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Global well-posedness for a fifth-order shallow water equation in Sobolev spaces ☆

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ABSTRACT

The Cauchy problem of a fifth-order shallow water equation

$$\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u - \partial_x^5 u = 0$$

is shown to be globally well-posed in Sobolev spaces $H^s(\mathbf{R})$ for $s > (6\sqrt{10} - 17)/4$. The proof relies on the I-method developed by Colliander, Keel, Staffilani, Takaoka and Tao. For this equation lacks scaling invariance, we reconsider the local result and pay special attention to the relationship between the lifespan of the local solution and the initial data. We prove the almost conservation law, and combine it with the local result to obtain the global well-posedness.

1. Introduction

We consider the global well-posedness for the Cauchy problem of a fifth-order shallow water equation

$$\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u - \partial_x^5 u = 0, \quad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

$$u(\cdot, 0) = u_0(\cdot) \in H^s(\mathbf{R}). \tag{1.2}$$

Eq. (1.1) is a higher-order modification of the following Camassa–Holm equation

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$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{1.3}$$

We may observe more precisely the connection between (1.1) and (1.3) by writing them into the following forms

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x u^2 + \left(1 - \partial_x^2\right)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0 \tag{1.4}$$

and

$$\partial_t u + \frac{1}{2}\partial_x u^2 + \left(1 - \partial_x^2\right)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0, \tag{1.5}$$

respectively.

The well-posedness for the initial value problem of Eq. (1.1) in Sobolev spaces has been investigated by several authors. For the periodic case, Himonas and Misiolek [12,13] proved the initial value problem of Eq. (1.1) is locally well-posed in $H^s(\mathbf{T})$ for $s > 1/2$, and Gorsky [9] proved a similar result for $s = 1/2$ under the restriction of small initial data, where $\mathbf{T}$ is the one-dimensional torus. For the non-periodic case, Himonas and Misiolek [14] proved that the initial value problem of Eq. (1.1) is locally well-posed in $H^s(\mathbf{R})$ for $s > 1/2$. Using the bilinear estimate method initiated by Bourgain [1], Byers [2] improved their result and proved the local well-posedness in $H^s(\mathbf{R})$ for $s > 1/4$. For both periodic and non-periodic cases, global well-posedness for Eq. (1.1) in H^1 follows from the local results and the H^1 energy conservation law. Recently, Wang and Cui [17] proved the global well-posedness for (1.1)–(1.2) in $H^s(\mathbf{R})$ for $s > (5\sqrt{7} - 10)/4$ by using the I-method, which is initially developed by Colliander, Keel, Staffilani, Takaoka and Tao [4–7]. Eq. (1.3) has been intensively studied by many authors, for instance, see [3,8,16,18] and references therein.

In this paper we use the I-method to prove the global well-posedness for (1.1)–(1.2) in $H^s(\mathbf{R})$ for $s > (6\sqrt{10} - 17)/4$ and thus improve the result in [17]. We reconsider the local well-posedness result and pay special attention to the relationship between the lifespan of the local solution and the initial data, which is important for equations that lack scaling invariance in the iteration process. This is done by proving the bilinear estimates involving an operator I, which will be defined below. Moreover, we prove the almost conservation law, which implies that the modified energy increases slowly and is used in the iteration process. However, there is still a gap in the regularity index between the condition $s > 1/4$ for the local well-posedness as far as we know and the condition $s > (6\sqrt{10} - 17)/4$ for the global well-posedness in this paper.

First we introduce some notations. We denote the Fourier transform in x and t of u by $\hat{u}$ or $\mathscr{F}u$, and in x by $\mathscr{F}_x u$. D_x^θ denotes the Fourier multiplier operator with symbol $|\xi|^\theta$. $\{W(t) = \exp(-t\partial_x^3)\}_{t\in\mathbf{R}}$ is the free Airy group defined by $\mathscr{F}_x(W(t)u_0)(\xi) = \exp(it\xi^3)\hat{u}_0(\xi)$. The notation $a+$ and $a-$ denote respectively $a + \varepsilon$ and $a - \varepsilon$ for an arbitrary small positive number ε. For any positive A and B, $A \sim B$ means that there exists a generic constant C such that $A \leqslant CB$ and $B \leqslant CA$. Denote $\langle\cdot\rangle = 1 + |\cdot|$, $\sigma = \tau - \xi^3$, $\sigma_j = \tau_j - \xi_j^3$, $j = 1, 2, 3$.

Let $\psi \in C_c^\infty(\mathbf{R})$ be a radially decreasing function with $\psi \equiv 1$ on $[0, 1]$ and $\operatorname{supp}\psi \subseteq [-1, 2]$, and $\psi_\delta(t) = \psi(t/\delta)$, $\delta > 0$.

Definition 1.1. For $s, b \in \mathbf{R}$, we define the space $X_{s,b}$ to be the completion of the Schwartz space $\mathscr{S}(\mathbf{R}^2)$ with respect to the norm

$$\|u\|_{X_{s,b}} = \left\| \langle\sigma\rangle^b \langle\xi\rangle^s \hat{u}(\xi, \tau)\right\|_{L_\xi^2 L_\tau^2}.$$

For any given $\delta > 0$, we define the function space $X_{s,b}^\delta$ to be the restriction of $X_{s,b}$ on $\mathbf{R} \times [0, \delta]$ with norm

$$\|u\|_{X^\delta_{s,b}} = \inf\big\{\|U\|_{X_{s,b}}\colon U \in X_{s,b},\ U|_{\mathbf{R}\times[0,\delta]} = u\big\}.$$

From Lemma 1.6 in [10], for $u \in X^\delta_{s,b}$, there exists an extension $\tilde{u} \in X_{s,b}$, i.e. $\tilde{u} = u$ on $[0,\delta]$, and for $s' \leqslant s$,

$$\|\tilde{u}\|_{X_{s',b}} = \|u\|_{X^\delta_{s',b}}. \tag{1.6}$$

Let $s < 1$ and $N \gg 1$ be fixed. We define the Fourier multiplier operator $I : H^s(\mathbf{R}) \to H^1(\mathbf{R})$ by $\mathscr{F}_x(Iu)(\xi) = m(\xi)\mathscr{F}_x u(\xi)$, where $m(\xi)$ is a smooth, radially monotone, even function satisfying

$$m(\xi) = \begin{cases} 1, & |\xi| \leqslant N; \\ (\frac{|\xi|}{N})^{s-1}, & |\xi| \geqslant 2N. \end{cases}$$

Then there exists a constant C independent of N such that

$$\|u\|_{H^s} \leqslant \|Iu\|_{H^1} \leqslant CN^{1-s}\|u\|_{H^s}; \tag{1.7}$$

$$\|u\|_{X_{s,b}} \leqslant \|Iu\|_{X_{1,b}} \leqslant CN^{1-s}\|u\|_{X_{s,b}}. \tag{1.8}$$

The main result of this paper is as follows.

Theorem 1.1. *The Cauchy problem of Eq.* (1.1) *is globally well-posed in* $H^s(\mathbf{R})$ *for* $s > (6\sqrt{10} - 17)/4$. *More precisely, for any given* $T \geqslant 1$, (1.1)–(1.2) *possesses a unique solution* $u \in C([0,T]; H^s(\mathbf{R}))$, *and the mapping* $u_0 \to u$ *belongs to* $C(H^s(\mathbf{R}); C([0,T]; H^s(\mathbf{R})))$. *Moreover, the solution u satisfies*

$$\sup_{t\in[0,T]} \big\|u(t)\big\|_{H^s} \leqslant CT^{\frac{5-\sqrt{10}}{5}\gamma(1-s)}\|u_0\|_{H^s}^{1+\gamma(1-s)},$$

where $\gamma = [(s - (6\sqrt{10} - 17)/4)^{-1}]+$.

This paper is arranged as follows. In Section 2, we establish linear and bilinear estimates, which will be used to prove the local well-posedness. In Section 3, we present the local result, giving the relationship between the lifespan of the local solution and the initial data. In Section 4, we prove the almost conservation law. In Section 5, we show the global well-posedness by an iteration process, and thus complete the proof of Theorem 1.1.

2. Linear and bilinear estimates

In this section, we establish linear and bilinear estimates, which will be used in the next section. In particular, we consider bilinear estimates involving the operator I.

First we recall some preliminary estimates. The following inequalities were established in [15]:

$$\|u\|_{L^4_x L^4_t} \leqslant C\|u\|_{X_{0,\frac{1}{3}+}}; \tag{2.1}$$

$$\|u\|_{L^4_x L^6_t} \leqslant C\|u\|_{X_{0,\frac{5}{12}+}}; \tag{2.2}$$

$$\big\|D_x^{\frac{1}{4}} u\big\|_{L^4_x L^3_t} \leqslant C\|u\|_{X_{0,\frac{1}{3}+}}. \tag{2.3}$$

From [10] or [11], we have

$$\left\| D_x^{\frac{1}{2}} I_-^{\frac{1}{2}}(u_1,u_2)\right\|_{L_x^2 L_t^2} \leqslant C\|u_1\|_{X_{0,\frac{1}{2}+}}\|u_2\|_{X_{0,\frac{1}{2}+}}, \tag{2.4}$$

where the operator I_-^{α} defined as

$$\mathscr{F}\big(I_-^{\alpha}(u_1,u_2)\big)(\xi,\tau) = \int\limits_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} |\xi_1-\xi_2|^{\alpha}\hat{u}_1(\xi_1,\tau_1)\hat{u}_2(\xi_2,\tau_2)\,d\xi_1\,d\tau_1.$$

Let $s\in\mathbf{R}$, $-1/2<b<b'\leqslant 0$ or $0\leqslant b<b'<1/2$, $\delta\in(0,1)$. Then

$$\|\psi_\delta u\|_{X_{s,b}} \leqslant C\delta^{b'-b}\|u\|_{X_{s,b'}}; \tag{2.5}$$

$$\|u\|_{X^\delta_{s,b}} \leqslant C\delta^{b'-b}\|u\|_{X^\delta_{s,b'}}. \tag{2.6}$$

For the proof of (2.5) we refer readers to [10], and (2.6) follows from (2.5).

From the linear estimates in [15], we can easily obtain their variant version.

Theorem 2.1. *Let $s\in\mathbf{R}$, $b>1/2$. Then the following inequalities hold*

$$\left\|W(t)u_0\right\|_{X^\delta_{s,b}} \leqslant C\|u_0\|_{H^s}; \tag{2.7}$$

$$\left\|\int_0^t W\big(t-t'\big)u\big(t'\big)\,dt'\right\|_{X^\delta_{s,b}} \leqslant C\|u\|_{X^\delta_{s,b-1}}. \tag{2.8}$$

Now we turn to establish bilinear estimates.

Theorem 2.2. *Assume $3/8<s<1$, $1/2<b\leqslant\min\{1,s+1/8\}$ and $0<\delta<1$, then it holds that*

$$\left\|I\big(1-\partial_x^2\big)^{-1}\partial_x(\partial_x u_1\partial_x u_2)\right\|_{X^\delta_{1,b-1}} \leqslant C\big(\delta^{1-b}+\delta^{\alpha-}N^{\beta}\big)\|Iu_1\|_{X^\delta_{1,b}}\|Iu_2\|_{X^\delta_{1,b}}, \tag{2.9}$$

where $\alpha=\min\{5/4-b,3/8+s-b\}$, $\beta=\min\{-1/4,3s-23/8\}$.

Proof. By duality and Plancherel identity, it suffices to show that for all $u\in X_{0,1-b}$, $u_j\in X^\delta_{0,b}$, $j=1,2$,

$$\begin{aligned}\Upsilon &\equiv \int\limits_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{m(\xi)}{m(\xi_1)m(\xi_2)}\frac{|\xi||\xi_1||\xi_2|}{\langle\xi\rangle\langle\xi_1\rangle\langle\xi_2\rangle}\big|\widehat{\psi_\delta u}(\xi,\tau)\big|\big|\hat{\tilde{u}}_1(\xi_1,\tau_1)\big|\big|\hat{\tilde{u}}_2(\xi_2,\tau_2)\big|\,d\xi\,d\tau\,d\xi_1\,d\tau_1\\ &\leqslant C\big(\delta^{1-b}+\delta^{\alpha-}N^{\beta}\big)\|u\|_{X_{0,1-b}}\|\tilde{u}_1\|_{X_{0,b}}\|\tilde{u}_2\|_{X_{0,b}},\end{aligned} \tag{2.10}$$

where $\tilde{u}_j$ is an extension of u_j. Without loss of generality, we may assume that $\widehat{\psi_\delta u}$, $\hat{\tilde{u}}_j$ are nonnegative. For simplicity of notations, we drop the $\sim$ over u_j in the following.

By symmetry of ξ_1 and ξ_2, we need only to control the integral in the domain

$$D=\big\{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2):\ \xi=\xi_1+\xi_2,\ \tau=\tau_1+\tau_2,\ |\xi_1|\leqslant|\xi_2|\big\}.$$

In order to estimate the integral, we split the domain of integration D into five regions:

$$D_1 = \{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2) \in D\colon |\xi_2| \leqslant 4N\};$$
$$D_2 = \{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2) \in D\colon |\xi_2| > 4N,\ |\xi_1| \ll |\xi_2|,\ |\xi_1| \leqslant 2N\};$$
$$D_3 = \{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2) \in D\colon |\xi_2| > 4N,\ |\xi_1| \ll |\xi_2|,\ |\xi_1| > 2N\};$$
$$D_4 = \{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2) \in D\colon |\xi_2| > 4N,\ |\xi_1| \sim |\xi_2|,\ |\xi| \leqslant 2N\};$$
$$D_5 = \{(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2) \in D\colon |\xi_2| > 4N,\ |\xi_1| \sim |\xi_2|,\ |\xi| > 2N\}.$$

We denote the integral restricted to these regions in their appearing order by Υ_1, Υ_2, Υ_3, Υ_4, Υ_5, respectively.

1. Estimate of Υ_1. By using Plancherel identity, (2.1), (2.5) and (2.6), we have

$$\begin{aligned}
|\Upsilon_1| &\leqslant C\int_{D_1} \widehat{\psi_\delta u}\,\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1 \\
&\leqslant C\|\psi_\delta u\|_{L_x^2L_t^2}\|u_1\|_{L_x^4L_t^4}\|u_2\|_{L_x^4L_t^4} \\
&\leqslant C\|\psi_\delta u\|_{X_{0,0}}\|u_1\|_{X_{0,\frac{1}{3}+}}\|u_2\|_{X_{0,\frac{1}{3}+}} \\
&\leqslant C\delta^{1-b}\|u\|_{X_{0,1-b}}\cdot\delta^{\frac{1}{6}-}\|u_1\|_{X_{0,\frac{1}{2}}}\cdot\delta^{\frac{1}{6}-}\|u_2\|_{X_{0,\frac{1}{2}}} \\
&\leqslant C\delta^{(\frac{4}{3}-b)-}\|u\|_{X_{0,1-b}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}.
\end{aligned}$$

2. Estimate of Υ_2. Noticing that $|\xi| \sim |\xi_2|$, and using (2.1), (2.5), (2.6), we have

$$\begin{aligned}
|\Upsilon_2| &\leqslant C\int_{D_2}\left(\frac{|\xi|}{N}\right)^{s-1}\left(\frac{|\xi_2|}{N}\right)^{1-s}\widehat{\psi_\delta u}\,\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1 \\
&\leqslant C\|\psi_\delta u\|_{L_x^2L_t^2}\|u_1\|_{L_x^4L_t^4}\|u_2\|_{L_x^4L_t^4} \\
&\leqslant C\delta^{(\frac{4}{3}-b)-}\|u\|_{X_{0,1-b}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}.
\end{aligned}$$

3. Estimate of Υ_3. Noticing $|\xi_1 \pm \xi_2| \sim |\xi_2|$, and using (2.4), (2.5), we have

$$\begin{aligned}
|\Upsilon_3| &\leqslant C\int_{D_3}\left(\frac{|\xi|}{N}\right)^{s-1}\left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s}\frac{|\xi||\xi_1||\xi_2|}{\langle\xi\rangle\langle\xi_1\rangle\langle\xi_2\rangle}\widehat{\psi_\delta u}\,\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1 \\
&\leqslant CN^{s-1}\int_{D_3}\frac{|\xi_1|^{1-s}}{|\xi|^{1-s}|\xi_2|^s}\widehat{\psi_\delta u}|\xi_1+\xi_2|^{\frac{1}{2}}|\xi_1-\xi_2|^{\frac{1}{2}}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1 \\
&\leqslant CN^{-1}\|\psi_\delta u\|_{L_x^2L_t^2}\left\|D_x^{\frac{1}{2}}I_-^{\frac{1}{2}}(u_1,u_2)\right\|_{L_x^2L_t^2} \\
&\leqslant CN^{-1}\delta^{1-b}\|u\|_{X_{0,1-b}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}.
\end{aligned}$$

4. Estimate of Υ_4.

$$|\Upsilon_4| \leqslant C\int_{D_4}\left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s}\frac{|\xi||\xi_1||\xi_2|}{\langle\xi\rangle\langle\xi_1\rangle\langle\xi_2\rangle}\widehat{\psi_\delta u}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

$$\leqslant CN^{2(s-1)}\int_{D_4}\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}|\xi|}{\langle\xi\rangle}\widehat{\psi_\delta u}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1.$$

Next we deal with the cases $7/8 \leqslant s < 1$ and $3/8 < s < 7/8$ in different ways.
(1) $7/8 \leqslant s < 1$. By using (2.2), (2.3), (2.5) and (2.6), we have

$$|\Upsilon_4| \leqslant CN^{-\frac{1}{4}}\int_{D_4}\widehat{\psi_\delta u}\hat{u}_1|\xi_2|^{\frac{1}{4}}\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

$$\leqslant CN^{-\frac{1}{4}}\|\psi_\delta u\|_{L_x^2L_t^2}\|u_1\|_{L_x^4L_t^6}\left\|D_x^{\frac{1}{4}}u_2\right\|_{L_x^4L_t^3}$$

$$\leqslant CN^{-\frac{1}{4}}\|\psi_\delta u\|_{X_{0,0}}\|u_1\|_{X_{0,\frac{5}{12}+}}\|u_2\|_{X_{0,\frac{1}{3}+}}$$

$$\leqslant CN^{-\frac{1}{4}}\delta^{1-b}\|u\|_{X_{0,1-b}}\cdot\delta^{\frac{1}{12}-}\|u_1\|_{X_{0,\frac{1}{2}}}\cdot\delta^{\frac{1}{6}-}\|u_2\|_{X_{0,\frac{1}{2}}}$$

$$\leqslant CN^{-\frac{1}{4}}\delta^{(\frac{5}{4}-b)-}\|u\|_{X_{0,1-b}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}.$$

(2) $3/8 < s < 7/8$. Since $\sigma - \sigma_1 - \sigma_2 = 3\xi\xi_1\xi_2$, one of the following cases always occurs:

$$\text{(a) } |\sigma| \geqslant |\xi\xi_1\xi_2|; \qquad \text{(b) } |\sigma_1| \geqslant |\xi\xi_1\xi_2|; \qquad \text{(c) } |\sigma_2| \geqslant |\xi\xi_1\xi_2|. \tag{2.11}$$

We consider the three cases separately. If (a) holds, for $s < 7/8$ and $7/8 - s \leqslant 1 - b$, the integral Υ_4 restricted to this domain is bounded by

$$CN^{2(s-1)}\int_{D_4}\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}|\xi|}{\langle\xi\rangle|\xi|^{\frac{7}{8}-s}|\xi_1|^{\frac{7}{8}-s}|\xi_2|^{\frac{7}{8}-s}}\langle\sigma\rangle^{\frac{7}{8}-s}\widehat{\psi_\delta u}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

$$\leqslant CN^{2(s-1)}\int_{D_4}\langle\sigma\rangle^{\frac{7}{8}-s}\widehat{\psi_\delta u}\hat{u}_1|\xi_2|^{\frac{1}{4}}\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

$$\leqslant CN^{2(s-1)}\left\|\langle\sigma\rangle^{\frac{7}{8}-s}\widehat{\psi_\delta u}\right\|_{L_\xi^2L_\tau^2}\|u_1\|_{L_x^4L_t^6}\left\|D_x^{\frac{1}{4}}u_2\right\|_{L_x^4L_t^3}$$

$$\leqslant CN^{2(s-1)}\|\psi_\delta u\|_{X_{0,\frac{7}{8}-s}}\|u_1\|_{X_{0,\frac{5}{12}+}}\|u_2\|_{X_{0,\frac{1}{3}+}}$$

$$\leqslant CN^{2(s-1)}\delta^{(\frac{3}{8}+s-b)-}\|u\|_{X_{0,1-b}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a).
5. Estimate of Υ_5.

$$|\Upsilon_5| \leqslant C\int_{D_5}\left(\frac{|\xi|}{N}\right)^{s-1}\left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s}\widehat{\psi_\delta u}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

$$\leqslant CN^{s-1}\int_{D_5}\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}}{|\xi|^{1-s}}\widehat{\psi_\delta u}\hat{u}_1\hat{u}_2\,d\xi\,d\tau\,d\xi_1\,d\tau_1.$$

Similarly, we deal with the cases $7/8 \leqslant s < 1$ and $3/8 < s < 7/8$ in different ways.

(1) $7/8 \leqslant s < 1$. We have

$$\begin{aligned}
|\Upsilon_5| &\leqslant CN^{-\frac{1}{4}} \int\limits_{D_5} \widehat{\psi_\delta u} \hat{u}_1 |\xi_2|^{\frac{1}{4}} \hat{u}_2 \, d\xi \, d\tau \, d\xi_1 \, d\tau_1 \\
&\leqslant CN^{-\frac{1}{4}} \|\psi_\delta u\|_{L^2_x L^2_t} \|u_1\|_{L^4_x L^6_t} \big\| D_x^{\frac{1}{4}} u_2 \big\|_{L^4_x L^3_t} \\
&\leqslant CN^{-\frac{1}{4}} \delta^{(\frac{5}{4}-b)-} \|u\|_{X_{0,1-b}} \|u_1\|_{X_{0,b}} \|u_2\|_{X_{0,b}}.
\end{aligned}$$

(2) $3/8 < s < 7/8$. We consider the three cases (a), (b), (c) in (2.11) separately. If (a) holds, the integral Υ_5 restricted to this domain is bounded by

$$\begin{aligned}
&CN^{s-1} \int\limits_{D_5} \frac{|\xi_1|^{1-s}|\xi_2|^{1-s}}{|\xi|^{1-s}|\xi|^{\frac{7}{8}-s}|\xi_1|^{\frac{7}{8}-s}|\xi_2|^{\frac{7}{8}-s}} \langle\sigma\rangle^{\frac{7}{8}-s} \widehat{\psi_\delta u} \hat{u}_1 \hat{u}_2 \, d\xi \, d\tau \, d\xi_1 \, d\tau_1 \\
&\leqslant CN^{3s-\frac{23}{8}} \int\limits_{D_5} \langle\sigma\rangle^{\frac{7}{8}-s} \widehat{\psi_\delta u} \hat{u}_1 |\xi_2|^{\frac{1}{4}} \hat{u}_2 \, d\xi \, d\tau \, d\xi_1 \, d\tau_1 \\
&\leqslant CN^{3s-\frac{23}{8}} \big\| \langle\sigma\rangle^{\frac{7}{8}-s} \widehat{\psi_\delta u} \big\|_{L^2_\xi L^2_\tau} \|u_1\|_{L^4_x L^6_t} \big\| D_x^{\frac{1}{4}} u_2 \big\|_{L^4_x L^3_t} \\
&\leqslant CN^{3s-\frac{23}{8}} \|\psi_\delta u\|_{X_{0,\frac{7}{8}-s}} \|u_1\|_{X_{0,\frac{5}{12}+}} \|u_2\|_{X_{0,\frac{1}{3}+}} \\
&\leqslant CN^{3s-\frac{23}{8}} \delta^{(\frac{3}{8}+s-b)-} \|u\|_{X_{0,1-b}} \|u_1\|_{X_{0,b}} \|u_2\|_{X_{0,b}}.
\end{aligned}$$

If (b) or (c) holds, the argument is similar to the case (a). The proof of Theorem 2.2 is complete. □

Theorem 2.3. *Assume* $0 \leqslant s < 1$, $1/2 < b \leqslant (2+s)/3$ *and* $0 < \delta < 1$, *then it holds that*

$$\big\| I\partial_x(u_1 u_2) \big\|_{X^\delta_{1,b-1}} \leqslant C\delta^{1-b} \|Iu_1\|_{X^\delta_{1,b}} \|Iu_2\|_{X^\delta_{1,b}}. \tag{2.12}$$

The proof of Theorem 2.3 is similar to that of Theorem 2.2 and so is omitted.

3. Local well-posedness result

In this section, we aim to present the local well-posedness result, especially focus our attention on the relationship between the lifespan of the local solution and the initial data.

Let $3/8 < s < 1$, $1/2 < b \leqslant \min\{(2+s)/3, s+1/8\}$, $0 < \delta < 1$. For $u_0 \in H^s(\mathbf{R})$, we define a ball in $X^\delta_{s,b}$

$$\mathscr{B} = \big\{ u \in X^\delta_{s,b}\colon\ Iu \in X^\delta_{1,b},\ \|Iu\|_{X^\delta_{1,b}} \leqslant 2C\|Iu_0\|_{H^1} \big\} \tag{3.1}$$

and a mapping

$$S(u) = W(t)u_0 - \int\limits_0^t W\big(t-t'\big) B(u,u)\big(t'\big) \, dt', \quad u \in \mathscr{B}, \tag{3.2}$$

where

$$B(u,u)=\frac{1}{2}\partial_x u^2+\left(1-\partial_x^2\right)^{-1}\partial_x\left[u^2+\frac{1}{2}(\partial_x u)^2\right].$$

For $u\in\mathscr{B}$, by using Theorems 2.1–2.3, we obtain

$$\begin{aligned}\left\|IS(u)\right\|_{X_{1,b}^{\delta}}&\leqslant C\|Iu_0\|_{H^1}+C\left(\delta^{1-b}+\delta^{\alpha-}N^{\beta}\right)\|Iu\|_{X_{1,b}^{\delta}}^2\\&\leqslant C\|Iu_0\|_{H^1}+C\left(\delta^{1-b}+\delta^{\alpha-}N^{\beta}\right)\left(2C\|Iu_0\|_{H^1}\right)^2,\end{aligned}$$

where α,β are defined as in Theorem 2.2. By choosing N suitable large and

$$\delta\sim\|Iu_0\|_{H^1}^{-\frac{1}{1-b}},\tag{3.3}$$

we have

$$C\delta^{1-b}\left(2C\|Iu_0\|_{H^1}\right)^2\leqslant\frac{1}{2}C\|Iu_0\|_{H^1};\tag{3.4}$$

$$C\delta^{\alpha-}N^{\beta}\left(2C\|Iu_0\|_{H^1}\right)^2\leqslant\frac{1}{2}C\|Iu_0\|_{H^1},\tag{3.5}$$

and thus

$$S(\mathscr{B})\subseteq\mathscr{B}.$$

In an analogous way, for $u,v\in\mathscr{B}$, by (3.4) and (3.5), we obtain

$$\begin{aligned}&\left\|IS(u)-IS(v)\right\|_{X_{1,b}^{\delta}}\\&\quad\leqslant C\left(\delta^{1-b}+\delta^{\alpha-}N^{\beta}\right)\left(\|Iu\|_{X_{1,b}^{\delta}}+\|Iv\|_{X_{1,b}^{\delta}}\right)\|Iu-Iv\|_{X_{1,b}^{\delta}}\\&\quad\leqslant\frac{1}{2}\|Iu-Iv\|_{X_{1,b}^{\delta}}.\end{aligned}$$

Therefore, S is a contraction mapping on $\mathscr{B}$. The fixed point u is the unique local solution to the Cauchy problem (1.1)–(1.2) in $X_{s,b}^{\delta}$, and furthermore

$$\|Iu\|_{X_{1,b}^{\delta}}\leqslant C\|Iu_0\|_{H^1}.\tag{3.6}$$

Theorem 3.1. *Assume* $3/8<s<1$, *then the Cauchy problem* (1.1)–(1.2) *is locally well-posed for initial data* $u_0\in H^s(\mathbf{R})$. *More precisely,* (1.1)–(1.2) *possesses a unique solution* u *satisfying* $Iu\in X_{1,b}^{\delta}\subseteq C([0,\delta];H^1(\mathbf{R}))$, *and the mapping* $u_0\to u$ *belongs to* $C(H^s(\mathbf{R});X_{s,b}^{\delta})\subseteq C(H^s(\mathbf{R});C([0,\delta];H^s(\mathbf{R})))$, *where* $1/2<b\leqslant\min\{(2+s)/3,s+1/8\}$. *Moreover, the lifespan and the solution satisfy* (3.3) *and* (3.6)*.*

4. Almost conservation law

The goal of this section is to prove the almost conservation law, which indicates that the modified energy $\|Iu(t)\|_{H^1}$ increases slowly provided that N is large and δ is small.

Assume u is the solution to the Cauchy problem (1.1)–(1.2) on $[0,\delta]$ in the sense of Theorem 3.1. Acting (1.4) with operator I, and then multiplying with $2(1-\partial_x^2)Iu$ and integrating in x, we obtain

$$\begin{aligned}\big\|Iu(\delta)\big\|_{H^1}^2-\|Iu_0\|_{H^1}^2 &= \int_0^\delta\int_{\mathbf{R}}\big(1-\partial_x^2\big)\partial_x(Iu)\big(Iu^2\big)\,dx\,dt\\ &\quad+2\int_0^\delta\int_{\mathbf{R}}\partial_x(Iu)\big(Iu^2\big)\,dx\,dt\\ &\quad+\int_0^\delta\int_{\mathbf{R}}\partial_x(Iu)\big(I(\partial_x u)^2\big)\,dx\,dt. \qquad (4.1)\end{aligned}$$

Noticing that

$$\int_0^\delta\int_{\mathbf{R}}\big[\big(1-\partial_x^2\big)\partial_x(Iu)(Iu)^2+\partial_x(Iu)(Iu)^2+\partial_x(Iu)(\partial_x Iu)^2\big]\,dx\,dt=0, \qquad (4.2)$$

we have

$$\begin{aligned}\big\|Iu(\delta)\big\|_{H^1}^2-\|Iu_0\|_{H^1}^2 &= \int_0^\delta\int_{\mathbf{R}}\big(1-\partial_x^2\big)\partial_x(Iu)\big[Iu^2-(Iu)^2\big]\,dx\,dt\\ &\quad+2\int_0^\delta\int_{\mathbf{R}}\partial_x(Iu)\big[Iu^2-(Iu)^2\big]\,dx\,dt\\ &\quad+\int_0^\delta\int_{\mathbf{R}}\partial_x(Iu)\big[I(\partial_x u)^2-(\partial_x Iu)^2\big]\,dx\,dt. \qquad (4.3)\end{aligned}$$

Inserting (4.2) into (4.1) enables us to take advantage of some internal cancellation.

Theorem 4.1. *Assume $s<1$, $b\geqslant\max\{(1/2)+,7/8-s\}$ and u is the solution to the Cauchy problem (1.1)–(1.2) on $[0,\delta]$ in the sense of Theorem 3.1. Then it holds that*

$$\big|\big\|Iu(\delta)\big\|_{H^1}^2-\|Iu_0\|_{H^1}^2\big|\leqslant C\delta^{\frac14-}N^{-2b-\frac14}\|Iu\|_{X_{1,b}^\delta}^3. \qquad (4.4)$$

Proof. We estimate the three integrals in (4.3) separately, and denote them in their appearing order by J_1, J_2, J_3, respectively.

First we estimate J_1. By Plancherel identity, it suffices to show that for all $u_j\in X_{0,b}^\delta$, $j=1,2,3$,

$$\int\limits_{\substack{\xi_1+\xi_2+\xi_3=0\\ \tau_1+\tau_2+\tau_3=0}} \frac{|m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \frac{|\xi_3|\langle\xi_3\rangle}{\langle\xi_1\rangle\langle\xi_2\rangle}$$
$$\cdot \left|\hat{\tilde{u}}_1(\xi_1,\tau_1)\right|\left|\hat{\tilde{u}}_2(\xi_2,\tau_2)\right|\left|\hat{\tilde{u}}_3(\xi_3,\tau_3)\right| d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$
$$\leqslant C\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\prod_{j=1}^{3}\|\tilde{u}_j\|_{X_{0,b}}, \tag{4.5}$$

where $\tilde{u}_j \in X_{0,b}$ is an extension of u_j. Without loss of generality, we may assume that $\hat{\tilde{u}}_j$ are nonnegative. For simplicity, we drop the $\sim$ over u_j in the following. By symmetry of ξ_1 and ξ_2, and noticing that $m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)$ vanishes when $|\xi_1|, |\xi_2| \leqslant N/2$, we restrict the domain of integration to

$$A = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi_3,\tau_3)\colon\ \xi_1+\xi_2+\xi_3=0,\ \tau_1+\tau_2+\tau_3=0,\ |\xi_1| \leqslant |\xi_2|,\ |\xi_2|>N/2\right\}. \tag{4.6}$$

In order to estimate the integral, we split the domain of integration A into three regions:

$$A_1 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi_3,\tau_3)\in A\colon\ |\xi_1| \ll |\xi_2|,\ |\xi_1| \leqslant N\right\};$$
$$A_2 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi_3,\tau_3)\in A\colon\ |\xi_1| \ll |\xi_2|,\ |\xi_1| > N\right\};$$
$$A_3 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi_3,\tau_3)\in A\colon\ |\xi_1| \sim |\xi_2|\right\}. \tag{4.7}$$

We denote the integral in (4.5) restricted to these regions in their appearing order by $J_{11},\ J_{12},\ J_{13}$, respectively.

Since $\sigma_1+\sigma_2+\sigma_3 = 3\xi_1\xi_2\xi_3$, one of the following cases always occurs:

$$\text{(a) } |\sigma_1| \geqslant |\xi_1\xi_2\xi_3|; \qquad \text{(b) } |\sigma_2| \geqslant |\xi_1\xi_2\xi_3|; \qquad \text{(c) } |\sigma_3| \geqslant |\xi_1\xi_2\xi_3|. \tag{4.8}$$

1. Estimate of J_{11}. By mean value theorem, there exists $\theta \in (0,1)$, such that

$$m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2) = m'(\theta\xi_1+\xi_2)\xi_1.$$

In the region of A_1, $|\theta\xi_1+\xi_2| \sim |\xi_2|$, thus

$$\frac{|m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} = \frac{|m(\xi_1+\xi_2)-m(\xi_2)|}{m(\xi_2)}$$
$$= \frac{|m'(\theta\xi_1+\xi_2)||\xi_1|}{m(\xi_2)} \leqslant C\frac{|\xi_1|}{|\xi_2|}. \tag{4.9}$$

If (a) holds, by Plancherel identity and (2.2), (2.3), (2.6), the integral J_{11} restricted to this domain is dominated by

$$C\int\limits_{A_1} \frac{|\xi_1|}{|\xi_2|}\frac{|\xi_3|\langle\xi_3\rangle}{\langle\xi_1\rangle\langle\xi_2\rangle}\frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$
$$\leqslant C\int\limits_{A_1} \frac{|\xi_1|^{1-b}|\xi_3|^{2-b}}{\langle\xi_1\rangle|\xi_2|^{2+b}}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$

$$\leqslant CN^{-2b-\frac{1}{4}} \left\| \langle\sigma_1\rangle^b \hat{u}_1 \right\|_{L^2_\xi L^2_\tau} \left\| D_x^{\frac{1}{4}} u_2 \right\|_{L^4_x L^3_t} \|u_3\|_{L^4_x L^6_t}$$

$$\leqslant CN^{-2b-\frac{1}{4}} \|u_1\|_{X_{0,b}} \|u_2\|_{X_{0,\frac{1}{3}+}} \|u_3\|_{X_{0,\frac{5}{12}+}}$$

$$\leqslant C\delta^{\frac{1}{4}-} N^{-2b-\frac{1}{4}} \prod_{j=1}^{3} \|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a).

2. Estimate of J_{12}. In the region of $A_2, m(\xi_1+\xi_2) \sim m(\xi_2)$, and thus

$$\frac{|m(\xi_1+\xi_2) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \leqslant \frac{\max\{m(\xi_1+\xi_2), m(\xi_2)\}}{m(\xi_1)m(\xi_2)} \leqslant \frac{C}{m(\xi_1)} \leqslant C\left(\frac{|\xi_1|}{N}\right)^{1-s}. \tag{4.10}$$

If (a) holds, for $2-b \leqslant b+5/4$, by (2.2), (2.3), (2.6), in this domain the integral J_{12} is bounded by

$$C\int_{A_2} \left(\frac{|\xi_1|}{N}\right)^{1-s} \frac{|\xi_3|\langle\xi_3\rangle}{\langle\xi_1\rangle\langle\xi_2\rangle} \frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b} \langle\sigma_1\rangle^b \hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$

$$\leqslant CN^{s-1} \int_{A_2} \frac{|\xi_3|^{2-b}}{|\xi_1|^{s+b}|\xi_2|^{1+b}} \langle\sigma_1\rangle^b \hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$

$$\leqslant CN^{-3b-\frac{1}{4}} \left\| \langle\sigma_1\rangle^b \hat{u}_1 \right\|_{L^2_\xi L^2_\tau} \left\| D_x^{\frac{1}{4}} u_2 \right\|_{L^4_x L^3_t} \|u_3\|_{L^4_x L^6_t}$$

$$\leqslant CN^{-3b-\frac{1}{4}} \|u_1\|_{X_{0,b}} \|u_2\|_{X_{0,\frac{1}{3}+}} \|u_3\|_{X_{0,\frac{5}{12}+}}$$

$$\leqslant C\delta^{\frac{1}{4}-} N^{-3b-\frac{1}{4}} \prod_{j=1}^{3} \|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a).

3. Estimate of J_{13}. In the region of A_3, we have

$$\frac{|m(\xi_1+\xi_2) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \leqslant C\left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s}. \tag{4.11}$$

If (a) holds, for $2-b \leqslant 2(s+b)+1/4$, we bound the integral J_{13} restricted to this domain by

$$C\int_{A_3} \left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s} \frac{|\xi_3|\langle\xi_3\rangle}{\langle\xi_1\rangle\langle\xi_2\rangle} \frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b} \langle\sigma_1\rangle^b \hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$

$$\leqslant CN^{2(s-1)} \int_{A_3} \frac{|\xi_3|^{1-b}\langle\xi_3\rangle}{|\xi_1|^{s+b}|\xi_2|^{s+b}} \langle\sigma_1\rangle^b \hat{u}_1\hat{u}_2\hat{u}_3\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2$$

$$\leqslant CN^{-3b-\frac{1}{4}} \left\| \langle\sigma_1\rangle^b \hat{u}_1 \right\|_{L^2_\xi L^2_\tau} \left\| D_x^{\frac{1}{4}} u_2 \right\|_{L^4_x L^3_t} \|u_3\|_{L^4_x L^6_t}$$

$$\leqslant CN^{-3b-\frac{1}{4}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,\frac{1}{3}+}}\|u_3\|_{X_{0,\frac{5}{12}+}}$$

$$\leqslant C\delta^{\frac{1}{4}-}N^{-3b-\frac{1}{4}}\prod_{j=1}^{3}\|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a). Therefore we obtain

$$|J_1| \leqslant C\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\|Iu\|^3_{X^\delta_{1,b}}.$$

The estimate of J_2 can be obtained similarly as for J_1, so we only present the result and omit the details

$$|J_2| \leqslant C\delta^{\frac{1}{4}-}N^{-2b-\frac{9}{4}}\|Iu\|^3_{X^\delta_{1,b}}.$$

Next we estimate J_3. By Plancherel identity, it suffices to show that for all $u_j \in X^\delta_{0,b}$, $j = 1, 2, 3$,

$$\int\limits_{\substack{\xi_1+\xi_2+\xi_3=0\\ \tau_1+\tau_2+\tau_3=0}} \frac{|m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)}\frac{|\xi_1||\xi_2||\xi_3|}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}$$

$$\cdot \left|\hat{\tilde{u}}_1(\xi_1,\tau_1)\right|\left|\hat{\tilde{u}}_2(\xi_2,\tau_2)\right|\left|\hat{\tilde{u}}_3(\xi_3,\tau_3)\right| d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$

$$\leqslant C\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\prod_{j=1}^{3}\|\tilde{u}_j\|_{X_{0,b}}, \tag{4.12}$$

where $\tilde{u}_j \in X_{0,b}$ is an extension of u_j. We may again assume that $\hat{\tilde{u}}_j$ are nonnegative. For simplicity, we drop the $\sim$ over u_j in the following. Similarly, we restrict the domain of integration to A defined in (4.6) and split it into three regions A_1, A_2, A_3 defined in (4.7). We denote the integral in (4.12) restricted to these regions in their appearing order by J_{31}, J_{32}, J_{33}, respectively.

1. Estimate of J_{31}. If the case (a) in (4.8) holds, noticing (4.9) and by Plancherel identity and (2.2), (2.3), (2.6), we bound the integral J_{31} restricted to this domain by

$$C\int\limits_{A_1}\frac{|\xi_1|}{|\xi_2|}\frac{|\xi_1||\xi_2||\xi_3|}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$

$$\leqslant C\int\limits_{A_1}\frac{|\xi_1|^{2-b}}{\langle\xi_1\rangle|\xi_2|^{1+b}|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$

$$\leqslant CN^{-3b-\frac{1}{4}}\left\|\langle\sigma_1\rangle^b\hat{u}_1\right\|_{L^2_\xi L^2_\tau}\left\|D_x^{\frac{1}{4}}u_2\right\|_{L^4_xL^3_t}\|u_3\|_{L^4_xL^6_t}$$

$$\leqslant CN^{-3b-\frac{1}{4}}\|u_1\|_{X_{0,b}}\|u_2\|_{X_{0,\frac{1}{3}+}}\|u_3\|_{X_{0,\frac{5}{12}+}}$$

$$\leqslant C\delta^{\frac{1}{4}-}N^{-3b-\frac{1}{4}}\prod_{j=1}^{3}\|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a).

2. Estimate of J_{32}. If (a) holds, noticing (4.10), we bound the integral J_{32} restricted to this domain by

$$C\int_{A_2}\left(\frac{|\xi_1|}{N}\right)^{1-s}\frac{|\xi_1||\xi_2||\xi_3|}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$
$$\leqslant CN^{s-1}\int_{A_2}\frac{|\xi_1|^{1-s-b}}{|\xi_2|^b|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$
$$\leqslant CN^{-3b-\frac14}\left\|\langle\sigma_1\rangle^b\hat{u}_1\right\|_{L^2_\xi L^2_\tau}\left\|D_x^{\frac14}u_2\right\|_{L^4_xL^3_t}\|u_3\|_{L^4_xL^6_t}$$
$$\leqslant C\delta^{\frac14-}N^{-3b-\frac14}\prod_{j=1}^{3}\|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a).

3. Estimate of J_{33}. If (a) holds, for $2(1-s-b)\leqslant 1/4$ and noticing (4.11), we bound the integral J_{33} restricted to this domain by

$$C\int_{A_3}\left(\frac{|\xi_1|}{N}\right)^{1-s}\left(\frac{|\xi_2|}{N}\right)^{1-s}\frac{|\xi_1||\xi_2||\xi_3|}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\frac{1}{|\xi_1|^b|\xi_2|^b|\xi_3|^b}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$
$$\leqslant CN^{2(s-1)}\int_{A_3}\frac{|\xi_1|^{1-s-b}|\xi_2|^{1-s-b}|\xi_3|^{1-b}}{\langle\xi_3\rangle}\langle\sigma_1\rangle^b\hat{u}_1\hat{u}_2\hat{u}_3\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2$$
$$\leqslant CN^{-2b-\frac14}\left\|\langle\sigma_1\rangle^b\hat{u}_1\right\|_{L^2_\xi L^2_\tau}\left\|D_x^{\frac14}u_2\right\|_{L^4_xL^3_t}\|u_3\|_{L^4_xL^6_t}$$
$$\leqslant C\delta^{\frac14-}N^{-2b-\frac14}\prod_{j=1}^{3}\|u_j\|_{X_{0,b}}.$$

If (b) or (c) holds, the argument is similar to case (a). Therefore we obtain

$$|J_3|\leqslant C\delta^{\frac14-}N^{-2b-\frac14}\|Iu\|^3_{X^\delta_{1,b}}.$$

This completes the proof of Theorem 4.1. □

5. Global well-posedness result

In this section, we prove the global well-posedness for the Cauchy problem of Eq. (1.1), i.e. for any given $T\geqslant 1$, (1.1)–(1.2) are well-posed on the time interval $[0,T]$.

For given initial data $u_0\in H^s(\mathbf{R})$,

$$\|Iu_0\|_{H^1}\leqslant C_1N^{1-s}\|u_0\|_{H^s}. \tag{5.1}$$

Theorem 3.1 shows that the solution u exists on $[0,\delta]$, with

$$\delta^{-1}\sim\|Iu_0\|_{H^1}^{\frac{1}{1-b}},\quad \frac12<b\leqslant\min\left\{\frac{2+s}{3},s+\frac18\right\} \tag{5.2}$$

and

$$\|Iu\|_{X^{\delta}_{1,b}} \leqslant C_2\|Iu_0\|_{H^1} \leqslant 2C_2\|Iu_0\|_{H^1}. \tag{5.3}$$

By Theorem 4.1, we have

$$\big\|Iu(\delta)\big\|^2_{H^1} \leqslant \|Iu_0\|^2_{H^1} + C_3\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\big(2C_2\|Iu_0\|_{H^1}\big)^3.$$

As long as

$$C_3\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\big(2C_2\|Iu_0\|_{H^1}\big)^3 \leqslant 3\|Iu_0\|^2_{H^1}, \tag{5.4}$$

we have

$$\big\|Iu(\delta)\big\|_{H^1} \leqslant 2\|Iu_0\|_{H^1}. \tag{5.5}$$

Thus we may consider the Cauchy problem of Eq. (1.1) with initial time $t=\delta$ and initial data $u(\delta)$, and repeat the above process to extend the solution to time $t=2\delta$.

To extend the local solution to the time interval $[0,T]$, we should iterate the process $[T\delta^{-1}]$ times, which can be done provided

$$C_3\delta^{\frac{1}{4}-}N^{-2b-\frac{1}{4}}\big(2C_2\|Iu_0\|_{H^1}\big)^3 T\delta^{-1} \leqslant 3\|Iu_0\|^2_{H^1}. \tag{5.6}$$

By (5.1) and (5.2), it suffices

$$C_4T\|u_0\|_{H^s}^{\frac{7-4b}{4(1-b)}+}N^{-2b-\frac{1}{4}+\frac{7-4b}{4(1-b)}(1-s)+} \leqslant 1, \tag{5.7}$$

which can be satisfied by choosing N sufficiently large and demanding that

$$-2b-\frac{1}{4}+\frac{7-4b}{4(1-b)}(1-s) < 0, \tag{5.8}$$

i.e.

$$s > \frac{8b^2-11b+6}{7-4b}. \tag{5.9}$$

As the right-hand side of (5.9) achieves its minimum $(6\sqrt{10}-17)/4$ when $b=(14-3\sqrt{10})/8$, and thus $s>(6\sqrt{10}-17)/4$. More precisely, by choosing

$$N = \big[C_4T\|u_0\|_{H^s}^{\frac{7-4b}{4(1-b)}+}\big]^{\overline{\frac{1}{2b+\frac{1}{4}-\frac{7-4b}{4(1-b)}(1-s)}}+} = CT^{\frac{5-\sqrt{10}}{5}\gamma}\|u_0\|^{\gamma}_{H^s},$$

where $\gamma = [(s-(6\sqrt{10}-17)/4)^{-1}]+$, from the iteration, we have

$$\begin{aligned}\sup_{t\in[0,T]}\big\|u(t)\big\|_{H^s} &\leqslant 2\|Iu_0\|_{H^1} \leqslant 2CN^{1-s}\|u_0\|_{H^s}\\ &\leqslant CT^{\frac{5-\sqrt{10}}{5}\gamma(1-s)}\|u_0\|^{1+\gamma(1-s)}_{H^s}.\end{aligned} \tag{5.10}$$

This completes the proof of Theorem 1.1.

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