^1,$$

then we can set $T(\varphi) = \infty$ for all $\varphi \in H^1$.

In Theorem 2, parts (a) and (b) correspond to the hypotheses of Theorem 4.3.1 of [2] and parts (c) and (d) correspond to Theorems 6.1.4 and 6.1.1 of [2], respectively.

Let us now formulate assumptions on the nonlinearity f that will permit us to discuss conditions under which time local/global well-posedness holds for (1).

(f2) There exists $\alpha \in [0, \frac{4-2b}{N-2})$ such that

$$\left|f(x,s)-f(x,t)\right| \leqslant C|x|^{-b}\left(1+|s|^{\alpha}+|t|^{\alpha}\right)|s-t| \quad \text{for } x \neq 0,\ |x| \leqslant 1,\ s,t \in \mathbb{R}.$$

(f3) There exists $\beta \in [0, \frac{4}{N-2})$ such that

$$\left|f(x,s)-f(x,t)\right| \leqslant C\left(1+|s|^{\beta}+|t|^{\beta}\right)|s-t| \quad \text{for } |x| \geqslant 1,\ s,t \in \mathbb{R}.$$

To write the nonlinearity in (1) in the notation of Theorem 2, simply define the superposition operator $g : H^1 \to H^{-1}$ by $w(x) \mapsto f(x, w(x))$ for $w \in H^1$. The following lemma then guarantees that the hypotheses in (a)–(c) are always satisfied under the assumptions (f2)–(f3), while that of (d) is satisfied provided some restriction is made on the exponents α and β in (f2)–(f3).

Lemma 3. *Suppose that f satisfies the hypotheses* (f2) *and* (f3)*. Then there exist g_i, $i = 1, \ldots, 4$, satisfying the hypotheses* (a)–(c) *of Theorem 2 such that $g = \sum_{i=1}^{4} g_i$. Furthermore, if $\alpha \in [0, \frac{4-2b}{N})$ and $\beta \in [0, \frac{4}{N})$ then the condition* (d) *is also satisfied.*

Proof. We only sketch the argument and invite the reader to complete the details following Appendix K of [4]. With a construction similar to that used in Remark 3.2.7 of [2], one can write $f = \sum_{i=1}^{4} f_i$ where the functions f_i, $i = 1, \ldots, 4$, satisfy the following properties:

$$\begin{aligned}
&f_i = \chi_{B(0,1)} f_i \text{ for } i = 1,2 \quad \text{and} \quad f_i = (1-\chi_{B(0,1)}) f_i \text{ for } i = 3,4,\\
&\left|f_1(x,s)-f_1(x,t)\right| \leqslant C|x|^{-b}|s-t| && \forall x \neq 0,\ \forall s,t \in \mathbb{R},\\
&\left|f_2(x,s)-f_2(x,t)\right| \leqslant C|x|^{-b}\left(|s|^{\alpha}+|t|^{\alpha}\right)|s-t| && \forall x \neq 0,\ \forall s,t \in \mathbb{R},\\
&\left|f_3(x,s)-f_3(x,t)\right| \leqslant C|s-t| && \forall x \in \mathbb{R}^N,\ \forall s,t \in \mathbb{R},\\
&\left|f_4(x,s)-f_4(x,t)\right| \leqslant C\left(|s|^{\beta}+|t|^{\beta}\right)|s-t| && \forall x \in \mathbb{R}^N,\ \forall s,t \in \mathbb{R}.
\end{aligned}$$

It is explained in Example 3.2.4 of [2] how similar inequalities for complex values of s, t are obtained from these ones and from (2).

For $i = 1, \ldots, 4$, define the superposition operator $g_i : H^1 \to H^{-1}$ by $w(x) \mapsto f_i(x, w(x))$ for $w \in H^1$. Then it is not difficult to prove Lemma 3 with arguments similar to those of Appendix K in [4]. Note that in the present context, the use of the Sobolev, Hölder and Gagliardo–Nirenberg inequalities involves the exponents α and β, whereas in the situation considered in [4], solely the exponent p was concerned. In particular, the condition $1 < p < 1 + \frac{4-2b}{N}$ required for part (d) of Theorem K.1 in [4] is replaced by the corresponding conditions on α and β, namely $\alpha \in [0, \frac{4-2b}{N})$ and $\beta \in [0, \frac{4}{N})$. $\square$

3. The auxiliary problem

We shall prove the existence of a C^1-branch of solutions for (4), making use of the auxiliary problem defined as follows. Let $\varphi \in C^{\infty}(\mathbb{R})$ be such that:

$$\varphi(-s) = \varphi(s) \quad \forall s \in \mathbb{R}, \qquad \varphi(s) = 1 \quad \text{if } |s| \leqslant s_0, \qquad \varphi(s) = 0 \quad \text{if } |s| \geqslant 2s_0.$$

Then define the truncation $\tilde{r}$ of r by

$$\tilde{r}(x,s) = \varphi(s) r(x,s) \tag{7}$$

for all $x \neq 0$ and all $s \in \mathbb{R}$. Now the auxiliary equation is

$$-\Delta u + \lambda u - \tilde{f}(x, u) = 0 \tag{8}$$

where

$$\tilde{f}(x, s) = V(x)|s|^{p-1}s + \tilde{r}(x, s). \tag{9}$$

Hence $\tilde{f}(x, s) = f(x, s)$ whenever $|s| \leqslant s_0$, so that L^∞-small solutions of (8) are also solutions of (4). It is clear that $\tilde{r}$ has the same regularity as r. The following properties of $\tilde{r}$ are easy consequences of (r1)–(r5), the fact that $s_0 \in (0, \frac{1}{2}]$ and the properties of φ. For all $x \in \mathbb{R}^N \setminus \{0\}$ and all $s, t \in \mathbb{R}$, we have

($\tilde{\mathrm{r}}$1) $|s\partial^2_{22}\tilde{r}(x, s)| \leqslant C|x|^{-b}|s|^{p-1}$,
($\tilde{\mathrm{r}}$2) $|x \cdot \partial^2_{21}\tilde{r}(x, s)| \leqslant C|x|^{-b}|s|^{p-1}$,
($\tilde{\mathrm{r}}$3) $|\partial_2\tilde{r}(x, s) - \partial_2\tilde{r}(x, t)| \leqslant C|x|^{-b}||s|^{p-2}s - |t|^{p-2}t|$,
($\tilde{\mathrm{r}}$4) $|\tilde{r}(x, s) - \tilde{r}(x, t)| \leqslant C|x|^{-b}||s|^{p-1}s - |t|^{p-1}t|$,
($\tilde{\mathrm{r}}$5) $|x \cdot \partial_1\tilde{r}(x, s) - x \cdot \partial_1\tilde{r}(x, t)| \leqslant C|x|^{-b}||s|^{p-1}s - |t|^{p-1}t|$,
($\tilde{\mathrm{r}}$6) $\lim_{k\to 0^+} k^{-2}\partial_2\tilde{r}(\frac{x}{k}, k^\theta s) = 0$,
($\tilde{\mathrm{r}}$7) $\lim_{k\to 0^+} k^{-(2+\theta)}(\frac{x}{k}) \cdot \partial_1\tilde{r}(\frac{x}{k}, k^\theta s) = 0$,
($\tilde{\mathrm{r}}$8) $\lim_{k\to 0^+} k^{-(2+\theta)}\tilde{r}(\frac{x}{k}, k^\theta s) = 0$.

We also define $\widetilde{R}(x, s) = x \cdot \partial_1\tilde{r}(x, s) - \theta s\partial_2\tilde{r}(x, s)$ which will be useful later. For all $x \in \mathbb{R}^N \setminus \{0\}$ and all $s, t \in \mathbb{R}$, we have $\widetilde{R}(x, \cdot) \in C^1(\mathbb{R})$ and

($\widetilde{\mathrm{R}}$1) $|\partial_2\widetilde{R}(x, s)| \leqslant C|x|^{-b}|s|^{p-1}$,
($\widetilde{\mathrm{R}}$2) $|\widetilde{R}(x, s) - \widetilde{R}(x, t)| \leqslant C|x|^{-b}||s|^{p-1}s - |t|^{p-1}t|$,
($\widetilde{\mathrm{R}}$3) $\lim_{k\to 0^+} k^{-(2+\theta)}\widetilde{R}(\frac{x}{k}, k^\theta s) = 0$.

4. Existence

Here and henceforth, let $H := H^1(\mathbb{R}^N, \mathbb{R})$ and H^{-1} denote its dual space. For $\lambda > 0$, we set $\lambda = k^2$, $k > 0$, and make the same change of variables as in Section 3 of [4]. Namely, we define the linear operator $\mathcal{S}(k) : H \to H$ by

$$\mathcal{S}(k)v(x) = k^{\frac{2-b}{p-1}} v(kx) \quad \text{for } k > 0. \tag{10}$$

Clearly $\mathcal{S} \in C((0, \infty), B(H, H))$ and $\mathcal{S}(k) : H \to H$ is invertible for all $k > 0$. Furthermore, if $u, v \in H$ are such that $u = \mathcal{S}(k)v$, then we have that

$$|u|^2_{L^2(\mathbb{R}^N)} = k^\varrho |v|^2_{L^2(\mathbb{R}^N)} \tag{11}$$

and

$$|\nabla u|^2_{L^2(\mathbb{R}^N)} = k^{\varrho+2}|\nabla v|^2_{L^2(\mathbb{R}^N)} \tag{12}$$

where $\varrho = [4 - 2b - N(p - 1)]/(p - 1)$.

On setting $u = \mathcal{S}(k)v$ and $y = kx \in \mathbb{R}^N$, (8) becomes

$$-\Delta v + v - k^{-(2+\theta)}\tilde{f}\big(y/k, k^\theta v\big) = 0. \tag{13}$$

By ($\tilde{r}$8), $k^{-(2+\theta)}\tilde{f}(y/k, k^\theta s) \to |y|^{-b}|s|^{p-1}s$ as $k \to 0^+$ for all $y \neq 0$ and all $s \in \mathbb{R}$. Therefore, we are led to the same limit problem as in [4]:

$$-\Delta v + v - |y|^{-b}|v|^{p-1}v = 0, \quad v \in H. \tag{14}$$

Note that this problem has been extensively studied by variational arguments in Section 2 of [4]. For completeness, we briefly recall the main results obtained by the variational approach to (14). One obtains a positive, spherically symmetric, radially decreasing solution of (14) by minimizing the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}\,dx$$

under the natural constraint

$$\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx - \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}\,dx = 0.$$

This constraint defines in H a C^2-manifold $\mathcal{N}$ of codimension 1, often called the *Nehari manifold*. Theorem 2.5 of [4] asserts that there exists in H a minimizer ψ of the problem $m = \inf\{J(u):\ u \in \mathcal{N}\}$, whose main properties are summarized in the following lemma.

Lemma 4.

(i) $\psi \in \mathcal{N}$ *and* $J(\psi) = m$.
(ii) ψ *is positive, spherically symmetric and radially strictly decreasing on* $\mathbb{R}^N$.
(iii) $\psi \in C(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$.
(iv) $\psi(x) \to 0$ *exponentially as* $|x| \to \infty$.
(v) $J''(\psi)[\psi, \psi] < 0$ *and* $J''(\psi)[\xi, \xi] \geqslant 0$ *for all* $\xi \in H$ *such that* $\langle \psi, \xi \rangle = 0$.

We naturally generalize Proposition 3.1 of [4] as follows.

Proposition 5. *Suppose V satisfies* (V1)–(V3), *r satisfies* (r1)–(r5) *and $\tilde{f}$ is defined by* (7) *and* (9). *Let ψ be the solution of* (14) *given by Theorem* 2.5 *of* [4]. *There exist $k_0 > 0$ and $v \in C([0, k_0), H) \cap C^1((0, k_0), H)$ such that $v(0) = \psi$ and $(k, v(k))$ is a nontrivial solution of* (13) *for all $k \in (0, k_0)$. Furthermore, $v(k) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for all $k \in (0, k_0)$ and $|v(k)|_{L^\infty(\mathbb{R}^N)}$ remains bounded as $k \to 0$.*

To prove Proposition 5, it is sufficient to extend Lemmas 3.1 and 3.2 of [4] and then use the implicit function theorem as in the proof of Proposition 3.1 of [4]. For convenience, we recall the definition and properties of the function F that appears in Lemma 3.1 of [4]. We can now forget the change of variables for a while, so we denote again by x the variable in $\mathbb{R}^N$.

Lemma 6. *(See Lemma 3.1 of [4].) Let the hypotheses* (V1)–(V3) *be satisfied and $F : \mathbb{R} \times H \to H^{-1}$ be defined by*

$$F(k, u) = \begin{cases} -\Delta u + u - k^{-b}V(x/k)|u|^{p-1}u & \text{if } k > 0, \\ -\Delta u + u - |x|^{-b}|u|^{p-1}u & \text{if } k = 0. \end{cases}$$

(i) $F \in C(\mathbb{R} \times H, H^{-1})$.
(ii) *For all* $k \in \mathbb{R}$, $F(k, \cdot) : H \to H^{-1}$ *is Fréchet differentiable and*

$$D_u F(k,u)v = \begin{cases} -\Delta v + v - pk^{-b} V(x/k)|u|^{p-1} v & \text{if } k > 0, \\ -\Delta v + v - p|x|^{-b}|u|^{p-1} v & \text{if } k = 0, \end{cases}$$

for all $u, v \in H$. *Moreover* $D_u F \in C(\mathbb{R} \times H, B(H, H^{-1}))$.
(iii) $F \in C^1((0,\infty) \times H, H^{-1})$.

In our context, we wish to apply an implicit function theorem to the function $\widetilde{F} : H \to H^{-1}$ defined by

$$\widetilde{F}(k,u) = \begin{cases} -\Delta u + u - |k|^{-(2+\theta)} \tilde{f}(x/|k|, |k|^{\theta} u) & \text{if } k \neq 0, \\ -\Delta u + u - |x|^{-b}|u|^{p-1} u & \text{if } k = 0. \end{cases}$$

Of course, $\widetilde{F}$ reduces to F when putting $\tilde{r} \equiv 0$.

Lemma 7. *Suppose* V *satisfies* (V1)–(V3), r *satisfies* (r1)–(r5) *and* $\tilde{r}$ *is defined by* (7).

(i) $\widetilde{F} \in C(\mathbb{R} \times H, H^{-1})$.
(ii) *For all* $k \in \mathbb{R}$, $\widetilde{F}(k, \cdot) : H \to H^{-1}$ *is Fréchet differentiable and*

$$D_u \widetilde{F}(k,u)v = \begin{cases} -\Delta v + v - |k|^{-2} \partial_2 \tilde{f}(x/|k|, |k|^{\theta} u) & \text{if } k \neq 0, \\ -\Delta v + v - p|x|^{-b}|u|^{p-1} v & \text{if } k = 0, \end{cases}$$

for all $u, v \in H$. *Moreover* $D_u \widetilde{F} \in C(\mathbb{R} \times H, B(H, H^{-1}))$.
(iii) $\widetilde{F} \in C^1((0,\infty) \times H, H^{-1})$.

To prove Lemma 7, it is convenient to introduce the function $\psi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x,k,s) = \begin{cases} |k|^{-(2+\theta)} \tilde{r}(x/|k|, |k|^{\theta} s) & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Then

$$k^{-(2+\theta)} \tilde{f}\big(x/k, k^{\theta} s\big) = k^{-b} V(x/k)|s|^{p-1} s + \psi(x,k,s) \tag{15}$$

and

$$k^{-2} \partial_2 \tilde{f}\big(x/k, k^{\theta} s\big) = pk^{-b} V(x/k)|s|^{p-1} + \partial_s \psi(x,k,s) \tag{16}$$

for all $k > 0$. We give some useful properties of ψ which are easily derived from the properties of $\tilde{r}$. In the sequel, we shall use alternatively ψ or $\tilde{r}$ depending on the context. For all $k \in \mathbb{R}$, $\psi(\cdot, k, \cdot)$ is a Carathéodory function. Also, for all $x \in \mathbb{R}^N \setminus \{0\}$, $\psi(x, \cdot, \cdot)$ satisfies the following.

(ψ1) $\psi(x, \cdot, \cdot) \in C(\mathbb{R} \times \mathbb{R})$.
(ψ2) $|\psi(x,k,s) - \psi(x,k,t)| \leqslant C|x|^{-b} \big||s|^{p-1}s - |t|^{p-1}t\big|$ for all $s, t \in \mathbb{R}$.
(ψ3) For all $k \in \mathbb{R}$, $\psi(x,k,\cdot) \in C^1(\mathbb{R})$ with

$$\partial_s \psi(x,k,s) = \begin{cases} |k|^{-2} \partial_2 \tilde{r}(x/|k|, |k|^{\theta} s) & \text{if } k \neq 0, \\ 0 & \text{if } k = 0 \end{cases}$$

and $\partial_s \psi(x, \cdot, \cdot) \in C(\mathbb{R} \times \mathbb{R})$.

(ψ4) $|\partial_s\psi(x,k,s)-\partial_s\psi(x,k,t)|\leqslant C|x|^{-b}||s|^{p-2}s-|t|^{p-2}t|$ for all $s,t\in\mathbb{R}$.
(ψ5) $\psi(x,\cdot,\cdot)\in C^1((0,\infty)\times\mathbb{R})$ with $\partial_k\psi(x,k,s)=-k^{-(3+\theta)}[(2+\theta)\tilde r(x/k,k^\theta s)+\widetilde R(x/k,k^\theta s)]$ for all $k>0$, $s\in\mathbb{R}$.
(ψ6) $|\partial_k\psi(x,k,s)-\partial_k\psi(x,k,t)|\leqslant Ck^{-1}|x|^{-b}||s|^{p-1}s-|t|^{p-1}t|$ for all $k>0$ and all $s,t\in\mathbb{R}$.
(ψ7) $\partial_k\psi(x,k,\cdot)\in C^1(\mathbb{R})$ and $|\partial^2_{sk}\psi(x,k,s)|\leqslant Ck^{-1}|x|^{-b}|s|^{p-1}$ for all $k>0$ and all $s\in\mathbb{R}$.
(ψ8) $\lim_{k\to0}\psi(x,k,s)=0$ for all $s\in\mathbb{R}$.
(ψ9) $\lim_{k\to0}k\partial_k\psi(x,k,s)=0$ for all $s\in\mathbb{R}$.

In the previous properties of ψ, C denotes various positive constants which do not depend on k.

Proof of Lemma 7. We follow closely the proof of Lemma 3.1 of [4]. We use Lemmas 14–16 given in Appendix A.

Set $S(k,u)=\psi(x,k,u)$ for all $(k,u)\in\mathbb{R}\times H$. By Lemma 16, it follows from (ψ3) and (ψ4) that for all $k\in\mathbb{R}$, $S(k,\cdot)\in C^1(H,H^{-1})$ with

$$D_uS(k,u)v=\partial_s\psi(x,k,u)v\quad\text{for all } u,v\in H.$$

Since $\widetilde F=F+S$ and according to the properties of F recalled in Lemma 6, the proof of Lemma 7 only requires to show that S has the following properties:

(i) $S\in C(\mathbb{R}\times H,H^{-1})$,
(ii) $D_uS\in C(\mathbb{R}\times H,B(H,H^{-1}))$,
(iii) $S\in C^1((0,\infty)\times H,H^{-1})$.

Proof of (i). Fix $(k,u)\in\mathbb{R}\times H$ and consider $(h,v)\in\mathbb{R}\times H$. Then

$$\|S(k,u)-S(h,v)\|_{H^{-1}}\leqslant\|S(k,u)-S(h,u)\|_{H^{-1}}+\|S(h,u)-S(h,v)\|_{H^{-1}}$$

where

$$S(h,u)-S(h,v)=\int_0^1\frac{d}{dt}S\big(h,tu+(1-t)v\big)\,dt=\int_0^1 D_uS\big(h,tu+(1-t)v\big)\,dt\,(u-v)$$

and so

$$\begin{aligned}\|S(h,u)-S(h,v)\|_{H^{-1}}&\leqslant\int_0^1\|D_uS\big(h,tu+(1-t)v\big)\|_{B(H,H^{-1})}\,dt\,\|u-v\|\\&\leqslant C\int_0^1\|tu+(1-t)v\|^{p-1}\,dt\,\|u-v\|\end{aligned}$$

by Lemma 16(ii). The constant C is independent of h by (ψ4). Then $\|S(h,u)-S(h,v)\|_{H^{-1}}\to0$ as $\|u-v\|\to0$ uniformly for $h\in\mathbb{R}$ as in the proof of (i) in Appendix I of [4].

For the first term, we have

$$\|S(k,u)-S(h,u)\|_{H^{-1}}\leqslant\sup_{\varphi\in H\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\psi(x,k,u)-\psi(x,h,u)||\varphi|\,dx}{\|\varphi\|}$$

where $\psi(x,k,u)-\psi(x,h,u)\to0$ as $h\to k$ for all $x\in\mathbb{R}^N\setminus\{0\}$ by (ψ1) and $|\psi(x,k,u)-\psi(x,h,u)|\leqslant C|x|^{-b}|u|^p$ by (ψ2). Hence, $\|S(k,u)-S(h,u)\|_{H^{-1}}\to0$ as $h\to k$ by Lemma 15.

Proof of (ii). Fix $(k,u)\in\mathbb{R}\times H$ and consider $(h,v)\in\mathbb{R}\times H$. We have

$$\begin{aligned}&\left\|D_uS(k,u)-D_uS(h,v)\right\|_{B(H,H^{-1})}\\&\quad\leqslant\left\|D_uS(k,u)-D_uS(h,u)\right\|_{B(H,H^{-1})}+\left\|D_uS(h,u)-D_uS(h,v)\right\|_{B(H,H^{-1})}.\end{aligned}$$

First, it follows by (ψ4) and Lemma 16(ii) that the functions $\{D_uS(h,\cdot):H\to B(H,H^{-1})\}_{h\in\mathbb{R}}$ are equicontinuous at u, and so $\|S(h,u)-S(h,v)\|_{H^{-1}}\to 0$ as $\|u-v\|\to 0$ uniformly for $h\in\mathbb{R}$.

On the other hand,

$$\left\|D_uS(k,u)-D_uS(h,u)\right\|_{B(H,H^{-1})}\leqslant\sup_{\varphi,\xi\in H\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\partial_s\psi(x,k,u)-\partial_s\psi(x,h,u)||\varphi||\xi|\,dx}{\|\varphi\|\|\xi\|}$$

where $\partial_s\psi(x,k,u)-\partial_s\psi(x,h,u)\to 0$ as $h\to k$ for all $x\in\mathbb{R}^N\setminus\{0\}$ by (ψ3) and $|\partial_s\psi(x,k,u)-\partial_s\psi(x,h,u)|\leqslant C|x|^{-b}|u|^{p-1}$ by (ψ4). Therefore we have that $\|S(k,u)-S(h,u)\|_{H^{-1}}\to 0$ as $h\to k$ by Lemma 14.

Proof of (iii). It only remains to prove that S is Fréchet differentiable with respect to k at $(k,u)\in(0,\infty)\times H$ and that D_kS is continuous on $(0,\infty)\times H$. In fact, we prove that

$$D_kS(k,u)v=\partial_k\psi(x,k,u)v\quad\text{for all }(k,u)\in(0,\infty)\times H,\ v\in H.$$

For $\varphi\in H$, we have

$$\begin{aligned}&\left\langle\frac{S(k+h,u)-S(k,u)}{h}-\partial_k\psi(x,k,u),\varphi\right\rangle_{H^{-1}\times H}\\&\quad=\int_{\mathbb{R}^N}\left\{\frac{\psi(x,k+h,u)-\psi(x,k,u)}{h}-\partial_k\psi(x,k,u)\right\}\varphi\,dx.\end{aligned}$$

By (ψ6) and the mean-value theorem, we have $|\partial_k\psi(x,k,u)|\leqslant C\frac{1}{k}|x|^{-b}|u|^p$ and

$$\left|\frac{\psi(x,k+h,u)-\psi(x,k,u)}{h}\right|\leqslant C\frac{2}{k}|x|^{-b}|u|^p\quad\text{if }h\geqslant-\frac{k}{2}.$$

Hence,

$$\frac{\psi(x,k+h,u)-\psi(x,k,u)}{h}-\partial_k\psi(x,k,u)\to 0$$

as $h\to 0$ for all $x\neq 0$ and

$$\left|\frac{\psi(x,k+h,u)-\psi(x,k,u)}{h}-\partial_k\psi(x,k,u)\right|\leqslant C\frac{3}{k}|x|^{-b}|u|^p,$$

so that

$$\begin{aligned}&\left\|\frac{S(k+h,u)-S(k,u)}{h}-\partial_k\psi(x,k,u)\right\|_{H^{-1}}\\&\quad\leqslant\sup_{\varphi\in H\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\frac{1}{h}\{\psi(x,k+h,u)-\psi(x,k,u)\}-\partial_k\psi(x,k,u)||\varphi|\,dx}{\|\varphi\|}\to 0\end{aligned}$$

as $h\to 0$ by Lemma 15.

For the continuity of $D_k S$, fix again $(k,u) \in (0,\infty) \times H$ and let $(h,v) \in (0,\infty) \times H$. We have

$$\left\| D_k S(k,u) - D_k S(h,v) \right\|_{H^{-1}} \leqslant \left\| D_k S(k,u) - D_k S(h,u) \right\|_{H^{-1}} + \left\| D_k S(h,u) - D_k S(h,v) \right\|_{H^{-1}}.$$

First,

$$\left\| D_k S(k,u) - D_k S(h,u) \right\|_{H^{-1}} \leqslant \sup_{\varphi \in H \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\partial_k \psi(x,k,u) - \partial_k \psi(x,h,u)||\varphi|\,dx}{\|\varphi\|}$$

where the right-hand side is proven to vanish as $h \to k$ using Lemma 15.

On the other hand, it follows from Lemma 16 that, for all $h > 0$, $D_k S(h,\cdot) \in C^1(H, H^{-1})$ with

$$D_u D_k S(h,u)v = \partial^2_{sk} \psi(x,h,u)v \quad \text{for all } u, v \in H.$$

Also, by (ψ7) and Lemma 16(ii),

$$\left\| D_u D_k S(h,u) \right\|_{B(H,H^{-1})} \leqslant C \frac{1}{h} \|u\|^{p-1} \quad \text{for all } u \in H.$$

Since

$$\begin{aligned} D_k S(h,u) - D_k S(h,v) &= \int_0^1 \frac{d}{dt} D_k S\big(h, tu + (1-t)v\big)\,dt \\ &= \int_0^1 D_u D_k S\big(h, tu + (1-t)v\big)\,dt\,(u-v), \end{aligned}$$

choosing $h \geqslant k/2$, we have that

$$\left\| D_k S(h,u) - D_k S(h,v) \right\|_{H^{-1}} \leqslant \int_0^1 C \frac{2}{k} \left\| tu + (1-t)v \right\|^{p-1} dt\,\|u - v\|$$

where $C > 0$ is a constant independent of h. The result then follows as in part (i). This concludes the proof of Lemma 7. □

The following lemma generalizes Lemma 3.2 of [4] and ensures that the solutions to (13) given by Proposition 5 remain bounded in $L^\infty(\mathbb{R}^N)$.

Lemma 8.

(i) *If $(k,v) \in (0,\infty) \times H$ satisfies (13), then $v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

(ii) *For any $M > 0$, there exists a constant $C(M)$ such that $|v|_{L^\infty} \leqslant C(M)$ for all solutions $(k,v) \in (0,\infty) \times H$ of (13) with $\|v\| \leqslant M$.*

Proof. The proof follows by the bootstrap argument given in Appendix J of [4]. If $(k,v) \in (0,\infty) \times H$ is a solution of (13), we have

$$-\Delta v + v = L_k(v) \quad \text{where } L_k(v) = k^{-b} V(x/k)|v|^{p-1}v + \psi(x,k,v).$$

By (ψ2) we have $|L_k(v)| \leqslant C|x|^{-b}|v|^p$ and the proof is then rigorously the same as that of Lemma 3.1 in Appendix J of [4] (i.e. the case $\psi \equiv 0$). $\square$

Proof of Proposition 5. Since $D_u\widetilde{F}(0,\psi) = D_uF(0,\psi)$ is an isomorphism by Proposition 2.3 of [4], an implicit function theorem can be applied to the function $\widetilde{F}$ in the same way as in the proof of Proposition 3.1 in [4] with the function F. To do this, simply use Lemmas 7 and 8 instead of Lemmas 3.1 and 3.2 of [4], respectively. The continuity of $v(k)$ and the boundedness of $|v(k)|_{L^\infty(\mathbb{R}^N)}$ as $k \to 0$ are given by Lemma 8. $\square$

Following the proof of Theorem 1.1(a) at the end of Section 3 in [4], it is now straightforward to prove existence for the modified problem. More precisely, the following result holds.

Lemma 9. *There exist $\lambda_0 > 0$ and $u \in C^1((0,\lambda_0), H^1(\mathbb{R}^N,\mathbb{R}))$ such that $(\lambda, u(\lambda))$ is a weak solution of* (8) *for all $\lambda \in (0,\lambda_0)$, $u_\lambda = u(\lambda) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Furthermore, the following limits exist and are finite*:

$$\lim_{\lambda\to 0} \lambda^{-\gamma}|\nabla u_\lambda|_{L^2(\mathbb{R}^N)} = L_1 > 0 \quad \text{where } \gamma = \frac{4-2b-(N-2)(p-1)}{2(p-1)} > 0,$$

$$\lim_{\lambda\to 0} \lambda^{-\gamma+1}|u_\lambda|_{L^2(\mathbb{R}^N)} = L_2 > 0,$$

$$\lim_{\lambda\to 0} \lambda^{-\delta}|u_\lambda|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{for any } \delta < \frac{2-b}{2(p-1)}.$$

4.1. *Proof of Theorem 1, part (a)*

By Lemma 9, there exists $\lambda^* \in (0,\lambda_0)$ such that $|u_\lambda|_{L^\infty} \leqslant s_0$ for all $\lambda \in (0,\lambda^*)$. Since Eqs. (4) and (8) coincide for $u \in H$ such that $|u|_{L^\infty} \leqslant s_0$, part (a) of Theorem 1 follows from Lemma 9, the positivity of solutions being given by Lemma 12(iii) below.

5. Stability

Let us now return to the time-dependent Schrödinger equation (1). Now that we have proved part (a) of Theorem 1, we can discuss orbital stability/instability of standing waves for (1).

Let $\psi_\lambda(t,x) = e^{i\lambda t}u_\lambda(x)$ be a standing wave solution of (1), u_λ, $\lambda \in (0,\lambda^*)$, being given by Theorem 1(a). ψ_λ is a periodic function of time and we define its *orbit* $\Theta(\lambda) := \{e^{i\theta}u(\lambda)\colon\ \theta \in \mathbb{R}\}$. The standing wave ψ_λ is said to be *orbitally stable* in $H^1(\mathbb{R}^N,\mathbb{C})$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if w is a solution of (1) with $\|w(0,\cdot) - u(\lambda)\|_{H^1(\mathbb{R}^N,\mathbb{C})} < \delta$, then

$$\sup_{t\geqslant 0}\inf_{\theta\in\mathbb{R}} \bigl\| w(t,\cdot) - e^{i\theta}u(\lambda)\bigr\|_{H^1(\mathbb{R}^N,\mathbb{C})} < \varepsilon. \tag{17}$$

It is said to be *unstable* if it is not stable. This definition requires the solution w to be defined globally in time. Roughly speaking, orbital stability means that if the initial data $w(0,\cdot)$ is close to $\Theta(\lambda)$, then the solution $w(t,\cdot)$ remains close to $\Theta(\lambda)$ for all $t \geqslant 0$. The proof of the stability/instability result for (1) relies on the general theory of orbital stability for Hamiltonian systems presented in [5]. As in Section 4 of [4], it is divided into two parts. We shall first verify that, under our hypotheses on the function f, the basic assumptions of [5] are satisfied. Then we will apply Theorem 3 of [5] to state when stability/instability occurs. In our context, this is done by checking when

$$\frac{d}{d\lambda}|u_\lambda|^2_{L^2} > 0/{<}0. \tag{18}$$

It is explained in details in [4] how to set (1) in Hamiltonian formalism in order to use Theorem 3 of [5], and how the explicit criterion (18) for stability/instability is obtained from the general theory.

(See also [7], where the general theory of orbital stability of standing waves and its application to nonlinear Schrödinger equations are exposed in great detail.) Therefore we will only indicate here what has to be modified to cover the more general situation we are dealing with. Let us recall that we consider (1) as a Hamiltonian system on the space $X = H \times H$ where again $H := H^1(\mathbb{R}^N, \mathbb{R})$. That is, we identify $H^1(\mathbb{R}^N, \mathbb{C})$ with X via $H^1(\mathbb{R}^N, \mathbb{C}) \ni w \leftrightarrow \varphi = (\operatorname{Re} w, \operatorname{Im} w) \in X$. On this space, the "energy" and "charge" for the auxiliary problem are respectively defined by

$$\widetilde{E}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx - \widetilde{G}(\varphi) \quad \text{and} \quad Q(\varphi) = \frac{1}{2}\int_{\mathbb{R}^N} |\varphi|^2\,dx$$

where $\widetilde{G}$ is defined by

$$\widetilde{G}(\varphi) = \int_{\mathbb{R}^N} h\bigl(x, w(x)\bigr)\,dx, \quad h(x,z) = \int_0^{|z|} f(x,s)\,ds \quad \text{for all } z \in \mathbb{C},$$

if $\varphi \in X$ is identified with $w \in H^1(\mathbb{R}^N, \mathbb{C})$. As explained in [4], it is not difficult to show that $\widetilde{E}, Q \in C^2(X, \mathbb{R})$. Then (1) is equivalent to

$$\frac{d}{dt}\varphi(t) = J\widetilde{E}'\bigl(\varphi(t)\bigr) \quad \text{where } J = \begin{pmatrix} 0 & R^{-1} \\ -R^{-1} & 0 \end{pmatrix}$$

and $R = -\Delta + 1 : H \to H^{-1}$ is the Riesz isomorphism.

Before we turn to the basic assumptions that must be verified to make use of Theorem 3 of [5], let us make two important

Remarks. (a) The various objects involved in the analysis depend on the functionals $\widetilde{E}, Q : X \to \mathbb{R}$ and thus on the function f. We shall keep in mind that $f(x,s) = \tilde{f}(x,s)$ for $|s| \leqslant s_0$. Indeed, since $|u_\lambda|_{L^\infty} \leqslant s_0$ for $\lambda \in (0, \lambda^*)$, we will infer several conditions depending on $f(x, |u_\lambda|)$ and $\partial_2 f(x, |u_\lambda|)$ from the corresponding conditions with $\tilde{f}(x, |u_\lambda|)$ and $\partial_2 \tilde{f}(x, |u_\lambda|)$.

(b) We will not use in this section the positivity of u_λ given by Theorem 1(a) since we prove this property in Lemma 12(iii) below.

5.1. The basic hypotheses of [5]

Assumption 1 of [5] concerns time well-posedness of the initial-value problem for (1), as well as conservation of charge and energy. Under the hypotheses (f2) and (f3), this is ensured by Lemma 3 and Theorem 2. Note that, without the additional requirements that $\alpha \in [0, \frac{4-2b}{N})$ and $\beta \in [0, \frac{4}{N})$, (f2) and (f3) only guarantee global well-posedness for small initial data. But this is sufficient in our context since we know by Theorem 1(a) that $\|u_\lambda\| \to 0$ as $\lambda \to 0$. Indeed, the issue of orbital stability/instability of a standing wave $\psi_\lambda(t,x) = e^{i\lambda t}u_\lambda(x)$ requires to consider the long-time behaviour of solutions $w(t,x)$ of (1) with initial data $w(0,\cdot)$ close to u_λ. Therefore, if $\lambda \in (0, \lambda^*)$ is small enough, we can restrict ourself to solutions $w(t,x)$ such that $w(0,\cdot)$ is small enough in $H^1(\mathbb{R}^N, \mathbb{C})$ to ensure global existence, according to Theorem 2(c).

Assumption 2 of [5] concerns the existence of a smooth branch $\lambda \mapsto \varphi_\lambda$ of nontrivial solutions of (4). This is ensured by part (a) of Theorem 1 on setting

$$\varphi_\lambda = \bigl(u(\lambda), 0\bigr) \quad \text{for } \lambda \in (0, \lambda^*).$$

In our context, the operator $H_\lambda : X \to X^*$ of [4] is replaced by

$$\widetilde{H}_\lambda = \widetilde{E}''(\varphi_\lambda) + \lambda Q''(\varphi_\lambda), \quad \lambda \in (0, \lambda^*),$$

given explicitly by

$$\widetilde{H}_\lambda = \begin{pmatrix} \mathcal{A}(\lambda) & 0 \\ 0 & \mathcal{B}(\lambda) \end{pmatrix}$$

where

$$\mathcal{A}(\lambda) = -\Delta + \lambda - \partial_2 f\big(x, |u_\lambda|\big) \quad \text{and} \quad \mathcal{B}(\lambda) = -\Delta + \lambda - \frac{f(x, |u_\lambda|)}{|u_\lambda|}. \tag{19}$$

In fact, for all $\lambda \in (0, \lambda^*)$, $|u_\lambda|_{L^\infty} \leqslant s_0$ and we have

$$\mathcal{A}(\lambda) = -\Delta + \lambda - \partial_2 \tilde{f}\big(x, |u_\lambda|\big) \quad \text{and} \quad \mathcal{B}(\lambda) = -\Delta + \lambda - \frac{\tilde{f}(x, |u_\lambda|)}{|u_\lambda|}. \tag{20}$$

To verify the spectral hypotheses (H1)–(H3) that appear in Assumption 3, we will rather work with the expressions (20) than with (19). Indeed, the proof of Lemma 11 below relies heavily on properties of several objects depending on $\tilde{f}$ that have been studied in Section 4.

Note that the function $g(x, \cdot)$ defined by

$$g(x, s) = \begin{cases} \tilde{f}(x, s)/s & \text{for } s > 0, \\ 0 & \text{for } s = 0 \end{cases}$$

is continuous on $[0, \infty)$ for all $x \in \mathbb{R}^N \setminus \{0\}$ by the definition of $\tilde{f}$ and by ($\tilde{\text{r}}$4).

Assumption 3 of [5] requires that, for all λ in some open interval $(a, b) \subset (0, \lambda_0)$, the following holds:

(H1) There exists $a_\lambda < 0$ such that $S(\widetilde{H}_\lambda) \cap (-\infty, 0) = \{a_\lambda\}$ and $\dim\ker(\widetilde{H}_\lambda - a_\lambda \widetilde{R}) = 1$.
(H2) $\ker \widetilde{H}_\lambda = \operatorname{span}\{T'(0)\varphi_\lambda\} = \operatorname{span}\{(0, u(\lambda))\}$.
(H3) $S(\widetilde{H}_\lambda) \setminus \{a_\lambda, 0\}$ is bounded away from 0 in $\mathbb{R}$.

Here, the *spectrum* $S(\widetilde{H}_\lambda)$ of $\widetilde{H}_\lambda$ is defined as in [4] and $\widetilde{R} = \left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$.

Most of the work done in Section 4 of [4] is devoted to check Assumption 3. A first step is to reduce the discussion on the spectrum of $\widetilde{H}$ to a discussion pertaining only on isolated eigenvalues of $\widetilde{H}$. This is done in Lemma 4.1 of [4]. The key ingredient is to remark that the operators $A(\lambda) - \mu R$ and $B(\lambda) - \mu R$ are Fredholm operators for μ small enough. We briefly explain why, in fact, such a result holds in our context.

By (V1), (V2), ($\tilde{\text{r}}$3) and ($\tilde{\text{r}}$4), we have that

$$\big|\partial_2 \tilde{f}\big(x, |u_\lambda|\big)\big| \leqslant C|x|^{-b}|u_\lambda|^{p-1} \quad \text{and} \quad \frac{|\tilde{f}(x, |u_\lambda|)|}{|u_\lambda|} \leqslant C|x|^{-b}|u_\lambda|^{p-1}.$$

Consequently, since $|u_\lambda|_{L^\infty(\mathbb{R}^N)}$ is bounded, it follows from the proof of Lemma 2.10 in [4] that the operators $C, D : H \to H^{-1}$ defined by

$$Cv = \partial_2 \tilde{f}\big(x, |u_\lambda|\big)v \quad \text{and} \quad Dv = \frac{\tilde{f}(x, |u_\lambda|)}{|u_\lambda|} v$$

are compact. Therefore, the following lemma is proven in the same way as Lemma 4.1 of [4].

Lemma 10. *Suppose that the hypotheses* (V1), (V2) *and* (r1)–(r3) *are satisfied and let* $\tilde f$ *be defined by* (7) *and* (9)*. Then for* $\lambda \in (0,\lambda^*)$ *the following holds.* $\widetilde H_\lambda - \mu\widetilde R : X \to X^*$ *is a Fredholm operator for all* $\mu < \min\{1,\lambda\}$*. Furthermore, for* $\lambda \in (0,\lambda^*)$ *and* $\xi < \min\{1,\lambda\}$, $S(\widetilde H_\lambda) \cap (-\infty,\xi)$ *contains only a finite number of points.*

Let us now explain how we generalize Lemma 4.2 of [4]. By analogy with the equivalence (4.9) in [4], for $\lambda = k^2$, we perform the change of variables (10), so that

$$\begin{aligned}&\begin{pmatrix}\mathcal{S}(k)u\\ \mathcal{S}(k)v\end{pmatrix} \in \ker(\widetilde H_{k^2} - \mu\widetilde R)\\ \iff\quad &\begin{pmatrix}u\\ v\end{pmatrix} \in \ker\begin{pmatrix} k^2\widetilde{\mathcal{A}}(k) - \mu(-k^2\Delta+1) & 0\\ 0 & k^2\widetilde{\mathcal{B}}(k) - \mu(-k^2\Delta+1)\end{pmatrix}\end{aligned} \tag{21}$$

where

$$\widetilde{\mathcal{A}}(k) = -\Delta + 1 - k^{-2}\partial_2\tilde f\big(y/k, k^\theta|v_k|\big), \qquad \widetilde{\mathcal{B}}(k) = -\Delta + 1 - k^{-(2+\theta)}\frac{\tilde f(y/k, k^\theta|v_k|)}{|v_k|}$$

for all $k \in (0,k^*)$, where $k^* = \sqrt{\lambda^*}$, and

$$\widetilde{\mathcal{A}}(0) = -\Delta + 1 - p|y|^{-b}|v_0|^{p-1}, \qquad \widetilde{\mathcal{B}}(0) = -\Delta + 1 - |y|^{-b}|v_0|^{p-1}.$$

These operators correspond to the operators $\widetilde A(k)$ and $\widetilde B(k)$ of [4]. Here $v_k \equiv v(k)$ denotes the function given by Proposition 5. We also define the operators corresponding to $M(k)$ and $L(k)$ in [4] by $\mathcal{L}(k), \mathcal{M}(k) : H \to H$,

$$\mathcal{L}(k) = R^{-1}\widetilde{\mathcal{A}}(k) \quad\text{and}\quad \mathcal{M}(k) = R^{-1}\widetilde{\mathcal{B}}(k), \quad k \in [0,k^*).$$

Since $\widetilde{\mathcal{A}}(0) = \widetilde A(0)$ and $\widetilde{\mathcal{B}}(0) = \widetilde B(0)$ the proof of parts (i) and (ii) of Lemma 4.2 of [4] remains unchanged. To prove that part (iii) and consequently part (iv) hold for the operators $\widetilde{\mathcal{A}}(k)$ and $\widetilde{\mathcal{B}}(k)$, we introduce the quadratic forms

$$\begin{aligned}\mathfrak{a}_k(u) &= \big\langle\widetilde{\mathcal{A}}(k)u, u\big\rangle_{H^{-1}\times H} = \big\langle\mathcal{L}(k)u, u\big\rangle_{H\times H} \quad \text{for } u \in H,\\ \mathfrak{b}_k(u) &= \big\langle\widetilde{\mathcal{B}}(k)u, u\big\rangle_{H^{-1}\times H} = \big\langle\mathcal{M}(k)u, u\big\rangle_{H\times H} \quad \text{for } u \in H.\end{aligned}$$

It is then enough to prove the following lemma to generalize parts (iii) and (iv) of Lemma 4.2 of [4].

Lemma 11.

(i) *Let* $\delta > 0$*. There exists* $k_1 \in (0,k^*)$ *such that, for all* $k \in (0,k_1]$,

$$\left|\frac{\mathfrak{a}_k(u) - \mathfrak{a}_0(u)}{\|u\|^2}\right| \leqslant \frac{\delta}{2} \quad \textit{for all } u \in H\setminus\{0\}.$$

(ii) *For all* $k \in (0,k^*)$*, there exists a compact linear operator* $C(k)$ *such that*

$$\mathfrak{a}_k(u) = \|u\|^2 - \big\langle C(k)u, u\big\rangle_{H^{-1}\times H} \quad \textit{for all } u \in H.$$

(iii) *Let $\delta > 0$. There exists $k_2 \in (0, k^*)$ such that, for all $k \in (0, k_2]$,*

$$\left| \frac{\mathfrak{b}_k(u) - \mathfrak{b}_0(u)}{\|u\|^2} \right| \leqslant \frac{\delta}{2} \quad \text{for all } u \in H \setminus \{0\}.$$

(iv) *For all $k \in (0, k^*)$, the function $k^{-(2+\theta)} \tilde{f}(x/k, k^\theta |v_k|)/|v_k|$ is bounded on compact subsets of $\mathbb{R}^N \setminus \{0\}$.*

Proof. (i) follows from Lemma 7(ii) since $\widetilde{\mathcal{A}}(k) = \widetilde{F}(k, v_k)$ and $v_k \to v_0$ as $k \to 0$.

(ii) Set $C(k)v = k^{-2} \partial_2 \tilde{f}(x/k, k^\theta |v_k|) v$ for $v \in H$. Since

$$\left| k^{-2} \partial_2 \tilde{f}\big(x/k, k^\theta |v_k|\big) \right| \leqslant C|x|^{-b} |v_k|^{p-1}$$

by (16) and (ψ4) and since v_k is bounded by Lemma 8, it follows from the proof of Lemma 2.10 in [4] that $C(k)$ is compact.

(iii) It suffices to prove that $\|\widetilde{\mathcal{B}}(k) - \widetilde{\mathcal{B}}(0)\|_{B(H,H^{-1})} \to 0$ as $k \to 0$. We introduce a function $T : \mathbb{R} \times H \to H^{-1}$ as follows. Set

$$\rho(x, k, s) = \begin{cases} \psi(x, k, s)/s & \text{for } s \neq 0, \\ 0 & \text{for } s = 0 \end{cases}$$

and define

$$T(k, u)v = \rho(x, k, u)v \quad \text{for all } (k, u) \in \mathbb{R} \times H, \ v \in H.$$

Note that, by (ψ1) and (ψ2), $\rho(x, \cdot, \cdot) \in C(\mathbb{R} \times \mathbb{R})$ for all $x \in \mathbb{R}^N \setminus \{0\}$. Now

$$\widetilde{\mathcal{B}}(k) = \widetilde{B}(k) + T\big(k, |v_k|\big)$$

where $\widetilde{B}(k)$ has been defined in [4]. Since $\widetilde{\mathcal{B}}(0) = \widetilde{B}(0)$, we already know that $\|\widetilde{B}(k) - \widetilde{\mathcal{B}}(0)\|_{B(H,H^{-1})} \to 0$ as $k \to 0$. Therefore, we only need to show that

$$\big\| T\big(k, |v_k|\big) \big\|_{B(H,H^{-1})} \to 0 \quad \text{as } k \to 0. \tag{22}$$

Since $\rho(x, 0, s) \equiv 0$ by the definition of ψ, we have that $T(0, v_0) = 0$. Hence, since $v_k \to v_0$ in H as $k \to 0$, (22) will hold if we prove that $T \in C(\mathbb{R} \times H, B(H, H^{-1}))$. The proof of this fact is analogous to that of (ii) in Lemma 7. Fix $(k, u) \in \mathbb{R} \times H$ and consider $(h, v) \in \mathbb{R} \times H$. We have

$$\big\| T(k, u) - T(h, v) \big\|_{B(H,H^{-1})} \leqslant \big\| T(k, u) - T(h, u) \big\|_{B(H,H^{-1})} + \big\| T(h, u) - T(h, v) \big\|_{B(H,H^{-1})}.$$

It follows from the properties of ψ that the function $P(\cdot, k, \cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by $P(x, k, s) = \int_0^s \rho(x, k, \sigma)\, d\sigma$ satisfies the hypotheses of Lemma 16 for all $k \in \mathbb{R}$. In particular, (ψ2) and (ψ4) imply that

$$\big| \partial_s P(x, k, s) - \partial_s P(x, k, t) \big| = \big| \rho(x, k, s) - \rho(x, k, t) \big| \leqslant C|x|^{-b} \big| |s|^{p-2}s - |t|^{p-2}t \big| \tag{23}$$

where $C > 0$ is independent of k. Therefore, by Lemma 16(ii), the functions $\{T(h, \cdot) : H \to B(H, H^{-1})\}_{h \in \mathbb{R}}$ are equicontinuous at u. This implies that $\|T(h, u) - T(h, v)\|_{B(H,H^{-1})} \to 0$ as $\|u - v\| \to 0$, uniformly for $h \in \mathbb{R}$.

On the other hand,

$$\big\| T(k, u) - T(h, u) \big\|_{B(H,H^{-1})} \leqslant \sup_{\varphi, \xi \in H \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\rho(x, k, u) - \rho(x, h, u)||\varphi||\xi|\, dx}{\|\varphi\| \|\xi\|}$$

where $\rho(x,k,u) - \rho(x,h,u) \to 0$ as $h \to k$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and

$$\left|\rho(x,k,u) - \rho(x,h,u)\right| \leqslant C|x|^{-b}|u|^{p-1}$$

by (23). Hence, $\|T(k,u) - T(h,u)\|_{B(H,H^{-1})} \to 0$ as $h \to k$ by Lemma 14.

(iv) By (15) and (ψ2) we have that

$$\left|k^{-(2+\theta)}\frac{\tilde{f}(x/k, k^{\theta}|v_k|)}{|v_k|}\right| \leqslant C|x|^{-b}|v_k|^{p-1}.$$

Since $v_k \in L^\infty(\mathbb{R}^N)$, this implies (iv). $\square$

Using Lemma 11, it is not difficult to adapt the proof of Lemma 4.2 of [4] and to prove

Lemma 12. (i) $\mathcal{L}(0): H \to H$ *is a self-adjoint isomorphism,* $\inf\sigma_{\mathrm{ess}}(\mathcal{L}(0)) \geqslant 1$ *and* $\gamma \equiv \inf\sigma(\mathcal{L}(0))$ *is a simple eigenvalue of* $\mathcal{L}(0)$ *with* $\ker\{\mathcal{L}(0) - \gamma I\} = \{\eta\}$ *and* $\gamma < 0$. *There exists* $\delta > 0$ *such that*

$$\mathfrak{a}_0(u) \geqslant \delta\|u\|^2 \quad \textit{for all } u \in \eta^{\perp} = \left\{u \in H\colon \langle\eta, u\rangle_{H\times H} = 0\right\}.$$

(ii) $\mathcal{M}(0): H \to H$ *is a bounded self-adjoint operator,* $\inf\sigma_{\mathrm{ess}}(\mathcal{M}(0)) \geqslant 1$ *and* $0 = \inf\sigma(\mathcal{M}(0))$ *is a simple eigenvalue of* $\mathcal{M}(0)$ *with* $\ker\mathcal{M}(0) = \{\psi\}$. *There exists* $\delta > 0$ *such that*

$$\mathfrak{b}_0(u) \geqslant \delta\|u\|^2 \quad \textit{for all } u \in \psi^{\perp} = \left\{u \in H\colon \langle\psi, u\rangle_{H\times H} = 0\right\}.$$

(iii) *There exist* $k_1 \in (0, k^*)$ *and* $\gamma(k) \equiv \gamma_k \in (-\infty, 0)$ *such that, for all* $k \in (0, k_1]$,

$$\left\{\mu \in \mathbb{R}\colon \ker\left[\widetilde{\mathcal{A}}(k) - \mu\left(-k^2\Delta + 1\right)\right] \neq \{0\}\right\} \cap (-\infty, 0] = \{\gamma_k\}$$

and

$$\left\{\mu \in \mathbb{R}\colon \ker\left[\widetilde{\mathcal{B}}(k) - \mu\left(-k^2\Delta + 1\right)\right] \neq \{0\}\right\} \cap (-\infty, 0] = \{0\}.$$

Furthermore, there exists $w_k \in H$ *such that* $\ker[\widetilde{\mathcal{A}}(k) - \gamma_k(-k^2\Delta + 1)] = \operatorname{span}\{w_k\}$, *whereas* $\ker\widetilde{\mathcal{B}}(k) = \operatorname{span}\{v_k\}$ *where* v_k *is given by Proposition* 5. *We also have that* $v_k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *with* $v_k(x) > 0$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$.

(iv) *For all* $k \in (0, k_1]$, $S(\widetilde{H}_{k^2}) \cap (-\infty, 0] = \{k^2\gamma_k, 0\}$ *and we have that*

$$\ker\left(\widetilde{H}_{k^2} - k^2\gamma_k\widetilde{R}\right) = \operatorname{span}\begin{pmatrix}\mathcal{S}(k)w_k\\ 0\end{pmatrix} \quad \textit{and} \quad \ker\widetilde{H}_{k^2} = \operatorname{span}\begin{pmatrix}0\\ \mathcal{S}(k)v_k\end{pmatrix}.$$

Remark. Part (iii) of Lemma 12 yields the positivity of the solutions v_k and consequently that of $u(\lambda)$, for $\lambda > 0$ small enough, which completes the proof of Theorem 1(a).

The discussion of the spectral properties of $\widetilde{H}$ is closed by

Lemma 13. *There exists* $\lambda_1 > 0$ *such that, for all* $\lambda \in (0, \lambda_1)$, *the conditions* (H1)–(H3) *are satisfied.*

Proof. Let $\lambda_1 = k_1$ and then set $a_\lambda = \lambda\gamma(\lambda^{1/2})$ for $\lambda \in (0, \lambda_1]$. Recalling that $u(\lambda) = S(\lambda^{1/2})v(\lambda^{1/2})$ we see that Lemmas 10 and 12 yield the result. $\square$

5.2. *The stability criterion*

For all $k \in (0, k_1)$, we have by (11) that

$$\left|u\big(k^2\big)\right|_{L^2}^2 = \left|\mathcal{S}(k)v_k\right|_{L^2}^2 = k^{\varrho}|v_k|_{L^2}^2$$

where $\varrho = [4-2b-N(p-1)]/(p-1)$. Hence

$$\begin{aligned}\frac{d}{dk}\left|u\big(k^2\big)\right|_{L^2}^2 &= \varrho k^{\varrho-1}|v_k|_{L^2}^2 + k^{\varrho}2\left\langle v_k, \frac{d}{dk}v_k\right\rangle_{L^2} \\ &= k^{\varrho-1}\left\{\varrho|v_k|_{L^2}^2 + 2k\left\langle v_k, \frac{d}{dk}v_k\right\rangle_{L^2}\right\}. \end{aligned} \tag{24}$$

Since $|v_k|_{L^2} \to |v_0|_{L^2} = |\psi|_{L^2} > 0$ as $k \to 0$, the right-hand side of (24) will have the same sign as ϱ for k small enough if we prove that $k\langle v_k, \frac{d}{dk}v_k\rangle_{L^2} \to 0$ as $k \to 0$. To prove this is precisely to generalize Lemma 4.4 of [4] to the present situation. Replacing F by $\widetilde{F} = F + S$, where S is defined in the proof of Lemma 7, and following the proof of Lemma 4.4 in [4], one only has to show that

$$\eta_k \equiv kD_kS(k, v_k) = k\partial_k\psi(x, k, v_k) \to 0 \quad \text{in } H^{-1} \text{ as } k \to 0.$$

We have

$$\begin{aligned}\|\eta_k\|_{H^{-1}} &\leqslant \sup_{\varphi\in H\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|k\partial_k\psi(x,k,v_k)||\varphi|\,dx}{\|\varphi\|} \\ &\leqslant \sup_{\varphi\in H\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|k\partial_k\psi(x,k,v_k) - k\partial_k\psi(x,h,v_0)||\varphi|\,dx}{\|\varphi\|} \\ &\quad + \sup_{\varphi\in H\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|k\partial_k\psi(x,h,v_0)||\varphi|\,dx}{\|\varphi\|}. \end{aligned} \tag{25}$$

By (ψ6),

$$\left|k\partial_k\psi(x,k,v_k) - k\partial_k\psi(x,h,v_0)\right| \leqslant C|x|^{-b}\left|v_k^p - v_0^p\right|$$

and the first term in the right-hand side of (25) is proved to vanish as $k \to 0$ as in the proof of Lemma 4.4 of [4].

On the other hand, we have that $k\partial_k\psi(x,h,v_0) \to 0$ as $k \to 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and that

$$\left|k\partial_k\psi(x,h,v_0)\right| \leqslant C|x|^{-b}v_0^p$$

by (ψ9) and (ψ6) respectively. Hence the second term in the right-hand side of (25) vanishes as $k \to 0$ as well by Lemma 15.

We are now in a position to prove Theorem 1(b).

5.3. *Proof of Theorem 1, part (b)*

We refer to Section 5.2 and note that

$\varrho > 0$ if and only if $p < 1 + \frac{4-2b}{N}$ and
$\varrho < 0$ if and only if $p > 1 + \frac{4-2b}{N}$.

It follows by (24) that there exists $\bar{k} \in (0, k^*)$ such that, for all $k \in (0, \bar{k})$,

$$\frac{d}{d\lambda}\left|u(\lambda)\right|_{L^2}^2 = \frac{dk}{d\lambda}\frac{d}{dk}\left|u\left(k^2\right)\right|_{L^2}^2 = \frac{1}{2k}\frac{d}{dk}\left|u\left(k^2\right)\right|_{L^2}^2$$

has the same sign as ϱ for $p \neq 1 + \frac{4-2b}{N}$. By Lemma 13, we can choose $\bar{k}$ so that the spectral hypotheses (H1)–(H3) are satisfied on the interval $(0, \bar{\lambda})$ where $\bar{\lambda} = \bar{k}^2$. Hence Theorem 1(b) follows from Theorem 3 of [5]. $\square$

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Appendix A. Technical lemmas

The following technical lemmas are adaptations of Lemma A.1(ii) and Lemma B.1 of [4]. Since the proofs use arguments very similar to those in [4], we will only sketch them. Lemmas 14 and 15 are used several times together with dominated convergence. Lemma 16 is useful to prove continuity and differentiability properties of some functions.

Lemma 14. *Suppose that $N \geqslant 3$, $b \in (0, 2)$ and $1 < p < 1 + (4 - 2b)/(N - 2)$. For $s \geqslant 0$, let $q_s : \mathbb{R}^N \to \mathbb{R}$ be such that*:

(a) *There exist a constant $C > 0$ and $v \in H$ such that, for $s \geqslant 0$,*

$$\left|q_s(x)\right| \leqslant C|x|^{-b}|v|^{p-1} \quad \text{for almost every } x \in \mathbb{R}^N.$$

(b) *$q_s(x) \to 0$ as $s \to 0$ for almost every $x \in \mathbb{R}^N$.*

Then

$$\sup_{\varphi,\xi \in H\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |q_s(x)||\varphi||\xi|\,dx}{\|\varphi\|\|\xi\|} \to 0 \quad \text{as } s \to 0.$$

Proof. Note that, for $\varphi, \xi \in H$,

$$\int_{\mathbb{R}^N} \left|q_s(x)\right||\varphi||\xi|\,dx \leqslant C \int_{\mathbb{R}^N} |x|^{-b}|v|^{p-1}|\varphi||\xi|\,dx.$$

Comparing the right-hand side of this inequality with the left-hand side of (A.1) in Lemma A.1 of [4] with $w = 0$, Lemma 14 is proven using the same arguments as those used to prove (A.3) in [4]. $\square$

Lemma 15 below follows by similar arguments.

Lemma 15. *Suppose that $N \geqslant 3$, $b \in (0, 2)$ and $1 < p < 1 + (4 - 2b)/(N - 2)$. For $s \geqslant 0$, let $r_s : \mathbb{R}^N \to \mathbb{R}$ be such that*:

(a) *There exist a constant $C > 0$ and $v \in H$ such that, for $s \geqslant 0$,*

$$\left|r_s(x)\right| \leqslant C|x|^{-b}|v|^{p} \quad \text{for almost every } x \in \mathbb{R}^N.$$

(b) *$r_s(x) \to 0$ as $s \to 0$ for almost every $x \in \mathbb{R}^N$.*

Then

$$\sup_{\varphi \in H \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |r_s(x)||\varphi|\,dx}{\|\varphi\|} \to 0 \quad \text{as } s \to 0.$$

Lemma 16. *Suppose that $N \geqslant 3$, $b \in (0,2)$ and $1 < p < 1 + (4-2b)/(N-2)$. Let $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $\psi(x,0) = 0$ for almost every $x \in \mathbb{R}^N$. In addition, suppose that*:

(a) *$\psi(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \mathbb{R}^N$.*
(b) *There exists a constant $C_0 > 0$ such that*

$$\left|\partial_s \psi(x,s) - \partial_s \psi(x,t)\right| \leqslant C_0 |x|^{-b} \left| |s|^{p-2}s - |t|^{p-2}t \right|$$

for all $s, t \in \mathbb{R}$ and almost every $x \in \mathbb{R}^N$.

For $u \in H$, set $\Psi(u) = \psi(x,u)$. Then the following statements hold.

(i) *$\Psi \in C(H, H^{-1})$ and*

$$\left\|\Psi(u)\right\|_{H^{-1}} \leqslant C_1 \|u\|^p \quad \text{for all } u \in H,$$

where $C_1 > 0$ depends only on N, b, p and C_0.
(ii) *$\Psi \in C^1(H, H^{-1})$ with $\Psi'(u) : H \to H^{-1}$ is given by*

$$\left\langle \Psi'(u)v, w \right\rangle_{H^{-1} \times H} = \int_{\mathbb{R}^N} \partial_s \psi(x,u) v w \, dx.$$

Moreover,

$$\left\|\Psi'(u)\right\|_{B(H,H^{-1})} \leqslant C_2 \|u\|^{p-1} \quad \text{for all } u \in H,$$

where $C_2 > 0$ depends only on N, b, p and C_0.

Proof. First note that (a) and (b) imply

$$\left|\psi(x,s) - \psi(x,t)\right| \leqslant C|x|^{-b} \left| |s|^{p-1}s - |t|^{p-1}t \right|$$

for almost every $x \in \mathbb{R}^N$, for all $s, t \in \mathbb{R}$. Now (i) follows easily from the proof of Lemma B.1(i) in [4].

In the same spirit as Lemma B.1 in [4], let $\Theta(u) : H \times H \to \mathbb{R}$ be the bounded, symmetric bilinear form defined by

$$\Theta(u)[v,w] = \int_{\mathbb{R}^N} \partial_s \psi(x,u) v w \, dx.$$

(The fact that $\Theta(u)$ is bounded follows by assumption (b) and the estimate (A.2) in Lemma A.1 of [4].) Then, as in part (ii) of Lemma B.1 of [4], there exists a unique operator $B(u) \in B(H, H^{-1})$ such that

$$\left\langle B(u)v, w \right\rangle_{H^{-1} \times H} = \Theta(u)[v,w] \quad \text{for all } v, w \in H.$$

Furthermore, there is a constant $C_2 > 0$, depending only on N, b, p and C_0, such that

$$\|B(u)\|_{B(H,H^{-1})} \leqslant C_2 \|u\|^{p-1} \quad \text{for all } u \in H.$$

It is not difficult then to follow the proof of Lemma B.1(iii) in [4] and, using the hypotheses (a) and (b), to prove that $\Psi \in C^1(H, H^{-1})$ and that $\Psi'(u) = B(u)$ for all $u \in H$. $\square$

References

[1] A. de Bouard, R. Fukuizumi, Stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, Ann. Henri Poincaré 6 (6) (2005) 1157–1177.
[2] T. Cazenave, Semilinear Schrödinger Equations, Courant Lect. Notes Math., Amer. Math. Soc., 2003.
[3] R. Fukuizumi, M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, J. Math. Kyoto Univ. 45 (1) (2005) 145–158.
[4] F. Genoud, C.A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: Existence and stability of standing waves, Discrete Contin. Dyn. Syst. 21 (1) (2008) 137–186.
[5] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry I, J. Funct. Anal. 74 (1) (1987) 160–197.
[6] L. Jeanjean, S. Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations, Adv. Differential Equations 11 (7) (2006) 813–840.
[7] C.A. Stuart, Lectures on the orbital stability of standing waves and application to the nonlinear Schrödinger equation, Milan J. Math., in press.